

LOCAL FARSIGHTEDNESS IN NETWORK FORMATION

Pierre de Callatay, Ana Mauleon, Vincent
Vannetelbosch

REPRINT | 3285

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Local farsightedness in network formation

Pierre de Callatay¹ | Ana Mauleon¹  | Vincent Vannetelbosch²

¹CORE/LIDAM and CEREC,
UCLouvain, Louvain-la-Neuve, Belgium

²CORE/LIDAM, UCLouvain, Louvain-la-Neuve, Belgium

Correspondence

Ana Mauleon, CORE/LIDAM,
UCLouvain, 1348 Louvain-la-Neuve,
Belgium.
Email: ana.mauleon@uclouvain.be

Funding information

Fonds De La Recherche Scientifique—
FNRS, Grant/Award Number: T.0143.18

Abstract

We propose the concept of local- k farsighted consistent network for analyzing network formation games where players only consider a limited number of feasible networks. A network g is said to be local- k farsightedly consistent if, for any network g' within the distance- k neighborhood of g , either g is not defeated by g' , or g defeats g' . We show that if the utility function is (componentwise) egalitarian or satisfies reversibility or excludes externalities across components, then local- k farsightedness is more likely to be a good proxy for what would happen when players have full knowledge of all feasible networks.

KEYWORDS

local farsightedness, networks, stability

JEL CLASSIFICATION

A14, C70, D20

1 | INTRODUCTION

Many social and economic networks do not emerge randomly but are rather the outcome of decisions or choices made by individuals or agents (e.g., R&D networks, trade networks, criminal networks, etc.). Alternative solution concepts for predicting which network structures are likely to emerge in the long run have been proposed in the literature depending on whether and how far agents anticipate that their action may also induce others to change the network relations they maintain.¹ On the one

¹See Goyal (2007) and Jackson (2008) for an introduction to social and economic networks. Mauleon and Vannetelbosch (2016) provide a comprehensive overview of the solution concepts for solving network formation games.

hand, pairwise stability (Jackson & Wolinsky, 1996) involves fully myopic players in the sense that they do not anticipate that others might react to their actions. Players are able to modify the network one link at a time, and choose to change the network if the resulting network implies higher payoffs for the deviating agents. On the other hand, farsighted pairwise stability (Jackson, 2008) involves perfectly farsighted players. Players fully anticipate the complete sequence of reactions that results from their own actions in the network.²

But, there are many network formation games where a farsightedly pairwise stable network fails to exist. In fact, requiring a network to be immune to any profitable farsighted deviation is very demanding. One solution is to adopt set-valued concepts like the vNM farsighted stable set (Herings et al., 2009; Mauleon et al., 2011; Ray & Vohra, 2015), the largest consistent set (Chwe, 1994; Page et al., 2005), the DEM farsighted stable set (Herings et al., 2009) or the farsighted absorbing set (de Callatay et al., 2023). Another solution is to compare networks by comparing their sets of farsighted defeating networks (de Callatay et al., 2024). A network is minimally farsighted unstable if there is no other network which is more farsightedly stable.³ However, both approaches are quite demanding in terms of computational complexity since they require that players have full knowledge of all feasible networks and their associated payoffs.

In this paper, we consider network formation games where players only consider a limited number of feasible networks. A network is locally farsighted stable if it is not farsightedly defeated by some network in its neighborhood. Precisely, a network g is said to be local- k farsightedly stable if there is no network within the distance- k neighborhood of g that farsightedly defeats g . A network farsightedly defeats another network if there is a farsighted outgoing improving path from the latter to the former network. A farsighted outgoing improving path from g to g' is a sequence of networks within the distance- k neighborhood of g that can emerge when players form or cut links based on the improvement the end network offers relative to the current network. The distance- k neighborhood of g consists of all networks that are at most at a distance k of g . The distance between g and g' is simply the number of links that g does have while g' does not, plus the number of links that g does not have while g' does. Similarly to farsighted pairwise stability (Jackson, 2008), a local- k farsightedly stable network often fails to exist.⁴

Hence, we propose the notion of local- k farsightedly consistent network. Local farsighted consistency requires that if a network g is farsightedly defeated by some network g' in its neighborhood then g' is itself farsightedly defeated by g . Precisely, a network g is local- k farsightedly consistent if any farsighted deviation is deterred: either because the deviating players are worse off at the end network g' or because there is a farsighted ingoing improving path to g from the end network g' . In other words, the threat of a farsighted deviation to g' is consistently deterred by a subsequent farsighted deviation leading back to g . Local- k farsighted consistency is a coarsening of local- k farsighted stability.

Our next objective is to question the robustness of the predictions made under local farsightedness. In general, some networks that could emerge in the long run when all players are farsighted may not be locally farsighted stable or consistent, and vice versa. Under which

²Kirchsteiger et al. (2016) provide experimental evidence suggesting that subjects are consistent with (limited) farsightedness. Most of the players anticipate a number of reactions by the other players to the actions they take themselves. See also Teteryatnikova and Tremewan (2020).

³A network is more farsightedly stable than another network if the set of farsighted defeating networks in the former is a proper subset of the set of farsighted defeating networks in the latter network.

⁴Another approach for overcoming the lack of farsighted stability is to require the consent of component members for adding or deleting links. See Diamantoudi and Xue (2003), Caulier et al. (2013a), and Caulier et al. (2013b).

circumstances is looking at local farsightedness a good proxy for what would happen when players have full knowledge of all feasible networks? On the contrary, when restricting the analysis to local farsightedness could lead to false predictions?

First, we show that, under the egalitarian utility function, the set of local- k farsightedly consistent networks is equal to the set of local- k farsightedly stable networks. In addition, strongly efficient networks are always local- k farsightedly consistent. A public good investment model illustrates this property. Second, in the case the utility function is componentwise egalitarian, we characterize a set of networks that are local- k farsightedly consistent independently of k . Under a componentwise egalitarian utility function, players belonging to the same component get the same utility and there are no externalities across components. Third, if any profitable farsighted deviation from a network g within its distance- k neighborhood is reversible, then g is local- k farsightedly consistent. We show that, the distance-based utility model of Bloch and Jackson (2007) satisfies this property, and a star network g^* is local- k farsightedly consistent if k is greater than the number of links in g^* , even though g^* is not local- k farsightedly stable. Fourth, we show that, if the utility function allows for externalities across components, then the size of the neighborhood matters much more for determining the farsighted stability of a network. To sum up, if the utility function is (componentwise) egalitarian, or satisfies reversibility, or excludes externalities across components, then local- k farsighted stability or consistency is more likely to be a good predictor for farsightedness.

Beside situations where players have limited knowledge about feasible networks, there are also situations where players typically have incomplete information about others' connections. Expectations about these connections might then influence players' decisions with whom to connect. Recently, Foerster et al. (2021) propose a solution concept for network formation games where individuals can form two types of links: public links observed by everyone and shadow links generally not observed by others.⁵ An alternative approach to local farsightedness is limited farsightedness where players only consider farsighted improving paths that consist of a limited/bounded number of pairwise/groupwise deviations (see Herings et al., 2019; Mauleon et al., 2023). Under local farsightedness the neighborhood reflects the information players have when taking their decisions and this neighborhood is invariant to pairwise or groupwise deviations. However, under limited farsightedness the horizon reflects the limited capacity of anticipating players have when considering chains of deviations and this horizon is not invariant to pairwise or groupwise deviations.

The paper is organized as follows. In Section 2 we introduce some notation and basic properties of networks. In Section 3 we define the concepts of local farsightedness. In Section 4 we look at general properties for identifying locally farsighted consistent networks. In Section 5 we consider farsighted group deviations and externalities. In Section 6 we conclude. Finally, in the appendix we investigate the relationships between our concepts of local farsightedness and set-valued concepts of farsightedness.

2 | NETWORKS

The set of players is denoted by $N = \{1, 2, \dots, n\}$, where n is the total number of players. A network g is a list of which pairs of players are linked to each other and $ij \in g$ indicates that i and j are linked under g . The complete network on the set of players $S \subseteq N$ is denoted by g^S

⁵Once players can have shadow links, some players may overestimate others' connections and hence under-connect (relative to stable networks under correct beliefs), while others may underestimate connections and hence over-connect.

and is equal to the set of all subsets of S of size 2. It follows in particular that the empty network is denoted by $g^\emptyset$. The set of all possible networks on N is denoted by $\mathcal{G}$ and consists of all subsets of g^N . The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network that results from deleting link ij from an existing network g is denoted $g - ij$. Let $N(g) = \{i | \text{there is } j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g . Let $N_i(g) = \{j \in N | ij \in g\}$ be the set of neighbors of player i in g . A star network, denoted by g_i^* , is a network such that there exists some player i (the center) who is linked to every other player $j \neq i$ (the peripherals) and that contains no other links (i.e., g is such that $N_i(g) = N \setminus \{i\}$ and $N_j(g) = \{i\}$ for all $j \in N \setminus \{i\}$). A path in a network g between i and j is a sequence of players $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g is connected if for all $i \in N$ and $j \in N \setminus \{i\}$, there exists a path in g connecting i and j . A nonempty network $h \subseteq g$ is a component of g , if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j , and for any $i \in N(h)$ and $j \in N(g) \setminus N(h)$, $ij \in g$ implies $ij \in h$. The set of components of g is denoted by $H(g)$. A component h of g is minimally connected if h has $\#N(h) - 1$ links (i.e., every pair of players in the component are connected by exactly one path). The partition of N induced by g is denoted by $\Pi(g)$, where $S \in \Pi(g)$ if and only if either there exists $h \in H(g)$ such that $S = N(h)$ or there exists $i \notin N(g)$ such that $S = \{i\}$.⁶ We denote by $S(i)$ the coalition $S \in \Pi(g)$ such that $i \in S$.

A network utility function is a mapping $u : \mathcal{G} \rightarrow \mathbb{R}^N$ that assigns to each network g a utility $u_i(g)$ for each player $i \in N$. A network $g \in \mathcal{G}$ is strongly efficient relative to u if it maximizes $\sum_{i \in N} u_i(g)$; that is, if $\sum_{i \in N} u_i(g) \geq \sum_{i \in N} u_i(g')$ for all $g' \in \mathcal{G}$. Let E be the set of strongly efficient networks. A network $g \in \mathcal{G}$ Pareto dominates a network $g' \in \mathcal{G}$ relative to u if $u_i(g) \geq u_i(g')$ for all $i \in N$, with strict inequality for at least one $i \in N$. A network $g \in \mathcal{G}$ is Pareto efficient relative to u if it is not Pareto dominated, and a network $g \in \mathcal{G}$ is Pareto dominant if it Pareto dominates any other network. A simple way to determine which networks are stable is Jackson and Wolinsky (1996) myopic notion of pairwise stability: a network g is pairwise stable with respect to u if and only if (i) for all $ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$, and (ii) for all $ij \notin g$, if $u_i(g) < u_i(g + ij)$ then $u_j(g) > u_j(g + ij)$.⁷

Let us define a notion of distance between two networks which will be crucial for defining local farsightedness. For $g, g' \subseteq g^N$, $g \neq g'$ we denote by

$$d(g, g') \equiv \#\{ij \in g^N | (ij \in g \wedge ij \notin g') \vee (ij \notin g \wedge ij \in g')\},$$

the distance between g and g' .⁸ That is, $d(g, g')$ is the number of links that g does have while g' does not, plus the number of links that g does not have while g' does. Hence,

$$1 \leq d(g, g') \leq \frac{n(n-1)}{2}.$$

For a given network g , let $\Delta(g, k) \subseteq \mathcal{G}$ be the distance- k neighborhood of g . It is the set of networks which are at a distance of at most k from g . Formally,

⁶Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

⁷Tercieux and Vannetelbosch (2006) provide a refinement of pairwise stability, p -pairwise stability, which allows them to characterize the stochastically stable networks.

⁸ $\wedge$ is the mathematical symbol for logical conjunction and $\vee$ is the mathematical symbol for logical disjunction.

$$\Delta(g, k) \equiv \{g' \in \mathcal{G} \mid d(g, g') \leq k, g' \neq g\}.$$

3 | LOCAL FARSIGHTEDNESS

3.1 | Locally farsighted stable networks

We first define the notion of local- k farsightedly stable networks. A local- k farsightedly stable network is a network g such that farsighted players do not have incentives to deviate from it looking forward towards some network within the neighborhood of g . Precisely, a network is a local- k farsightedly stable network if there is no farsighted outgoing improving path emanating out of it within its distance- k neighborhood.

A farsighted outgoing improving path from g to g' is a sequence of networks that can emerge when farsighted players form or delete links based on the improvement the end network offers relative to the current network.⁹ All networks along the sequence belong to the distance- k neighborhood of g . Since we only allow for pairwise deviations, each network in the sequence differs from the previous one in that either a new link is formed between two players or an existing link is deleted. If a link ij is deleted, then it must be that either player i or player j prefers the end network to the current network. If a link is added between player i and player j , then both player i and player j must prefer the end network to the current network.

Definition 1. A farsighted outgoing improving path $(g_1, \dots, g_L)$ from a network g to a network $g' \in \Delta(g, k)$ within the distance- k neighborhood of g is a finite sequence of networks $g_1, \dots, g_L$ with $g_1 = g$ and $g_L = g'$ such that for any $l \in \{1, \dots, L-1\}$, $g_{l+1} \in \Delta(g, k)$ and either

- (i) $g_{l+1} = g_l - ij$ for some ij such that $u_i(g_L) > u_i(g_l)$ or $u_j(g_L) > u_j(g_l)$; or
- (ii) $g_{l+1} = g_l + ij$ for some ij such that $u_i(g_L) > u_i(g_l)$ and $u_j(g_L) \geq u_j(g_l)$.

If there exists a farsighted outgoing improving path from a network g to a network g' within the distance- k neighborhood of g , then we write $g \rightarrow_k g'$. For a given network $g \in \mathcal{G}$, let $\phi_k^{\text{out}}(g) \subseteq \Delta(g, k)$ be the set of all networks within the distance- k neighborhood of g that can be reached from g by a farsighted improving path. That is,

$$\phi_k^{\text{out}}(g) = \{g' \in \Delta(g, k) \mid g \rightarrow_k g'\}.$$

In other words, $\phi_k^{\text{out}}(g)$ is the set of networks within the distance- k neighborhood of g that farsightedly defeat g .

Definition 2. A network $g \in \mathcal{G}$ is local- k farsightedly stable if $\phi_k^{\text{out}}(g) = \emptyset$.

⁹See Jackson (2008) or Herings et al. (2009) for the original definition of a farsighted improving path. Jackson and Watts (2002) define the notion of improving path in the case that all players are myopic. Herings et al. (2020) and Luo et al. (2021) extend this notion to a mixed population composed of both myopic and farsighted players.

Notice that $\phi_{k-1}^{\text{out}}(g) \subseteq \phi_k^{\text{out}}(g)$ for $k > 1$ and if $\phi_k^{\text{out}}(g) = \emptyset$ then $\phi_{k-1}^{\text{out}}(g) = \emptyset$. For $k \geq n(n-1)/2$, $\Delta(g, k)$ is simply $\mathcal{G} \setminus \{g\}$ and $\phi_k^{\text{out}}(g) = \phi_{k+1}^{\text{out}}(g) = \phi_{\infty}^{\text{out}}(g)$, where $\phi_{\infty}^{\text{out}}(g)$ gives us the set of all networks that farsightedly defeat g .

Let $P(k)$ be the set of local- k farsightedly stable networks. Notice that $P(k) \subseteq P(k-1)$. Unfortunately, $P(k)$ might turn to be empty. Since $\mathcal{G}$ is finite, there exists k' such that $P(k) = P(k+1) = P(\infty)$ for $k > k'$. Obviously, $P(1)$ is the set of pairwise stable networks (Jackson & Wolinsky, 1996) and $P(\infty)$ is the set of farsightedly pairwise stable networks (Jackson, 2008). There is no guarantee that $P(1)$ is nonempty.

Remark 1. $P(1) \supseteq P(k) \supseteq P(k+1) \supseteq P(\infty)$.

Remark 1 tells us that $P(k+1)$ refines $P(k)$. Hence, emptiness of $P(k)$ (or instability) is more likely to become a problem when k increases. We now provide an example that illustrates this fact: some networks are pairwise stable while none of them remains farsightedly stable once the distance- k neighborhood increases.

Example 1. Consider a situation where three players can form links. The payoffs they obtain from the different network configurations are given in Figure 1. For $k = 1$ we have:

$$\begin{aligned} \phi_1^{\text{out}}(g_0) &= \{g_1, g_2, g_3\}, \\ \phi_1^{\text{out}}(g_1) &= \{g_4, g_5\}, \\ \phi_1^{\text{out}}(g_2) &= \{g_4, g_6\}, \\ \phi_1^{\text{out}}(g_3) &= \{g_5, g_6\}, \\ \phi_1^{\text{out}}(g_4) &= \phi_1^{\text{out}}(g_5) = \phi_1^{\text{out}}(g_6) = \emptyset, \\ \phi_1^{\text{out}}(g_7) &= \{g_4, g_5, g_6\}. \end{aligned}$$

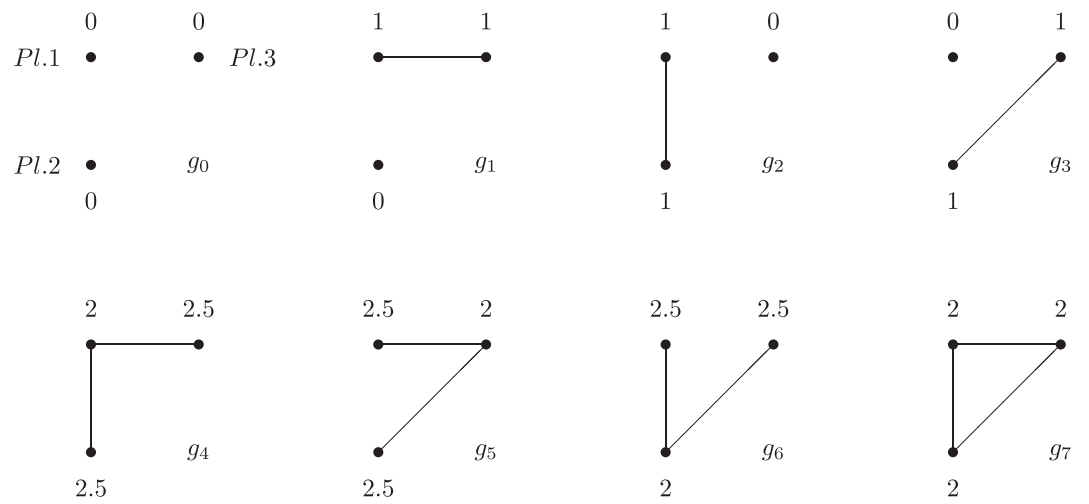


FIGURE 1 An example with three players.

Hence, $P(1) = \{g_4, g_5, g_6\}$. For $k \geq 2$ we have:

$$\begin{aligned}\phi_k^{\text{out}}(g_0) &= \{g_1, g_2, g_3, g_4, g_5, g_6\}, \\ \phi_k^{\text{out}}(g_1) &= \phi_k^{\text{out}}(g_2) = \phi_k^{\text{out}}(g_3) = \phi_k^{\text{out}}(g_7) = \{g_4, g_5, g_6\}, \\ \phi_k^{\text{out}}(g_4) &= \{g_5, g_6\}, \\ \phi_k^{\text{out}}(g_5) &= \{g_4, g_6\}, \\ \phi_k^{\text{out}}(g_6) &= \{g_4, g_5\}.\end{aligned}$$

So, there is no local- k farsightedly stable network for $k \geq 2$, that is, $P(k) = \emptyset$ for $k \geq 2$.

3.2 | Locally farsighted absorbing networks

We now define the notion of local- k farsightedly absorbing networks. A network g is a local- k farsightedly absorbing network if from any network within its distance- k neighborhood there is a farsighted ingoing improving path leading to g .

Definition 3. A farsighted ingoing improving path $(g_1, \dots, g_L)$ to a network g from a network $g' \in \Delta(g, k)$ within the distance- k neighborhood of g is a finite sequence of networks $g_1, \dots, g_L$ with $g_1 = g'$ and $g_L = g$ such that for any $l \in \{1, \dots, L-1\}$, $g_{l+1} \in \Delta(g, k)$ and either

- (i) $g_{l+1} = g_l - ij$ for some ij such that $u_i(g_L) > u_i(g_l)$ or $u_j(g_L) > u_j(g_l)$; or
- (ii) $g_{l+1} = g_l + ij$ for some ij such that $u_i(g_L) > u_i(g_l)$ and $u_j(g_L) \geq u_j(g_l)$.

If there exists a farsighted ingoing improving path from a network g' to a network g within the distance- k neighborhood of g , then we write $g \leftarrow_k g'$. For a given network $g \in \mathcal{G}$, let $\phi_k^{\text{in}}(g) \subseteq \Delta(g, k)$ be the set of all networks within the distance- k neighborhood of g from which there is a farsighted ingoing improving path leading to g . That is,

$$\phi_k^{\text{in}}(g) = \{g' \in \Delta(g, k) | g \leftarrow_k g'\}.$$

In other words, $\phi_k^{\text{in}}(g)$ is the set of networks within the distance- k neighborhood of g that are farsightedly defeated by g .

Definition 4. A network $g \in \mathcal{G}$ is local- k farsightedly absorbing if $\phi_k^{\text{in}}(g) = \Delta(g, k)$.

For $k \geq n(n-1)/2$, $\Delta(g, k)$ is simply $\mathcal{G} \setminus \{g\}$ and $\phi_k^{\text{in}}(g) = \phi_{k+1}^{\text{in}}(g) = \phi_\infty^{\text{in}}(g)$, where $\phi_\infty^{\text{in}}(g)$ gives us the set of all networks that are farsightedly defeated by g .

Let $A(k)$ be the set of local- k farsightedly absorbing networks. Obviously, $A(1) \subseteq P(1)$ since a local-1 farsightedly absorbing network is pairwise stable. Suppose now that k becomes very large. If g is local- ∞ farsightedly absorbing then any other network $g' \neq g$ cannot be

local- ∞ farsightedly stable since $\phi_{\infty}^{\text{out}}(g') \neq \emptyset$ for any $g' \neq g$. Hence, (i) if $A(\infty) = \{g\}$ then $P(\infty) \subseteq A(\infty) = \{g\}$ and (ii) if $\#A(\infty) \geq 2$ then $P(\infty) = \emptyset$.

Example 1 (Continued). We reconsider the situation where three players can form links of Figure 1. For $k = 1$ we have:

$$\begin{aligned}\phi_1^{\text{in}}(g_0) &= \phi_1^{\text{in}}(g_7) = \emptyset, \\ \phi_1^{\text{in}}(g_1) &= \phi_1^{\text{in}}(g_2) = \phi_1^{\text{in}}(g_3) = \{g_0\}, \\ \phi_1^{\text{in}}(g_4) &= \{g_1, g_2, g_7\} = \Delta(g_4, 1), \\ \phi_1^{\text{in}}(g_5) &= \{g_1, g_3, g_7\} = \Delta(g_5, 1), \\ \phi_1^{\text{in}}(g_6) &= \{g_2, g_3, g_7\} = \Delta(g_6, 1).\end{aligned}$$

Hence, we get $A(1) = \{g_4, g_5, g_6\}$. For $k = 2$ we have:

$$\begin{aligned}\phi_2^{\text{in}}(g_0) &= \phi_2^{\text{in}}(g_7) = \emptyset, \\ \phi_2^{\text{in}}(g_1) &= \phi_2^{\text{in}}(g_2) = \phi_2^{\text{in}}(g_3) = \{g_0\}, \\ \phi_2^{\text{in}}(g_4) &= \{g_0, g_1, g_2, g_5, g_6, g_7\} = \Delta(g_4, 2), \\ \phi_2^{\text{in}}(g_5) &= \{g_0, g_1, g_3, g_4, g_6, g_7\} = \Delta(g_5, 2), \\ \phi_2^{\text{in}}(g_6) &= \{g_0, g_2, g_3, g_4, g_5, g_7\} = \Delta(g_6, 2).\end{aligned}$$

Hence, we get $A(2) = \{g_4, g_5, g_6\}$. For $k \geq 3$ we have:

$$\begin{aligned}\phi_k^{\text{in}}(g_0) &= \phi_k^{\text{in}}(g_7) = \emptyset, \\ \phi_k^{\text{in}}(g_1) &= \phi_k^{\text{in}}(g_2) = \phi_k^{\text{in}}(g_3) = \{g_0\}, \\ \phi_k^{\text{in}}(g_4) &= \{g_0, g_1, g_2, g_3, g_5, g_6, g_7\} = \Delta(g_4, k) = \mathcal{G} \setminus \{g_4\}, \\ \phi_k^{\text{in}}(g_5) &= \{g_0, g_1, g_2, g_3, g_4, g_6, g_7\} = \Delta(g_5, k) = \mathcal{G} \setminus \{g_5\}, \\ \phi_k^{\text{in}}(g_6) &= \{g_0, g_1, g_2, g_3, g_4, g_5, g_7\} = \Delta(g_6, k) = \mathcal{G} \setminus \{g_6\}.\end{aligned}$$

Hence, the set of local- k farsightedly absorbing networks is given by $A(k) = \{g_4, g_5, g_6\}$ for $k \geq 1$.

We now provide an example that illustrates the nonmonotonicity of the set of local- k farsightedly absorbing networks with respect to k .

Example 2. Consider another situation where three players can form links. The payoffs they obtain from the different network configurations are given in Figure 2. For $k = 1$ we have:

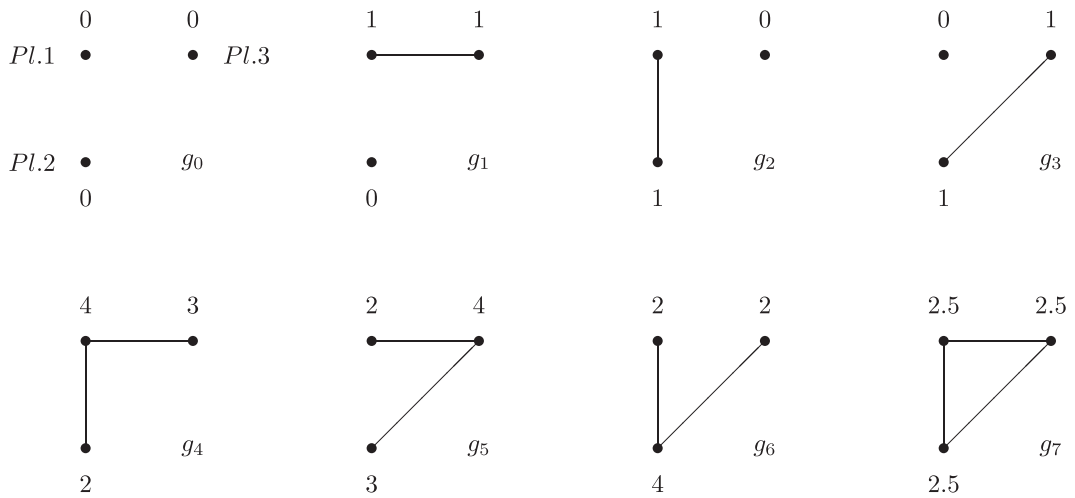


FIGURE 2 Another example with three players.

$$\begin{aligned}
 \phi_1^{\text{in}}(g_0) &= \emptyset, \\
 \phi_1^{\text{in}}(g_1) &= \phi_1^{\text{in}}(g_2) = \phi_1^{\text{in}}(g_3) = \{g_0\}, \\
 \phi_1^{\text{in}}(g_4) &= \{g_1, g_2, g_7\} = \Delta(g_4, 1), \\
 \phi_1^{\text{in}}(g_5) &= \{g_1, g_3, g_7\} = \Delta(g_5, 1), \\
 \phi_1^{\text{in}}(g_6) &= \{g_2, g_3\} \subsetneq \Delta(g_6, 1), \\
 \phi_1^{\text{in}}(g_7) &= \{g_6\} \subsetneq \Delta(g_7, 1).
 \end{aligned}$$

Hence, we get $A(1) = \{g_4, g_5\}$. For $k = 2$ we have:

$$\begin{aligned}
 \phi_2^{\text{in}}(g_0) &= \emptyset, \\
 \phi_2^{\text{in}}(g_1) &= \phi_2^{\text{in}}(g_2) = \phi_2^{\text{in}}(g_3) = \{g_0\}, \\
 \phi_2^{\text{in}}(g_4) &= \{g_0, g_1, g_2, g_6, g_7\} \subsetneq \Delta(g_4, 2), \\
 \phi_2^{\text{in}}(g_5) &= \{g_0, g_1, g_3, g_4, g_6, g_7\} = \Delta(g_5, 2), \\
 \phi_2^{\text{in}}(g_6) &= \{g_0, g_2, g_3\} \subsetneq \Delta(g_6, 2), \\
 \phi_2^{\text{in}}(g_7) &= \{g_1, g_2, g_3, g_4, g_5, g_6\} = \Delta(g_7, 2).
 \end{aligned}$$

Hence, we obtain $A(2) = \{g_5, g_7\}$. For $k \geq 3$ we have:

$$\begin{aligned}
 \phi_k^{\text{in}}(g_0) &= \emptyset, \\
 \phi_k^{\text{in}}(g_1) &= \phi_k^{\text{in}}(g_2) = \phi_k^{\text{in}}(g_3) = \{g_0\}, \\
 \phi_k^{\text{in}}(g_4) &= \{g_0, g_1, g_2, g_3, g_5, g_6, g_7\} = \Delta(g_4, k) = \mathcal{G} \setminus \{g_4\}, \\
 \phi_k^{\text{in}}(g_5) &= \{g_0, g_1, g_2, g_3, g_4, g_6, g_7\} = \Delta(g_5, k) = \mathcal{G} \setminus \{g_5\}, \\
 \phi_k^{\text{in}}(g_6) &= \{g_0, g_1, g_2, g_3, g_4, g_5, g_7\} = \Delta(g_6, k) = \mathcal{G} \setminus \{g_6\}, \\
 \phi_k^{\text{in}}(g_7) &= \{g_0, g_1, g_2, g_3, g_4, g_5, g_6\} = \Delta(g_7, k) = \mathcal{G} \setminus \{g_7\}.
 \end{aligned}$$

Hence, we get $A(k) = \{g_4, g_5, g_6, g_7\}$ for $k \geq 3$. Thus, $A(1) \not\subseteq A(2) \subseteq A(k)$ for $k \geq 3$. So, the set of local- k farsightedly absorbing networks may depend in a nonmonotonic way on the distance k .

3.3 | Locally farsighted consistent networks

We now define the notion of local- k farsightedly consistent networks. A local- k farsightedly consistent network is a network g such that any farsighted deviation from g within its distance- k neighborhood is deterred. Either because the deviating players are worse off at the end network g' or because there is a farsighted ingoing improving path to g from the end network g' . In other words, the threat of a farsighted deviation to g' is consistently deterred by a subsequent farsighted deviation leading back to g .

Definition 5. A network $g \in \mathcal{G}$ is local- k farsightedly consistent if for any $g' \in \Delta(g, k)$ either $g' \notin \phi_k^{\text{out}}(g)$ or $g' \in \phi_k^{\text{in}}(g)$.

Let $F(k)$ be the set of local- k farsightedly consistent networks. The next remark tells us that the set of local- k farsightedly consistent networks is a coarsening of both the set of local- k farsightedly stable networks and the set of local- k farsightedly absorbing networks.

Remark 2. $P(k) \subseteq F(k)$ and $A(k) \subseteq F(k)$ for all $k \geq 1$.

We now provide an example that illustrates the notion and shows the nonmonotonicity of the set of local- k farsightedly consistent networks with respect to k .

Example 2 (Continued). We reconsider the situation where three players can form links of Figure 2. For $k = 1$ we have:

$$\begin{aligned}\phi_1^{\text{out}}(g_0) &= \{g_1, g_2, g_3\}, \\ \phi_1^{\text{out}}(g_1) &= \phi_1^{\text{out}}(g_7) = \{g_4, g_5\}, \\ \phi_1^{\text{out}}(g_2) &= \{g_4, g_6\}, \\ \phi_1^{\text{out}}(g_3) &= \{g_5, g_6\}, \\ \phi_1^{\text{out}}(g_4) &= \phi_1^{\text{out}}(g_5) = \emptyset, \\ \phi_1^{\text{out}}(g_6) &= \{g_7\}.\end{aligned}$$

Hence, we get $P(1) = \{g_4, g_5\}$ and $F(1) = \{g_4, g_5\}$. For $k = 2$ we have:

$$\begin{aligned}\phi_2^{\text{out}}(g_0) &= \{g_1, g_2, g_3, g_4, g_5, g_6\}, \\ \phi_2^{\text{out}}(g_1) &= \phi_2^{\text{out}}(g_7) = \{g_4, g_5\}, \\ \phi_2^{\text{out}}(g_2) &= \phi_2^{\text{out}}(g_5) = \{g_4, g_6, g_7\}, \\ \phi_2^{\text{out}}(g_3) &= \phi_2^{\text{out}}(g_4) = \{g_5, g_6, g_7\}, \\ \phi_2^{\text{out}}(g_6) &= \{g_4, g_5, g_7\}.\end{aligned}$$

Hence, we obtain $P(2) = \emptyset$ and $F(2) = \{g_5, g_7\}$. For instance, g_5 is local-2 farsightedly consistent since farsighted deviations to g_4, g_6 and g_7 are deterred by the threat of subsequent farsighted deviations leading back to g_5 . Indeed, $\phi_2^{\text{out}}(g_5) = \{g_4, g_6, g_7\}$ while $g_5 \in \phi_2^{\text{out}}(g_4)$, $g_5 \in \phi_2^{\text{out}}(g_6)$ and $g_5 \in \phi_2^{\text{out}}(g_7)$. For $k \geq 3$ we have:

$$\begin{aligned}\phi_k^{\text{out}}(g_0) &= \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}, \\ \phi_k^{\text{out}}(g_1) &= \phi_k^{\text{out}}(g_2) = \phi_k^{\text{out}}(g_3) = \{g_4, g_5, g_6, g_7\}, \\ \phi_k^{\text{out}}(g_4) &= \{g_5, g_6, g_7\}, \\ \phi_k^{\text{out}}(g_5) &= \{g_4, g_6, g_7\}, \\ \phi_k^{\text{out}}(g_6) &= \{g_4, g_5, g_7\}, \\ \phi_k^{\text{out}}(g_7) &= \{g_4, g_5, g_6\}.\end{aligned}$$

Hence, $P(k) = \emptyset$ and $F(k) = \{g_4, g_5, g_6, g_7\}$ for $k \geq 3$. Thus, the set of local- k farsightedly consistent networks may depend in a nonmonotonic way on the distance k .

We now provide an example that illustrates the property that the set of locally farsighted consistent networks is a coarsening of the set of locally farsighted stable networks and the set of locally farsighted absorbing networks. In Figure 3 we depict the relationship between the three notions of local farsightedness.

Example 3. Consider the situation where three players can form links and where the payoffs are given in Figure 4. For $k \geq 3$ we have:

$$\begin{aligned}\phi_k^{\text{out}}(g_0) &= \phi_k^{\text{out}}(g_3) = \{g_4, g_7\} \subseteq \phi_k^{\text{in}}(g_0) = \phi_k^{\text{in}}(g_3) = \{g_1, g_2, g_4, g_5, g_6, g_7\}, \\ \phi_k^{\text{out}}(g_1) &= \phi_k^{\text{out}}(g_2) = \{g_0, g_3, g_4, g_7\} \not\subseteq \phi_k^{\text{in}}(g_1) = \phi_k^{\text{in}}(g_2) = \emptyset, \\ \phi_k^{\text{out}}(g_4) &= \phi_k^{\text{out}}(g_7) = \{g_0, g_3\} \subseteq \phi_k^{\text{in}}(g_4) = \phi_k^{\text{in}}(g_7) = \{g_0, g_1, g_2, g_3, g_5, g_6\}, \\ \phi_k^{\text{out}}(g_5) &= \phi_k^{\text{out}}(g_6) = \{g_0, g_3, g_4, g_7\} \not\subseteq \phi_k^{\text{in}}(g_5) = \phi_k^{\text{in}}(g_6) = \emptyset.\end{aligned}$$

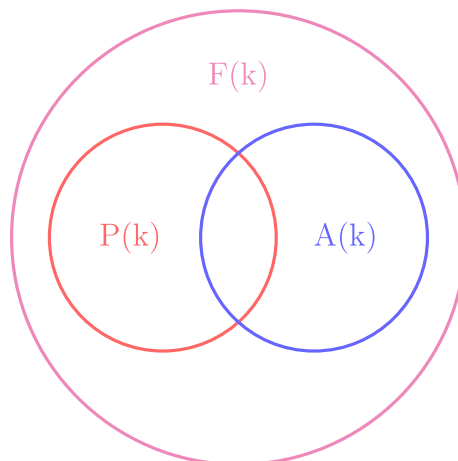


FIGURE 3 Relationship between the alternative concepts of local farsightedness. [Color figure can be viewed at wileyonlinelibrary.com]

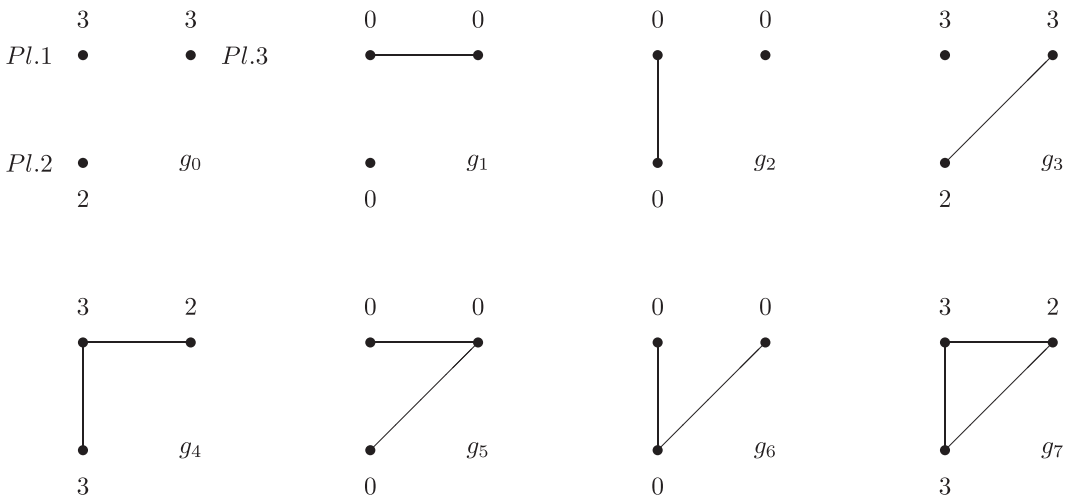


FIGURE 4 $F(k)$ is a strict coarsening of $P(k)$ and $A(k)$.

Hence, we get $P(k) = \emptyset$ and $A(k) = \emptyset$ for $k \geq 3$. However, $F(k) = \{g_0, g_3, g_4, g_7\}$ for $k \geq 3$. Thus, $F(k)$ is a strict coarsening of $P(k)$ and $A(k)$ in this example.

Proposition 1. A network $g \in \mathcal{G}$ is local- k farsightedly consistent if and only if $\phi_k^{\text{out}}(g) \subseteq \phi_k^{\text{in}}(g)$.

Proof. ($\Leftarrow$) Suppose that $\phi_k^{\text{out}}(g) \not\subseteq \phi_k^{\text{in}}(g)$. Then, there exists $g' \in \Delta(g, k)$ such that $g' \in \phi_k^{\text{out}}(g)$ and $g' \notin \phi_k^{\text{in}}(g)$, and so it follows that g is not a local- k farsightedly consistent network, a contradiction. ($\Rightarrow$) Suppose that g is not a local- k farsightedly consistent network. Then, there exists $g' \in \Delta(g, k)$ such that $g' \in \phi_k^{\text{out}}(g)$ and $g' \notin \phi_k^{\text{in}}(g)$, and so it follows that $\phi_k^{\text{out}}(g) \not\subseteq \phi_k^{\text{in}}(g)$, a contradiction. $\square$

Proposition 1 tells us that a network g is local- k farsightedly consistent if and only if the set of networks within the distance- k neighborhood of g that farsightedly defeat g is included in the set of networks within the distance- k neighborhood of g that are farsightedly defeated by g . We now provide an example that illustrates Proposition 1.

Example 4. Consider the situation where three players can form links and where the payoffs are given in Figure 5. For $k = 1$ we have:

$$\begin{aligned}\phi_1^{\text{out}}(g_0) &= \{g_3\} \quad \phi_1^{\text{in}}(g_0) = \{g_1, g_2\}, \\ \phi_1^{\text{out}}(g_3) &= \emptyset \quad \phi_1^{\text{in}}(g_3) = \{g_0, g_5, g_6\}.\end{aligned}$$

From Proposition 1, since $\phi_1^{\text{out}}(g_0) = \{g_3\} \not\subseteq \{g_1, g_2\} = \phi_1^{\text{in}}(g_0)$ and $\phi_1^{\text{out}}(g_3) = \emptyset$, we have that g_3 is a local-1 farsightedly consistent network while g_0 is not. For $k = 2$ we have:

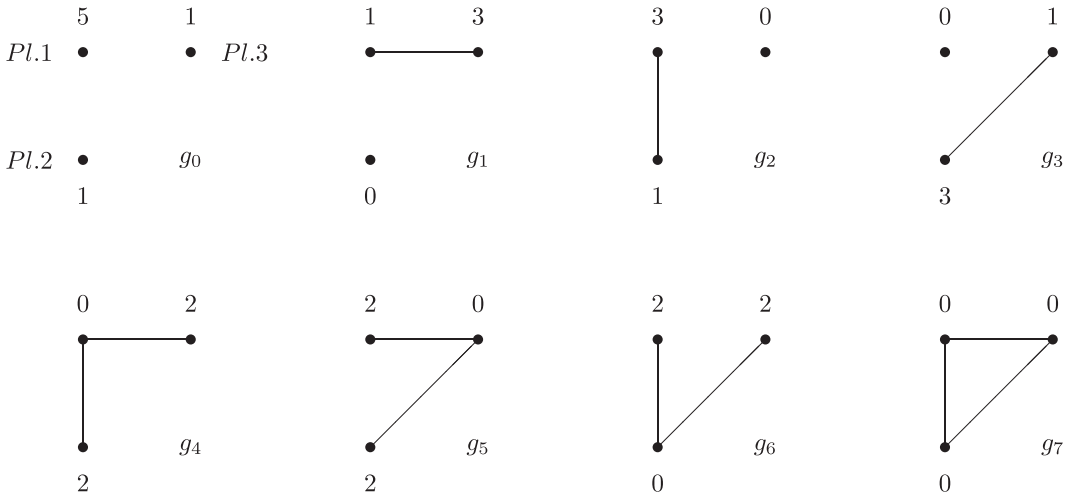


FIGURE 5 Another example with three players.

$$\begin{aligned}\phi_2^{\text{out}}(g_0) &= \{g_3\} & \phi_2^{\text{in}}(g_0) &= \{g_1, g_2, g_3, g_4, g_5, g_6\}, \\ \phi_2^{\text{out}}(g_3) &= \{g_0, g_1\} & \phi_2^{\text{in}}(g_3) &= \{g_0, g_2, g_5, g_6, g_7\}.\end{aligned}$$

For $k \geq 3$ we have:

$$\begin{aligned}\phi_k^{\text{out}}(g_0) &= \{g_3\} & \phi_k^{\text{in}}(g_0) &= \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}, \\ \phi_k^{\text{out}}(g_3) &= \{g_0, g_1\} & \phi_k^{\text{in}}(g_3) &= \{g_0, g_2, g_5, g_6, g_7\}.\end{aligned}$$

From Proposition 1, since $\phi_k^{\text{out}}(g_0) \subseteq \phi_k^{\text{in}}(g_0)$ and $\phi_k^{\text{out}}(g_3) \not\subseteq \phi_k^{\text{in}}(g_3)$ for $k \geq 2$, we have that g_0 is a local- k farsightedly consistent network for $k \geq 2$ while g_3 is not. Hence, $F(1) = \{g_3\}$ while $F(k) = \{g_0\}$ for $k \geq 2$.

Chwe (1994) and Page et al. (2005) introduce the notions of farsighted consistent set and largest farsighted consistent set. Similarly to our notion of locally farsighted consistent network, the notion of farsighted consistent set questions the credibility of any profitable deviation. However, the farsighted consistent set is a set-valued concept. Hence, a farsighted consistent set G requires that profitable deviations from any network $g \in G$ to some network g' are deterred by the farsighted threat of ending at some network $g'' \in (\phi_\infty^{\text{out}}(g') \cap G)$ (g'' may differ from g) where players who deviate first are worse off or equally well off. In addition, the farsighted consistent set only contemplates one-shot deviations while our notion of locally farsighted consistent network considers farsighted deviations.¹⁰ Finally, the farsighted consistent set allows for groupwise deviations while

¹⁰In the same spirit, Haghpahan and Siegel (2023) propose a notion of stability for market segmentations where a segmentation i is stable if and only if i blocks any other segmentation j that blocks i . One major difference with respect to our notion of locally farsighted consistent network is that we allow for farsighted blocking.

our notion of locally farsighted consistent network focuses on pairwise deviations. In Section 5 we extend our definition to groupwise deviations.¹¹

In the appendix, we study the relationship between local farsightedness and set-valued concepts such as the largest pairwise consistent set, the vNM pairwise farsightedly stable set, the pairwise farsightedly stable and the farsightedly absorbing set.¹²

4 | GENERAL PROPERTIES

We next investigate general properties that could help us for identifying locally farsighted consistent networks.

4.1 | Reversibility

Definition 6. A farsighted outgoing improving path $(g_1, g_2, \dots, g_{L-1}, g_L)$ from a network $g_1 = g$ to a network $g_L = g' \in \Delta(g, k)$ within the distance- k neighborhood of g is reversible if the finite sequence of networks $(g_L, g_{L-1}, \dots, g_2, g_1)$ is a farsighted ingoing improving path to g from g' .

Let $\Phi_k(g)$ be the set of all networks within the distance- k neighborhood of g that can be reached from g by a farsighted outgoing improving path that is reversible.

Proposition 2. *If for any $g' \in \Delta(g, k)$, either $g' \notin \phi_k^{\text{out}}(g)$ or $g' \in \Phi_k(g)$, then g is a local- k farsightedly consistent network.*

Proof. Suppose that g is not a local- k farsightedly consistent network. Then, there exists $g' \in \Delta(g, k)$ such that $g' \in \phi_k^{\text{out}}(g)$ and $g' \notin \phi_k^{\text{in}}(g)$ and so $g' \notin \Phi_k(g)$, a contradiction. $\square$

Proposition 2 tells us that, if for any network g' within the distance- k neighborhood of g , either g' does not farsightedly defeat g or there exists a reversible farsighted outgoing improving path from g to g' , then g is a local- k farsightedly consistent network. The next corollary simply says that if no network can be reached from g by a farsighted outgoing improving that is not reversible, then g is a local- k farsightedly consistent network.

Corollary 1. *If $\phi_k^{\text{out}}(g) = \Phi_k(g)$ then g is a local- k farsightedly consistent network.*

Example 1 (Continued). *We reconsider the situation where three players can form links of Figure 1. Take for instance the network g_4 . We had $\phi_1^{\text{in}}(g_4) = \{g_1, g_2, g_7\} = \Delta(g_4, 1)$, $\phi_2^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_5, g_6, g_7\} = \Delta(g_4, 2)$, $\phi_k^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_3, g_5, g_6, g_7\} = \Delta(g_4, k) = \mathcal{G} \setminus \{g_4\}$*

¹¹Obviously, if g is local- ∞ groupwise farsightedly consistent then $\{g\}$ is a farsighted consistent set and belongs to the largest consistent set. However, the converse may not be true.

¹²Other approaches to farsightedness can be found in Xue (1998), Herings et al. (2004, 2019, 2023), Mauleon and Vannetelbosch (2004), Dutta et al. (2005), Page and Wooders (2009), Dutta and Vohra (2017), Ray and Vohra (2019), Kimya (2020), Bloch and van den Nouweland (2020).

for $k \geq 3$. In addition, $\phi_1^{\text{out}}(g_4) = \emptyset$, $\phi_k^{\text{out}}(g_4) = \{g_5, g_6\}$ for $k \geq 2$. There is a unique farsighted outgoing improving path from g_4 to g_5 consisting of the sequence (g_4, g_1, g_5) . There is also a unique farsighted outgoing improving path from g_4 to g_6 consisting of the sequence (g_4, g_2, g_6) . Both farsighted outgoing improving paths are reversible. Indeed, the sequence (g_5, g_1, g_4) is a farsighted ingoing improving path to g_4 from g_5 within $\Delta(g_4, k)$ and the sequence (g_6, g_2, g_4) is a farsighted ingoing improving path to g_4 from g_6 within $\Delta(g_4, k)$. Hence, $\phi_k^{\text{out}}(g_4) = \Phi_k(g_4)$ for $k \geq 2$. Similarly for g_5 and g_6 . So, $\phi_k^{\text{out}}(g_5) = \Phi_k(g_5)$ and $\phi_k^{\text{out}}(g_6) = \Phi_k(g_6)$ for $k \geq 2$.

Example 5 Distance-based utility model (Bloch & Jackson, 2007). Example 1 is a three-player version of the distance-based model. In general, if player i is connected to player j by a path of t links, then player i receives a benefit of $b(t)$ from her indirect connection with player j . It is assumed that $b(t) \geq b(t+1) > 0$ for any t . Each direct link $ij \in g$ results in a benefit $b(1)$ and a cost c to both i and j . This cost can be interpreted as the time a player must spend with another player in order to maintain a direct link. Player i 's distance-based payoff from a network g is given by

$$u_i(g) = \sum_{j \neq i} b(t(ij)) - \#N_i(g) \cdot c,$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j), $c \geq 0$ is a cost per link, and b is a nonincreasing function. The symmetric connections model ($b(t) = \delta^t$) of Jackson and Wolinsky (1996) is a special case of distance-based payoffs. Remember that a star network g_i^* is a network such that there exists some player i (the center) who is linked to every other player $j \neq i$ (the peripherals) and that contains no other links (i.e., g is such that $N_i(g) = N \setminus \{i\}$ and $N_j(g) = \{i\}$ for all $j \in N \setminus \{i\}$). For intermediate costs, $b(1) - b(2) < c < b(1)$, all star networks are the strongly efficient networks. In Figure 1, the three-player case for $b(1) = 3$, $b(2) = 1.5$, and $c = 2$ is depicted.

Proposition 3. Consider the distance-based utility model in the case $b(1) - b(2) < c < b(1)$. If g is a star network then g is local- k farsightedly consistent for $k \geq n - 1$.

Proof. Take any network $g \neq g_i^*$ such that $g \in \Delta(g_i^*, k)$ and $k \geq n - 1$. We show that $\phi_k^{\text{in}}(g_i^*) \ni g$. (i) Suppose $g \neq g_j^*$ ($j \neq i$). From g , looking forward to g_i^* (where they obtain their highest possible payoff), players ($\neq i$) delete all their links successively (deleting first the links that are not in g_i^*) to reach the empty network $g^\emptyset$. Since $k \geq n - 1$ and g_i^* has exactly $n - 1$ links, $g^\emptyset \in \Delta(g_i^*, k)$. Hence, along any sequence $(g_1 = g, g_2, \dots, g_L = g^\emptyset)$ with $g_l = g_{l-1} - jk$ such that $jk \in g_{l-1}$, we have that $g_l \in \Delta(g_i^*, k)$ for $l = 1, \dots, L$. From $g^\emptyset$, players have incentives (since $b(1) > c$) to add links successively to build the star network g_i^* with player i in the center. (ii) Suppose $g = g_j^*$ ($j \neq i$). From g , looking forward to g_i^* , player j deletes all her links successively (deleting first the links that are not in g_i^*) to reach the empty network. From $g^\emptyset$, players have incentives (since $b(1) > c$)

to add links successively to build the star network g_i^* with player i in the center. Hence, $g_i^* \in A(k) \subseteq F(k)$. $\square$

The following corollary follows from the proof of Proposition 3. It shows that farsighted outgoing improving paths from a star network to another star network are reversible.

Corollary 2. *Consider the distance-based utility model in the case $b(1) - b(2) < c < b(1)$. If $k \geq 2(n - 2)$ then $g_i^* \in \Phi_k(g_j^*)$ for $j \neq i, i, j \in N$.*

Notice that, in the case the cost for maintaining a link is low enough and the strongly efficient network is the complete one, the complete network is local- k farsightedly consistent for $k \geq 1$. Similarly, in the case the cost is large enough and the strongly efficient network is the empty one, the empty network is local- k farsightedly consistent for $k \geq 1$.

4.2 | Egalitarian utility function

Suppose that the utility function u is such that, for any given network, all players get the same payoff: $u_i(g) = u_j(g)$ for all $i, j \in N$. With the egalitarian utility function u , each player's payoff depends on the network but not on the specific role she plays within the network. The next proposition shows that, under the egalitarian utility function, (i) the set of local- k farsightedly consistent networks is equal to the set of local- k farsightedly stable networks, (ii) increasing the size of the neighborhood refines the set of locally farsighted consistent networks, and (iii) only the strongly efficient networks remain locally farsighted consistent for k large enough.

Proposition 4. *Take any u such that $u_i(g) = u_j(g)$ for all $i, j \in N$. We have*

- (i) $F(k) = \{g \in \mathcal{G} \mid \phi_k^{\text{out}}(g) = \emptyset\} \equiv P(k)$.
- (ii) $F(k) \supseteq F(k + 1)$.
- (iii) $F(k) = E$ for $k \geq n(n - 1)/2$.

Proof.

- (i) Given the egalitarian utility function u , we have that $u_i(g_l) < u_i(g_L)$ for all $i \in N, l = 1, \dots, L - 1$, along any farsighted (outgoing or ingoing) improving path $(g_1, \dots, g_L)$. Hence, if $g \in \phi_k^{\text{out}}(g')$ then $u_i(g') < u_i(g)$ for all $i \in N$ and $g \notin \phi_k^{\text{in}}(g')$. So, any profitable farsighted deviation from g' to g cannot be deterred by the threat of going back to the original network g' . Thus, a network g is a local- k farsightedly consistent network if and only if $\phi_k^{\text{out}}(g) = \emptyset$. So, $F(k) = P(k)$.
- (ii) Since $P(k + 1) \subseteq P(k)$ and $F(k) = P(k)$ for $k \geq 1$, it follows that $F(k) \supseteq F(k + 1)$.
- (iii) Take $k \geq n(n - 1)/2$. Hence, $\Delta(g, k) = \mathcal{G} \setminus \{g\}$ for every $g \in \mathcal{G}$. Given the egalitarian utility function u , any strongly efficient network $g \in E$ Pareto dominates any

network $g' \notin E$. That is, $u_i(g) > u_i(g')$ for all $g \in E$, $g' \notin E$, for all $i \in N$. Hence, we have that $\phi_k^{\text{out}}(g) = \emptyset$ for all $g \in E$ and $\phi_k^{\text{out}}(g') \cap E \neq \emptyset$ for all $g' \in \mathcal{G} \setminus E$. So, $F(k) = E$ for $k \geq n(n-1)/2$. $\square$

Example 6 Public good investment model. We consider a public good investment model to illustrate the above proposition. Each player or city decides about the links she wants to form with other players. Every player shares equally the total cost for the provision of the public good and every player can only benefit from the public good if the whole community is connected. Let c be the cost of forming a link between two players; $0 < c < 1$. Thus, player i 's payoff is simply given by

$$u_i(g) = \begin{cases} a \cdot (n-1) - c \cdot \#g, & \text{if } \Pi(g) = \{N\}, \\ -c \cdot \#g, & \text{if } \Pi(g) \neq \{N\}, \end{cases}$$

where $\#g$ denotes the number of links in the network g and $a \geq 1$. In Figure 6, we have depicted the 3-player case.

Proposition 5 shows that, for $k < n-1$, the local- k farsightedly consistent networks in the public good investment model are the empty network and all minimally connected networks connecting all players. Once $k \geq n-1$, the empty network is no more local- k farsightedly consistent.

Proposition 5. *The set of local- k farsightedly consistent networks in the public good investment model is*

- (i) $F(k) = \{g^\emptyset\} \cup \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\}$ if $k < n-1$, and
- (ii) $F(k) = \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\}$ if $k \geq n-1$.

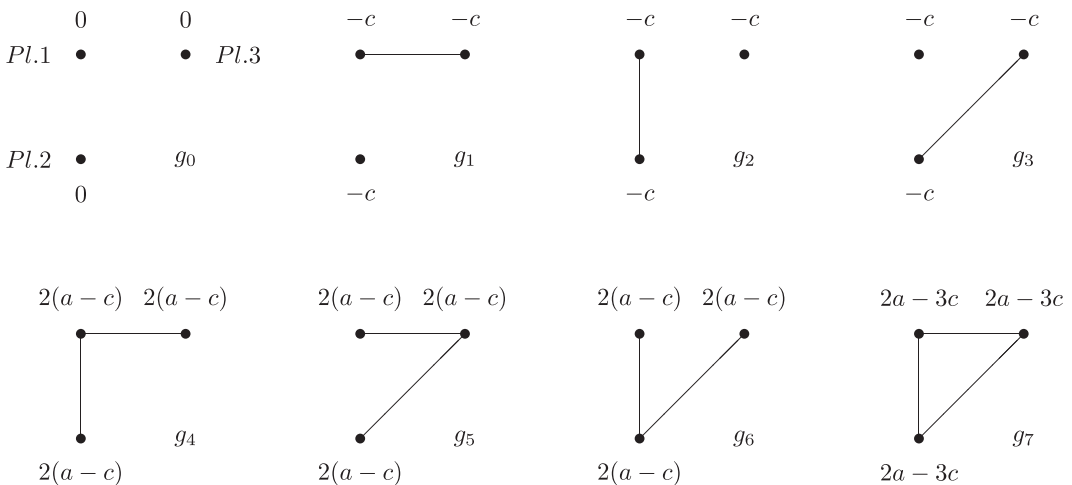


FIGURE 6 The public good investment game with three players.

Proof. In the public good investment model, the utility function is an egalitarian utility function. Thus, any strongly efficient network $g \in E$ Pareto dominates any network $g' \notin E$. That is, $u_i(g) > u_i(g')$ for all $g \in E, g' \notin E$, for all $i \in N$. We get that g is strongly efficient if and only if g is a minimally connected network; that is, $E = \{g \in \mathcal{G} \mid \#g = n - 1 \text{ and } \Pi(g) = \{N\}\}$. For any $g \notin E \cup \{g^\emptyset\}$ we have $u_i(g - kl) > u_i(g)$ for all $i \in N$, for any $kl \in g$. From Proposition 4 we have that $\phi_k^{\text{out}}(g) = \emptyset$ for all $g \in E$ and $g \notin F(k)$ if $\phi_k^{\text{out}}(g) \neq \emptyset$. Hence, $g \notin F(k)$ if $g \notin E \cup \{g^\emptyset\}$.

- (i) Take $k < n - 1$. We have that $\phi_k^{\text{out}}(g^\emptyset) = \emptyset$ and $\phi_k^{\text{in}}(g^\emptyset) = \Delta(g^\emptyset, k)$ since $\Delta(g^\emptyset, k) \cap E = \emptyset$. Hence, $F(k) = \{g^\emptyset\} \cup \{g \in \mathcal{G} \mid \#g = n - 1 \text{ and } \Pi(g) = \{N\}\} = \{g^\emptyset\} \cup E$.
- (ii) Take $k \geq n - 1$. Since $u_i(g^\emptyset) < u_i(g)$ for all $i \in N$, for all $g \in E$, and $\Delta(g^\emptyset, k) \cap E \neq \emptyset$, we have that $\phi_k^{\text{out}}(g^\emptyset) \cap E \neq \emptyset$ and $\phi_k^{\text{in}}(g^\emptyset) = \Delta(g^\emptyset, k) \setminus E$. Hence, $g^\emptyset \notin F(k)$, and so $F(k) = \{g \in \mathcal{G} \mid \#g = n - 1 \text{ and } \Pi(g) = \{N\}\} = E$. $\square$

5 | GROUP DEVIATIONS AND EXTERNALITIES

5.1 | Local farsightedness with group deviations

The notion of local- k farsightedly consistent network only considers deviations by at most a pair of players at a time. It might be that some group of players could all be made better off by some complicated reorganization of their links, which is not accounted for under local- k farsighted consistency. A network $g' \in \mathcal{G}$ is obtainable from $g \in \mathcal{G}$ via deviations by group $S \subseteq N$ if (i) $ij \in g'$ and $ij \notin g$ implies $\{i, j\} \subseteq S$, and (ii) $ij \in g$ and $ij \notin g'$ implies $\{i, j\} \cap S \neq \emptyset$.¹³

Definition 7. A groupwise farsighted outgoing improving path $(g_1, \dots, g_L)$ from a network g to a network $g' \in \Delta(g, k)$ within the distance- k neighborhood of g is a finite sequence of networks $g_1, \dots, g_L$ with $g_1 = g$ and $g_L = g'$ such that for any $l \in \{1, \dots, L - 1\}$, $g_{l+1} \in \Delta(g, k)$ is obtainable from g_l via deviations by $S_l \subseteq N$, $u_i(g_L) \geq u_i(g_l)$ for all $i \in S_l$ and $u_i(g_L) > u_i(g_l)$ for some $i \in S_l$.

For a given network $g \in \mathcal{G}$, let $\hat{\phi}_k^{\text{out}}(g)$ be the set of networks within the distance- k neighborhood of g that can be reached from g by a groupwise farsighted outgoing improving path.

Definition 8. A groupwise farsighted ingoing improving path $(g_1, \dots, g_L)$ to a network g from a network $g' \in \Delta(g, k)$ within the distance- k neighborhood of g is a finite sequence of networks $g_1, \dots, g_L$ with $g_1 = g'$ and $g_L = g$ such that for any $l \in \{1, \dots, L - 1\}$, $g_{l+1} \in \Delta(g, k)$ is obtainable from g_l via deviations by $S_l \subseteq N$, $u_i(g_L) \geq u_i(g_l)$ for all $i \in S_l$ and $u_i(g_L) > u_i(g_l)$ for some $i \in S_l$.

¹³Dutta and Mutuswami (1997), Jackson and van den Nouweland (2005) extend the notion of pairwise stability to group deviations, while Luo et al. (2022) propose the notion of coalition-proof stability for predicting the networks that could emerge when group deviations are allowed. A network g is strongly stable if for any $S \subseteq N$, g' that is obtainable from g via deviations by S , and $i \in S$ such that $u_i(g') > u_i(g)$, there exists $j \in S$ such that $u_j(g') < u_j(g)$.

For a given network $g \in \mathcal{G}$, let $\hat{\phi}_k^{\text{in}}(g)$ be the set of networks within the distance- k neighborhood of g from which there is a groupwise farsighted outgoing improving path leading to g .

Definition 9. A network $g \in \mathcal{G}$ is local- k groupwise farsightedly consistent if for any $g' \in \Delta(g, k)$ either $g' \notin \hat{\phi}_k^{\text{out}}(g)$ or $g' \in \hat{\phi}_k^{\text{in}}(g)$.

Let $\hat{F}(k)$ be the set of local- k groupwise farsightedly consistent networks.

5.2 | Componentwise egalitarian utility function

Let

$$g(S) = \left\{ g \subseteq g^S \mid \frac{\sum_{i \in N(g)} u_i(g)}{\#N(g)} \geq \frac{\sum_{i \in N(g')} u_i(g')}{\#N(g')} \quad \forall g' \subseteq g^S, g' \neq \emptyset \right\}$$

be the set of networks with the highest average payoff out of those that can be formed by players in $S \subseteq N$.

Suppose that the utility function u is componentwise egalitarian. That is, u is such that (i) $u_i(g) = u_j(g)$ for all $i, j \in S \in \Pi(g)$ and (ii) $u_i(g) = u_i(h)$ with $h \in H(g)$ and $i \in N(h)$. With a componentwise egalitarian utility function u , (i) players belonging to the same component get the same utility and (ii) there are no externalities across components (i.e., payoffs of players belonging to a component in a given network do not depend on the structure of other components). Given the componentwise egalitarian utility function u , we find a network $\hat{g}$ through the following algorithm due to Banerjee (1999). Pick some $h_1 \in g(N)$. Next, pick some $h_2 \in g(N \setminus N(h_1))$. At stage l pick some $h_l \in g(N \setminus \cup_{m \leq l-1} N(h_m))$. Since N is finite this process stops after a finite number L of stages. The union of the components picked in this way defines a network $\hat{g}$. We denote by $\hat{G}$ the set of all networks that can be found through this algorithm.¹⁴

The next proposition shows that if (i) players belonging to the same component get the same utility, (ii) there are no externalities across components, (iii) players prefer to be more connected inside their component, and (iv) players are not indifferent between components of different sizes, then every network obtained through the above algorithm is locally groupwise farsightedly consistent whatever the size of the neighborhood.

Proposition 6. Take any u such that

- (i) $u_i(g) = u_j(g)$ for all $i, j \in S \in \Pi(g)$,
- (ii) $u_i(g) = u_i(h)$ with $h \in H(g)$ and $i \in N(h)$,
- (iii) $u_i(h) > u_i(h')$ if $\#h > \#h'$ and $N(h) = N(h')$ with $h \in H(g), h' \in H(g')$ and $i \in N(h)$,

¹⁴More than one network may be picked up through this algorithm since players may be permuted or even be indifferent between components of different sizes.

(iv) $u_i(g) \neq u_i(g')$ if $\#S(i) \neq \#S'(i)$ with $S(i) \in \Pi(g)$, $S'(i) \in \Pi(g')$.

Every $g \in \hat{G}$ is a local- k groupwise farsightedly consistent network for $k \geq 1$.

All the proofs not in the main text can be found in the appendix. Except if the algorithm delivers only single component networks, any network $g \in \hat{G}$ is not strongly stable. Indeed, from any network $g \in \hat{G}$ there exists an obtainable deviation to some network g' by some coalition $S = S_1 \cup S_2$ composed of players belonging to the best component of g , namely $S_1 \subsetneq N(h_1)$, and others who are worse off than in h_1 with $\#S_2 = \#(N(h_1) \setminus S_1)$. In g' players belonging to S form a component h'_1 that has the same architecture as h_1 , and players belonging to S_2 are strictly better off while players in S_1 are equally well off. However, Proposition 6 tells us that, once players become farsighted, such deviations are no more credible.¹⁵

5.3 | Local public good networks with externalities

We consider now a local public good network model with externalities across components. Each municipality or player decides about the links she wants to form with other players. Every player shares common facilities with other players they are connected to. There is a fixed amount A for the whole provision of the public good financed by the government. Costs for the provision of the public good are shared equally among the connected players. Let c be the cost of forming a link between two players; $0 < c < 1$ and $c \cdot n(n+1) < A$. Thus, player i 's payoff is simply given by

$$u_i(g) = \frac{A}{[\#S(i)] \cdot [\#\Pi(g) + 1]^2} - c \cdot \frac{\#g + 1}{\#S(i)},$$

where $S(i)$ is the coalition $S \in \Pi(g)$ such that $i \in S$.¹⁶ In Figure 7 we have depicted the 3-player case for $A = 144$. This utility function satisfies some general properties (P1–P5) that are useful for characterizing the networks that are locally groupwise farsighted consistent. P1 (*Positive spillovers*) states that, in any network, linking two components increases the payoffs of the players that do not belong to those components. P2 (*Negative association*) states that, in any network, players belonging to smaller components obtain greater payoffs than players belonging to larger components. P3 (*No redundant links*) states that, in any network, players belonging to any component increase their payoffs by deleting links without modifying the groups of connected players. P4 (*Free-riding*) states that if a player leaves her component to become isolated, then she is better off. P5 (*Efficiency*) states that every minimally connected

¹⁵Notice that if a coalition only deviates when all members of the coalition are strictly better off at the end network, then every network $g \in \hat{G}$ is strongly stable and would already be locally groupwise farsightedly consistent under only conditions (i) and (ii) of Proposition 6.

¹⁶The utility function reflects the fact that having disconnected municipalities may lead to managing inefficiencies like the duplication of tasks or facilities. The formulation of the cost is chosen so that each player in a minimally connected component simply bears a cost equal to c .

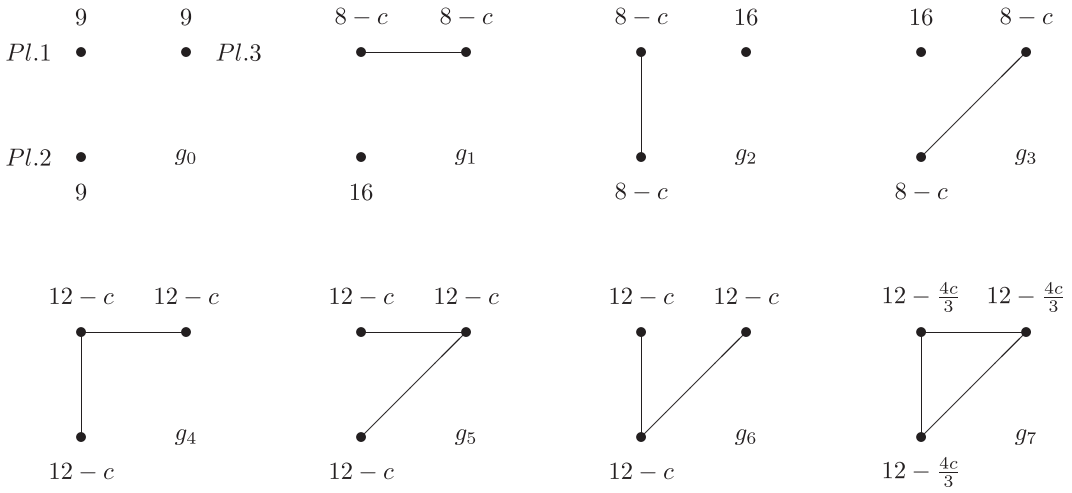


FIGURE 7 The local public good model with three players.

network is strongly efficient. Formally, the utility function satisfies the following five properties.

- P1 In any network g , we have that $u_i(g + jk) > u_i(g)$ if $S(i) \neq S(j) \neq S(k)$ and $S(i), S(j), S(k) \in \Pi(g)$.
- P2 In any network g , we have that $u_i(g) < u_j(g)$ if and only if $\#S(i) > \#S(j)$.
- P3 In any network g , we have that $u_i(g - ij) > u_i(g)$ if $\Pi(g - ij) = \Pi(g)$.
- P4 In any network g , we have that $u_i(g') > u_i(g)$ if $\Pi(g') = \Pi(g) \setminus \{S_i\} \cup \{S_i \setminus \{i\}, \{i\}\}$ for all $i \in S_i, S_i \in \Pi(g)$.
- P5 The set of strongly efficient networks is $E = \{g \in \mathcal{G} \mid \#g = n - 1 \text{ and } \Pi(g) = \{N\}\}$.

The next proposition shows that, in the local public good network model under P1–P5, every minimally connected network connecting all players is locally farsightedly consistent when k is large enough.

Proposition 7. *Consider the local public good network model under P1–P5. If $k \geq (n - 1)(n - 2)/2$ then every $g \in \{g \in \mathcal{G} \mid \#g = n - 1 \text{ and } \Pi(g) = \{N\}\}$ is a local- k groupwise farsightedly consistent network.*

6 | CONCLUSION

We have proposed the concept of local- k farsightedly consistent network for analyzing network formation games where players do not have full knowledge of the set of feasible networks. Precisely, players only consider a limited number of feasible networks when they decide about the links they want to form or keep with other players. A network g is said to be local- k farsightedly consistent if, for any network g' within the distance- k neighborhood of g , either g is not defeated by g' , or g defeats g' . That is, either there is no farsighted deviation to g' or the farsighted deviation to g' is deterred by the threat of a subsequent farsighted deviation that

leads back to g . We have shown that if the utility function is (componentwise) egalitarian or satisfies reversibility or excludes externalities across components, then local- k farsightedness is more likely to be a good proxy for what would happen when players have full knowledge of all feasible networks. Finally, we have studied the relationship with set-valued concepts of farsightedness.

ACKNOWLEDGMENTS

Ana Mauleon and Vincent Vannetelbosch are, respectively, Research Director and Senior Research Associate of the National Fund for Scientific Research (FNRS). Financial support from the Fonds de la Recherche Scientifique—FNRS research grant T.0143.18 is gratefully acknowledged.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interest.

ORCID

Ana Mauleon  <http://orcid.org/0000-0002-4218-2127>

REFERENCES

- Banerjee, S. (1999) *Efficiency and stability in economic networks*. Mimeograph.
- Bloch, F. & Jackson, M.O. (2007) The formation of networks with transfers among players. *Journal of Economic Theory*, 133, 83–110.
- Bloch, F. & van den Nouweland, A. (2020) Farsighted stability with heterogeneous expectations. *Games and Economic Behavior*, 121, 32–54.
- Caulier, J.-F., Mauleon, A. & Vannetelbosch, V. (2013a) Contractually stable networks. *International Journal of Game Theory*, 42, 483–499.
- Caulier, J.-F., Mauleon, A., Sempere-Monerris, J.J. & Vannetelbosch, V. (2013b) Stable and efficient coalitional networks. *Review of Economic Design*, 17, 249–271.
- Chwe, M.S. (1994) Farsighted coalitional stability. *Journal of Economic Theory*, 63, 299–325.
- de Callataÿ, P., Mauleon, A. & Vannetelbosch, V. (2024) *Minimal farsighted instability*. CORE Discussion Paper 2021-12, UCLouvain, Belgium.
- de Callataÿ, P., Mauleon, A. & Vannetelbosch, V. (2023) Myopic-farsighted absorbing networks. *Theory and Decision*, 94, 405–422.
- Diamantoudi, E. & Xue, L. (2003) Farsighted stability in hedonic games. *Social Choice and Welfare*, 21, 39–61.
- Dutta, B., Ghosal, S. & Ray, D. (2005) Farsighted network formation. *Journal of Economic Theory*, 122, 143–164.
- Dutta B. & Mutuswami, S. (1997) Stable networks. *Journal of Economic Theory*, 76, 322–344.
- Dutta, B. & Vohra, R. (2017) Rational expectations and farsighted stability. *Theoretical Economics*, 12, 1191–1227.
- Foerster, M., Mauleon, A. & Vannetelbosch, V. (2021) Shadow links. *Journal of Economic Theory*, 197, 105325.
- Goyal, S. (2007) *Connections: an introduction to the economics of networks*. Princeton University Press: Princeton, NJ, USA.
- Haghpahan, N. & Siegel, R. (2023) *A theory of stable market segmentations*. Working Paper, Department of Economics, Penn State University, USA.
- Herings, P.J.J., Mauleon, A. & Vannetelbosch, V. (2004) Rationalizability for social environments. *Games and Economic Behavior*, 49, 135–156.
- Herings, P.J.J., Mauleon, A. & Vannetelbosch, V. (2009) Farsightedly stable networks. *Games and Economic Behavior*, 67, 526–541.
- Herings, P.J.J., Mauleon, A. & Vannetelbosch, V. (2019) Stability of networks under horizon- K farsightedness. *Economic Theory*, 68, 177–201.

- Herings, P.J.J., Mauleon, A. & Vannetelbosch, V. (2020) Matching with myopic and farsighted players. *Journal of Economic Theory*, 190, 105125.
- Herings, P.J.J., Mauleon, A. & Vannetelbosch, V. (2023) Social rationalizability with mediation. *Dynamic Games and Applications*, 13, 440–461.
- Jackson, M.O. (2008) *Social and economic networks*. Princeton University Press: Princeton, NJ, USA.
- Jackson, M.O. & van den Nouweland, A. (2005) Strongly stable networks. *Games and Economic Behavior*, 51, 420–444.
- Jackson, M.O. & Watts, A. (2002) The evolution of social and economic networks. *Journal of Economic Theory*, 106, 265–295.
- Jackson, M.O. & Wolinsky, A. (1996) A strategic model of social and economic networks. *Journal of Economic Theory*, 71, 44–74.
- Kimya, M. (2020) Equilibrium coalition behavior. *Theoretical Economics*, 15, 669–714.
- Kirchsteiger, G., Mantovani, M., Mauleon, A. & Vannetelbosch, V. (2016) Limited farsightedness in network formation. *Journal of Economic Behavior and Organization*, 128, 97–120.
- Luo, C., Mauleon, A. & Vannetelbosch, V. (2021) Network formation with myopic and farsighted players. *Economic Theory*, 71, 1283–1317.
- Luo, C., Mauleon, A. & Vannetelbosch, V. (2022) Coalition-proof stable networks. *Review of Economic Design*, 26, 185–209.
- Mauleon, A., Sempere-Monerris, J.J. & Vannetelbosch, V. (2023) Limited farsightedness in R&D networks formation. *Dynamic Games and Applications*, 13, 549–565.
- Mauleon, A. & Vannetelbosch, V. (2004) Farsightedness and cautiousness in coalition formation games with positive spillovers. *Theory and Decision*, 56, 291–324.
- Mauleon, A. & Vannetelbosch, V. (2016) Network formation games. In: Bramoullé, Y., Galeotti, A. & Rogers, B.W. (Eds.), *The Oxford Handbook of the Economics of Networks*, Oxford University Press, UK.
- Mauleon, A., Vannetelbosch, V. & Vergote, W. (2011) von Neumann Morgenstern farsightedly stable sets in two-sided matching. *Theoretical Economics*, 6, 499–521.
- Page Jr., F.H., Wooders, M. & Kamat, S. (2005) Networks and farsighted stability. *Journal of Economic Theory*, 120, 257–269.
- Page Jr., F.H. & Wooders, M. (2009) Strategic basins of attraction, the path dominance core, and network formation games. *Games and Economic Behavior*, 66, 462–487.
- Ray, D. & Vohra, R. (2015) The farsighted stable set. *Econometrica*, 83, 977–1011.
- Ray, D. & Vohra, R. (2019) Maximality in the farsighted stable set. *Econometrica*, 87(5), 1763–1779.
- Tercieux, O. & Vannetelbosch, V. (2006) A characterization of stochastically stable networks. *International Journal of Game Theory*, 34, 351–369.
- Teteryatnikova, M. & Tremewan, J. (2020) Myopic and farsighted stability in network formation games: an experimental study. *Economic Theory*, 69, 987–1021.
- Xue, L. (1998) Coalitional stability under perfect foresight. *Economic Theory*, 11, 603–627.

How to cite this article: de Callataï, P., Mauleon, A. & Vannetelbosch, V. (2024) Local farsightedness in network formation. *International Journal of Economic Theory*, 1–28.
<https://doi.org/10.1111/ijet.12396>

APPENDIX A: RELATIONSHIP WITH SET-VALUED CONCEPTS

A.1. The largest pairwise consistent set

The largest pairwise consistent set has been defined in Chwe (1994) for general social environments. By considering a network as a social environment and by allowing only pairwise deviations, Herings et al. (2009) define the largest pairwise consistent set. A set of networks G is a pairwise consistent set if all profitable pairwise deviations from any network $g \in G$ to an

adjacent network $g + ij$ (or $g - ij$) are deterred by a credible threat of ending worse off or equally well off. Credibility requires that there exists a farsighted improving path from the adjacent network leading to some network $g' \in G$ where both i and j are not better off.

Definition 10. G is a pairwise consistent set if for all $g \in G$,

- (ia) for all $ij \notin g$, there exists $g' \in G$, where $g' = g + ij$ or $g' \in \phi_{\infty}^{\text{out}}(g + ij) \cap G$, such that $u_i(g') < u_i(g)$ or $u_j(g') < u_j(g)$ or $(u_i(g'), u_j(g')) = (u_i(g), u_j(g))$,
- (ib) for all $ij \in g$, there exists $g', g'' \in G$, where $g' = g - ij$ or $g' \in \phi_{\infty}^{\text{out}}(g - ij) \cap G$, and $g'' = g - ij$ or $g'' \in \phi_{\infty}^{\text{out}}(g - ij) \cap G$, such that $u_i(g') \leq u_i(g)$ and $u_j(g'') \leq u_j(g)$.

The largest pairwise consistent set is the pairwise consistent set that contains any pairwise consistent set.

The set G is a pairwise consistent set if both external and internal deviations are deterred. The largest pairwise consistent set is the set that contains any pairwise consistent set. It follows from the results in Chwe (1994) that the largest pairwise consistent set exists, is nonempty, and satisfies external stability.

Two networks g and g' are adjacent if they differ by one link. The utility function u exhibits no indifference if for any g and g' that are adjacent either g defeats g' or g' defeats g .

Proposition 8. *Suppose that u exhibits no indifference. If g is local- k farsightedly consistent then $\{g\}$ is a pairwise consistent set and g belongs to the largest pairwise consistent set.*

Proof. Since u exhibits no indifference, we have that a local- k farsightedly consistent network g defeats (i) $g + ij$ for all $ij \notin g$ and (ii) $g - ij$ for all $ij \in g$. Thus, $g \in \phi_{\infty}^{\text{out}}(g + ij)$, $g + ij \in \phi_1^{\text{in}}(g)$, $g \in \phi_{\infty}^{\text{out}}(g - ij)$, and $g - ij \in \phi_1^{\text{in}}(g)$. So, $\{g\}$ is a pairwise consistent set and g belongs to the largest pairwise consistent set. $\square$

A.2. The vNM pairwise farsightedly stable set

Incorporating the notion of farsighted improving paths into the original definition of the von Neumann-Morgenstern stable set, Herings et al. (2009) obtain the vNM pairwise farsightedly stable set. A set of networks is a vNM pairwise farsightedly stable set if (i) there is no farsighted improving path between networks within the set (i.e., the internal stability condition); (ii) there is a farsighted improving path from any network outside the set to some network within the set (i.e., the external stability condition).

Definition 11. The set G is a vNM pairwise farsightedly stable set if (i) for all $g \in G$, $\phi_{\infty}^{\text{out}}(g) \cap G = \emptyset$ and (ii) for all $g' \in \mathcal{G} \setminus G$, $\phi_{\infty}^{\text{out}}(g') \cap G \neq \emptyset$.

Notice that vNM pairwise farsightedly stable sets do not always exist. In Example 4 we have that $\{g_0\}$ is the unique vNM farsightedly stable set while g_0 is the unique local- k farsightedly

consistent for $k \geq 2$ but is not local- k farsightedly consistent for $k = 1$ (g_3 is the unique local- k farsightedly consistent for $k = 1$).

Proposition 9. *If $\{g\}$ is a vNM pairwise farsightedly stable set then there exists $\bar{k}$ such that g is local- k farsightedly consistent for $k \geq \bar{k}$.*

Proof. Take $\bar{k} = n(n-1)/2$. Suppose that g is not a local- k farsightedly consistent network. Hence, there exists g' such that $g' \in \phi_k^{\text{out}}(g)$ and $g' \notin \phi_k^{\text{in}}(g)$. Since $\Delta(g, k) = \mathcal{G} \setminus \{g\}$ and $\Delta(g', k) = \mathcal{G} \setminus \{g'\}$, we have that $g \notin \phi_k^{\text{out}}(g') = \phi_\infty^{\text{out}}(g')$, and so $\{g\}$ is not a vNM pairwise farsightedly stable set, a contradiction. $\square$

A.3. The DEM pairwise farsightedly stable set

Herings et al. (2009) introduce the DEM pairwise farsightedly stable set.¹⁷ It is obtained by requiring: (D) deterrence of external deviations, (E) external stability, and (M) minimality. More precisely, a set of networks G is DEM pairwise farsightedly stable if (D) all possible pairwise deviations from any network $g \in G$ to a network outside G are deterred by a credible threat of ending worse off or equally well off, (E) there exists a farsighted improving path from any network outside the set leading to some network in the set, and (M) there is no proper subset of G satisfying (D) and (M).

Definition 12. A set of networks $G \subseteq \mathcal{G}$ is DEM pairwise farsightedly stable if

(D) for all $g \in G$,

(i) for all $ij \notin g$ such that $g + ij \notin G$, there exists $g' \in \phi_\infty^{\text{out}}(g + ij) \cap G$ such that $(u_i(g'), u_j(g')) = (u_i(g), u_j(g))$ or $u_i(g') < u_i(g)$ or $u_j(g') < u_j(g)$,

(ii) for all $ij \in g$ such that $g - ij \notin G$, there exists $g', g'' \in \phi_\infty^{\text{out}}(g - ij) \cap G$ such that $u_i(g') \leq u_i(g)$ and $u_j(g'') \leq u_j(g)$,

(E) for all $g' \in \mathcal{G} \setminus G$, $\phi_\infty^{\text{out}}(g') \cap G \neq \emptyset$.

(M) there does not exist $G' \subsetneq G$ such that G' satisfies (D) and (E).

DEM pairwise farsightedly stable sets need not to be consistent in the sense of Chwe (1994) since internal deviations need not to be deterred. However, a pairwise consistent set does not necessarily satisfies the external stability condition. Only the largest pairwise consistent set is guaranteed to satisfy external stability.

Corollary 4 in Herings et al. (2009) tells us that the set $\{g\}$ is a DEM pairwise farsightedly stable set if and only if it is a vNM pairwise farsightedly stable set. Hence, together with Proposition 9 we obtain the following corollary.

¹⁷Herings et al. (2009) originally called it the pairwise farsightedly stable set. They later on call it DEM farsightedly stable set to avoid the confusion with the vNM farsightedly stable set.

Corollary 3. *If $\{g\}$ is a DEM pairwise farsightedly stable set then there exists $\bar{k}$ such that g is local- k farsightedly consistent for $k \geq \bar{k}$.*

A.4. The farsightedly absorbing set

de Callataï et al. (2023) propose the concept of farsightedly absorbing set. A set of networks is said to be farsightedly absorbing if it satisfies the following three conditions: no external deviation, external stability and minimality. That is, (i) there is no farsighted improving path from networks within the set to some networks outside the set; (ii) there is a farsighted improving path from any network outside the set to some network within the set; and (iii) there is no proper subset satisfying (i) and (ii).

Definition 13. A set of networks $G \subseteq \mathcal{G}$ is a farsightedly absorbing set if

- (i) for every $g \in G$, it holds that $\phi_{\infty}^{\text{out}}(g) \cap (\mathcal{G} \setminus G) = \emptyset$,
- (ii) for every $g \in \mathcal{G} \setminus G$, it holds that $\phi_{\infty}^{\text{out}}(g) \cap G \neq \emptyset$,
- (iii) there does not exist $G' \subsetneq G$ such that G' satisfies (i) and (ii).

Proposition 10. *If $\{g\}$ is a farsightedly absorbing set then g is a local- k farsightedly consistent network for $k \geq 1$.*

Proof. Since $\{g\}$ is a farsightedly absorbing set, we have that $\phi_{\infty}^{\text{out}}(g) = \emptyset$. Given that, in general, $\phi_{k-1}^{\text{out}}(g) \subseteq \phi_k^{\text{out}}(g)$ for $k > 1$ and $\phi_{k-1}^{\text{out}}(g) = \emptyset$ if $\phi_k^{\text{out}}(g) = \emptyset$, it follows that g is a local- k farsightedly consistent network for $k \geq 1$. $\square$

APPENDIX B: PROOFS

Proof of Proposition 6. Take any $g = \bigcup_{l=1}^L h_l \in \widehat{G}$ and any $g' \in \Delta(g, k)$ such that $u_i(g') \neq u_i(g)$ for some $i \in N$. Otherwise, $g' \notin \phi_k^{\text{out}}(g)$. We show in a constructive way that $g' \in \phi_k^{\text{in}}(g)$. Notice that each component h_l is fully connected, that is, $\#h_l = \#N(h_l)(\#N(h_l) - 1)/2$, for $l = 1, \dots, L$.

Step 1.1: Let $T_1 \subseteq N$ be the set of players i such that $u_i(g') \neq u_i(g)$, $i \in N$. Pick some $j^* \in T_1$ and let $\hat{i}^1 = j^*$. If $h_1 \in H(g')$ then go to Step 1.2 with $g_1 = g'$ and $\hat{i}^2 = j^*$. Otherwise, looking forward towards g , the coalition $S_1 = \{\hat{i}^1\} \cup N(h_1)$ deviates jointly from g' to:

- (i) either $g' - \{ij | i \in N(h_1), ij \in g' \text{ and } ij \notin h_1\} + \{ij | i \in N(h_1), ij \in h_1 \text{ and } ij \notin g'\} - k_1^1 k_2^1$ with $k_1^1, k_2^1 \in N(h_1)$ if $\hat{i}^1$ is already linked to some $i \in N(h_1)$ in g' and so all players belonging to S_1 are worse off in g' than in g ;
- (ii) or $g' + \hat{i}^1 l_1 - \{ij | i \in N(h_1), ij \in g' \text{ and } ij \notin h_1\}$ with $l_1, k_1^1, k_2^1 \in N(h_1)$ if $\hat{i}^1$ is not already linked to some $i \in N(h_1)$ in g' .

That is, $\hat{i}^1$ links to some $l_1 \in N(h_1)$ (if not yet linked) and all players belonging to $N(h_1)$ delete their links that are in g' but not in g and build the missing links that are not in g' but are in g , and two players $k_1^1, k_2^1 \in N(h_1)$ delete (or do not build) the link between them. Player $\hat{i}^1$ is strictly better off in g compared to g' while players in $N(h_1)$ are at least as well off.

Next, if player $\hat{i}^1 \notin N(h_1)$ then player l_1 cuts the link to player $\hat{i}^1$. We reach the network $g_1 = g' - \{ijli \in N(h_1), ij \in g' \text{ and } ij \notin h_1\} + \{ijli \in N(h_1), ij \in h_1 \text{ and } ij \notin g'\} - k_1^1 k_2^1$ with $k_1^1, k_2^1 \in N(h_1)$ where $h_1 - k_1^1 k_2^1 \in H(g_1)$. Along the sequence from g' to g_1 all networks belong to $\Delta(g, k)$. Notice that at g_1 all players in $N(h_1)$ are worse off than at g since the link $k_1^1 k_2^1$ is missing. Let $\hat{i}^2 = k_1^1$ and go to Step 1.2.

Step 1.2: If $h_2 \in H(g_1)$ then go to Step 1.3 with $g_2 = g_1$ and $\hat{i}^3 = \hat{i}^2$. Otherwise, looking forward towards g , the coalition $S_2 = \{\hat{i}^2\} \cup N(h_2)$ deviates jointly from g_1 to:

- (i) either $g_1 - \{ijli \in N(h_2), ij \in g_1 \text{ and } ij \notin h_2\} + \{ijli \in N(h_2), ij \in h_2 \text{ and } ij \notin g_1\} - k_1^2 k_2^2$ with $k_1^2, k_2^2 \in N(h_2)$ if $\hat{i}^2$ is already linked to some $i \in N(h_2)$ in g_1 and so all players belonging to S_2 are worse off in g_1 than in g ;
- (ii) or $g_1 + \hat{i}^2 l_2 - \{ijli \in N(h_2), ij \in g_1 \text{ and } ij \notin h_2\} + \{ijli \in N(h_2), ij \in h_2 \text{ and } ij \notin g_1\} - k_1^2 k_2^2$ with $l_2, k_1^2, k_2^2 \in N(h_2)$ if $\hat{i}^2$ is not already linked to some $i \in N(h_2)$ in g_1 .

That is, $\hat{i}^2$ links to some $l_2 \in N(h_2)$ (if not yet linked) and all players belonging to $N(h_2)$ delete their links that are in g_1 but not in g and build the missing links that are not in g_1 but are in g , and two players $k_1^2, k_2^2 \in N(h_2)$ delete (or do not build) the link between them. Player $\hat{i}^2$ is strictly better off in g compared to g_1 while players in $N(h_2)$ are at least as well off.

Next, if player $\hat{i}^2 \notin N(h_2)$ then player l_2 cuts the link to player $\hat{i}^2$. We reach the network $g_2 = g_1 - \{ijli \in N(h_2), ij \in g_1 \text{ and } ij \notin h_2\} + \{ijli \in N(h_2), ij \in h_2 \text{ and } ij \notin g_1\} - k_1^2 k_2^2$ with $k_1^2, k_2^2 \in N(h_2)$ where $h_2 - k_1^2 k_2^2 \in H(g_2)$. Along the sequence from g_1 to g_2 all networks belong to $\Delta(g, k)$. Notice that at g_2 all players in $N(h_2)$ are worse off than at g since the link $k_1^2 k_2^2$ is missing. Let $\hat{i}^3 = k_1^2$ and go to Step 1.3.

Step 1. l : If $h_l \in H(g_{l-1})$ then go to Step 1. $l + 1$ with $g_l = g_{l-1}$ and $\hat{i}^{l+1} = \hat{i}^l$. Otherwise, looking forward towards g , the coalition $S_l = \{\hat{i}^l\} \cup N(h_l)$ deviates jointly from g_{l-1} to:

- (i) either $g_{l-1} - \{ijli \in N(h_l), ij \in g_{l-1} \text{ and } ij \notin h_l\} + \{ijli \in N(h_l), ij \in h_l \text{ and } ij \notin g_{l-1}\} - k_1^l k_2^l$ with $k_1^l, k_2^l \in N(h_l)$ if $\hat{i}^l$ is already linked to some $i \in N(h_l)$ in g_{l-1} and so all players belonging to S_l are worse off in g_{l-1} than in g ;
- (ii) or $g_{l-1} + \hat{i}^l l_l - \{ijli \in N(h_l), ij \in g_{l-1} \text{ and } ij \notin h_l\} + \{ijli \in N(h_l), ij \in h_l \text{ and } ij \notin g_{l-1}\} - k_1^l k_2^l$ with $l_l, k_1^l, k_2^l \in N(h_l)$ if $\hat{i}^l$ is not already linked to some $i \in N(h_l)$ in g_{l-1} .

That is, $\hat{i}^l$ links to some $l_l \in N(h_l)$ (if not yet linked) and all players belonging to $N(h_l)$ delete their links that are in g_{l-1} but not in g and build the missing links that are not in g_{l-1} but are in g , and two players $k_1^l, k_2^l \in N(h_l)$ delete (or do not build) the link between them. Player $\hat{i}^l$ is strictly better off in g compared to g_{l-1} while players in $N(h_l)$ are at least as well off.

Next, if player $\hat{i}^l \notin N(h_l)$ then player l_l cuts the link to player $\hat{i}^l$. We reach the network $g_l = g_{l-1} - \{ijli \in N(h_l), ij \in g_{l-1} \text{ and } ij \notin h_l\} + \{ijli \in$

$N(h_l), ij \in h_l$ and $ij \notin g_{l-1}\} - k_1^l k_2^l$ with $k_1^l, k_2^l \in N(h_l)$, where $h_l - k_1^l k_2^l \in H(g_l)$. Along the sequence from g_{l-1} to g_l all networks belong to $\Delta(g, k)$. Notice that at g_l all players in $N(h_l)$ are worse off than at g since the link $k_1^l k_2^l$ is missing. Let $i^{l+1} = k_1^l$ and go to Step 1. $l + 1$.

We proceed so until we reach g_L at Step L , where $h_l - k_1^l k_2^l \in H(g_L)$ for $l = 1, \dots, L$. Thus, $g_L = \bigcup_{l=1}^L (h_l - k_1^l k_2^l)$, where $h_l \in H(g)$ and $k_1^l, k_2^l \in N(h_l)$ for $l = 1, \dots, L$.

Step 2.1: From g_L , looking forward towards g , players $k_1^l, k_2^l \in N(h_l)$ add the missing link between them to form $g_{L+1} = g_L + k_1^l k_2^l$ and so we have $h_l \in H(g_{L+1})$.

Step 2. l : From g_{L+l-1} , looking forward towards g , players $k_1^l, k_2^l \in N(h_l)$ add the missing link between them to form $g_{L+l} = g_{L+l-1} + k_1^l k_2^l$ and so we have $h_l \in H(g_{L+l})$; and so on until we reach the network $g = \bigcup_{l=1}^L h_l$.

Step 2. L : From g_{L+L-1} , looking forward towards g , players $k_1^L, k_2^L \in N(h_L)$ add the missing link between them to form $g_{L+L} = g_{L+L-1} + k_1^L k_2^L$. So we have $h_L \in H(g_{L+L}) = H(g)$ and we have reached the network $g = \bigcup_{l=1}^L h_l$.

Thus, we have build a farsightedly ingoing improving path to $g \in \widehat{G}$ from any $g' \in \Delta(g, k)$ such that $u_i(g') \neq u_i(g)$ for some $i \in N$; so $g' \in \widehat{\phi}_k^{\text{in}}(g)$. If $u_i(g') = u_i(g)$ for all $i \in N$ then $g' \notin \widehat{\phi}_k^{\text{out}}(g)$. Hence, every $g \in \widehat{G}$ is a local- k farsightedly consistent network for $k \geq 1$. $\square$

Proof of Proposition 7. Take any g such that $g \in \{g \in \mathcal{G} \mid \#g = n - 1 \text{ and } \Pi(g) = \{N\}\}$. Since $k \geq (n - 1)(n - 2)/2$, we have that $g^\emptyset \in \Delta(g, k)$. In addition, $g' \in \Delta(g, k)$ for g' such that $\#g' = (n - 1)(n - 2)/2$ and $\Pi(g') = \{N \setminus \{j\}, \{j\}\}$. From P4 we have that $\widehat{\phi}_k^{\text{out}}(g) \neq \emptyset$. From P3, if $g' \notin E$ and $\Pi(g') = \{N\}$, then $g' \notin \widehat{\phi}_k^{\text{out}}(g)$. Thus, it is sufficient to show that, for any $g' \notin E$ such that $\Pi(g') \neq \{N\}$ and $g' \in \Delta(g, k)$, then $g' \in \widehat{\phi}_k^{\text{in}}(g)$. We show it in two steps.

[Step A.] Given P2 and P4, the players belonging to the largest component of g' are worse off than in g . Moreover, all players prefer g to $g^\emptyset$ (i.e., when all players are isolated).

[Step B.] From g' we build a groupwise farsighted ingoing improving path to g as follows. The improving path consists of a sequence of moves where at each move one player belonging to the largest component in the current network deviates to become an isolated player, until the empty network $g^\emptyset$ is reached. Since $k \geq (n - 1)(n - 2)/2$, it is guaranteed that along the sequence every network belongs to $\Delta(g, k)$. From the empty network $g^\emptyset$, all the players deviate jointly to g with $g \in \{g \in \mathcal{G} \mid \#g = n - 1 \text{ and } \Pi(g) = \{N\}\}$. By (A) and (B) we have that, for any $g' \notin E$ such that $\Pi(g') \neq \{N\}$ and $g' \in \Delta(g, k)$, then $g' \in \widehat{\phi}_k^{\text{in}}(g)$. So, g is a local- k groupwise farsightedly consistent network. $\square$