

Efficiency and accuracy of LHLL, HLLS and Augmented Roe's scheme with energy balance on arbitrary topography

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Abstract

This paper compares three different 1D numerical schemes used to simulate transient flows in rivers. All solve the shallow-water equations in cross-sections of arbitrary shape. Two are based on the HLL scheme and one on Roe's approach. The goal of the work presented hereafter is to identify the best scheme to be extended to sediment transport in rivers. This scheme should satisfy the following conditions which are necessary to have an accurate model for sediment transport: it should, in pure hydrodynamics, (1) be well-balanced and (2) give a correct prediction of both water level and discharge even with a coarse mesh.

Tests of water at rest and steady flows in a convergent/divergent channel are used to compare those schemes.

The results revealed that HLL based schemes are unable to provide accurate results without a proper mesh refinement. The best identified scheme is, finally, the Augmented Roe's scheme with energy balance, even though it has shown some limitations in the treatment of shocks.

Keywords: Finite-volumes; 1D models; Arbitrary cross-sections; ; well-balanced

1. Introduction

Even though floods are not the deadliest natural hazards, they affect a lot of people and generate heavy damages. These damages mostly come from fast flows with intense sediment transport. These flows can cause important changes in the river morphology due to both erosion and deposition of sediments. Besides, natural floods can be aggravated by the failure of a dam or a dyke. Modeling this kind of flows presents a major difficulty, that consists of the strong influence of the irregular topographies on the computed results. Therefore, it is of paramount importance to calculate the source terms accurately.

Moreover, to study these flows, on the field, 1D models are often preferred. This is mainly due to two advantages of 1D models over 2D models. First, the data needed for a 1D model does not have to be as detailed as for 2D models. Thus, the data collection is less expensive. Then, 1D model are faster which is important in case of crisis.

An accurate model should be able to provide acceptable results for the bed level after a dam break but should also give good results in pure hydrodynamic flows. To asses the accuracy of a model in hydrodynamics, two properties should be checked: (1) the well-balanced property, meaning that water at rest stays at rest, and (2) the discharge and water should be accurately predicted in steady flows using a coarse mesh.

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Most popular numerical schemes solve an approximate local Riemann problem to compute the intercell fluxes. Among them, two families can be identified: (1) Roe-type schemes [8], solving this approximate Riemann problem exactly, and (2) HLL-type schemes [3], solving the approximate problem in an approximate way, ie. with approximate values for the waves celerities. One of the latest development in Roe's scheme for arbitrary topography is the Augmented Roe's solver with energy balance [5]. This modification of the classical Roe's scheme adds a stationary wave to represent the source terms who are computed considering the energy conservation. For HLL scheme, Petaccia et al. [7] presented a Lateralized HLL scheme for arbitrary sections. This scheme uses Nujic [6] variation for the mass flux and, also, a distribution of the source terms on the fluxes over each side of the interface according to the celerities [1][2]. Lastly, Murillo and Garcia-Navarro [4] developed a HLL based model which takes the source terms into account considering an intermediary state. This model, called HLLS, is currently only adapted to rectangular and prismatic sections.

The main goal of the work presented hereafter is, first, to extend the HLLS scheme to arbitrary topographies, then to compare its behavior with the Augmented Roe with energy balance and the Lateralized HLL schemes.

This work is presented as follow. In section 2, the governing equations and the different models are presented. Then, in section 3, these models are compared to analytical solutions for water at rest and steady flows.

2. Numerical Model

2.1. Governing equations

The one-dimensional shallow-water equations describe the mass and momentum conservation of water. Written for cross-sections of arbitrary shape, the governing equations read:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} Q = 0 \quad (1)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} + gI_1 \right) = g[I_2 + A(S_0 - S_f)] \quad (2)$$

where Q is the discharge, A the wetted area, g the gravity, S_0 the bed slope, I_1 the hydrostatic pressure (see Figure 1) defined by

$$I_1 = \int_0^h [h - \eta] b(x, \eta) d\eta \quad (3)$$

with the water depth h and the width b . I_2 represents the lateral pressures due to the longitudinal width variation.

$$I_2 = \int_0^h [h - \eta] \frac{\partial b}{\partial x} d\eta \quad (4)$$

The friction is represented using Manning formula

$$S_f = n^2 \frac{Q^2 P^{4/3}}{A^{10/3}} \quad (5)$$

with the wetted perimeter P and the roughness coefficient n . Finally, the bed slope is represented by

$$S_0 = -\frac{dz}{dx} \quad (6)$$

The system of equations (1) and (2) will be hereafter written in vector form as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S \quad (7)$$

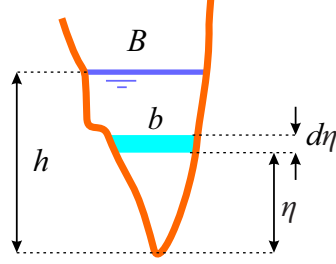


Figure 1: Definition of the hydrostatic pressure

with U the vector of conserved variables $(A, Q)^T$, F the vector of fluxes $(Q, Q^2/A + gI_1)^T$ and S the source terms $(0, g[I_2 + A(S_0 - S_f)])^T$

System (7) is discretized using a first-order finite volume scheme

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} (F_{i-1/2}^* - F_{i+1/2}^*) \quad (8)$$

The next section describes the calculation of the intercell fluxes $F_{i\pm 1/2}^*$

2.2. Fluxes computation

2.2.1. Lateralized HLL

This scheme was presented by Petaccia et al. [7] for arbitrary cross-sections. The mass flux is written considering Nujic [6] variation. The momentum flux considers a difference between the left and the right side of an interface taking into account the source terms as in [1]

2.2.2. Augmented Roe's solver with energy balance

This model was developed by Murillo and García-Navarro [5] for arbitrary cross-sections. It includes the source terms by adding a stationary wave to the system, and the source terms are computed using a combination of two possible expressions ensuring energy conservation.

2.2.3. HLLS

This model, developed by Murillo and García-Navarro [4], also includes an additional wave to represent the source terms. The approach presented by the authors considers rectangular and prismatic sections. It is here extended to arbitrary topography.

The fluxes are defined as

$$F_{i-1/2,L}^* = \frac{\lambda^+ F_{i-1} - \lambda^- F_i + \lambda^+ \lambda^- (U_i - U_{i-1}) + \lambda^- (S - \lambda^+ H)}{\lambda^+ - \lambda^-} \quad (9)$$

$$F_{i-1/2,R}^* = \frac{\lambda^+ F_{i-1} - \lambda^- F_i + \lambda^+ \lambda^- (U_i - U_{i-1}) + \lambda^+ (S - \lambda^- H)}{\lambda^+ - \lambda^-} \quad (10)$$

with the source terms written as

$$S = \begin{bmatrix} 0 \\ S_g + S_\tau \end{bmatrix} = \begin{bmatrix} 0 \\ g\bar{A}\delta z + \delta I_1 + S_\tau \end{bmatrix} \quad (11)$$

$$H = \frac{-S}{\lambda^+ \lambda^-} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (12)$$

with S_τ defined using Manning equation. The celerities used in this scheme are Roe's ones.

$$\lambda^+ = \tilde{u} + \tilde{c} \quad (13)$$

$$\lambda^- = \tilde{u} - \tilde{c} \quad (14)$$

3. Results

3.1. Water at rest

The first test focuses on the well-balanced property in a trapezoidal channel with varying bed level and width [5]. The shape of the sections is defined by

$$B(x) = B_0(x) + 0.2(B_0(x) - 9.5)h \quad (15)$$

with B_0 the width of the channel at $h = 0$. It is illustrated, along with the bed elevation, in Figure 2. As can be seen in Figure 3, all three schemes are able to maintain the water at rest even with a coarse mesh.

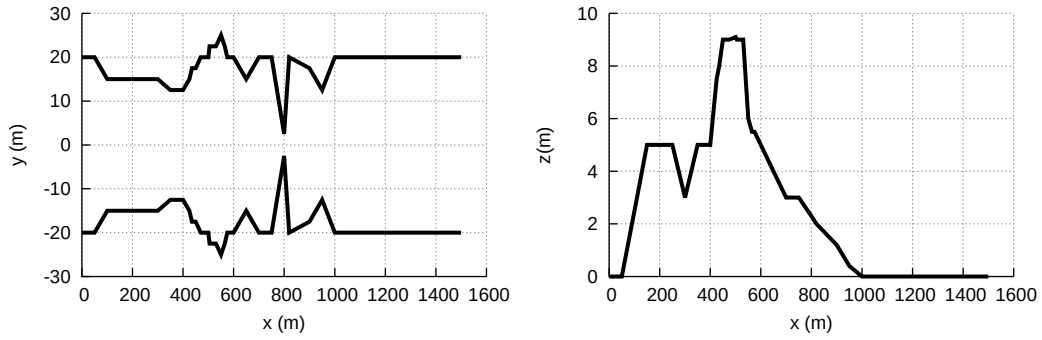


Figure 2: Top view and bed elevation

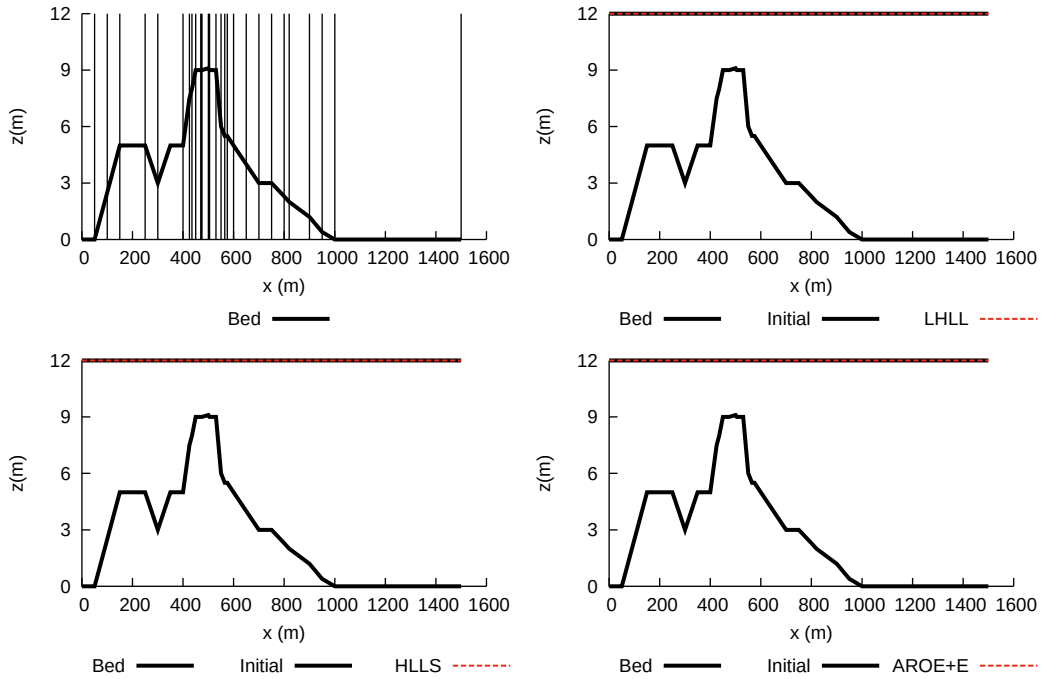


Figure 3: Mesh refinement (top left), water level after 1000 s for the LHLL scheme (top right), HLLS scheme (bottom left) and Augmented Roe's scheme with energy balance (bottom right)

3.2. Steady flow in a convergent-divergent channel

Test cases of a flow in a convergent-divergent channel are interesting to assess the behavior of numerical schemes as it can be compared to analytical solutions. In the present case, the domain is frictionless, 25 m long. The bed is horizontal and the width (see Figure 4) is described by

$$B = \min(0.8 + 0.05(x - 10)^2, 1) \quad (16)$$

Different steady flows are obtained by changing the boundary conditions: (1) a transcritical flow without a shock, (2) a transcritical flow with a shock

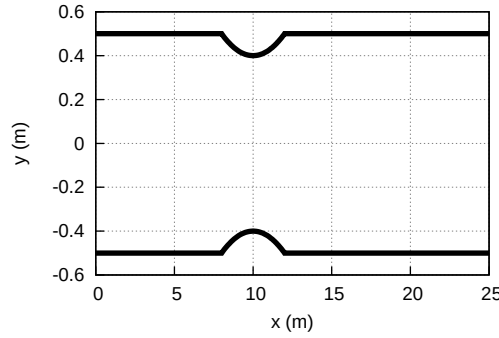


Figure 4: Top view

3.2.1. Transcritical case without shock

To consider a transcritical flow, the following boundary conditions are selected. Upstream, the discharge is imposed at $4.42 \text{ m}^3/\text{s}$. The flow being supercritical, nothing is imposed downstream. Two grid refinement are used: $dx = 0.5 \text{ m}$ and $dx = 0.05 \text{ m}$. The results are presented in Figures 5 to 7. All three schemes give accurate predictions of the water level with the finer grid refinement. The HLLS (Figure 6) and Augmented Roe's with energy balance (Figure 7) schemes are also able to conserve the discharge. However, only the Augmented Roe's with energy balance scheme predicts the exact solutions for z, q, Fr with a low grid refinement ($dx = 0.5 \text{ m}$). This phenomenon takes its roots in two characteristics of this scheme: the imposed energy conservation and the Froude number that is exactly equal to one at the narrowest section.

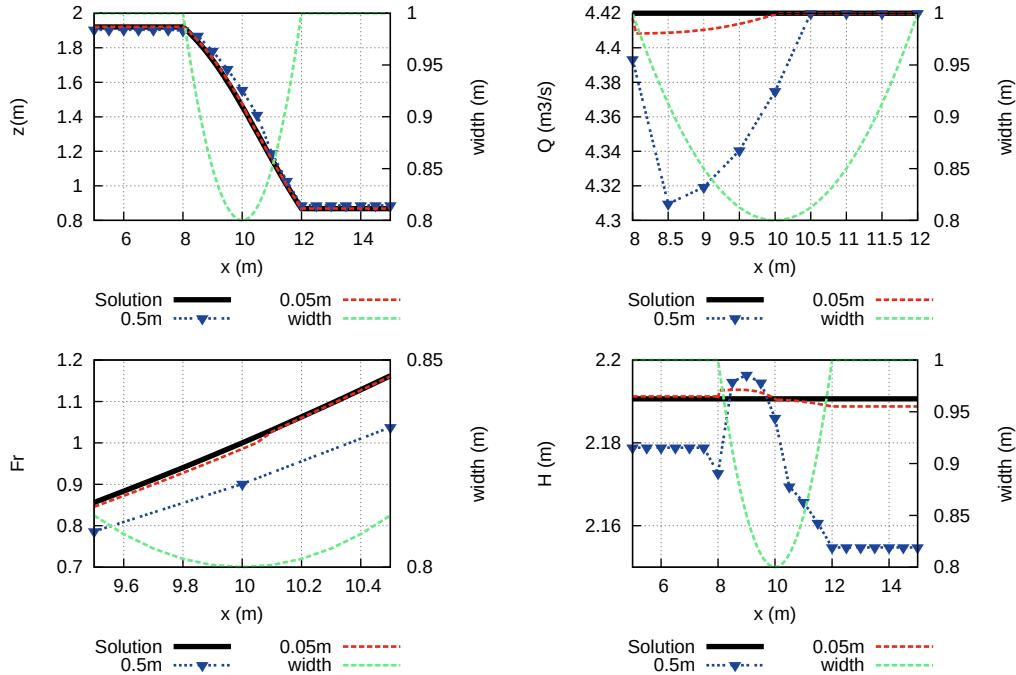


Figure 5: Transcritical case without shock: results obtained for the water level (top left), the discharge (top right), the Froude number (bottom left) and the mechanical energy (bottom right) using LHL scheme

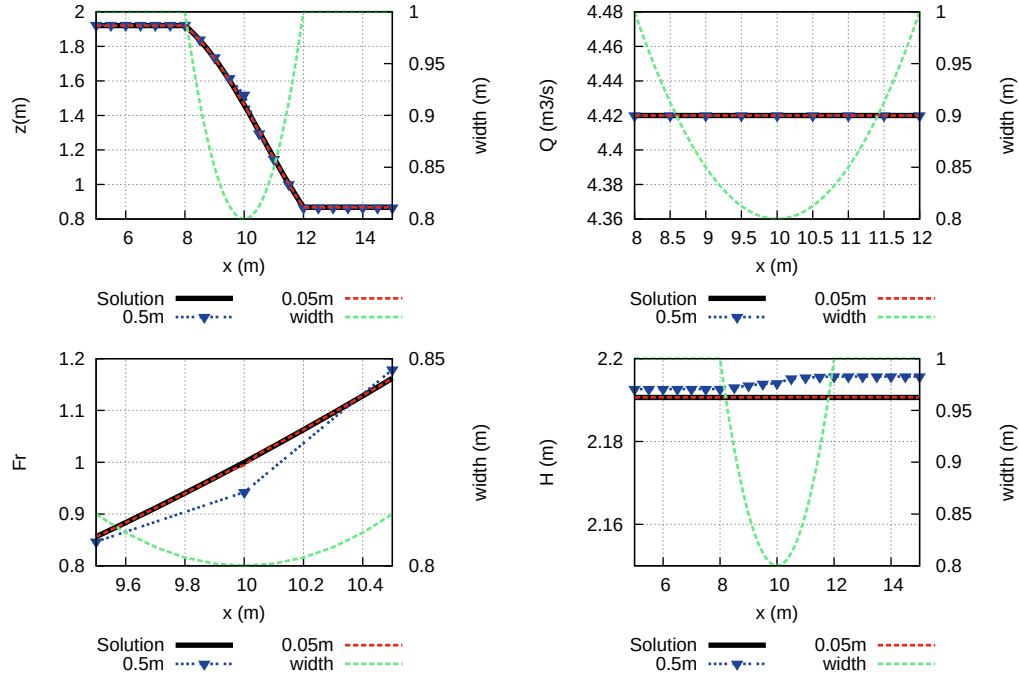


Figure 6: Transcritical case without shock: results obtained for the water level (top left), the discharge (top right), the Froude number (bottom left) and the mechanical energy (bottom right) using HLLS scheme

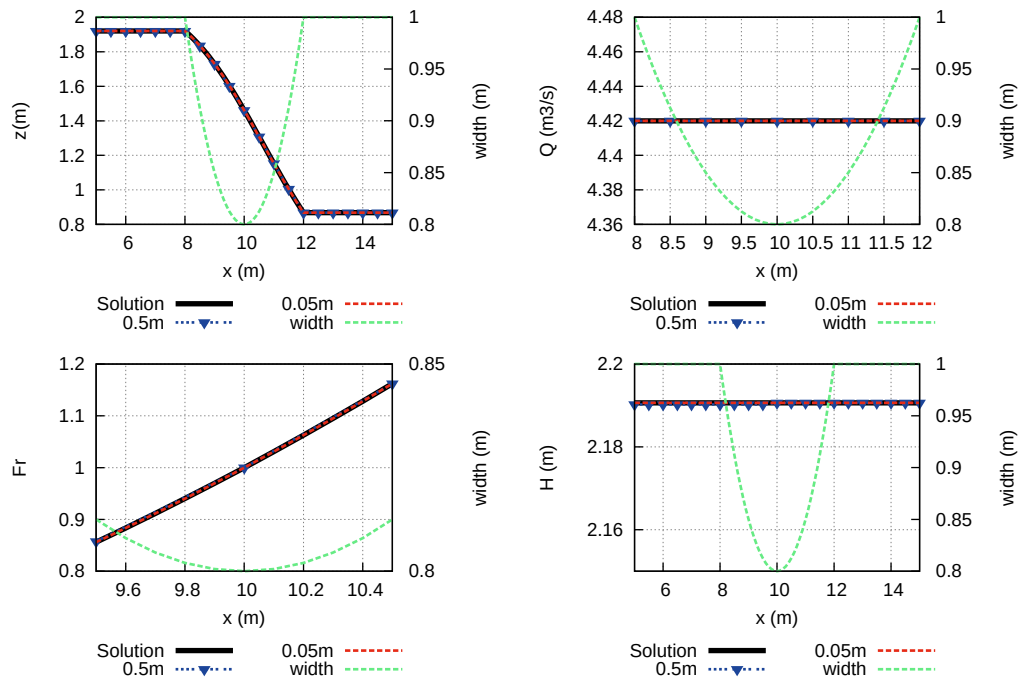


Figure 7: Transcritical case without shock: results obtained for the water level (top left), the discharge (top right), the Froude number (bottom left) and the mechanical energy (bottom right) using Augmented Roe's scheme with energy balance

3.2.2. Transcritical case with shock

This case illustrates a transcritical flow followed by a jump. Upstream, the discharge is imposed at $4.42 \text{ m}^3/\text{s}$ and, downstream, the water level is fixed at 1.9 m . The position of the jump can be computed analytically, it should be located at $x = 10.899 \text{ m}$. Two grid refinement are used: $dx = 0.025 \text{ m}$ and $dx = 0.05 \text{ m}$. The results are presented in Figures 8 to 10. All three schemes are able to globally represent the water level. However, the Augmented Roe's scheme with energy balance (Figure 10) does not predict the correct position of the jump. Considering the discharge, both HLLS (Figure 9) and Augmented Roe's scheme with energy balance (Figure 10) provide conservation in every cell except at the jump location.

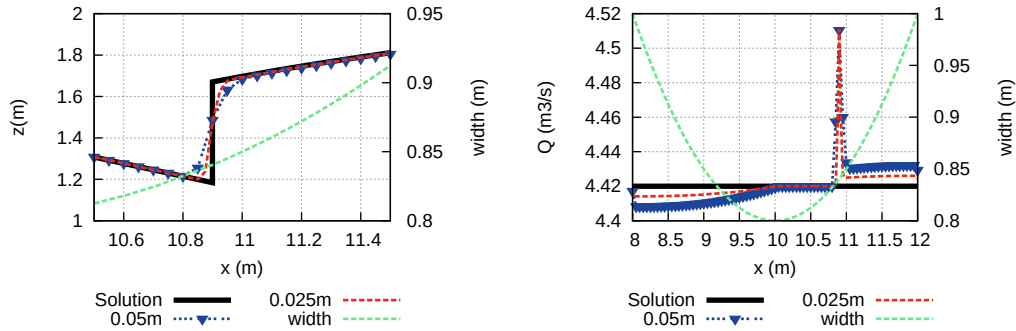


Figure 8: Transcritical case with shock: results obtained for the water level (left) and the discharge (right) using LHLL scheme

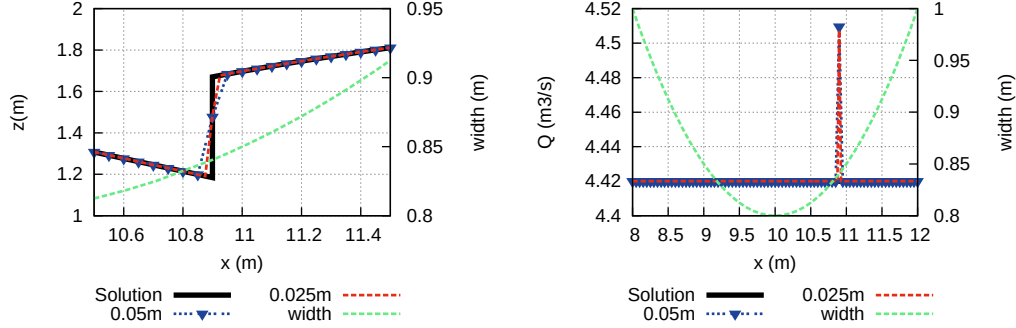


Figure 9: Transcritical case with shock: results obtained for the water level (left) and the discharge (right) using HLLS scheme

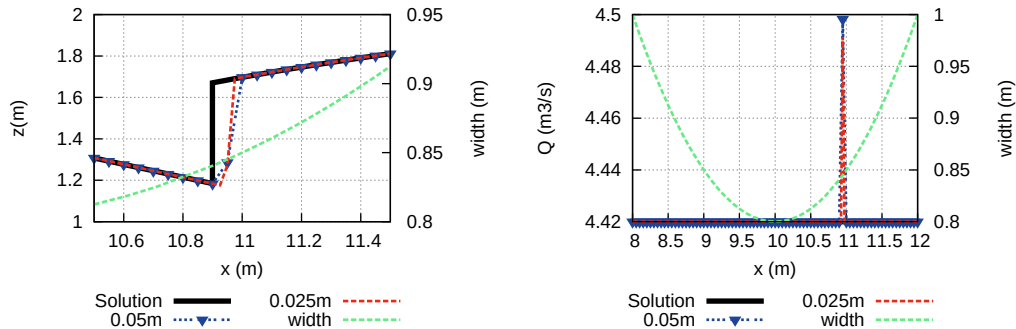


Figure 10: Transcritical case with shock: results obtained for the water level (left) and the discharge (right) using Augmented Roe's scheme with energy balance

3.3. Summary

The results are summarized in Table 1. It can be observed that the HLLS is an improvement of the LHLL scheme. Also, the Augmented Roe's scheme with energy conservation gives the best results, one of its limitations being the prediction of the position of the shock. This phenomenon can be explained when looking at the way the energy conservation is computed [5]. Two expressions of the source terms are combined using a weight coefficient. This weight is defined by the energy conservation. However, in the case of a shock, energy is dissipated. To represent that, the weight is arbitrarily set to zero. Changing the way the weight is determined in case of a shock could possibly solve this problem.

Scheme	Rest	Large dx	Discharge	Water level	Energy	Jump	$Fr = 1$
LHLL	ok			ok		ok	
HLLS	ok		ok	ok		ok	
AROE+E	ok	ok	ok	ok	ok		ok

Table 1: Summary of the capabilities of each scheme

4. Conclusions

An extension of the HLLS model to complex sections has been presented. It also has been compared to two other schemes, the LHLL and the Augmented Roe's with energy balance.

The LHLL scheme gives good results for the water level with a fine enough mesh. However, the discharge is not correctly conserved. This scheme that has the advantage of being very simple for the calculation of the wave celerities, could thus be used for cases where the water level need to be known and where an accurate prediction of the discharge is not mandatory. This excludes all cases with sediment transport as the transport capacity is determined as a function of the discharge.

The Augmented Roe's scheme with energy balance provides promising results for both the water level and the discharge. It is also capable of giving a correct value of the Froude number in the narrowest section. This, combined with the energy balance property, allows the model to give accurate results even with a coarse mesh. Still, this model currently suffers one limitation: it is not able to predict the correct position of the jump.

The HLLS model shows one clear improvement with respect to the LHLL: the conservation of the discharge. Nevertheless, its incapacity to give the exact solution, for example a Froude number equal to one, makes it less precise than the Augmented Roe's scheme with energy balance. Moreover, adding the energy balance of the Augmented Roe's scheme to the HLLS scheme would not further improve the results. It was shown that the HLLS does not provide the exact Froude number and the energy balance technique requires Froude number calculations.

To be used for sediment transport modeling in rivers, a numerical scheme should, in hydrodynamics, give a correct prediction of the water level and the discharge. Moreover, the mesh size that can be provided for real rivers is often quite coarse, due to the limited number of available cross-sections. Obtaining the data to establish a fine mesh would often be too expensive. So, HLL based schemes currently appear not to fulfill all these conditions. It is thus better to use the Augmented Roe's scheme with energy balance as a basis to develop a sediment transport model. Future works should focus on this extension but, also, pay a closer attention to the position of the jump.

5. Acknowledgments

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