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Abstract

Efron (1965) studied the stochastic increasingness of the vector of independent random variables entering a sum, given the value of the sum. Precisely, he proved that log-concavity for the distributions of the random variables ensures that the vector becomes larger (in the sense of the usual multivariate stochastic order) when the sum is known to increase. This result is known as Efron's "monotonicity property". Under the condition that the random variables entering in the sum have density functions with bounded second derivatives, we investigate whether Efron's monotonicity property generalizes when sums involve a large number of terms to which a central-limit theorem applies.

Keywords: stochastic increasingness, stochastic dominance, log-concavity, central-limit theorem.

1 Introduction and motivation

Let $X_1, \ldots, X_n$ be independent and centered real-valued random variables, with density functions $f_1, \ldots, f_n$ and variances $\sigma_1^2, \ldots, \sigma_n^2$ respectively. Define $X_{i,n} = X_i / s_n$ where $s_n^2 = \sum_{i=1}^n \sigma_i^2$, and let

$$S_n = \sum_{i=1}^{n} X_{i,n} = \frac{1}{s_n} \sum_{i=1}^{n} X_i$$

be their sum. Efron (1965) assumed that density functions f_i are log-concave (or equivalently belong to the class of Polya frequency functions of order 2) and proved that, for any real measurable function G_n on $\mathbb{R}^n$ which is non-decreasing in each of its arguments,

$$\alpha \mapsto \mathrm{E}[G_n(X_{1,n}, \ldots, X_{n,n}) | S_n = F_{S_n}^{-1}(\alpha)]$$

is a non-decreasing function over $(0, 1)$, where $F_{S_n}^{-1}$ denotes the quantile function of S_n. This result is known as Efron's "monotonicity property". Let us make two remarks on this property. First, Efron (1965) rather considered random variables that are neither centered nor normalized (such that the variance of the sum is equal to 1), but these choices do not change the result since log-concavity is position-invariant as well as scale-invariant. Second he studied the increasingness of the function $s \mapsto \mathrm{E}[G_n(X_{1,1}, \ldots, X_{n,n}) | S_n = s]$, but this also does not change the increasingness property since $F_{S_n}^{-1}$ is a non-decreasing function.

Efron's "monotonicity property" is equivalent to say that, for any $0 < \alpha_1 \leq \alpha_2 < 1$, the random vector $(X_{i,n})_{i=1,\ldots,n} | S_n = F_{S_n}^{-1}(\alpha_2)$ has usual multivariate stochastic dominance over the random vector $(X_{i,n})_{i=1,\ldots,n} | S_n = F_{S_n}^{-1}(\alpha_1)$ as defined in Section 6.B in Shaked and Shanthikumar (2007). The usual multivariate stochastic dominance is a possible multivariate generalization of the univariate first-order stochastic dominance. Another famous generalization yields the upper orthant order that is strictly weaker than the usual multivariate stochastic order, because it considers n-variate functions of the form $G_n(x_1, \ldots, x_n) = \prod_{i=1}^n g_i(x_i)$ where the g_i's are univariate nonnegative increasing functions (see Section 6.G.1 in Shaked and Shanthikumar (2007)). For the upper orthant order, increasing the value of the sum S_n from $F_{S_n}^{-1}(\alpha_1)$ to $F_{S_n}^{-1}(\alpha_2)$ makes all terms $X_{i,n}$ larger in the sense that the conditional survival functions never cross, the surface of the conditional survival function given $S_n = F_{S_n}^{-1}(\alpha_2)$ lying entirely above the surface of the one given $S_n = F_{S_n}^{-1}(\alpha_1)$, i.e. for all $\alpha_1 \leq \alpha_2$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$,

$$\mathrm{P}[\cap_{i=1}^n \{X_{i,n} > t_i\} | S_n = F_{S_n}^{-1}(\alpha_1)] \leq \mathrm{P}[\cap_{i=1}^n \{X_{i,n} > t_i\} | S_n = F_{S_n}^{-1}(\alpha_2)]. \tag{1.1}$$

Thus the probability that all terms are large, in the sense that their exceed thresholds t_i, increases with the value $F_{S_n}^{-1}(\alpha)$ of the sum S_n. Finally note that we know from Denuit and Muller (2002, Theorem 3.2) that, for all $\alpha_1 \leq \alpha_2$,

$$\begin{aligned}
& \mathrm{E}\left[G_n(X_{1,n}, \ldots, X_{n,n}) | S_n = F_{S_n}^{-1}(\alpha_1)\right] \leq \mathrm{E}[G_n(X_{1,n}, \ldots, X_{n,n}) | S_n = F_{S_n}^{-1}(\alpha_2)] \\
& \text{for all non-decreasing function } G_n \\
\Leftrightarrow\; & \mathrm{E}\left[G_n(X_{1,n}, \ldots, X_{n,n}) | S_n = F_{S_n}^{-1}(\alpha_1)\right] \leq \mathrm{E}[G_n(X_{1,n}, \ldots, X_{n,n}) | S_n = F_{S_n}^{-1}(\alpha_2)] \\
& \text{for all infinitely differentiable function } G_n \text{ such that } \partial G_n(x) / \partial x_i > 0 \,,
\end{aligned}$$

so that we can equivalently consider the class of non-decreasing function G_n or the subclass of infinitely differentiable function G_n such that $\partial G_n(x)/\partial x_i > 0$.

When $n = 2$, Saumard and Wellner (2018) extend Efron's monotonicity property to the case of general measures on $\mathbb{R}^2$. In this paper we want to extend Efron's monotonicity property for distributions which are not log-concave but have density functions with bounded second derivatives and satisfy a central-limit theorem. Asymptotic properties derived in Denuit and Robert (2020) in the special case $G_n(x) = x_i$ suggest that Efron's monotonicity property may be generally valid in this setting. This is in contrast with sums comprising a limited number of terms where restrictive conditions must be imposed on the distribution of each term, such as log-concavity for instance. By letting the number of random variables in the sum increases, the distribution of S_n will get closer and closer to the standard Gaussian distribution which is log-concave, and we prove that the increasingness property will be satisfied for finite-dimensional vectors of the form $(X_{i_1,n}, \ldots, X_{i_d,n})$, $d \in \mathbb{N}_*$, $i_1, \ldots, i_d \in \mathbb{N}_*^d$, as n is large and for intervals of α in $(0, 1)$.

The remainder of this paper is organized as follows. Section 2 gathers the main results of the paper. We first introduce a representation formula established by Saumard and Wellner (2014) to explain why the assumption of log-concave density functions provides a natural condition for Efron's monotonicity property. We then give sufficient conditions for Efron's monotonicity property to be asymptotically valid as n is large. The proofs are gathered in Section 3.

The following notation is adopted throughout the text. For two positive functions g_1 and g_2 defined in a neighborhood of $l \in \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, we write $g_1 = o(g_2)$ provided $\lim_{x\to l} g_1(x)/g_2(x) = 0$, and $g_1 = O(g_2)$ provided $\lim_{x\to l} |g_1(x)/g_2(x)| < \infty$ without specifying l if its value is clear.

2 Main results

2.1 Efron's monotonicity property

Saumard and Wellner (2018) note that Efron's monotonicity property is equivalent to non-decreasingness in α of the functions

$$\alpha \mapsto \mathrm{E}[g_i(X_{i,n})|S_n = F_{S_n}^{-1}(\alpha)] \text{ for } i = 1, \ldots, n, \tag{2.1}$$

for every non-decreasing functions g_i, when $n = 2$. An elegant representation formula for the derivative of $\mathrm{E}[g_i(X_{i,n})|S_n = s]$ with respect to s is established in the proof of Proposition 6.7 of their survey paper, Saumard and Wellner (2014). This formula appears to be intimately connected to the problem investigated in the present paper. We use it to provide the following proposition whose proof is given in Section 3.1.

Proposition 2.1. *Let $S_n^{(-i)} = S_n - X_{i,n}$. Assume that, for $j \in \{1, \ldots, n\}\backslash\{i\}$, the density functions f_j are differentiable such that $||f_i'||_\infty < \infty$. Let g_i be a real-valued function such* *that* $\mathrm{E}[|g_i(X_{i,n})|] < \infty$. *Then, for* $\alpha \in (0,1)$,

$$\frac{\partial}{\partial \alpha}\mathrm{E}[g_i(X_{i,n})|S_n = F_{S_n}^{-1}(\alpha)] = \frac{1}{f_{S_n}\left(F_{S_n}^{-1}(\alpha)\right)}\mathrm{Cov}\left[g_i(X_{i,n}), \frac{f'_{S_n^{(-i)}}(S_n^{(-i)})}{f_{S_n^{(-i)}}(S_n^{(-i)})}\middle| S_n = F_{S_n}^{-1}(\alpha)\right]. \tag{2.2}$$

If moreover the functions g_i *are non-decreasing differentiable functions such that* $g_i'(0) > 0$, *and, for* $i = 1, \ldots, n$, *the density functions* f_i *are log-concave, then, for any* $\alpha \in (0,1)$,

$$\inf_{i=1,\ldots,n} \frac{\partial}{\partial \alpha}\mathrm{E}[g_i(X_{i,n})|S_n = F_{S_n}^{-1}(\alpha)] > 0. \tag{2.3}$$

Given $S_n = X_{i,n} + S_n^{(-i)} = F_{S_n}^{-1}(\alpha)$, the random variables $X_{i,n}$ and $S_n^{(-i)}$ are negatively related. This has been formally established by Joag-Dev and Proschan (1983) when $X_{i,n}$ and $S_n^{(-i)}$ both possess log-concave probability density functions (see Theorem 2.8 in that paper establishing that, given their sum, $X_{i,n}$ and $S_n^{(-i)}$ are negatively associated). Assuming that $S_n^{(-i)}$ possesses a log-concave density, that is, $\ln f_{S_n^{(-i)}}$ is a concave function, is equivalent to assume that $f'_{S_n^{(-i)}}/f_{S_n^{(-i)}}$ decreases. In particular, it is well-known, that, if, for $j \in \{1, \ldots, n\}\backslash\{i\}$, the density functions f_j are log-concave, the density function of $S_n^{(-i)}$ is also log-concave. Therefore, given $S_n = F_{S_n}^{-1}(\alpha)$, $g_i(X_{i,n})$ and $f'_{S_n^{(-i)}}(S_n^{(-i)})/f_{S_n^{(-i)}}(S_n^{(-i)})$ are positively related so that it is reasonable to expect that the covariance appearing in formula (2.2) is positive (see Section 3.1 for more details). Hence, formula (2.2) shows that inequality (2.3) directly follows in that case.

2.2 Efron's asymptotic monotonicity property

In this sub-section, we study the validity of the monotonicity established in Proposition 2.1 from an asymptotic point of view, i.e. as the number of terms entering the sum tends to infinity. The intuition is that (2.3) may still be valid when $S_n^{(-i)}$ is a sum of many terms obeying some central-limit theorem so that its distribution tends to the Normal one, which is known to be log-concave.

For a distribution function F (or its probability density function f), let us denote its characteristic function by

$$\phi(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}F(x), \quad t \in \mathbb{R}.$$

If f is twice-differentiable with a bounded second derivative, its characteristic function satisfies

$$|\phi(t)| \leq \frac{A}{|t|} \tag{2.4}$$

for all real t, $|t| \geq R$ for some positive constants A and R, since f has bounded variations. F belongs to the class J introduced by Smith (1953) (with $\alpha = 1$). Let us denote the characteristic functions of X_1, X_2, ... by ϕ_1, ϕ_2, ... We say that the sequence of random variables contains a smoothing subsequence if there exists a subsequence ϕ_{n_1}, ϕ_{n_2}, ... such that for some positive A^*, R^*,

$$\left|\phi_{n_i}(t)\right| \leq \frac{A^*}{|t|}, \quad j \geq 1,$$

for all real t, $|t| \geq R^*$. We may, without loss of generality, assume that $R^* > 2A^*$. Let n^* be the number of members of the smoothing subsequence in $X_1, X_2, ..., X_n$.

Let Φ denote the distribution function of the standard Normal distribution, and let φ denote the associated probability density function. The next result establishes the asymptotic property of non-decreasingness in α of the functions $\alpha \mapsto \mathrm{E}\left[g_i(X_{i,n})\middle|S_n = F_{S_n}^{-1}(\alpha)\right]$ for a family of random variables with density functions with bounded second derivatives and belonging to the Gaussian stable domain of attraction. The proof of the result is given in Section 3.2.

Proposition 2.2. *Assume that the following conditions hold true:*

- *<u>Condition A:</u> For all $i \geq 1$, f_i is twice differentiable such that $||f_i''||_\infty < \infty$.*
- *<u>Condition B:</u> There exists a smoothing subsequence such that $\underline{\lim}_{n\to\infty} n^*/s_n^2 > 0$.*
- *<u>Condition C:</u> For all $i \geq 1$, $\mathrm{E}\left[|X_i|^3\right] < \infty$, and there exist positive constants γ and Γ such that, for all n, we have*

$$s_n^2 \geq n\gamma \text{ and } \sum_{i=1}^{n} \mathrm{E}\left[|X_i|^3\right] \leq n\Gamma.$$

Assume that, for all $i \geq 1$ the function g_i is twice differentiable such that $||g_i''||_\infty < \infty$. Then, for any $\alpha \in (0,1)$,

$$\lim_{n\to\infty} s_n^2 \frac{\partial}{\partial\alpha} \mathrm{E}\left[g_i(X_{i,n})\middle|S_n = F_{S_n}^{-1}(\alpha)\right] = \frac{\sigma_i^2 g_i'(0)}{\varphi\left(\Phi^{-1}(\alpha)\right)}.$$

Moreover, if $\inf_{i\geq 0} \sigma_i^2 g_i'(0) > 0$, then, for any $\alpha \in (0,1)$,

$$\underline{\lim}_{n\to\infty} \inf_{i=1,\ldots,n} s_n^2 \frac{\partial}{\partial\alpha} \mathrm{E}\left[g_i(X_{i,n})\middle|S_n = F_{S_n}^{-1}(\alpha)\right] > 0.$$

We now establish an equivalent of Efron's monotonicity property for finite-dimensional vectors of the form $(X_{i_1,n}, \ldots, X_{i_d,n})$, $d \in \mathbb{N}_*$, $i_1, \ldots, i_d \in \mathbb{N}_*^d$, as n is large for α in $(0,1)$. The proof of the result is given in Section 3.3.

Proposition 2.3. *Assume that Conditions A, B and C of Proposition 2.2 hold true. Let $d \in \mathbb{N}_*$, $i_1, \ldots, i_d \in \mathbb{N}_*^d$. Assume that $G_d : \mathbb{R}^d \to \mathbb{R}$ is a non-decreasing and twice differentiable function such that $\sum_{i,j} ||\partial^2 G_d(x)/\partial x_i \partial x_j||_\infty < \infty$. Then, for any $\alpha \in (0,1)$,*

$$\lim_{n\to\infty} s_n^2 \frac{\partial}{\partial\alpha} \mathrm{E}\left[G_d(X_{i_1,n}, \ldots, X_{i_d,n})\middle|S_n = F_{S_n}^{-1}(\alpha)\right] = \frac{\sum_{j=1}^{d} \sigma_{i_j}^2 \partial G_d(0,\ldots,0)/\partial x_j}{\varphi\left(\Phi^{-1}(\alpha)\right)}.$$

Proposition 2.3 provides a local version of Efron's monotonicity property in that it proves the increasingness property for intervals of α included in $(0, 1)$ (by continuity) for sufficiently large n, but not for the whole interval $(0, 1)$ as in Efron (1965).

For the uniform property, it seems necessary to consider some classes of density functions. Such a class could be the class of s-concave density functions. Let

$$M_s(a, b; \theta) = \begin{cases} ((1-\theta)\, a^s + \theta b^s)^{1/s}, & s \neq 0, a, b > 0, \\ 0, & s < 0, ab = 0, \\ a^{1-\theta} b^{\theta}, & s = 0, \\ a \wedge b & s = -\infty. \end{cases}$$

A density f is called s-concave, that is, $f \in \mathcal{P}_s$, if for all $x, y \in \mathbb{R}$ and $\theta \in (0, 1)$, $f((1-\theta)x + \theta y) \geq M_s(f(x), f(y); \theta)$. This definition goes back to Avriel (1972). The function classes $\mathcal{P}_s$ are nested in s in that for every $r > 0 > s$, we have $\mathcal{P}_r \subset \mathcal{P}_0 \subset \mathcal{P}_s \subset \mathcal{P}_{-\infty}$, where $\mathcal{P}_0$ is the class of log-concave densities. For example, Student distributions with ν degrees of freedom belong to $\mathcal{P}_{-1/(\nu+1)}$ and also to $\mathcal{P}_s$ for any $s < -1/(\nu+1)$.

Uhrin (1984) studies the property of stability through convolution of s-concave densities. He proves that the convolution of an α-concave density with a β-concave density is $(1/\alpha + 1/\beta + 1)^{-1}$-concave if $\alpha + \beta \geq 0$ and $\alpha\beta/(\alpha + \beta) \geq -1$. Let us assume that f_i are β_i-concave densities, for $i \geq 1$, such that the sequence $(\alpha_n)_{n\geq 1}$ defined by $\alpha_1 = \beta_1$ and

$$\alpha_{n+1}^{-1} = \alpha_n^{-1} + \beta_n^{-1} + 1, \quad n \geq 1,$$

satisfies $\alpha_n + \beta_n \geq 0$ and $\alpha_n\beta_n/(\alpha_n + \beta_n) \geq -1$ for all $n \geq 2$. Then S_n has an α_n-concave density. If, for all $i \geq 1$, $\beta_i > 0$, it is clear that the constraints are all satisfied and that $\alpha_n \to 0$ as $n \to \infty$. But, in this case, all densities f_i are log-concave. More interestingly there exist sequences $(\beta_n)_{n\geq 1}$ for which an infinite number of β_i are negative, the sequence $(\alpha_n)_{n\geq 1}$ satisfies the previous constraints and $\alpha_n \to 0$ as $n \to \infty$. For such sequences, it should be possible to find an increasing sequence of intervals converging to $(0, 1)$ such that Efron's monotonicity property would hold over these intervals.

3 Proofs

3.1 Proof of Proposition 2.1

We first begin by a lemma whose result has been first provided in Saumard and Wellner (2014), but under different assumptions.

Lemma 3.1. *Let X and Y be two independent real-valued random variables with density functions f_X and f_Y. Assume that f_Y is differentiable such that $\|f'_Y\|_\infty < \infty$ and that g is a non-decreasing function such that* $\mathrm{E}[|g(X)|] < \infty$, *then*

$$\frac{\partial}{\partial s}\mathrm{E}[g(X)|X+Y=s] = \mathrm{Cov}\left[g(X), \frac{f'_Y(Y)}{f_Y(Y)}\Big| X+Y=s\right]. \tag{3.1}$$

Proof: Let us first note that

$$\mathrm{E}[g(X)|X+Y=s] = \frac{1}{f_{X+Y}(s)} \int_{-\infty}^{\infty} g(x) f_X(x) f_Y(s-x) \mathrm{d}x.$$

Since

$$|g(x) f_X(x) f'_Y(s-x)| \le \|f'_Y\|_\infty |g(x)| f_X(x)$$

and $\mathrm{E}[|g(X)|] < \infty$, we can interchange integration and differentiation and we obtain

$$\begin{aligned}
& \frac{\partial}{\partial s}\mathrm{E}[g(X)|X+Y=s] \\
= & \int_{-\infty}^{\infty} \frac{g(x) f_X(x) f'_Y(s-x)}{f_{X+Y}(s)} \mathrm{d}x - \frac{f'_{X+Y}(s)}{(f_{X+Y}(s))^2} \int_{-\infty}^{\infty} g(x) f_X(x) f_Y(s-x) \mathrm{d}x \\
= & \mathrm{E}\left[g(X) \frac{f'_Y(Y)}{f_Y(Y)} \Big| X+Y=s \right] - \int_{-\infty}^{\infty} \frac{f_X(x) f'_Y(s-x)}{f_{X+Y}(s)} \mathrm{d}x \int_{-\infty}^{\infty} \frac{g(x) f_X(x) f_Y(s-x)}{f_{X+Y}(s)} \mathrm{d}x \\
= & \mathrm{E}\left[g(X) \frac{f'_Y(Y)}{f_Y(Y)} \Big| X+Y=s \right] - \mathrm{E}\left[\frac{f'_Y(Y)}{f_Y(Y)} \Big| X+Y=s \right] \mathrm{E}\big[g(X) \big| X+Y=s \big]
\end{aligned}$$

which corresponds to (3.1). □

Saumard and Wellner (2014) noticed in the proof of their Proposition 6.7 that the covariance in (3.1) can be equivalently rewritten as

$$\begin{aligned}
& \frac{1}{2}\mathrm{E}\left[\big(g(X_1) - g(X_2)\big) \left(\frac{f'_Y(Y_1)}{f_Y(Y_1)} - \frac{f'_Y(Y_2)}{f_Y(Y_2)} \right) \Big| X_1 + Y_1 = s, X_2 + Y_2 = s \right] \\
= & \frac{1}{4}\mathrm{E}\left[\big(g(X_1) - g(X_2)\big) \left(\frac{f'_Y(Y_1)}{f_Y(Y_1)} - \frac{f'_Y(Y_2)}{f_Y(Y_2)} \right) \mathrm{I}[X_1 > X_2] \Big| X_1 + Y_1 = s, X_2 + Y_2 = s \right]
\end{aligned}$$

where (X_1, Y_1) and (X_2, Y_2) are independent copies of (X, Y). In the latter expression, the inequality $Y_1 < Y_2$ must hold true because $X_1 > X_2$ and the sums $X_1 + Y_1$ and $X_2 + Y_2$ are constrained to be equal to some fixed value s. If f_Y is assumed to be log-concave, the ratio f'_Y/f_Y is decreasing and the second factor in the last conditional expectation must thus be positive. This implies that $\frac{\partial}{\partial s}\mathrm{E}[g(X)|X+Y=s] \ge 0$.

To get formula (2.2), choose $X = X_{i,n}$, $Y = S_n^{(-i)}$ and differentiate the conditional expectation with respect to α where $s = F_{S_n}^{-1}(\alpha)$. To get formula (2.3), note that, for $i = 1, \dots, n$, $f_{S_n^{(-i)}}$ is log-concave since log-concavity is preserved under convolution and that $||f'_{S_n^{(-i)}}||_\infty \le \min_{j \ne i} ||f'_{X_{j,n}}||_\infty$. Since $g'_i(0) > 0$ for $i = 1, \dots, n$, the functions g_i are not constant and therefore the inequalities are strict.

3.2 Proof of Proposition 2.2

We first begin by a series of lemmas.

Lemma 3.2. *Consider a couple of real-valued random variables (Z, X) with joint probability density function $f_{(Z,X)}$. Assume that $f_{Z|X=x}$ is a differentiable function for any x such that $\sup_x ||f'_{Z|X=x}||_\infty < \infty$, and that $\mathrm{E}[|X|] < \infty$. Then, we have*

$$f_{Z+\varepsilon X}(z) = f_Z(z) - \varepsilon \mathrm{E}\left[X f'_{Z|X}(z) \right] + o(\varepsilon) \ \ \textit{as } \varepsilon \to 0.$$

Proof: Let us first note that

$$\begin{aligned} f_{Z+\varepsilon X}(z) &= \int_{-\infty}^{\infty} f_{(Z,\varepsilon X)}\left(z-u,u\right)\mathrm{d}u \\ &= \varepsilon\int_{-\infty}^{\infty} f_{(Z,\varepsilon X)}\left(z-\varepsilon x,\varepsilon x\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f_{(Z,X)}\left(z-\varepsilon x,x\right)\mathrm{d}x. \end{aligned}$$

Since

$$f_{(Z,X)}\left(z-\varepsilon x,x\right)=f_X\left(x\right)f_{Z|X=x}\left(z-\varepsilon x\right)$$

and

$$\frac{\partial}{\partial\varepsilon}f_{(Z,X)}\left(z-\varepsilon x,x\right)=-xf_X\left(x\right)f'_{Z|X=x}\left(z-\varepsilon x\right),$$

we have

$$\left|\frac{\partial}{\partial\varepsilon}f_{(Z,X)}\left(z-\varepsilon x,x\right)\right|\leq\sup_x||f'_{Z|X=x}||_\infty\left|x\right|f_X\left(x\right).$$

Moreover $\mathrm{E}[|X|]<\infty$ and we deduce that

$$\varepsilon\mapsto f_{Z+\varepsilon X}(z)$$

is a differentiable function and its first derivative is given by

$$\varepsilon\mapsto-\int_{-\infty}^{\infty}x\frac{\partial}{\partial z}f_{(Z,X)}\left(z-\varepsilon x,x\right)\mathrm{d}x.$$

By Taylor's theorem, we have

$$f_{Z+\varepsilon X}(z)=f_Z(z)-\varepsilon\mathrm{E}\left[Xf'_{Z|X}(z)\right]+o\left(\varepsilon\right)\ \text{ as }\varepsilon\to0.$$

as announced. □

Let $w=(w_1,w_2,\ldots,w_n)$ be a vector of constants and let us consider the weighted sums

$$S_w=\sum_{i=1}^n w_iX_i,$$

with distribution function F_{S_w}, probability density function f_{S_w} and quantile function $F_{S_w}^{-1}$.

Lemma 3.3. *Assume that f_1, f_2, ..., f_n are differentiable functions such that, for $i=1,\ldots,n$, $\|f_i'\|_\infty<\infty$. We have*

$$\begin{aligned} &\frac{\partial}{\partial\alpha}\mathrm{E}\left[g_i\left(w_iX_i\right)|S_w=F_{S_w}^{-1}(\alpha)\right] \\ =\ &\frac{1}{f_{S_w}^2\left(F_{S_w}^{-1}(\alpha)\right)}\Big(\mathrm{E}\left[g_i\left(w_iX_i\right)f'_{S_w|X_i}(F_{S_w}^{-1}(\alpha))\right]\Big)-\frac{f'_{S_w}(F_{S_w}^{-1}(\alpha))}{f_{S_w}(F_{S_w}^{-1}(\alpha))}\mathrm{E}[g_i\left(w_iX_i\right)f_{S_w|X_i}(F_{S_w}^{-1}(\alpha))]\Big). \end{aligned}$$

Proof: We have

$$\begin{aligned}\mathrm{E}\left[g_i\left(w_i X_i\right)|S_w = F_{S_w}^{-1}(\alpha)\right] &= \frac{1}{f_{S_w}\left(F_{S_w}^{-1}(\alpha)\right)}\mathrm{E}\left[g_i\left(w_i X_i\right) f_{\Sigma_{j\neq i} w_j X_j}\left(F_{S_w}^{-1}(\alpha) - w_i X_i\right)\right] \\ &= \frac{1}{f_{S_w}\left(F_{S_w}^{-1}(\alpha)\right)}\mathrm{E}\left[g_i\left(w_i X_i\right) f_{S_w|X_i}(F_{S_w}^{-1}(\alpha))\right].\end{aligned}$$

By differentiating with respect to α, we get

$$\begin{aligned}&\frac{\partial}{\partial\alpha}\mathrm{E}\left[g_i\left(w_i X_i\right)|S_w = F_{S_w}^{-1}(\alpha)\right] \\ = &\frac{1}{f_{S_w}^2\left(F_{S_w}^{-1}(\alpha)\right)}\Bigg(\mathrm{E}\left[g_i\left(w_i X_i\right) f'_{S_w|X_i}(F_{S_w}^{-1}(\alpha))\right]\Bigg) - \frac{f'_{S_w}(F_{S_w}^{-1}(\alpha))}{f_{S_w}(F_{S_w}^{-1}(\alpha))}\mathrm{E}[g_i\left(w_i X_i\right) f_{S_w|X_i}(F_{S_w}^{-1}(\alpha))]\Bigg)\end{aligned}$$

and the result follows. □

Lemma 3.4. *Assume that f_1, f_2, ..., f_n are twice differentiable functions such that, for $i = 1, \ldots, n$, $\|f_i''\|_\infty < \infty$. Define*

$$S_w^{(-i)} = S_w - w_i X_i = \sum_{j\neq i} w_j X_j.$$

Then,

$$\lim_{w_i\to 0}\frac{1}{w_i^2}\frac{\partial}{\partial\alpha}\mathrm{E}[w_i X_i|S_w = F_{S_w}^{-1}(\alpha)] = -\frac{\sigma_i^2}{f_{S_w^{(-i)}}(F_{S_w^{(-i)}}^{-1}(\alpha))}\frac{\partial^2}{\partial s^2}\ln f_{S_w^{(-i)}}(s)\Bigg|_{s=F_{S_w^{(-i)}}^{-1}(\alpha)}$$

holds true for any probability level $\alpha \in (0,1)$.

Proof: Since

$$\mathrm{P}\left[S_w \leq s\right] = \mathrm{P}\left[S_w^{(-i)} + w_i X_i \leq s\right] = \mathrm{E}\left[F_{S_w^{(-i)}}(s - w_i X_i)\right]$$

and $f_{S_w^{(-i)}}$ is a bounded function, we can proceed as in the proof of Lemma 3.2 and write, as $w_i \to 0$,

$$F_{S_w}(s) = F_{S_w^{(-i)}}(s) - w_i\mathrm{E}\left[X_i\right] f_{S_w^{(-i)}}(s) + o\left(w_i\right). \tag{3.2}$$

From Lemma 3.2, we also have

$$f_{S_w}(s) = f_{S_w^{(-i)}}(s) - w_i\mathrm{E}\left[X_i\right] f'_{S_w^{(-i)}}(s) + o\left(w_i\right) \tag{3.3}$$

and since $\|f_i''\|_\infty < \infty$, we derive

$$f'_{S_w}(s) = f'_{S_w^{(-i)}}(s) - w_i\mathrm{E}\left[X_i\right] f''_{S_w^{(-i)}}(s) + o\left(w_i\right). \tag{3.4}$$

Moreover $\alpha = F_{S_w}\left(F_{S_w}^{-1}(\alpha)\right)$ and we deduce from (3.2) that

$$F_{S_w}^{-1}(\alpha) = F_{S_w^{(-i)}}^{-1}(\alpha) + w_i\mathrm{E}\left[X_i\right] + o\left(w_i\right). \tag{3.5}$$

Because X_i and $S_w^{(-i)}$ are mutually independent, we can write

$$f_{S_w|X_i}(s) = f_{S_w^{(-i)}}(s - w_iX_i) \quad \text{and} \quad f'_{S_w|X_i}(s) = f'_{S_w^{(-i)}}(s - w_iX_i).$$

This yields

$$\begin{aligned}
\mathrm{E}\left[X_i f_{S_w|X_i}(s)\right] &= \mathrm{E}\left[X_i f_{S_w^{(-i)}}(s - w_iX_i)\right] = \int_{-\infty}^{\infty} x f_{S_w^{(-i)}}(s - w_ix) f_i(x)\,dx \\
\mathrm{E}\left[X_i f'_{S_w|X_i}(s)\right] &= \mathrm{E}\left[X_i f'_{S_w^{(-i)}}(s - w_iX_i)\right] = \int_{-\infty}^{\infty} x f'_{S_w^{(-i)}}(s - w_ix) f_i(x)\,dx.
\end{aligned}$$

Since

$$\begin{aligned}
\left|x^2 f'_{S_w^{(-i)}}(s - w_ix) f_i(x)\right| &\leq \left\|f'_{S_w^{(-i)}}\right\|_\infty x^2 f_i(x) \\
\left|x^2 f''_{S_w^{(-i)}}(s - w_ix) f_i(x)\right| &\leq \left\|f''_{S_w^{(-i)}}\right\|_\infty x^2 f_i(x)
\end{aligned}$$

and $\mathrm{E}\left[X_i^2\right] < \infty$, we deduce that $w_i \mapsto \mathrm{E}\left[X_i f_{S_w|X_i}(s)\right]$ and $w_i \mapsto \mathrm{E}\left[X_i f'_{S_w|X_i}(s)\right]$ are differentiable functions and that

$$\begin{aligned}
\mathrm{E}\left[X_i f_{S_w|X_i}(s)\right] &= \mathrm{E}\left[X_i f_{S_w^{(-i)}}(s)\right] - w_i \mathrm{E}\left[X_i^2 f_{S_w^{(-i)}}(s)\right] + o(w_i) \\
\mathrm{E}\left[X_i f'_{S_w|X_i}(s)\right] &= \mathrm{E}\left[X_i f'_{S_w^{(-i)}}(s)\right] - w_i \mathrm{E}\left[X_i^2 f'_{S_w^{(-i)}}(s)\right] + o(w_i).
\end{aligned}$$

It follows from (3.5) and (3.4) that

$$\begin{aligned}
\mathrm{E}\left[w_iX_i f_{S_w|X_i}(F_{S_w}^{-1}(\alpha))\right] &= \mathrm{E}\left[w_iX_i\right] f_{S_w^{(-i)}}\left(F_{S_w}^{-1}(\alpha)\right) - \mathrm{E}\left[(w_iX_i)^2\right] f'_{S_w^{(-i)}}\left(F_{S_w}^{-1}(\alpha)\right) + o\left(w_i^2\right) \\
&= \mathrm{E}\left[w_iX_i\right] f_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) - \mathrm{Var}\left[w_iX_i\right] f'_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) + o\left(w_i^2\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left[w_iX_i f'_{S_w|X_i}(F_{S_w}^{-1}(\alpha))\right] &= \mathrm{E}\left[w_iX_i\right] f'_{S_w^{(-i)}}\left(F_{S_w}^{-1}(\alpha)\right) - \mathrm{E}\left[(w_iX_i)^2\right] f''_{S_w^{(-i)}}\left(F_{S_w}^{-1}(\alpha)\right) + o\left(w_i^2\right) \\
&= \mathrm{E}\left[w_iX_i\right] f'_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) - \mathrm{Var}\left[w_iX_i\right] f''_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) + o\left(w_i^2\right).
\end{aligned}$$

Going back to (3.3), (3.4) and (3.5), we also have

$$f_{S_w}(F_{S_w}^{-1}(\alpha)) = f_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) + o(w_i)$$

and

$$f'_{S_w}(F_{S_w}^{-1}(\alpha)) = f'_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) + o(w_i).$$

These allow us to write

$$\begin{aligned}
&\mathrm{E}\left[w_iX_i | S_w = F_{S_w}^{-1}(\alpha)\right] \\
= \;& \frac{1}{f_{S_w}\left(F_{S_w}^{-1}(\alpha)\right)} w_i \int_{-\infty}^{\infty} x f_{S_w|X_i=x}\left(F_{S_w}^{-1}(\alpha)\right) f_{X_i}(x)\,\mathrm{d}x \\
= \;& \frac{1}{f_{S_w}\left(F_{S_w}^{-1}(\alpha)\right)} \mathrm{E}\left[w_iX_i f_{S_w|X_i}(F_{S_w}^{-1}(\alpha))\right] \\
= \;& \frac{\mathrm{E}\left[w_iX_i\right] f_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) - \mathrm{Var}\left[w_iX_i\right] f'_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) + o\left(w_i^2\right)}{f_{S_w^{(-i)}}\left(F_{S_w^{(-i)}}^{-1}(\alpha)\right) + o(w_i)}.
\end{aligned}$$

Now,

$$
\begin{aligned}
& \mathrm{E}\left[w_i X_i f'_{S_w|X_i}(F^{-1}_{S_w}(\alpha))\right] - f'_{S_w}(F^{-1}_{S_w}(\alpha))\mathrm{E}\left[w_i X_i | S_w = F^{-1}_{S_w}(\alpha)\right] \\
= \; & \mathrm{E}\left[w_i X_i\right] f'_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) - \mathrm{Var}\left[w_i X_i\right] f''_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) + o\left(w_i^2\right) \\
& - \frac{f'_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) + o\left(w_i\right)}{f_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) + o\left(w_i\right)} \left(\mathrm{E}\left[w_i X_i\right] f_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) - \mathrm{Var}\left[w_i X_i\right] f'_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) + o\left(w_i^2\right)\right) \\
= \; & -\mathrm{Var}\left[w_i X_i\right] \left(f''_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) - \frac{\left(f'_{S_w^{(-i)}}\left(F^{-1}_{S_w}(\alpha)\right)\right)^2}{f_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right)} \right) + o\left(w_i^2\right) \\
= \; & -\mathrm{Var}\left[w_i X_i\right] f_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) \left(\frac{\partial^2 \ln f_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right)}{\partial s^2} \right) + o\left(w_i^2\right).
\end{aligned}
$$

It then follows from Lemma 3.3 that

$$
\lim_{w_i \to 0} \frac{1}{w_i^2} \frac{\partial \mathrm{E}[w_i X_i | S_w = F^{-1}_{S_w}(\alpha)]}{\partial \alpha} = -\frac{\mathrm{Var}\left[X_i\right]}{f_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right)} \left. \frac{\partial^2 \ln f_{S_w^{(-i)}}(s)}{\partial s^2} \right|_{s = F^{-1}_{S_w^{(-i)}}(\alpha)},
$$

which ends the proof. $\square$

Lemma 3.5. *Assume that f_1, f_2, ..., f_n are twice differentiable functions such that, for $i = 1, \ldots, n$, $\|f_i''\|_\infty < \infty$.*

1. Assume that g_i is a twice differentiable function with $\|g_i''\|_\infty < \infty$. Then,

$$
\lim_{w_i \to 0} \frac{1}{w_i^2} \frac{\partial}{\partial \alpha} \mathrm{E}[g_i(w_i X_i) | S_w = F^{-1}_{S_w}(\alpha)] = -g_i'(0) \frac{\sigma_i^2}{f_{S_w^{(-i)}}(F^{-1}_{S_w^{(-i)}}(\alpha))} \left. \frac{\partial^2}{\partial s^2} \ln f_{S_w^{(-i)}}(s) \right|_{s = F^{-1}_{S_w^{(-i)}}(\alpha)} \tag{3.6}
$$

holds true for any probability level $\alpha \in (0, 1)$.

2. Let $d \in \{2, \ldots, n-1\}$. Assume that $G_d : \mathbb{R}^d \to \mathbb{R}$ is twice differentiable such that $\sum_{i,j} ||\partial^2 G_d(x) / \partial x_i \partial x_j||_\infty < \infty$. Let $i_1, \ldots, i_d \in \{1, \ldots, n\}^d$, $i_1 \neq \ldots \neq i_d$ and define

$$
S_w^{(-(i_1,\ldots,i_d))} = \sum_{j \neq i_1, \ldots, i_d} w_j X_j.
$$

Assume that $\omega = w_{i_1} = \ldots = w_{i_d}$. We have

$$
\begin{aligned}
& \lim_{\omega \to 0} \frac{1}{\omega^2} \frac{\partial}{\partial \alpha} \mathrm{E}\left[G_d\left(w_{i_1} X_{i_1}, \ldots, w_{i_d} X_{i_d}\right) | S_w = F^{-1}_{S_w}(\alpha)\right] & (3.7) \\
= \; & -\sum_{i=1}^{d} \frac{\partial G_d}{\partial x_i}(0) \frac{\sigma_i^2}{f_{S_w^{(-(i_1,\ldots,i_d))}}(F^{-1}_{S_w^{(-(i_1,\ldots,i_d))}}(\alpha))} \left. \frac{\partial^2}{\partial s^2} \ln f_{S_w^{(-(i_1,\ldots,i_d))}}(s) \right|_{s = F^{-1}_{S_w^{(-(i_1,\ldots,i_d))}}(\alpha)} & (3.8)
\end{aligned}
$$

holds true for any probability level $\alpha \in (0, 1)$.

Proof:

1. By Lemma 3.3, we have

$$\begin{aligned}
& \frac{\partial}{\partial\alpha}\mathrm{E}\left[g_i\left(w_iX_i\right)|S_w = F_{S_w}^{-1}(\alpha)\right] \\
= \ & \frac{1}{f_{S_w}^2\left(F_{S_w}^{-1}(\alpha)\right)}\Big(\mathrm{E}\left[g_i\left(w_iX_i\right)f'_{S_w|X_i}(F_{S_w}^{-1}(\alpha))\right]\Big) - \frac{f'_{S_w}(F_{S_w}^{-1}(\alpha))}{f_{S_w}(F_{S_w}^{-1}(\alpha))}\mathrm{E}[g_i\left(w_iX_i\right)f_{S_w|X_i}(F_{S_w}^{-1}(\alpha))]\Big).
\end{aligned}$$

Since g_i is a twice differentiable function with $\|g_i''\|_\infty < \infty$, we get

$$\begin{aligned}
& \frac{\partial}{\partial\alpha}\mathrm{E}\left[g_i\left(w_iX_i\right)|S_w = F_{S_w}^{-1}(\alpha)\right] \\
= \ & g_i'(0)\frac{\partial}{\partial\alpha}\mathrm{E}\left[w_iX_i|S_w = F_{S_w}^{-1}(\alpha)\right] + \frac{1}{2}g_i''(0)\frac{\partial}{\partial\alpha}\mathrm{E}\left[w_i^2X_i^2|S_w = F_{S_w}^{-1}(\alpha)\right] + o\left(w_i^2\right).
\end{aligned}$$

By the same reasoning as in the proof of Lemma 3.4, it is possible to show that

$$\frac{\partial}{\partial\alpha}\mathrm{E}\left[w_i^2X_i^2|S_w = F_{S_w}^{-1}(\alpha)\right] = o\left(w_i^2\right)$$

which is enough to conclude.

2. With the same arguments as developed in Lemma 3.3, we have

$$\begin{aligned}
& \mathrm{E}\left[G_d\left(w_{i_1}X_{i_1},\ldots,w_{i_d}X_{i_d}\right)|S_w = F_{S_w}^{-1}(\alpha)\right] \\
= \ & \frac{1}{f_{S_w}\left(F_{S_w}^{-1}(\alpha)\right)}\mathrm{E}\left[G_d\left(w_{i_1}X_{i_1},\ldots,w_{i_d}X_{i_d}\right)f_{S_w|X_{i_1},\ldots,X_{i_d}}(F_{S_w}^{-1}(\alpha))\right].
\end{aligned}$$

By differentiating with respect to α, we get

$$\begin{aligned}
& \frac{\partial}{\partial\alpha}\mathrm{E}\left[G_d\left(w_{i_1}X_{i_1},\ldots,w_{i_d}X_{i_d}\right)|S_w = F_{S_w}^{-1}(\alpha)\right] \\
= \ & \frac{1}{f_{S_w}^2\left(F_{S_w}^{-1}(\alpha)\right)}\Big(\mathrm{E}\left[G_d\left(w_{i_1}X_{i_1},\ldots,w_{i_d}X_{i_d}\right)f'_{S_w|X_{i_1},\ldots,X_{i_d}}(F_{S_w}^{-1}(\alpha))\right]\Big) \\
& -\frac{f'_{S_w}(F_{S_w}^{-1}(\alpha))}{f_{S_w}(F_{S_w}^{-1}(\alpha))}\mathrm{E}[G_d\left(w_{i_1}X_{i_1},\ldots,w_{i_d}X_{i_d}\right)f_{S_w|X_{i_1},\ldots,X_{i_d}}(F_{S_w}^{-1}(\alpha))]\Big).
\end{aligned}$$

Since G_d is a twice differentiable function with $\sum_{i,j}||\partial^2G_d\left(x\right)/\partial x_i\partial x_j||_\infty < \infty$, we get, with $\omega = w_{i_1} = \ldots = w_{i_d}$,

$$\begin{aligned}
& \frac{\partial}{\partial\alpha}\mathrm{E}\left[G_d\left(\omega X_{i_1},\ldots,\omega X_{i_d}\right)|S_w = F_{S_w}^{-1}(\alpha)\right] \\
= \ & \sum_{j=1}^{d}\frac{\partial G_d}{\partial x_j}\left(0,\ldots,0\right)\frac{\partial}{\partial\alpha}\mathrm{E}\left[\omega X_{i_j}|S_w = F_{S_w}^{-1}(\alpha)\right] \\
& +\frac{1}{2}\sum_{j,k}\frac{\partial^2G_d}{\partial x_j\partial x_k}\left(0\ldots,0\right)\frac{\partial}{\partial\alpha}\mathrm{E}\left[\omega^2X_{i_j}X_{i_k}|S_w = F_{S_w}^{-1}(\alpha)\right] + o\left(\omega^2\right).
\end{aligned}$$

By the same reasoning as in the proof of Lemma 3.4, we can show that

$$\lim_{\omega\to 0}\frac{1}{\omega^2}\frac{\partial \mathrm{E}[\omega X_i|S_w=F_{S_w}^{-1}(\alpha)]}{\partial\alpha} = -\frac{\sigma_i^2}{f_{S_w^{(-(i_1,\ldots,i_d))}}(F^{-1}_{S_w^{(-(i_1,\ldots,i_d))}}(\alpha))}\frac{\partial^2}{\partial s^2}\ln f_{S_w^{(-(i_1,\ldots,i_d))}}(s)\Big|_{s=F^{-1}_{S_w^{(-(i_1,\ldots,i_d))}}(\alpha)},$$

and that

$$\frac{\partial}{\partial\alpha}\mathrm{E}\left[\omega^2 X_{i_j}X_{i_k}|S_w=F_{S_w}^{-1}(\alpha)\right]=o\left(\omega^2\right)$$

which is enough to conclude the proof. □

Let H_n be the distribution function of the S_n. According to Lyapunov's condition and Condition C, $\lim_{n\to\infty} H_n(x)=\Phi(x)$ for all x (see e.g. Petrov (1995)). The next technical lemma proves that the two first derivatives of H_n also converge to the derivatives of Φ and provides an upper bound for the rate of convergence.

Lemma 3.6. *Assume Conditions A, B and C stated in Proposition 2.2 hold true. Then there exist positive constants C_0, C_1, and C_2 such that, for n sufficiently large, the inequalities*

$$\begin{aligned}
\left|h_n(x)-\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\right| &\le C_0 n^{-1/2}\\
\left|h_n'(x)+x\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\right| &\le C_1 n^{-1/2}\\
\left|h_n''(x)-(x^2-1)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\right| &\le C_2 n^{-1/2}
\end{aligned}$$

are all valid.

Proof: From Petrov (1956), for all real t, we have

$$\left|\phi_{n_j}(t)\right|\le\Omega(t),\ j\ge 1,\text{ where }\Omega(t)=\begin{cases}A^*/|t| & ,\ |t|\ge R^*\\ e^{-\gamma t^2} & ,\ |t|<R^*\end{cases},$$

where γ is a positive constant. The characteristic function of H_n is

$$\theta_n(t)=\prod_{j=1}^{n}\phi_j\left(\frac{t}{s_n}\right).$$

By virtue of Condition B, there exists $\mu>0$ such that for all sufficiently large n, $n^*>\mu s_n^2$. Since $|\Omega(t)|\le 1$ for all t, it follows that

$$|\theta_n(t)|\le\left(\Omega\left(\frac{t}{s_n}\right)\right)^{\mu s_n^2}.$$

Let us show that $t^2\theta_n(t)$ is absolutely integrable for all sufficiently large n. This will imply the existence of the following integrals

$$\begin{aligned}
h_n(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tx}\theta_n(t)\,\mathrm{d}t\\
h_{1,n}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(-\mathrm{i}t)\mathrm{e}^{-\mathrm{i}tx}\theta_n(t)\,\mathrm{d}t\\
h_{2,n}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}t^2\mathrm{e}^{-\mathrm{i}tx}\theta_n(t)\,\mathrm{d}t
\end{aligned}$$

and the validity of the following formulas $h_n'(x) = h_{1,n}(x)$ and $h_n''(x) = h_{2,n}(x)$.

Let us denote Liapunov's fraction by

$$L_n = \frac{1}{s_n^3} \sum_{i=1}^{n} \mathrm{E}\left[|X_i|^3\right].$$

We know from a lemma in Petrov (1956) and Condition B that the inequality $n^{-1/2} \leq L_n \leq (\Gamma/\gamma^{3/2}) n^{-1/2}$ holds true. Following Petrov (1956), we have

$$\begin{aligned} \int_{-\infty}^{\infty} t^2 \left|\theta_n(t)\right| dt &\leq \int_{-\infty}^{\infty} t^2 \left|\theta_n(t) - \mathrm{e}^{-t^2/2}\right| dt + \int_{-\infty}^{\infty} t^2 \mathrm{e}^{-t^2/2} dt \\ &\leq R_1 + R_2 + 2 \int_{-\infty}^{\infty} t^2 \mathrm{e}^{-t^2/2} dt \end{aligned}$$

where

$$R_1 = \int_{|t| \leq (24L_n)^{-1}} t^2 \left|\theta_n(t) - \mathrm{e}^{-t^2/2}\right| dt \quad R_2 = \int_{|t| > (24L_n)^{-1}} t^2 \left|\theta_n(t)\right| dt.$$

It is known (see, for instance Lemma 1 in Esseen (1945)), that if $|t| \leq (24L_n)^{-1}$, then

$$\left|\theta_n(t) - \mathrm{e}^{-t^2/2}\right| \leq c_1 L_n |t|^3 \mathrm{e}^{-t^2/4}$$

where c_1 is a positive constant. It follows that

$$R_1 \leq c_1 L_n \int_{|t| \leq (24L_n)^{-1}} |t|^5 \mathrm{e}^{-t^2/4} \mathrm{d}t \leq c_2 L_n$$

for some positive constant c_2. We have

$$\begin{aligned} R_2 &\leq \int_{|t| > (24L_n)^{-1}} t^2 \left(\Omega\left(\frac{t}{s_n}\right)\right)^{\mu s_n^2} \mathrm{d}t \\ &= s_n^3 \int_{|t| > (s_n 24 L_n)^{-1}} t^2 \left(\Omega(t)\right)^{\mu s_n^2} \mathrm{d}t. \end{aligned}$$

Let ε be a constant satisfying $0 < \varepsilon < (24\Gamma/\gamma^{1/2})^{-1}$. Obviously,

$$\frac{1}{24 s_n L_n} > \varepsilon$$

and

$$R_2 \leq s_n^3 \int_{|t| > \varepsilon} t^2 \left(\Omega(t)\right)^{\mu s_n^2} \mathrm{d}t.$$

Let us write

$$\int_{|t| > \varepsilon} t^2 \left(\Omega(t)\right)^{\mu s_n^2} \mathrm{d}t = \int_{\varepsilon < |t| < R} t^2 \left(\Omega(t)\right)^{\mu s_n^2} \mathrm{d}t + \int_{|t| \geq R} t^2 \left(\Omega(t)\right)^{\mu s_n^2} \mathrm{d}t = T_1 + T_2.$$

We have

$$T_1 \le \int_{\varepsilon<|t|<R} t^2 \mathrm{e}^{-\gamma\mu s_n^2 t^2} dt = O\left(s_n^{-p}\right)$$

for all $p > 0$. Also,

$$\begin{aligned} T_2 &\le \int_{|t|\ge R} t^2 \left(\frac{A}{|t|}\right)^{\mu s_n^2} \mathrm{d}t \\ &\le \left(\frac{A}{R}\right)^{\mu s_n^2} R^4 \int_{|t|\ge R} t^{-2} \mathrm{d}t = O\left(s_n^{-m}\right) \end{aligned}$$

for all $m > 0$. It follows that

$$R_2 \le c_3 L_n$$

for some positive constant c_3. We deduce that $t^2\theta_n(t)$ is absolutely integrable for all sufficiently large n.

Let us now consider the pointwise convergence of $h_{2,n}(x)$ to $\varphi''(x)$. We have

$$\begin{aligned} h_{2,n}(x) - (x^2-1)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} t^2 \mathrm{e}^{-\mathrm{i}tx}(\theta_n(t) - \mathrm{e}^{-t^2/2})\mathrm{d}t \\ &= \frac{1}{2\pi}(I_1 + I_2 + I_3) \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{|t|\le(24L_n)^{-1}} t^2 \mathrm{e}^{-\mathrm{i}tx}(\theta_n(t) - \mathrm{e}^{-t^2/2})\mathrm{d}t \\ I_2 &= \int_{|t|>(24L_n)^{-1}} t^2 \mathrm{e}^{-\mathrm{i}tx}\theta_n(t)\,\mathrm{d}t \\ I_3 &= -\int_{|t|>(24L_n)^{-1}} t^2 \mathrm{e}^{-\mathrm{i}tx}\mathrm{e}^{-t^2/2}\mathrm{d}t. \end{aligned}$$

We obviously have $|I_1| \le R_1 \le c_2 L_n$ and $|I_2| \le R_2 \le c_3 L_n$ for large n. Moreover

$$|I_3| \le \int_{|t|>(24L_n)^{-1}} t^2 \mathrm{e}^{-t^2/2}\mathrm{d}t \le \frac{1}{6}L_n^{-1}\mathrm{e}^{-L_n^{-1}/48} \le c_4 L_n$$

for some positive constant c_4. It follows

$$\left| h_n''(x) - (x^2-1)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2} \right| \le \frac{c_2+c_3+c_4}{2\pi} L_n.$$

Condition C then ensures that $L_n(\delta) \le (\Gamma/\gamma^{3/2})n^{-1/2}$ and

$$\left| h_n''(x) - (x^2-1)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2} \right| \le \frac{(c_2+c_3+c_4)\Gamma}{2\pi\gamma^{3/2}} n^{-1/2} = C_2 n^{-1/2}.$$

It is possible to prove in the same way that

$$\begin{aligned} \left| h_n(x) - \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \right| &\leq C_0 n^{-1/2} \\ \left| h_n'(x) + x \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \right| &\leq C_1 n^{-1/2} \end{aligned}$$

for some positive constants C_0 and C_1. This ends the proof. □

The results stated in Proposition 2.2 now follow from Lemma 3.5 1) and Lemma 3.6. First note that, by Theorem 5.4 in Petrov (1995), we have

$$\left| F_{S_n^{(-i)}}(x) - \Phi(x) \right| \leq A L_n^{(-i)}$$

for some positive constant A (that does not depend on i and n) and where

$$L_n^{(-i)} = \frac{\sum_{j \in \{1,..,n\} \setminus \{i\}} \mathrm{E}\left[|X_j|^3 \right]}{\left(\sum_{j \in \{1,..,n\} \setminus \{i\}} \sigma_j^2 \right)^{3/2}}.$$

Moreover there exists a positive constant B (that does not depend on i and n) such that, for large n, $L_n^{(-i)} \leq B n^{-1/2}$, and we deduce that, for $x \in \mathbb{R}$ and $\alpha \in (0,1)$

$$F_{S_n^{(-i)}}(x) = \Phi(x) + O\left(n^{-1/2}\right)$$

and

$$F^{-1}_{S_n^{(-i)}}(\alpha) = \Phi^{-1}(\alpha) + O\left(n^{-1/2}\right).$$

Second, notice that $\frac{\partial^2}{\partial s^2} \ln \varphi(s) = -1$, then, by Lemma 3.6, we deduce that

$$\frac{\partial^2}{\partial s^2} \ln F^{-1}_{S_n^{(-i)}}(s) \bigg|_{s = F^{-1}_{S_n^{(-i)}}(\alpha)} = 1 + O\left(n^{-1/2}\right)$$

and

$$f_{S_w^{(-i)}}\left(F^{-1}_{S_w^{(-i)}}(\alpha)\right) = \varphi\left(\Phi^{-1}(\alpha)\right) + O\left(n^{-1/2}\right)$$

where $O\left(n^{-1/2}\right)$ are uniform for $i = 1, ..., n$. Let us now choose $w_i = 1/s_n$ in Lemma 3.5 1), we deduce that

$$s_n^2 \frac{\partial}{\partial \alpha} \mathrm{E}\left[g_i(X_{i,n}) \middle| S_n = F^{-1}_{S_n}(\alpha) \right] = g_i'(0) \frac{\sigma_i^2}{\varphi\left(\Phi^{-1}(\alpha)\right)} + o(1)$$

since, as $w_i \to 0$,

$$\frac{1}{w_i^2} \frac{\partial}{\partial \alpha} \mathrm{E}[g_i(w_i X_i) | S_w = F^{-1}_{S_w}(\alpha)] = -g_i'(0) \frac{\sigma_i^2}{f_{S_w^{(-i)}}(F^{-1}_{S_w^{(-i)}}(\alpha))} \frac{\partial^2}{\partial s^2} \ln f_{S_w^{(-i)}}(s) \bigg|_{s = F^{-1}_{S_w^{(-i)}}(\alpha)} + o(1),$$

where $o(1)$ are uniform for $i = 1, ..., n$.

Since $\inf_{i \geq 0} \sigma_i^2 g_i'(0) > 0$, we finally deduce that, for any $\alpha \in (0,1)$,

$$\underline{\lim}_{n \to \infty} \inf_{i=1,\ldots,n} s_n^2 \frac{\partial}{\partial \alpha} \mathrm{E}\left[g_i(X_{i,n}) \middle| S_n = F^{-1}_{S_n}(\alpha) \right] > 0.$$

3.3 Proof of Proposition 2.3

The results stated in Proposition 3.3 follow from Lemma 3.5 2) and Lemma 3.6. Let $S_n^{(-(i_1,\ldots,i_d))} = S_n - \sum_{j=1,\ldots,d} X_{i_j,n}$. In the same way as in the previous proof, we deduce that

$$\frac{\partial^2}{\partial s^2} \ln F^{-1}_{S_n^{(-(i_1,\ldots,i_d))}}(s)\Big|_{s=F^{-1}_{S_n^{(-(i_1,\ldots,i_d))}}(\alpha)} = 1 + O\left(n^{-1/2}\right)$$

and

$$f_{S_n^{(-(i_1,\ldots,i_d))}}(F^{-1}_{S_n^{(-(i_1,\ldots,i_d))}}(\alpha)) = \varphi\left(\Phi^{-1}(\alpha)\right) + O\left(n^{-1/2}\right)$$

where $O\left(n^{-1/2}\right)$ are uniform for $i = 1, \ldots, n$. Let us now choose $w = 1/s_n$ in Lemma 3.5 2), we deduce that

$$s_n^2 \frac{\partial}{\partial \alpha} \mathrm{E}\left[G_d\left(X_{i_1,n}, \ldots, wX_{i_d,n}\right) | S_n = F^{-1}_{S_n}(\alpha)\right] = \sum_{j=1}^{d} \frac{\partial G_d}{\partial x_j}(0, \ldots, 0) \frac{\sigma^2_{i_j}}{\varphi\left(\Phi^{-1}(\alpha)\right)} + o(1).$$

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