

ALGEBRAICALLY COHERENT CATEGORIES

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ABSTRACT. We call a finitely complete category *algebraically coherent* when the change-of-base functors of its fibration of points are *coherent*, which means that they preserve finite limits and jointly strongly epimorphic pairs of arrows. We give examples of categories satisfying this condition; for instance, coherent categories, *categories of interest*, and (compact) Hausdorff algebras over a semi-abelian algebraically coherent theory. We study equivalent conditions in the context of semi-abelian categories, as well as some of its consequences: including amongst others, strong protomodularity, and normality of Higgins commutators for normal subobjects, and in the varietal case, fibre-wise algebraic cartesian closedness.

1. INTRODUCTION

The aim of this article is to study a condition which recently arose in some loosely interrelated categorical-algebraic investigations [46, 47, 23, 22]: we ask of a semi-abelian category [38] that the change-of-base functors of its fibration of points are *coherent*, which means that they preserve finite limits and jointly strongly epimorphic pairs of arrows.

Despite its apparent simplicity, this property—which we shall call *algebraic coherence*—has some important consequences. For instance, any algebraically coherent semi-abelian category satisfies the so-called *Smith is Huq* condition (SH) [4, 48]. In fact (see Section 4) it also satisfies the *strong protomodularity* condition as well as the conditions (SSH), which is a strong version of (SH), and (NH), *normality of Higgins commutators of normal subobjects*—studied in [9, 4], [49] and [20, 21], respectively. Nevertheless, there are many examples including all *categories of interest* [51] (Theorem 3.14). In particular, the categories of groups, non-unitary (commutative) rings, Lie algebras over a commutative ring with unit, Poisson algebras and associative algebras are all examples, as well as the categories of (compact) Hausdorff groups, Lie algebras etc. Knowing that a category is not only semi-abelian, but satisfies these additional conditions is crucial for many results in categorical algebra leading to applications in (co)homology theory. For instance, the description of internal crossed modules [35] becomes simpler when (SH) holds [48, 34]; the theory of universal central extensions depends on the validity of both (SH) and (NH) [19, 31]; and under (SH) higher central extensions admit a characterisation in terms of binary commutators which helps in the interpretation of (co)homology groups [57, 56].

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The concept of *algebraically coherent* category is meant to be an algebraic version of the classical concept of *coherent* category [42], as explained by a certain formal parallel between Topos Theory and Categorical Algebra [36]. The key idea is that notions which in Topos Theory are expressed by properties of the basic fibration $\text{cod}: \text{Arr}(\mathcal{C}) \rightarrow \mathcal{C}$ may have a meaningful counterpart in Categorical Algebra when the basic fibration is replaced by the fibration of points $\P_{\mathcal{C}} = \text{cod}: \text{Pt}(\mathcal{C}) \rightarrow \mathcal{C}$. That is to say, the slice categories $(\mathcal{C} \downarrow X)$ are replaced by the categories $\text{Pt}_X(\mathcal{C})$ of points over X in $\mathcal{C}$, of which the objects are split epimorphisms with a chosen splitting $(p: Y \rightarrow X, s: X \rightarrow Y)$, $ps = 1_X$. A successful example of this parallel is the second author’s notion of *algebraically cartesian closed* category—see [29, 14] and related works. The present paper provides a new example: while a coherent category is a regular category $\mathcal{C}$ where every change-of-base functor of the basic fibration $\text{cod}: \text{Arr}(\mathcal{C}) \rightarrow \mathcal{C}$ is coherent, an algebraically coherent category is a finitely complete category $\mathcal{C}$ where the same property holds for the fibration of points $\text{cod}: \text{Pt}(\mathcal{C}) \rightarrow \mathcal{C}$. As a consequence, certain results carry over from Topos Theory to Categorical Algebra for purely formal reasons: for instance, in parallel with the long-established [42, Lemma 1.5.13], any locally algebraically cartesian closed category is algebraically coherent (Theorem 3.4). Note that this procedure (replacing the basic fibration with the fibration of points) is indeed necessary, because while a semi-abelian category [38] may or may not be algebraically coherent—see Section 3 for a list of examples—it is never coherent, unless it is trivial (Proposition 2.10).

In Section 2 we give the basic definitions, we characterise algebraic coherence in terms of the kernel functor alone (Proposition 2.19, Theorem 2.29) and we study the condition’s stability properties: closure under slices and coslices (Proposition 2.21), points (Corollary 2.22), and (regular epi)-reflections (Proposition 2.23). In Section 3 we give examples, non-examples and counterexamples. The major results here are Theorem 3.14 proving that all *categories of interest* are algebraically coherent, and Theorem 3.10 which says that categories of (compact) Hausdorff algebras over a semi-abelian algebraically coherent theory are still algebraically coherent. In Section 4 we focus on categorical-algebraic consequences of algebraic coherence, mostly in the semi-abelian context. We show that (SH), (NH), (SSH) and strong proto-modularity are all consequences of algebraic coherence (see Theorems 4.15, 4.16 and 4.18). Furthermore, in the varietal case, algebraic coherence implies fibre-wise algebraic cartesian closedness (FWACC) (see Theorem 4.20), meaning that centralisers exist in the fibres of the fibration of points. Section 5 focuses on the higher-order Higgins commutator and a proof of the *Three Subobjects Lemma for normal subobjects* (Theorem 5.1). Section 6 is devoted to pullbacks along “surjections” in the Mal’tsev context. It contains a characterisation of Mal’tsev categories amongst finitely complete ones (Theorem 6.8) which extends the characterisation of regular Mal’tsev categories in terms of the *relative Mal’tsev axiom* [24] to a non-regular context, and which is then used to prove the section’s main result, Theorem 6.10. The final section gives a short summary of results that hold in the semi-abelian context.

2. DEFINITIONS, FIRST RESULTS AND EQUIVALENT CONDITIONS

Recall that a cospan (f, g) over an object Z in an arbitrary category is called a

- (a) **jointly strongly epimorphic pair** when for each commutative diagram

$$\begin{array}{ccccc}
 & & M & & \\
 & \nearrow f' & \downarrow m & \nwarrow g' & \\
 X & \xrightarrow{f} & Z & \xleftarrow{g} & Y
 \end{array}$$

if m is a monomorphism, then there exists a unique morphism $\varphi: Z \rightarrow M$ such that $m\varphi = \phi$;

- (b) **jointly extremal-epimorphic pair** when for each commutative diagram

$$\begin{array}{ccccc}
 & & M & & \\
 & \nearrow f' & \downarrow m & \nwarrow g' & \\
 X & \xrightarrow{f} & Z & \xleftarrow{g} & Y
 \end{array}$$

if m is a monomorphism, then m is an isomorphism.

Like with extremal epimorphisms and strong epimorphisms (see for instance [4]) we have

Lemma 2.1. *Let $\mathcal{C}$ be an arbitrary category and let (f, g) be a cospan over an object Z . If the pair (f, g) is jointly strongly epimorphic, then it is jointly extremal-epimorphic. If $\mathcal{C}$ has pullbacks then (f, g) is jointly extremal-epimorphic if and only if it is jointly strongly epimorphic.* $\square$

Lemma 2.2. *In an arbitrary category, let $(f: X \rightarrow Z, g: Y \rightarrow Z)$ be a cospan over Z and let $e: W \rightarrow X$ be a strong epimorphism.*

- (a) (f, g) is jointly extremal-epimorphic if and only if (fe, g) is jointly extremal-epimorphic;
- (b) (f, g) is jointly strongly epimorphic if and only if (fe, g) is jointly strongly epimorphic. $\square$

Lemma 2.3. *For each commutative diagram*

$$\begin{array}{ccccc}
 K & \xrightarrow{f'} & M & \xleftarrow{g'} & L \\
 & \searrow f & \downarrow & \swarrow g & \\
 & & Z & &
 \end{array}$$

in an arbitrary category, $M \leq Z$ is the join of $K \leq Z$ and $L \leq Z$ if and only if (f', g') is jointly extremal-epimorphic. In particular (f, g) is jointly extremal-epimorphic when the diagram above with $M = Z$ is a join. $\square$

Lemma 2.4. *For each diagram*

$$\begin{array}{ccccc}
 & & X + Y & & \\
 & \nearrow \iota_X & \downarrow \begin{pmatrix} f \\ g \end{pmatrix} & \nwarrow \iota_Y & \\
 X & \xrightarrow{f} & Z & \xleftarrow{g} & Y
 \end{array}$$

in a category with binary coproducts, f and g are jointly extremal-epimorphic / jointly strongly epimorphic if and only if $\begin{pmatrix} f \\ g \end{pmatrix}$ is an extremal epimorphism / strong epimorphism. $\square$

Since in the rest of the paper all categories considered will have finite limits we will freely interchange “jointly strongly epimorphic” and “jointly extremal-epimorphic” (see Lemma 2.1 above). We shall call a pullback-stable strong epimorphism **stably strong**.

Definition 2.5. A functor between categories with finite limits is called **coherent** if it preserves finite limits and jointly strongly epimorphic pairs.

Since a morphism is monic if and only if its kernel pair is the discrete equivalence relation, it follows that any functor which preserves kernel pairs, preserves monomorphisms. In particular every coherent functor preserves monomorphisms. Note that in a regular category a morphism f is a regular epimorphism if and only if (f, f) is a jointly strongly epimorphic pair. It easily follows that a coherent functor between regular categories is always **regular**, that is, it preserves finite limits and regular epimorphisms.

The next proposition shows that in the regular case, the above definition coincides with the one given in Section A.1.4 of [42].

Proposition 2.6. A regular functor between regular categories with binary joins of subobjects is coherent if and only if it preserves binary joins of subobjects.

Proof. Note that by Lemma 2.2 (b) a cospan (f, g) in a regular category is jointly strongly epimorphic if and only if the cospan $(\text{Im}(f), \text{Im}(g))$ is jointly strongly epimorphic. Note also that any regular functor preserves (regular epi, mono)-factorisations. Therefore the proof follows from Lemma 2.3: under either condition diagrams of the form as in Lemma 2.3 are preserved. $\square$

Proposition 2.7. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between categories with finite limits and binary coproducts. The following are equivalent:

- (i) F is coherent;
- (ii) F preserves strong epimorphisms and the comparison morphism

$$\begin{pmatrix} F(\iota_X) \\ F(\iota_Y) \end{pmatrix} : F(X) + F(Y) \rightarrow F(X + Y)$$

is a strong epimorphism for all $X, Y \in \mathcal{C}$.

When in addition $\mathcal{C}$ is pointed, these condition are further equivalent to:

- (iii) F preserves strong epimorphisms and joins;
- (iv) F preserves strong epimorphisms and joins of the form

$$X \xrightarrow{\iota_1} X + Y \xleftarrow{\iota_2} Y.$$

Proof. For any jointly strongly epimorphic cospan (f, g) over an object Z consider the diagram

$$\begin{array}{ccccc} & & F(X) + F(Y) & & \\ & \nearrow \iota_{F(X)} & \downarrow \begin{pmatrix} F(\iota_X) \\ F(\iota_Y) \end{pmatrix} & \nwarrow \iota_{F(Y)} & \\ & & F(X + Y) & & \\ & \nearrow F(\iota_X) & \downarrow F\begin{pmatrix} f \\ g \end{pmatrix} & \nwarrow F(\iota_Y) & \\ F(X) & \xrightarrow{F(f)} & F(Z) & \xleftarrow{F(g)} & F(Y). \end{array}$$

Suppose that (ii) holds. It follows from Lemma 2.4 and the fact that F preserves strong epimorphisms that $F\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ is a strong epimorphism. Therefore the vertical composite $F\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \left(\begin{smallmatrix} F(\iota_X) \\ F(\iota_Y) \end{smallmatrix}\right) = \left(\begin{smallmatrix} F(f) \\ F(g) \end{smallmatrix}\right)$ is a strong epimorphism and so according to Lemma 2.4 the cospan $(F(f), F(g))$ is jointly strongly epimorphic. This proves that (ii) implies (i). Since (iii) follows trivially from (i), and (iv) from (iii), it remains only to show that (iv) implies (ii). However this follows from Lemma 2.3 and 2.4. $\square$

Definition 2.8. A regular category with finite coproducts $\mathcal{C}$ is **coherent** in the sense of [42] (and called a **pre-logos** in [26]) if and only if, for any morphism $f: X \rightarrow Y$ in $\mathcal{C}$, the change-of-base functor $f^*: (\mathcal{C} \downarrow Y) \rightarrow (\mathcal{C} \downarrow X)$ is coherent.

The categories **Gp** and **Ab** (all groups, abelian groups) are well-known not to be coherent. In fact, the only semi-abelian (or, more generally, unital) coherent category is the trivial one. Recall from [8, 4] that a pointed finitely complete category is **unital** when for any pair of objects X, Y the cospan

$$X \xrightarrow{(1_X, 0)} X \times Y \xleftarrow{(0, 1_Y)} Y$$

is jointly strongly epimorphic.

Lemma 2.9. *Let $\mathcal{C}$ be a unital category. For each object X in $\mathcal{C}$ the pullback functor $[1_X, 1_X]^*: (\mathcal{C} \downarrow (X \times X)) \rightarrow (\mathcal{C} \downarrow X)$ is coherent if and only if X is a zero object.*

Proof. In the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longleftarrow & 0 \\ \downarrow & & \downarrow (1_X, 1_X) & & \downarrow \\ X & \xrightarrow{(1_X, 0)} & X \times X & \xleftarrow{(0, 1_X)} & X \end{array}$$

the two squares are pullbacks and $((1_X, 0), (0, 1_X))$ is a jointly strongly epimorphic cospan in $\mathcal{C}$, and hence in $(\mathcal{C} \downarrow (X \times X))$. It follows that $0 \rightarrow X$ is a strong epimorphism, so that X is isomorphic to 0. $\square$

Proposition 2.10. *If a unital category is coherent, then it is trivial.*

Proof. The proof follows trivially from Lemma 2.9. $\square$

However, we will see that in a unital category certain change of base functors are always coherent.

Lemma 2.11. *Let $\mathcal{C}$ be a unital category. If (f, g) and (f', g') are jointly strongly epimorphic cospans over Z and Z' respectively, then $(f \times f', g \times g')$ is a jointly strongly epimorphic cospan over $Z \times Z'$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & & & T' & & \\ & & & & \downarrow n' & & \\ & & & & S & & \\ & & & & \downarrow m & & \\ & & & & T & & \\ & & & & \downarrow n & & \\ X & \xrightarrow{f} & Z & \xleftarrow{g} & Y & & \\ \uparrow (1_X, 0) & & \uparrow (1_Z, 0) & & \uparrow (1_Y, 0) & & \\ X \times X' & \xrightarrow{f \times f'} & Z \times Z' & \xleftarrow{g \times g'} & Y \times Y' & & \\ \uparrow (0, 1_{X'}) & & \uparrow (0, 1_{Z'}) & & \uparrow (0, 1_{Y'}) & & \\ X' & \xrightarrow{f'} & Z' & \xleftarrow{g'} & Y' & & \\ & & & & \downarrow n' & & \\ & & & & T' & & \end{array}$$

where m monomorphism of cospans and the monomorphisms of cospans n and n' are obtained by pullback. Since (f, g) and (f', g') are jointly strongly epimorphic cospans it follows that n and n' are isomorphisms, respectively. Therefore since $\mathcal{C}$ is unital it follows that m is an isomorphism as required. $\square$

As an immediate corollary we obtain:

Lemma 2.12. *Let $\mathcal{C}$ be a unital category. For each object X in $\mathcal{C}$ the functor $X \times (-): \mathcal{C} \rightarrow \mathcal{C}$ is coherent and hence such are the change-of-base functors $\mathcal{C} \rightarrow (\mathcal{C} \downarrow X)$ and $\mathcal{C} \rightarrow \mathbf{Pt}_X(\mathcal{C})$ along $X \rightarrow 0$.* $\square$

2.13. Algebraically coherent categories. Considering that even the most basic algebraic categories are never coherent, it is natural to consider an algebraic variant of the concept, which involves change-of-base functors of the *fibration of points* instead of the *basic fibration*:

Definition 2.14. A category with finite limits is called **algebraically coherent** or **(CAlg)** if and only if for every morphism $f: X \rightarrow Y$ in $\mathcal{C}$, the change-of-base functor

$$f^*: \mathbf{Pt}_Y(\mathcal{C}) \rightarrow \mathbf{Pt}_X(\mathcal{C})$$

is coherent.

This definition means that given a cospan (u, v) in $\mathbf{Pt}_Y(\mathcal{C})$ and its pullback

$$\begin{array}{ccccc}
 & & A'' & \xrightarrow{\bar{u}} & A & \xleftarrow{\bar{v}} & A' \\
 & \swarrow g'' & \uparrow & & \uparrow g & & \swarrow g' \\
 B'' & \xrightarrow{u} & B & \xleftarrow{v} & B' & & \\
 \uparrow t'' & & \uparrow t & & \uparrow t' & & \\
 & \searrow q'' & \downarrow s'' & & \downarrow s & & \searrow q' \\
 & & X & \xrightarrow{f} & X & \xrightarrow{f} & X \\
 & & \downarrow p'' & & \downarrow p & & \downarrow p' \\
 & & Y & \xrightarrow{f} & Y & \xrightarrow{f} & Y
 \end{array} \quad (\mathbf{A})$$

along a morphism f in $\mathcal{C}$, if (u, v) is a jointly strongly epimorphic pair, then also the pair $(\bar{u}, \bar{v})$ is jointly strongly epimorphic. Note that we can interpret those conditions in $\mathcal{C}$ itself:

Lemma 2.15. *Any jointly strongly epimorphic pair in a category of points $\mathbf{Pt}_X(\mathcal{C})$ is still jointly strongly epimorphic when considered in $(\mathcal{C} \downarrow X)$ or even $\mathcal{C}$.*

Proof. Consider such a pair (u, v) in $\mathbf{Pt}_X(\mathcal{C})$ and a subobject m in $\mathcal{C}$.

$$\begin{array}{ccccc}
 & & M & & \\
 & \swarrow \bar{u} & \downarrow m & \nwarrow \bar{v} & \\
 A'' & \xrightarrow{u} & A & \xleftarrow{v} & A' \\
 \uparrow s'' & & \uparrow s & & \uparrow s' \\
 X & \xrightarrow{f} & X & \xrightarrow{f} & X
 \end{array}$$

Then, clearly, pm is split by $\bar{u}s'' = \bar{v}s'$: $X \rightarrow M$, thus m , $\bar{u}$ and $\bar{v}$ become morphisms of points. $\square$

2.16. Kernel functors. Lemma 4.3 (c) will tell us that any change-of-base functor along a stably strong epimorphism (and in particular along regular epimorphisms in a regular category) *reflects* jointly strongly epimorphic pairs. However, we first explore the protomodular case, where *all* change-of-base functors reflect jointly strongly epimorphic pairs. Using this result we will prove that when $\mathcal{C}$ is a pointed protomodular category, algebraic coherence can be expressed in terms of kernel functors alone.

Lemma 2.17. *If $\mathcal{C}$ is a protomodular category, then the change-of-base functors reflect jointly strongly epimorphic pairs.*

Proof. Consider a cospan (u, v) in $\mathbf{Pt}_Y(\mathcal{C})$ and morphism $f: X \rightarrow Y$ in $\mathcal{C}$. Since $\mathcal{C}$ is protomodular, the pairs (g'', t'') , (g, t) and (g', t') in the induced diagram **(A)** are jointly strongly epimorphic [4, Lemma 3.1.22]. Assuming that $(\bar{u}, \bar{v})$ is a jointly strongly epimorphic pair, we see that (u, v) is also jointly strongly epimorphic. $\square$

Lemma 2.18. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ be functors. If GF is coherent and G reflects jointly strongly epimorphic pairs, then F is coherent.*

Proof. Let (u, v) be a jointly strongly epimorphic cospan. Since GF is coherent and G reflects jointly strongly epimorphic cospans, it follows that $(GF(u), GF(v))$ and hence $(F(u), F(v))$ is a jointly strongly epimorphic cospan. $\square$

Proposition 2.19. *A protomodular category $\mathcal{C}$ with an initial object is algebraically coherent if and only if the change-of-base functors along each morphism from the initial object are coherent. In particular a pointed protomodular category is algebraically coherent if and only if the kernel functors are coherent.*

Proof. Since by Lemma 2.17 every change-of-base functor reflects jointly monomorphic pairs, the non-trivial implication follows from Lemma 2.18 applied to the commutative triangle

$$\begin{array}{ccc} \mathbf{Pt}_Y(\mathcal{C}) & \xrightarrow{f^*} & \mathbf{Pt}_X(\mathcal{C}) \\ & \searrow \downarrow !_Y^* & \swarrow \downarrow !_X^* \\ & \mathbf{Pt}_0(\mathcal{C}) & \end{array}$$

where $f: X \rightarrow Y$ is an arbitrary morphism in $\mathcal{C}$ and 0 is the initial object in $\mathcal{C}$. $\square$

2.20. Stability properties. Next we will show that if a category is algebraically coherent, then so are its slice and coslice categories and so is any full subcategory which is closed under products and subobjects.

Proposition 2.21. *If a category $\mathcal{C}$ is algebraically coherent, then, for any X in $\mathcal{C}$, the categories $(\mathcal{C} \downarrow X)$ and $(X \downarrow \mathcal{C})$ are also algebraically coherent.*

Proof. Given a morphism in the slice category $(\mathcal{C} \downarrow X)$, so a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ & \searrow \alpha & \swarrow \beta \\ & X & \end{array}$$

in $\mathcal{C}$, there are isomorphisms of categories

$$\mathbf{Pt}_{(Y, \alpha)}(\mathcal{C} \downarrow X) \cong \mathbf{Pt}_Y(\mathcal{C}) \quad \text{and} \quad \mathbf{Pt}_{(Z, \beta)}(\mathcal{C} \downarrow X) \cong \mathbf{Pt}_Z(\mathcal{C})$$

making the diagram

$$\begin{array}{ccc} \mathrm{Pt}_{(Z,\beta)}(\mathcal{C} \downarrow X) & \xrightarrow{\cong} & \mathrm{Pt}_Z(\mathcal{C}) \\ (f \downarrow X)^* \downarrow & & \downarrow f^* \\ \mathrm{Pt}_{(Y,\alpha)}(\mathcal{C} \downarrow X) & \xrightarrow{\cong} & \mathrm{Pt}_Y(\mathcal{C}) \end{array}$$

commute. It follows that $(f \downarrow X)^*$ is coherent whenever f^* is. A similar argument holds for the coslice category $(X \downarrow \mathcal{C})$. $\square$

Corollary 2.22. *If a category $\mathcal{C}$ is algebraically coherent, then any fibre $\mathrm{Pt}_X(\mathcal{C})$ is also algebraically coherent.*

Proof. Since $\mathrm{Pt}_X(\mathcal{C}) = ((X, 1_X) \downarrow (\mathcal{C} \downarrow X))$, this follows from Proposition 2.21. $\square$

Proposition 2.23. *If $\mathcal{B}$ is a full subcategory of an algebraically coherent category $\mathcal{C}$ closed under finite products and subobjects, then $\mathcal{B}$ is algebraically coherent. In particular, any (regular epi)-reflective subcategory of an algebraically coherent category is algebraically coherent.*

Proof. We have to show that, for any morphism $f: X \rightarrow Y$ in $\mathcal{B}$, the change-of-base functor $f^*: \mathrm{Pt}_Y(\mathcal{B}) \rightarrow \mathrm{Pt}_X(\mathcal{B})$ is coherent. Since—the category $\mathcal{B}$ being closed under products and subobjects in $\mathcal{C}$ —this functor is a restriction of the change-of-base functor $f^*: \mathrm{Pt}_Y(\mathcal{C}) \rightarrow \mathrm{Pt}_X(\mathcal{C})$, it suffices to note that cospans in $\mathcal{B}$ are jointly strongly epimorphic in $\mathcal{B}$ if and only if they are in $\mathcal{C}$. $\mathcal{B}$ being closed under subobjects in $\mathcal{C}$, this is indeed the case. $\square$

It is worth spelling out what Proposition 2.7 means in a pointed protomodular category with pushouts.

Proposition 2.24. *A pointed protomodular category with pushouts $\mathcal{C}$ is algebraically coherent if and only if for every diagram of split extensions of the form*

$$\begin{array}{ccccc} H & \xrightarrow{\quad} & K & \xleftarrow{\quad} & L \\ \downarrow h & & \downarrow & & \downarrow l \\ A & \xrightarrow{\iota_A} & A +_X C & \xleftarrow{\iota_C} & C \\ \uparrow s' \downarrow p' & & \uparrow s \downarrow p & & \uparrow s'' \downarrow p'' \\ X & \xlongequal{\quad} & X & \xlongequal{\quad} & X \end{array} \quad (\mathbf{B})$$

the induced arrow $H + L \rightarrow K$ is a strong epimorphism.

Proof. This is a combination of Proposition 2.7 (i) $\Leftrightarrow$ (ii) and Proposition 2.19. $\square$

This result may be rephrased as follows. Note the resemblance with the strong protomodularity condition (cf. Theorem 4.18).

Corollary 2.25. *A homological category with pushouts is algebraically coherent if and only if for every diagram such as (B), K is the join of H and L in $A +_X C$. $\square$*

2.26. Coherence in terms of the functors $X\mathfrak{b}(-)$. We end this section with a characterisation of algebraic coherence in terms of the action comonad $X\mathfrak{b}(-)$. Recall from [15] that $X\mathfrak{b}(-): \mathcal{C} \rightarrow \mathcal{C}$ takes an object Y and sends it to the kernel in the short exact sequence

$$0 \longrightarrow X\mathfrak{b}Y \xrightarrow{\kappa_{X,Y}} X + Y \xleftarrow[\iota_1]{\begin{pmatrix} 1_X \\ 0 \end{pmatrix}} X \longrightarrow 0.$$

This functor is part of a comonad on $\mathcal{C}$, induced by the adjunction $(X + (-)) \dashv \text{Ker}$, for which the algebras are called **internal X -actions** and which gives rise to an equivalence $\text{Pt}_X(\mathcal{C}) \simeq \mathcal{C}^{X\flat(-)}$.

Lemma 2.27. *If $\mathcal{C}$ is a pointed algebraically coherent category with binary coproducts, then for any object X , the functor $X\flat(-): \mathcal{C} \rightarrow \mathcal{C}$ preserves jointly strongly epimorphic pairs.*

Proof. This follows from the fact that kernel functors are coherent while left adjoints preserves jointly strongly epimorphic pairs. $\square$

Lemma 2.28. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ be functors such that $\mathcal{C}$ has binary coproducts and F preserves them, F preserves jointly strongly epimorphic pairs, G preserves finite limits and strong epimorphisms, and for every D in $\mathcal{D}$ there exists a strong epimorphism $F(C) \rightarrow D$. GF preserves jointly strongly epimorphic pairs if and only if G is coherent.*

Proof. The “if” part follows from the fact that the composite of functors which preserve jointly strongly epimorphic pairs, preserves jointly strongly epimorphic pairs. For the “only if” part let (g_1, g_2) be a jointly strongly epimorphic cospan and construct the diagram

$$\begin{array}{ccccc} F(C_1) & \xrightarrow{F(\iota_1)} & F(C_1 + C_2) & \xleftarrow{F(\iota_2)} & F(C_2) \\ e_1 \downarrow & & \downarrow e & & \downarrow e_2 \\ D_1 & \xrightarrow{g_1} & D & \xleftarrow{g_2} & D_2 \end{array}$$

where e_1 and e_2 are arbitrary strong epimorphism existing by assumption, and e is induced by the coproduct. Since (g_1, g_2) is jointly strongly epimorphic, e is necessarily strong by Lemma 2.4. Therefore, since G preserves extremal epimorphisms and GF is coherent it follows that

$$(G(e)GF(\iota_1), G(e)GF(\iota_2)) = (G(g_1)G(e_1), G(g_2)G(e_2))$$

is a jointly strongly epimorphic cospan, and so $(G(g_1), G(g_2))$ is jointly strongly epimorphic by Lemma 2.2. $\square$

One situation where this lemma applies is when $F \dashv G$ is an adjunction with a strongly epimorphic counit. Taking $(X + (-)) \dashv \text{Ker}$ we find, for instance:

Theorem 2.29. *Let $\mathcal{C}$ be a protomodular category with binary coproducts in which strong epimorphisms are pullback-stable. $\mathcal{C}$ is algebraically coherent if and only if for every X , the functor $X\flat(-): \mathcal{C} \rightarrow \mathcal{C}$ preserves jointly strongly epimorphic pairs.* $\square$

Corollary 2.30. *Let $\mathcal{C}$ be a regular protomodular category with binary coproducts. $\mathcal{C}$ is algebraically coherent if and only if for every X , the functor $X\flat(-): \mathcal{C} \rightarrow \mathcal{C}$ preserves jointly strongly epimorphic pairs.* $\square$

In the article [47], the authors consider a variation on this condition, and assume that the functors $X\flat(-)$ preserve jointly epimorphic pairs.

3. EXAMPLES, NON-EXAMPLES AND COUNTEREXAMPLES

Proposition 3.1. *Any finitely complete naturally Mal'tsev category [41] is algebraically coherent.*

Proof. If $\mathcal{C}$ is naturally Mal'tsev, then for any object X of $\mathcal{C}$, the category $\mathbf{Pt}_X(\mathcal{C})$ of points over B is naturally Mal'tsev, pointed and finitely complete, hence it is additive by the proposition in [41]. As a consequence, the change-of-base functors $f^*: \mathbf{Pt}_Y(\mathcal{C}) \rightarrow \mathbf{Pt}_X(\mathcal{C})$ all preserve binary sums. $\square$

Examples 3.2. The following are algebraically coherent: all abelian categories, all additive categories, all affine categories in the sense of [17].

Note, however, that some of the results we shall prove in Section 4 apply only to semi-abelian categories, so need not apply to all the examples above. On the other hand, being semi-abelian is not enough for algebraic coherence.

Examples 3.3. Not all semi-abelian (or even strongly semi-abelian) varieties are algebraically coherent. We list some, together with the consequence of algebraic coherence which they lack: (commutative) loops and digroups (since by the results in [4, 9, 34] they do not satisfy (SH), see Theorem 4.15 below), non-associative rings (or algebras in general), Jordan algebras (since as explained in [20, 21] they need not satisfy (NH), see Theorem 4.15), and Heyting semilattices (which, as explained in [49], form an arithmetical [4, 52] Moore category [55] that does not satisfy (SSH), see Theorem 4.16).

It is well known [42, Lemma 1.5.13] that *any finitely cocomplete locally cartesian closed category is coherent*. We find the following algebraic version of this classical result. We recall from [29, 14] that a finitely complete category $\mathcal{C}$ is said to be **locally algebraically cartesian closed** (satisfies condition **(LACC)**) when, for every $f: X \rightarrow Y$ in $\mathcal{C}$, the change-of-base functor $f^*: \mathbf{Pt}_Y(\mathcal{C}) \rightarrow \mathbf{Pt}_X(\mathcal{C})$ is a left adjoint.

Theorem 3.4. *Any locally algebraically cartesian closed regular category with push-outs is algebraically coherent.*

Proof. This is a consequence of the fact that under (LACC), the change-of-base functors preserve limits and colimits. $\square$

Example 3.5. The category of cocommutative Hopf algebras over a field K of characteristic zero is semi-abelian as explained in [43]. It is also locally algebraically cartesian closed by Proposition 5.3 in [29], being the category of internal groups in the category of cocommutative coalgebras, which is cartesian closed as shown in [1, Theorem 5.3]. Incidentally, via 4.4 in [6], the same argument suffices to show that the category $\mathbf{HopfAlg}_{K, \text{coc}}$ has representable object actions. More generally, the same holds for internal groups in the category of cocommutative comonoids in any so-called *admissible* (symmetric) monoidal closed category, which is cartesian closed by 3.2 in [54].

Before treating further algebraic examples, let us first consider those given by Topos Theory.

Proposition 3.6. *Any coherent category is algebraically coherent.*

Proof. This is an immediate consequence of Lemma 2.15. $\square$

Examples 3.7. This provides us with all elementary toposes as examples (sets, finite sets, sheaves, etc.).

Example 3.8. The dual of the category of pointed sets is semi-abelian [10] algebraically coherent. One way to verify this is by the dual of the condition of Proposition 2.24 in the category $\mathbf{Set}_*$. Given two elements of $A \times_X C$, it suffices to check all relevant cases to see that it is still possible to separate them after X has been collapsed. The same argument is valid to prove that $\mathcal{E}_*^{\text{op}}$ is algebraically

coherent when $\mathcal{E}$ is any boolean topos: the existence of complements allows us to express the cokernel of a monomorphism $m: M \rightarrow X$ in $\mathbf{Pt}_1(\mathcal{E})$ as a disjoint union $(X \setminus M) \sqcup 1$. Indeed, in the diagram in $\mathcal{E}$

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ X \setminus M & \longrightarrow & X & \longrightarrow & X/M \end{array}$$

each square is simultaneously a pullback and a pushout—see [42]. Being given the opposite

$$\begin{array}{ccccc} (A \setminus X) \sqcup 1 & \xleftarrow{\bar{f}} & (B \setminus X) \sqcup 1 & \xrightarrow{\bar{g}} & (C \setminus X) \sqcup 1 \\ \uparrow & & \uparrow & & \uparrow \\ A & \xleftarrow{f} & B & \xrightarrow{g} & C \\ \uparrow p' & & \uparrow p & & \uparrow p'' \\ X & \xleftarrow{s'} & X & \xrightarrow{s} & X \end{array}$$

of diagram **(B)** in $\mathbf{Pt}_1(\mathcal{E})$, we now have to prove that $\bar{f}$ and $\bar{g}$ are jointly (strongly) monic when such are f and g . Being given $b, b' \in (B \setminus X) \sqcup 1$ such that $\bar{f}(b) = \bar{f}(b')$ and $\bar{g}(b) = \bar{g}(b')$, we shall see that $b = b'$. Without loss of generality, as follows from $\mathcal{E}$ being lextensive, we may assume that one of the three cases

- (a) $b, b' \in B \setminus X$;
- (b) $b \in B \setminus X$ and $b' \in 1$;
- (c) $b \in 1$ and $b' \in B \setminus X$

is satisfied, of which only the first leads to further work. Things are fine if either $\bar{f}(b) = \bar{f}(b')$ or $\bar{g}(b) = \bar{g}(b')$ is outside 1. When, however, both $\bar{f}(b) = 1$ and $\bar{g}(b) = 1$, then $f(b) = fps(b)$ and $g(b) = gps(b)$, which proves that $b = ps(b) \in X$.

Example 3.9. The category **Top** of topological spaces and continuous maps is not coherent, because it is not even regular.

In fact, **Top** it is not *algebraically* coherent either, since the change-of-base functors of the fibration of points need not be regular. To see this, let us consider the following variation on Counterexample 2.4.5 in [3]. A is the set $\{a, b, c, d\}$ with topology generated by $\{\{a, b\}\}$, the topology on $B = \{l, m, n\}$ is generated by $\{\{l, m\}\}$, while $C = \{x, y, z\}$ and $D = \{i, j\}$ carry the indiscrete topology. Consider the continuous maps $f: A \rightarrow C$, $s: D \rightarrow A$, $p: C \rightarrow D$ and $g: B \rightarrow D$ given as follows:

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline f & x & y & y & z \end{array} \quad \begin{array}{c|cc} & i & j \\ \hline s & d & c \end{array} \quad \begin{array}{c|ccc} & x & y & z \\ \hline p & i & j & i \end{array} \quad \begin{array}{c|ccc} & l & m & n \\ \hline g & i & i & i \end{array}$$

Then f is actually a regular epimorphism $(pf, s) \rightarrow (p, fs)$ in $\mathbf{Pt}_D(\mathbf{Top})$. However, its image g^*f through the change-of-base functor $g^*: \mathbf{Pt}_D(\mathbf{Top}) \rightarrow \mathbf{Pt}_B(\mathbf{Top})$ is a surjection, but not a regular epimorphism. Indeed, the subset

$$\{(a, (x, l)), (a, (x, m))\} = (g^*f)^{-1}(\{(x, l), (x, m)\})$$

of $A \times_C (C \times_D B)$ is open in the subspace topology coming from $A \times (C \times_D B)$, where

$$\{a, b\} \times \{(x, l), (x, m), (z, l), (z, m)\}$$

is open as a product of an open in A with an open in $C \times_D B$. Yet $\{(x, l), (x, m)\}$ itself is not open in $C \times_D B$, which shows us that $C \times_D B$ does not carry the quotient topology induced by g^*f .

In general, (compact) Hausdorff algebras over an algebraically coherent semi-abelian theory are still algebraically coherent.

Theorem 3.10. *Let $\mathbb{T}$ be a theory such that $\mathbf{Set}^{\mathbb{T}}$ is an algebraically coherent semi-abelian variety. Then the homological category $\mathbf{Haus}^{\mathbb{T}}$ and the semi-abelian category $\mathbf{HComp}^{\mathbb{T}}$ are algebraically coherent.*

Proof. The category $\mathbf{Haus}^{\mathbb{T}}$ is homological and $\mathbf{HComp}^{\mathbb{T}}$ is semi-abelian by the results in [5], so we may use Proposition 2.24 to show their algebraic coherence.

Let us consider a diagram like **(B)** in $\mathbf{Haus}^{\mathbb{T}}$. Since $\mathbf{Set}^{\mathbb{T}}$ is algebraically coherent, we have $H \vee L = K$. Given a subset $S \subset K$ which is open in the final topology on K induced by $H, L \subset K$, we have to prove that S is open in the subspace topology induced by $K \subset A +_X C$. By definition, $H \cap S$ and $L \cap S$ are open in H and in L , respectively. Since for Hausdorff algebras kernels are closed [5, Proposition 26], $(H \cap S) \cup (A \setminus H)$ and $(L \cap S) \cup (C \setminus L)$ are open in A and in C , respectively. Hence $S \cup ((A +_X C) \setminus K)$ is open in $A +_X C$, which carries the final topology.

Since limits in $\mathbf{HComp}$ are computed in $\mathbf{Top}$, this proof also works for compact Hausdorff algebras. $\square$

To make full use of this result, we need further examples of algebraically coherent semi-abelian varieties of algebras. One class of such are the *categories of interest* in the sense of [51].

Definition 3.11. A **category of interest** is a variety of universal algebras whose theory contains a unique constant 0, a set Ω of finitary operations and a set of identities $\mathbb{E}$ such that:

- (COI1) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$, where Ω_i is the set of i -ary operations;
- (COI2) $\Omega_0 = \{0\}$, $- \in \Omega_1$ and $+ \in \Omega_2$, where Ω_i is the set of i -ary operations, and $\mathbb{E}$ includes the group laws for 0, $-$, $+$; define $\Omega'_1 = \Omega_1 \setminus \{-\}$, $\Omega'_2 = \Omega_2 \setminus \{+\}$;
- (COI3) for any $* \in \Omega'_2$, the set Ω'_2 contains $*^{\text{op}}$ defined by $x *^{\text{op}} y = y * x$;
- (COI4) for any $\omega \in \Omega'_1$, $\mathbb{E}$ includes the identity $\omega(x + y) = \omega(x) + \omega(y)$;
- (COI5) for any $* \in \Omega'_2$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;
- (COI6) for any $\omega \in \Omega'_1$ and $* \in \Omega'_2$, $\mathbb{E}$ includes the identity $\omega(x) * y = \omega(x * y)$;
- (COI7) for any $* \in \Omega'_2$, $\mathbb{E}$ includes the identity $x + (y * z) = (y * z) + x$;
- (COI8) for any $*, * \in \Omega'_2$, there exists a word w such that $\mathbb{E}$ includes the identity

$$(x * y) * z = w(x * _1 (y * _1 z), \dots, x * _m (y * _m z), y * _{m+1} (x * _{m+1} z), \dots, y * _n (x * _n z))$$

where $*_1, \dots, *_n$ and $*_1, \dots, *_n$ are operations in Ω'_2 .

Lemma 3.12. *Let $\mathcal{C}$ be a variety of universal algebras whose theory contains a unique constant 0, a set of finitary operations Ω , and a set of identities $\mathbb{E}$ such that (COI1)–(COI5) of Definition 3.11 hold. For every B in $\mathcal{C}$ define $\mathcal{C}_B$ to be a new variety whose theory contains a unique constant 0, a set of finitary operations Ω_B , and a set of identities $\mathbb{E}_B$ such that:*

- (a) $\Omega_B = \Omega_{B_0} \cup \Omega_{B_1} \cup \Omega_{B_2}$, where Ω_{B_i} is the set of i -ary operations;
- (b) $\Omega_{B_0} = \Omega_0$, $\Omega_{B_2} = \Omega_2$ and $\Omega_{B_1} = \Omega_1 \sqcup \Theta_1$ where $\Theta_1 = \{u_{b,*} \mid b \in B, * \in \Omega_2\}$;
- (c) $\mathbb{E}_B$ has the same identities as in $\mathbb{E}$ but in addition for each $u_{b,*}$ in Θ_1 the identity $u_{b,*}(x + y) = u_{b,*}(x) + u_{b,*}(y)$.

The functor $I_B: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}_B$ sending a split epimorphism

$$A \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} B$$

to the kernel of α with all operations induced by those on A except for the unary operations $u_{b,*}$ which are defined by

$$u_{b,*}(x) = \begin{cases} \beta(b) + x - \beta(b) & \text{if } * = + \\ \beta(b) * x & \text{otherwise} \end{cases}$$

is such that $\overline{\mathcal{C}_B} = I_B(\text{Pt}_B(\mathcal{C}))$ is a subvariety of $\mathcal{C}_B$ and $I_B: \text{Pt}_B(\mathcal{C}) \rightarrow \overline{\mathcal{C}_B}$ is an equivalence of categories.

Moreover if conditions (COI6)–(COI8) of Definition 3.11 also hold, then for every n -ary word w of $\overline{\mathcal{C}_B}$ there exists an m -ary word w' of $\mathcal{C}$ and unary words $u_{i,1}, u_{i,2}, \dots, u_{i,m_i}$ of $\overline{\mathcal{C}_B}$ for each i in $\{1, \dots, n\}$ such that

$$w(x_1, \dots, x_n) = w'(u_{1,1}(x_1), \dots, u_{1,m_1}(x_1), u_{2,1}(x_2), \dots, u_{2,m_2}(x_2), \dots, u_{n,1}(x_n), \dots, u_{n,m_n}(x_n)).$$

Proof. For a semi-abelian category, kernel functors are always faithful, since they preserve equalisers and reflect isomorphisms. Hence the functor I_B is faithful too, because the kernel functor factors through it. Since the kernel functor reflects limits it follows that I_B does too. This proves that $\text{Pt}_B(\mathcal{C})$ is closed under limits in $\mathcal{C}_B$.

For each X in $\mathcal{C}_B$ we can define all operations in Ω on $X \times B$ as follows:

$$\begin{aligned} 0 &= (0, 0) && \text{is the unique constant} \\ u(x, b) &= (u(x), u(b)) && \text{for each } u \text{ in } \Omega'_1 \\ -(x, b) &= (u_{-b,+}(-x), -b) \\ (x, b) + (y, c) &= (x + u_{b,+}(y), b + c) \\ (x, b) * (y, c) &= (x * y + u_{b,*}(y) + u_{c,*^{\text{op}}}(x), b * c) && \text{for each } * \text{ in } \Omega'_2. \end{aligned}$$

These operations are such that the maps $\pi_2: X \times B \rightarrow B$ and $[0, 1_B]: B \rightarrow X \times B$ preserve them. If

$$X = I_B(A \xrightleftharpoons[\beta]{\alpha} B)$$

then the map $\varphi: X \times B \rightarrow A$ defined by $\varphi(x, b) = x + \beta(b)$ is a bijection which preserves all operations. Indeed

$$\varphi(u(x, b)) = \varphi(u(x), u(b)) = u(x) + \beta(u(b)) = u(\varphi(x, b))$$

$$\begin{aligned} \varphi((x, b) + (y, c)) &= x + u_{b,+}(y) + \beta(b + c) \\ &= x + \beta(b) + y - \beta(b) + \beta(b) + \beta(c) \\ &= \varphi(x, b) + \varphi(y, c) \end{aligned}$$

$$\begin{aligned} \varphi((x, b) * (y, c)) &= x * y + u_{b,*}(y) + u_{c,*^{\text{op}}}(x) + \beta(b * c) \\ &= x * y + \beta(b) * y + x * \beta(c) + \beta(b) * \beta(c) \\ &= (x + \beta(b)) * (y + \beta(c)) \\ &= \varphi(x, b) * \varphi(y, c). \end{aligned}$$

Next we will show that for each $f: X \rightarrow X'$ in $\mathcal{C}_B$ the map $f \times 1_B: X \times B \rightarrow X' \times B$ which trivially makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{(1_X, 0)} & X \times B \xrightleftharpoons[(0, 1_B)]{\pi_2} B \\ f \downarrow & & \downarrow f \times 1_B \parallel \\ X' & \xrightarrow{(1_{X'}, 0)} & X' \times B \xrightleftharpoons[(0, 1_B)]{\pi_2} B \end{array}$$

commute also preserves the operations defined above. We have

$$\begin{aligned}(f \times 1_B)(u(x, b)) &= (f \times 1_B)(u(x), u(b)) = (f(u(x)), u(b)) \\ &= u((f \times 1_B)(x, b))\end{aligned}$$

$$\begin{aligned}(f \times 1_B)((x, b) + (y, c)) &= (f \times 1_B)(x + u_{b,+}(y), b + c) \\ &= (f(x + u_{b,+}(y)), b + c) \\ &= (f(x), b) + (f(y), c) \\ &= (f \times 1_B)(x, b) + (f \times 1_B)(y, c)\end{aligned}$$

$$\begin{aligned}(f \times 1_B)((x, b) * (y, c)) &= (f \times 1_B)(x * y + u_{b,*}(y) + u_{c,*^{\text{op}}}(x), b * c) \\ &= (f(x * y + u_{b,*}(y) + u_{c,*^{\text{op}}}(x)), b * c) \\ &= (f(x), b) * (f(y), c) \\ &= (f \times 1_B)(x, b) * (f \times 1_B)(y, c).\end{aligned}$$

This means that I_B is full, and also that $\text{Pt}_B(\mathcal{C})$ is closed under monomorphisms and quotients in $\mathcal{C}_B$. Indeed, $f \times 1_B$ is a monomorphism or a regular epimorphism as soon as f is.

It is easy to check that

$$\begin{array}{ll}0 + x = x & \text{using (COI2)} \\ 0 * x = 0 & \text{when } * \neq + \text{ using (COI5)} \\ -(x + y) = (-x) + (-y) & \text{using (COI2)} \\ -(x * y) = (-x) * y & \text{when } * \neq + \text{ using (COI2), (COI2) and (COI5)}\end{array}$$

and for each u in Ω'_1

$$\begin{array}{ll}u(x + y) = u(x) + u(y) & \text{using (COI4)} \\ u(x * y) = u(x) * y & \text{using (COI6)}\end{array}$$

which means that for each n -ary word w from $\mathcal{C}$ there exists an n -ary word w' built using only operations from Ω_2 and unary words $u_1, \dots, u_n$ which are composites of operations from Ω_1 such that $w(x_1, \dots, x_n) = w'(u_1(x_1), \dots, u_n(x_n))$. It is also easy to check that for each $u_{b,*}$ in Θ_1

$$\begin{array}{ll}u_{b,*}(x + y) = u_{b,*}(x) + u_{b,*}(y) & \text{using (COI2) for } * = + \text{ and (COI5) otherwise} \\ u_{b,*}(x * y) = x * y & \text{when } * = + \text{ and } * \neq + \text{ using (COI2), (COI7).}\end{array}$$

When $* \neq +$ and $* \neq +$, according to (COI3) and (COI8) and what was proved above, there exists a word w built using only operations from Ω_2 and unary words $u_1, \dots, u_n$ which are composites of operations from Ω_1 such that

$$\begin{aligned}u_{b,*}(x * y) &= w(u_1(x *_{*1} (u_{b,*_1}(y))), \dots, u_m(x *_{*m} ((u_{b,*_m}(y))))) \\ &\quad u_{m+1}(y *_{*m+1} (u_{b,*_{m+1}}(x))), \dots, u_n(y_n *_{*n} (u_{b,*_n}(x)))) \\ &= w(x *_{*1} (u_{b,*_1}(u_1(y))), \dots, x *_{*m} ((u_{b,*_m}(u_m(y))))) \\ &\quad y *_{*m+1} (u_{b,*_{m+1}}(u_{m+1}(x))), \dots, y_n *_{*n} (u_{b,*_n}(u_n(X)))).\end{aligned}$$

The final claim follows by induction. $\square$

Lemma 3.13. *Let $U: \mathcal{B} \rightarrow \mathcal{C}$ be a forgetful functor between varieties such that for each n -ary word w in $\mathcal{B}$ there exists an m -ary word w' in $\mathcal{C}$ and unary words*

$u_{i,1}, u_{i,2}, \dots, u_{i,m_i}$ in $\mathcal{B}$ for each $i \in \{1, \dots, n\}$ such that

$$w(x_1, \dots, x_n) = w'(u_{1,1}(x_1), \dots, u_{1,m_1}(x_1), \\ u_{2,1}(x_2), \dots, u_{2,m_2}(x_2), \dots, u_{n,1}(x_n), \dots, u_{n,m_n}(x_n)).$$

The functor U is coherent.

Proof. Since every element of $U(X+Y)$ is of the form $a = w(x_1, \dots, x_k, y_{k+1}, \dots, y_n)$ for some n -ary word w from $\mathcal{B}$, where $x_1, \dots, x_k$ are in X and $y_{k+1}, \dots, y_n$ are in Y , it follows by assumption that there exist a word w' from $\mathcal{C}$ and $u_{i,1}, u_{i,2}, \dots, u_{i,m_i}$ for each $i \in \{1, \dots, n\}$ in $\mathcal{B}$ such that

$$w(x_1, \dots, x_n) = w'(u_{1,1}(x_1), \dots, u_{1,m_1}(x_1), \\ u_{2,1}(x_2), \dots, u_{2,m_2}(x_2), \dots, u_{n,1}(x_n), \dots, u_{n,m_n}(x_n)).$$

Therefore, since each $u_i(x_i)$ is in X and each $u_i(y_i)$ is in Y it follows that a is in the image of

$$\begin{pmatrix} U(\iota_1) \\ U(\iota_2) \end{pmatrix} : U(X) + U(Y) \rightarrow U(X + Y)$$

and so U is coherent by Lemma 2.7. $\square$

Theorem 3.14. *Every category of interest is algebraically coherent.*

Proof. The proof is a consequence of Lemma 3.12 and 3.13 because any kernel functor $\text{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$ factors into an equivalence $I_B : \text{Pt}_B(\mathcal{C}) \rightarrow \overline{\mathcal{C}_B}$ followed by a coherent functor $U : \overline{\mathcal{C}_B} \rightarrow \mathcal{C}$. $\square$

Examples 3.15. The categories of groups and non-unital (Boolean) rings are algebraically coherent semi-abelian categories, as are the categories of associative algebras, Lie algebras, Leibniz algebras, Poisson algebras over a commutative ring with unit, all *varieties of groups* in the sense of [50], and all categories of compact Hausdorff spaces over such.

Proposition 3.16. *If $\mathcal{C}$ is a semi-abelian algebraically coherent category and X is an object of $\mathcal{C}$, then the category $\text{Act}_X(\mathcal{C}) = \mathcal{C}^{X^{\text{b}}(-)}$ of X -actions in $\mathcal{C}$ is semi-abelian algebraically coherent.*

Proof. This is an immediate consequence of Proposition 2.22, using the equivalence between actions and points from [15], see also 2.26. $\square$

Proposition 3.17. *If $\mathcal{C}$ is algebraically coherent, then so is any category of diagrams in $\mathcal{C}$. In particular, the category $\text{RG}(\mathcal{C})$ of reflexive graphs in $\mathcal{C}$ is algebraically coherent.*

If, moreover, $\mathcal{C}$ is exact Mal'tsev, then also the category $\text{Cat}(\mathcal{C})$ of internal categories (= internal groupoids) in $\mathcal{C}$ and the category $\text{Gp}(\mathcal{C})$ of internal groups in $\mathcal{C}$ are algebraically coherent. As a consequence, the category $\text{Eq}(\mathcal{C})$ of (effective) equivalence relations in $\mathcal{C}$ is algebraically coherent.

If, moreover, $\mathcal{C}$ is semi-abelian then, by equivalence, the categories $\text{PMod}(\mathcal{C})$ and $\text{XMod}(\mathcal{C})$ of (pre)crossed modules in $\mathcal{C}$ are algebraically coherent.

Proof. Since in a functor category, limits and colimits are pointwise, the passage to categories of diagrams in $\mathcal{C}$ is obvious. Now assume that $\mathcal{C}$ is exact Mal'tsev. Since the category of internal categories of $\mathcal{C}$ is (regular epi)-reflective in $\text{RG}(\mathcal{C})$, we have that $\text{Cat}(\mathcal{C})$ is algebraically coherent by Proposition 2.23. By the same result, also $\text{Gp}(\mathcal{C})$ is algebraically coherent. In turn, following [27, 7], we see that the category $\text{Eq}(\mathcal{C})$ is (regular epi)-reflective in $\text{Cat}(\mathcal{C})$. The final claim in the semi-abelian context now follows from the results of [35]. $\square$

Examples 3.18. Crossed modules (of groups, rings, Lie algebras, etc.); n -cat-groups, for all n [44]; groups in a coherent category.

Proposition 3.19. *If $\mathcal{C}$ is an algebraically coherent exact Mal'tsev category, then*

- (a) *the category $\text{Arr}(\mathcal{C})$ of arrows in $\mathcal{C}$,*
- (b) *its full subcategory $\text{Ext}(\mathcal{C})$ determined by the extensions (= regular epimorphisms), and*
- (c) *the category $\text{CExt}_{\mathcal{B}}(\mathcal{C})$ of $\mathcal{B}$ -central extensions [37] in $\mathcal{C}$, for any Birkhoff subcategory $\mathcal{B}$ of $\mathcal{C}$,*

are all algebraically coherent.

Proof. (a) follows from Proposition 3.17 since $\text{Arr}(\mathcal{C})$ is a category of diagrams in $\mathcal{C}$. So does (b), because $\text{Ext}(\mathcal{C})$ and $\text{Eq}(\mathcal{C})$ are equivalent categories. (c) now follows from (b) by Proposition 2.23. $\square$

Examples 3.20. Inclusions of normal subgroups (considered as a full subcategory of $\text{Arr}(\mathbf{Gp})$); central extensions of groups, Lie algebras, crossed modules, etc.; discrete fibrations of internal categories (considered as a full subcategory of $\text{Arr}(\text{Cat}(\mathcal{C}))$) in an algebraically coherent semi-abelian category $\mathcal{C}$ [27, Theorem 3.2].

Proposition 3.21. *Any sub-quasivariety (in particular, any subvariety) of an algebraically coherent variety is algebraically coherent.*

Proof. Since any sub-quasivariety is a (regular epi)-reflective subcategory [45], this follows from Proposition 2.23. $\square$

Examples 3.22. n -nilpotent or n -solvable groups, rings, Lie algebras etc.; torsion-free (abelian) groups, reduced rings.

3.23. Coherence of the functor $X \diamond (-)$, normality of unions, and two-nilpotency. Consider a cospan $(k: K \rightarrow X, l: L \rightarrow X)$ in $\mathcal{C}$. Following [47], we compute the **Higgins commutator** $[K, L] \leq X$ as in the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K \diamond L & \xrightarrow{\iota_{K,L}} & K + L & \xrightarrow{\sigma_{K,L}} & K \times L \longrightarrow 0 \\
 & & \downarrow & & \downarrow \begin{pmatrix} k \\ l \end{pmatrix} & & \\
 & & [K, L] & \longrightarrow & X & &
 \end{array} \tag{C}$$

where $\iota_{K,L}$ is the kernel of $\sigma_{K,L}$ and $[K, L]$ is its regular image through $\begin{pmatrix} k \\ l \end{pmatrix}$.

In contrast with Lemma 2.27, even in a semi-abelian algebraically coherent category $\mathcal{C}$, the **co-smash product** functors $X \diamond (-): \mathcal{C} \rightarrow \mathcal{C}$ for $X \in \mathcal{C}$ introduced in [18, 47] need not preserve jointly strongly epimorphic pairs in general. Indeed, this would imply that Higgins commutators in $\mathcal{C}$ distribute over joins, but this property fails in $\mathbf{Gp}$, as the following example shows.

Example 3.24. Let us consider the symmetric group $X = S_4$ and its subgroups

$$K = \langle (12) \rangle, \quad L = \langle (23) \rangle \quad \text{and} \quad M = \langle (34) \rangle.$$

Then $L \vee M = \langle (23), (34) \rangle$, $[K, L] = \langle (123) \rangle$, and $[K, M] = 0$, while $[K, L \vee M]$ is the alternating group A_4 . That is:

$$[K, L \vee M] \neq [K, L] \vee [K, M].$$

On the other hand, this condition on the functors $X \diamond (-)$ *does* imply algebraic coherence. It is well known that, for any $X, Y \in \mathcal{C}$, the object $X \bowtie Y$ decomposes as $(X \diamond Y) \vee Y$. As a consequence, if $X \diamond (-)$ preserves jointly strongly epimorphic pairs, then so does the functor $X \bowtie (-)$; algebraic coherence of $\mathcal{C}$ now follows from Corollary 2.30.

If a semi-abelian category $\mathcal{C}$ is **two-nilpotent**—which means [32] that every *ternary* co-smash product $X \diamond Y \diamond Z$, which may be obtained as the kernel in the short exact sequence

$$0 \longrightarrow X \diamond Y \diamond Z \rightrightarrows (X + Y) \diamond Z \longrightarrow (X \diamond Z) \times (Y \diamond Z) \longrightarrow 0,$$

is trivial—then Higgins commutators in $\mathcal{C}$ do also distribute over joins by Proposition 2.22 of [34], see also [33]. Hence it follows that all functors $X \diamond (-): \mathcal{C} \rightarrow \mathcal{C}$ preserve jointly strongly epimorphic pairs. One example of this situation is the category $\text{Nil}_2(\mathbf{Gp})$ of groups of nilpotency class at most 2. More generally, this happens in the **two-nilpotent core** $\text{Nil}_2(\mathcal{C})$ of any semi-abelian category $\mathcal{C}$, which is the Birkhoff subcategory of $\mathcal{C}$ determined by the **two-nilpotent objects**: those X for which $[X, X, X] = 0$ where, given three subobjects $K, L, M \leq X$ represented by monomorphisms k, l and m , the **ternary commutator** $[K, L, M] \leq X$ is the image of the composite

$$K \diamond L \diamond M \xrightarrow{\iota_{K,L,M}} K + L + M \xrightarrow{\begin{pmatrix} k \\ l \\ m \end{pmatrix}} X.$$

Thus we proved:

Theorem 3.25. *Any two-nilpotent semi-abelian category is algebraically coherent.* $\square$

In any semi-abelian category $\mathcal{C}$, the Huq commutator $[K, L]_X$ of two subobjects $K, L \leq X$ is the normal closure of the Higgins commutator $[K, L]$, so by the results in [33] it may be obtained as the join $[K, L] \vee [[K, L], X] \triangleleft X$. We see that if $\mathcal{C}$ is two-nilpotent, then Huq commutators distribute over joins of subobjects. Hence if it is, moreover, a variety, it is *algebraically cartesian closed* (ACC) by [30]. We will, however, prove a stronger result for categories which are merely algebraically coherent: see Theorem 4.20 below.

Examples 3.26. $\text{Nil}_2(\mathcal{C})$ for any semi-abelian category $\mathcal{C}$; modules over a square ring [2].

3.27. Monoids. We end this section with some partial algebraic coherence properties for monoids.

Proposition 3.28. *If X is a monoid satisfying the quasi-identity $xy = 1 \Rightarrow yx = 1$, then the kernel functor $\text{Ker}: \text{Pt}_X(\mathbf{Mon}) \rightarrow \mathbf{Mon}$ is coherent.*

Proof. We consider a diagram such as (B) in the category of monoids. We need to show that any element k of K written as a product $k = a_1 c_1 \cdots a_n c_n$ of elements of A and C in $A +_X C$ may be written as a product of elements of H and L in K . We prove this by induction on the length of the product $a_1 c_1 \cdots a_n c_n$.

When $k = ac$, first note that since $p(ac) = 1$ it follows that $p''(c)p'(a) = 1$ and so $s'(p''(c))s''(p'(a)) = 1$. Hence

$$k = ac = as'(p''(c)) \cdot s''(p'(a))c,$$

where $as'(p''(c)) \in H$ and $s''(p'(a))c \in K$.

If $k = a_1 c_1 a_2 \cdots a_n c_n$, then $p(a_1 c_1 a_2 \cdots a_n) = p''(c_n)^{-1}$. Hence

$$\begin{aligned} k &= a_1 c_1 a_2 \cdots a_n c_n \\ &= a_1 c_1 a_2 \cdots a_n s'(p''(c_n)) \cdot s''(p(a_1 c_1 a_2 \cdots a_n))c_n \end{aligned}$$

is a product of two elements of K , where the first has length $n - 1$ and the second is in L . $\square$

As a consequence, both the category $\mathbf{MonC}$ of monoids with cancellation and the category $\mathbf{CMon}$ of commutative monoids have coherent kernel functors.

Remark 3.29. Although we shall not explore this further here, it is worth noting that the category of *all* monoids is **relatively** algebraically coherent: if we replace the fibration of points in Definition 2.14 by the *fibration of Schreier points* considered in [16], all kernel functors $\text{Ker}: \text{SPt}_X(\text{Mon}) \rightarrow \text{Mon}$ will be coherent. To see this, it suffices to modify the proof of Proposition 3.28 as follows.

If $k = ac$, use [16, Lemma 2.1.6] to write a as hx with $h \in H$ and $x \in X$. Then $k = h \cdot xc$ where $1 = p(k) = p(h) \cdot p(xc) = p(xc)$, so that $xc \in L$.

If $k = a_1 c_1 a_2 \cdots a_n c_n$, write a_1 as hx with $h \in H$ and $x \in X$. Then $k = h \cdot (xc_1) a_2 \cdots a_n c_n$ where $1 = p(k) = p(h) \cdot p((xc_1) a_2 \cdots a_n c_n) = p((xc_1) a_2 \cdots a_n c_n)$. Hence the induction hypothesis may be used on the product $(xc_1) a_2 \cdots a_n c_n$.

4. CATEGORICAL-ALGEBRAIC CONSEQUENCES

We begin this section by showing that a pointed Mal'tsev category which is algebraically coherent is necessarily protomodular—a straightforward generalisation of Theorem 3.10 in [12].

Theorem 4.1. *Let $\mathcal{C}$ be a pointed algebraically coherent category. If $\mathcal{C}$ is a Mal'tsev category, then it is protomodular.*

Proof. Let

$$X \xrightarrow{\kappa} A \xrightleftharpoons[\beta]{\alpha} B$$

be an arbitrary split extension. Since the diagram

$$\begin{array}{ccccc} A & \xleftarrow{\pi_2} & B \times A & \xrightarrow{1_B \times \alpha} & B \times B \\ \beta \updownarrow \alpha & & (1_B, \beta) \updownarrow \alpha \pi_2 & & (1_B, 1_B) \updownarrow \pi_2 \\ B & \xlongequal{\quad} & B & \xlongequal{\quad} & B \end{array}$$

is a product in the unital category $\text{Pt}_B(\mathcal{C})$, it follows that the morphisms

$$\begin{array}{ccccc} A & \xrightarrow{(\alpha, 1_A)} & B \times A & \xleftarrow{1_B \times \beta} & B \times B \\ \beta \updownarrow \alpha & & (1_B, \beta) \updownarrow \alpha \pi_2 & & (1_B, 1_B) \updownarrow \pi_2 \\ B & \xlongequal{\quad} & B & \xlongequal{\quad} & B \end{array}$$

are jointly strongly epimorphic in $\text{Pt}_B(\mathcal{C})$. Hence Lemma 2.15 implies that they are jointly strongly epimorphic in $\mathcal{C}$. Therefore, since in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & A & \xrightleftharpoons[\beta]{\alpha} & B \\ \kappa \downarrow & & \downarrow (0, 1_A) & & \downarrow (0, 1_B) \\ A & \xrightarrow{(\alpha, 1_A)} & B \times A & \xrightleftharpoons[\beta]{\alpha} & B \times B \\ \beta \updownarrow \alpha & & (1_B, \beta) \updownarrow \pi_1 & & (1_B, 1_B) \updownarrow \pi_1 \\ B & \xlongequal{\quad} & B & \xlongequal{\quad} & B \end{array}$$

the top split extension is obtained by applying the kernel functor to the bottom split extension in $\text{Pt}_B(\mathcal{C})$, it follows that κ and β are jointly strongly epimorphic. Hence $\mathcal{C}$ is protomodular by Proposition 3.9 in [13]. $\square$

Lemma 4.2. *Let $\mathcal{C}$ be an arbitrary category with pullbacks. If $s: D \rightarrow B$ is a split monomorphism and $\text{Pt}_D(\mathcal{C})$ is protomodular, then $s^*: \text{Pt}_B(\mathcal{C}) \rightarrow \text{Pt}_D(\mathcal{C})$ reflects isomorphisms.*

Proof. Again by Proposition 3.9 in [13], it is sufficient to show that for each split pullback

$$\begin{array}{ccc} C & \xrightarrow{r} & A \\ \delta \downarrow & \gamma & \downarrow \beta \\ D & \xrightarrow{s} & B \end{array}$$

the morphisms r and β are jointly strongly epimorphic. However this is an immediate consequence of Lemma 2.15, because if f is a splitting of s , then the morphism r in the diagram

$$\begin{array}{ccccc} C & \xrightarrow{r} & A & \xleftarrow{\alpha} & B \\ \delta \downarrow & \gamma & \downarrow \beta s & f \alpha & \downarrow s \\ D & \xlongequal{\quad} & D & \xlongequal{\quad} & D \end{array}$$

is the kernel of α in $\mathbf{Pt}_D(\mathcal{C})$. $\square$

The following variation is well known, so we omit its proof.

Lemma 4.3. *Let $\mathcal{C}$ be an arbitrary category with pullbacks and let $q: D \rightarrow B$ be a stably strong epimorphism. Then the functor $q^*: (\mathcal{C} \downarrow B) \rightarrow (\mathcal{C} \downarrow D)$ and hence the functor $q^*: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathbf{Pt}_D(\mathcal{C})$ reflects:*

- (a) *isomorphisms;*
- (b) *monomorphisms;*
- (c) *jointly strongly epimorphic cospans.*

$\square$

In a category $\mathcal{C}$ with a terminal object 1 , we call an object D **inhabited** when it has global support: the unique morphism $D \rightarrow 1$ is a stably strong epimorphism. We write $\mathbf{Inh}(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ determined by the inhabited objects.

Lemma 4.4. *Let $\mathcal{C}$ be a category with a terminal object. Let D be an inhabited object for which $\mathbf{Pt}_D(\mathcal{C})$ is protomodular. For every morphism $q: D \rightarrow B$ the pullback functor $q^*: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathbf{Pt}_D(\mathcal{C})$ reflects isomorphisms.*

Proof. Let $q: D \rightarrow B$ be a morphism in $\mathcal{C}$ such that $D \rightarrow 1$ is a stably strong epimorphism, and $\mathbf{Pt}_D(\mathcal{C})$ is protomodular. The result follows from Lemma 4.2 and 4.3 since q can be factored as in the diagram

$$\begin{array}{ccc} D & \xrightarrow{q} & B \\ & \searrow (1_D, q) & \nearrow \pi_2 \\ & D \times B & \end{array}$$

where $(1_D, q)$ is a split monomorphism and π_2 , being a pullback of $D \rightarrow 1$, is a stably strong epimorphism. $\square$

We obtain a generalisation of Theorem 3.11 in [12]:

Theorem 4.5. *Let $\mathcal{C}$ be a Mal'tsev category such that, for any $X \in \mathcal{C}$, $\mathbf{Pt}_X(\mathcal{C})$ is algebraically coherent. Then the category $\mathbf{Inh}(\mathcal{C})$ is protomodular. In particular, if every object in $\mathcal{C}$ admits a stably strong epimorphism to the terminal object, then $\mathcal{C}$ is protomodular.*

Proof. This follows from Theorem 4.1 and Lemma 4.4. $\square$

Remark 4.6. The above theorem together with Proposition 2.22 implies that a Mal'tsev algebraically coherent category can only have an inhabited initial object when it is a protomodular category.

4.7. Higgins commutators, normal subobjects and normal closures. We now describe the effect of coherent functors on Higgins commutators (see 3.23), normal subobjects and normal closures.

Proposition 4.8. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a coherent functor between regular pointed categories with binary coproducts. Then F preserves Higgins commutators of arbitrary cospans.*

Proof. Consider a cospan $(k: K \rightarrow X, l: L \rightarrow X)$ in $\mathcal{C}$ and the induced diagram (C). Since F is coherent, it preserves finite limits and the comparison morphism $F(K) + F(L) \rightarrow F(K + L)$ is a regular epimorphism. Hence, the leftmost vertical arrow in the diagram

$$\begin{array}{ccccc} F(K) \diamond F(L) & \xrightarrow{\iota_{F(K), F(L)}} & F(K) + F(L) & \xrightarrow{\sigma_{F(K), F(L)}} & F(K) \times F(L) \\ \downarrow & \lrcorner & \downarrow & & \downarrow \cong \\ F(K \diamond L) & \xrightarrow{F(\iota_{K, L})} & F(K + L) & \xrightarrow{F(\sigma_{K, L})} & F(K \times L) \end{array}$$

is a regular epimorphism. Finally, applying F to Diagram (C) and pasting with the left hand square above, we obtain the square

$$\begin{array}{ccc} F(K) \diamond F(L) & \xrightarrow{\iota_{F(K), F(L)}} & F(K) + F(L) \\ \downarrow & & \downarrow \begin{pmatrix} F(k) \\ F(l) \end{pmatrix} \\ F([K, L]) & \xrightarrow{\quad} & F(X) \end{array}$$

showing us that $F([K, L]) \cong [F(K), F(L)]$. $\square$

Recall from [33, 47] that, for any subobject $K \leq X$ in a semi-abelian category, its normal closure in X may be obtained as the join $K \vee [K, X]$.

Corollary 4.9. *Any coherent functor between semi-abelian categories preserves normal closures.*

Proof. This is Proposition 2.6 combined with Proposition 4.8. $\square$

However, this result can be proved in a more general context (which, for instance, includes all ideal determined categories [39]) via the following variation on Lemma 4.10 of [49].

Proposition 4.10. *Let $\mathcal{C}$ be a pointed regular category with finite coproducts in which the regular image of a normal monomorphism is normal. For a monomorphism $m: M \rightarrow X$, the monomorphism $\bar{m}: M_X \rightarrow X$ in the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X \bowtie M & \xrightarrow{\kappa_{X, M}} & X + M & \xrightleftharpoons[\iota_1]{\begin{pmatrix} 1_X \\ 0 \end{pmatrix}} & X \longrightarrow 0 \\ & & \downarrow \theta & & \downarrow \begin{pmatrix} 1_X \\ m \end{pmatrix} & & \\ & & M_X & \xrightarrow{\bar{m}} & X & & \end{array}$$

where θ is a regular epimorphism, is the normal closure of m .

Proof. First note that $\bar{m}$ may be obtained as the image of $\kappa_{X, M}$ along $\begin{pmatrix} 1_X \\ m \end{pmatrix}$. This arrow is normal, being the regular image of a normal monomorphism. Let $\eta_M: M \rightarrow X \bowtie M$ be the unique morphism such that

$$\kappa_{X, M} \eta_M = \iota_2: M \rightarrow X + M.$$

Since $\overline{m}\theta\eta_M = \begin{pmatrix} 1_X \\ m \end{pmatrix} \kappa_{X,M}\eta_M = \begin{pmatrix} 1_X \\ m \end{pmatrix} \iota_2 = m$ it follows that m factors through $\overline{m}$. It remains to show that $\overline{m}$ is the smallest normal monomorphism through which m factors.

Let $k: K \rightarrow X$ be a normal monomorphism and let $f: X \rightarrow Y$ be a morphism such that k is the kernel of f . Consider the diagram

$$\begin{array}{ccccc}
 K & & & & \\
 \eta_K \downarrow & \searrow \iota_2 & & & \\
 X \bowtie K & \xrightarrow{\kappa_{X,K}} & X + K & \xrightleftharpoons[\iota_1]{\begin{pmatrix} 1_X \\ 0 \end{pmatrix}} & X \\
 \varphi \downarrow & \begin{pmatrix} 1_X & 1_X \\ k & 0 \end{pmatrix} \downarrow & \downarrow & & \parallel \\
 K & \xrightarrow{(k,0)} & X \times_Y X & \xrightleftharpoons[(1_X, 1_X)]{\pi_2} & X \\
 \parallel & \pi_1 \downarrow & \downarrow & & \downarrow f \\
 K & \xrightarrow{k} & X & \xrightarrow{f} & Y
 \end{array}$$

where

- η_K is the unique morphism making the triangle at the top commute;
- the bottom right square is a pullback;
- $(k, 0)$ is the kernel of π_2 ;
- φ is the unique morphism making the top a morphism of split extensions.

Since $(k, 0)$ is a monomorphism it follows that $\varphi\eta_K = 1_K$ and so in the commutative diagram

$$\begin{array}{ccc}
 X \bowtie K & \xrightarrow{\kappa_{X,K}} & X + K \\
 \varphi \downarrow & & \downarrow \begin{pmatrix} 1_X \\ k \end{pmatrix} \\
 K & \xrightarrow{k} & X
 \end{array}$$

$k\varphi$ is the (regular epi, mono)-factorisation of $\begin{pmatrix} 1_X \\ k \end{pmatrix} \kappa_{X,K}$. Now suppose that there exists $t: M \rightarrow K$ such that $kt = m$. Since there exists a unique morphism $X \bowtie t: X \bowtie M \rightarrow X \bowtie K$ making the diagram

$$\begin{array}{ccccc}
 X \bowtie M & \xrightarrow{\kappa_{X,M}} & X + M & \xrightleftharpoons[\iota_1]{\begin{pmatrix} 1_X \\ 0 \end{pmatrix}} & X \\
 X \bowtie t \downarrow & & 1_X + t \downarrow & & \parallel \\
 X \bowtie K & \xrightarrow{\kappa_{X,K}} & X + K & \xrightleftharpoons[\iota_1]{\begin{pmatrix} 1_X \\ 0 \end{pmatrix}} & X
 \end{array}$$

a morphism of split extensions, functoriality of regular images tells us that $\overline{m}$ factors through k

$$\begin{array}{ccccc}
 & X \bowtie M & \xrightarrow{\kappa_{X,M}} & X + M & \\
 X \bowtie t \swarrow & \downarrow \theta & \searrow 1_X + t & & \downarrow \begin{pmatrix} 1_X \\ m \end{pmatrix} \\
 X \bowtie K & \xrightarrow{\kappa_{X,K}} & X + K & & \\
 \varphi \downarrow & \downarrow & \downarrow \begin{pmatrix} 1_X \\ k \end{pmatrix} & & \parallel \\
 & M_X & \xrightarrow{\overline{m}} & X & \\
 \swarrow \text{dotted} & \downarrow & & & \\
 K & \xrightarrow{k} & X & &
 \end{array}$$

as required. $\square$

Proposition 4.11. *Let F be a functor between pointed regular categories with finite coproducts in which the regular image of a normal monomorphism is normal. If F is coherent, then F preserves normal closure.*

Proof. Let $m: M \rightarrow X$ be a monomorphism. Consider the diagram

$$\begin{array}{ccccc}
 F(X) \bowtie f(M) & \xrightarrow{\kappa_{F(X), F(M)}} & F(X) + F(M) & \xrightleftharpoons[\iota_1]{\begin{pmatrix} 1_{F(X)} \\ 0 \end{pmatrix}} & F(X) \\
 \downarrow h & & \downarrow \begin{pmatrix} F(\iota_1) \\ F(\iota_2) \end{pmatrix} & & \parallel \\
 F(X) \bowtie M & \xrightarrow{F(\kappa_{X, M})} & F(X + M) & \xrightleftharpoons[F(\iota_1)]{F\left(\begin{pmatrix} 1_X \\ 0 \end{pmatrix}\right)} & F(X) \\
 \downarrow F(\theta) & & \downarrow F\left(\begin{pmatrix} 1_X \\ m \end{pmatrix}\right) & & \\
 F(M_X) & \xrightarrow{F(\overline{m})} & F(X), & &
 \end{array}$$

where

- $\overline{m}\theta$ is the regular image of $\begin{pmatrix} 1_X \\ m \end{pmatrix} \kappa$;
- h is the unique morphism making the upper part of the diagram into a morphism of split extensions—which exists since F preserves limits.

Proposition 4.10 tells us that $\overline{m}$ is the normal closure of m . Since F is coherent it follows by Proposition 2.7 that the dotted middle arrow is a regular epimorphism. Hence h is a regular epimorphism, because the top left square is a pullback. Since F preserves (regular epi, mono)-factorisations it follows that $F(\overline{m})(F(\theta)h)$ is the (regular epi, mono)-factorisation of

$$F\left(\begin{pmatrix} 1_X \\ m \end{pmatrix}\right) \left(\begin{pmatrix} F(\iota_1) \\ F(\iota_2) \end{pmatrix} \right) \kappa_{F(X), F(M)} = \left(\begin{pmatrix} 1_{F(X)} \\ F(m) \end{pmatrix} \right) \kappa_{F(X), F(M)}$$

and so, by Proposition 4.10, $F(\overline{m})$ is the normal closure of $F(m)$. $\square$

Recall that functor between homological categories is said to be **sequentially exact** if it preserves short exact sequences.

Corollary 4.12. *Any regular functor which preserves normal closures and normal epimorphisms preserves all cokernels.*

Proof. It suffices to preserve cokernels of arbitrary monomorphisms, which are in fact the cokernels of their normal closures. Those are preserved since the functor under consideration is sequentially exact, because it preserves finite limits and normal epimorphisms. $\square$

Recall from [47] that for a pair of subobjects in a normal unital category with binary coproducts, their Huq commutator is the normal closure of the Higgins commutator. Thus we find:

Corollary 4.13. *Let F be a coherent functor between normal unital categories with binary coproducts. If F preserves normal closures, then F preserves Huq commutators of arbitrary cospans.*

Proof. This follows from Proposition 4.8. $\square$

Theorem 4.14. *Let $\mathcal{C}$ be an algebraically coherent category with pushouts. For any morphism $f: X \rightarrow Y$, consider the change-of-base functor $f^*: \text{Pt}_Y(\mathcal{C}) \rightarrow \text{Pt}_X(\mathcal{C})$. Then*

- (a) f^* preserves Higgins commutators of arbitrary cospans.

If, in addition, $\mathcal{C}$ is ideal-determined [39], then

- (b) f^* preserves normal closures;

- (c) f^* preserves all cokernels;
- (d) f^* preserves Huq commutators of arbitrary cospans.

Proof. Apply the previous results to the coherent functor f^* . In particular, (a), (b), (c) and (d) follow from Proposition 4.8, Proposition 4.11, Corollary 4.12 and Corollary 4.13, respectively. $\square$

Theorem 4.15. *Any algebraically coherent semi-abelian category satisfies (SH) and (NH).*

Proof. This is (d) in Theorem 4.14 combined with Theorem 6.5 in [21]. $\square$

In the article [49], the authors consider a strong version of the *Smith is Huq* condition, asking that the kernel functors

$$\text{Ker}: \text{Pt}_X(\mathcal{C}) \rightarrow \mathcal{C}$$

reflect Huq commutativity of arbitrary cospans (rather than just pairs of normal subobjects). We write this condition (SSH). Of course (SSH) $\Rightarrow$ (SH). On the other hand, as shown in [49], (SSH) is implied by (LACC). This is a consequence of Theorem 3.4 in combination with the following result.

Theorem 4.16. *If $\mathcal{C}$ is an algebraically coherent semi-abelian category, then the kernel functors $\text{Ker}: \text{Pt}_X(\mathcal{C}) \rightarrow \mathcal{C}$ reflect Huq commutators. Hence the category $\mathcal{C}$ satisfies (SSH).*

Proof. We may combine (d) in Theorem 4.14 with Lemma 6.4 in [21]. We find precisely the definition of (SSH) as given in [49]. $\square$

Lemma 4.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be pointed categories with finite limits such that normal closures of monomorphisms exist in $\mathcal{C}$, and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a conservative functor. If F preserves normal closures, then F reflects normal monomorphisms.*

Proof. Let $m: M \rightarrow X$ be a morphism such that $F(m)$ is normal. Using that F preserves limits and reflects isomorphisms, it is easily seen that m is a monomorphism. Now let $n: N \rightarrow X$ be the normal closure of m and $i: M \rightarrow N$ the unique factorisation $m = ni$. The monomorphism $F(m)$ being normal, we see that $F(i)$ is an isomorphism: $F(i)$ is the unique factorisation of $F(m)$ through its normal closure $F(n)$. Since F reflects isomorphisms, i is an isomorphism, and m is normal. $\square$

Theorem 4.18. *Any algebraically coherent semi-abelian category is strongly protomodular [9].*

Proof. This follows from Lemma 4.17. $\square$

Lemma 4.19. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{C} \rightarrow \mathcal{D}$ be functors between categories with finite limits and binary coproducts such that $GF = 1_{\mathcal{C}}$ and G reflects isomorphisms. If F and G are coherent, then F preserves binary coproducts.*

Proof. Since F is coherent, by Proposition 2.7 (ii) the induced morphism

$$f := \begin{pmatrix} F(\iota_1) \\ F(\iota_2) \end{pmatrix} : F(A) + F(B) \rightarrow F(A + B)$$

is a strong epimorphism. It follows by the universal property of the coproduct that the diagram

$$\begin{array}{ccccc} A + B & \xlongequal{\quad} & & & A + B \\ \parallel & & & & \parallel \\ GF(A) + GF(B) & \xrightarrow{g := \begin{pmatrix} G(\iota_1) \\ G(\iota_2) \end{pmatrix}} & G(F(A) + F(B)) & \xrightarrow{G(f)} & GF(A + B) \end{array}$$

commutes, and so since G is coherent, by Proposition 2.7 (ii), that the morphism g is an isomorphism. This means that $G(f)$ is an isomorphism and hence—since G reflects isomorphisms—that f is an isomorphism as required. $\square$

It was shown in [14] that a pointed Mal'tsev category is (FWACC) if and only if each fibre of the fibration of points has centralisers.

Theorem 4.20. *Let $\mathcal{C}$ be a regular Mal'tsev category.*

- (a) *If $\mathcal{C}$ is algebraically coherent, then the change-of-base along any split epimorphism preserves finite colimits.*
- (b) *When $\mathcal{C}$ is, in addition, a cocomplete well-powered category in which filtered colimits commute with finite limits—for instance, $\mathcal{C}$ could be a variety—then if $\mathcal{C}$ is algebraically coherent, it is fibre-wise algebraically cartesian closed (FWACC).*

Proof. The first statement is an immediate consequence of Lemma 4.19, while statement (b) follows from (a) via Theorem 4.3 in [29]. $\square$

5. DECOMPOSITION OF THE TERNARY COMMUTATOR

It is known [32] that for normal subgroups K , L and M of a group X ,

$$[K, L, M] = [[K, L], M] \vee [[L, M], K] \vee [[M, K], L]$$

where the commutator on the left is defined as in 3.23. This result is valid in any algebraically coherent semi-abelian category. This gives us a categorical version of the so-called *Three Subgroups Lemma*, valid for *normal* subobjects of a given object. (Note that the usual Three Subgroups Lemma for groups works for arbitrary subobjects.)

First note that in the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (K \bowtie L) \diamond (K \bowtie M) & \xrightarrow{\quad} & (K \bowtie L) + (K \bowtie M) & \longrightarrow & (K \bowtie L) \times (K \bowtie M) \longrightarrow 0 \\ & & \downarrow \nabla & & \downarrow \nabla & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & K \bowtie (L + M) & \longrightarrow & (K \bowtie L) \times (K \bowtie M) \longrightarrow 0 \end{array}$$

the middle arrow, and hence also the induced left hand side arrow, are regular epimorphisms by algebraic coherence. Hence also in the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \longrightarrow & (K \bowtie L) \diamond (K \bowtie M) & \rightleftarrows & L \diamond M \longrightarrow 0 \\ & & \downarrow \nabla & & \downarrow \nabla & & \parallel \\ 0 & \longrightarrow & K \diamond L \diamond M & \longrightarrow & A & \rightleftarrows & L \diamond M \longrightarrow 0 \end{array}$$

we have a vertical regular epimorphism on the left: indeed, $K \diamond L \diamond M$ is the kernel of $K \diamond (L + M) \rightarrow (K \diamond L) \times (K \diamond M)$, and $L \diamond M$ is the kernel of $L + M \rightarrow L \times M$.

Again using algebraic coherence, via Proposition 2.22 in [34] we may decompose B into components which together cover $K \diamond L \diamond M$. We have that $K \bowtie M = (K \diamond M) \vee M$ in $K + M$ and $K \bowtie L = (K \diamond L) \vee L$ in $K + L$, so $(K \bowtie L) \diamond (K \bowtie M)$ is covered by

$$L \diamond (K \bowtie M) + (K \diamond L) \diamond (K \bowtie M) + L \diamond (K \diamond L) \diamond (K \bowtie M),$$

which by further decomposition gives us a regular epimorphism from

$$\begin{aligned} & (L \diamond M) + L \diamond (K \diamond M) + L \diamond M \diamond (K \diamond M) \\ & + (K \diamond L) \diamond M + (K \diamond L) \diamond (K \diamond M) + (K \diamond L) \diamond M \diamond (K \diamond M) \\ & + L \diamond (K \diamond L) \diamond M + L \diamond (K \diamond L) \diamond (K \diamond M) + L \diamond (K \diamond L) \diamond M \diamond (K \diamond M) \end{aligned}$$

to $(K \bowtie L) \diamond (K \bowtie M)$. Note that all morphisms to $(K \bowtie L) \diamond (K \bowtie M)$ already lift over B , except for the one with domain $L \diamond M$, whose intersection with B is trivial. Considering K , L and M as subobjects of X now, we take their images to see that

$$\begin{aligned} [K, L, M] &= [L, [K, M]] \vee [L, M, [K, M]] \vee [[K, L], M] \\ &\quad \vee [[K, L], [K, M]] \vee [[K, L], M, [K, M]] \vee [L, [K, L], M] \\ &\quad \vee [L, [K, L], [K, M]] \vee [L, [K, L], M, [K, M]] \\ &= [L, [K, M]] \vee [[K, L], M] \end{aligned}$$

in X , by Proposition 2.21 in [34], using (SH) in the form of [34, Theorem 4.6], using (NH) and the fact that K , L and M are normal. Thus we find:

Theorem 5.1 (Three Subobjects Lemma for normal subobjects). *If K , L and M are normal subobjects of an object X in an algebraically coherent semi-abelian category, then*

$$[K, L, M] = [[K, L], M] \vee [[M, K], L].$$

In particular, $[[L, M], K] \leq [[K, L], M] \vee [[M, K], L]$. $\square$

As a consequence, in any algebraically coherent semi-abelian category, the two natural, but generally non-equivalent, definitions of *two-nilpotent object*— X such that either $[X, X, X]$ or $[[X, X], X]$ vanishes, see also 3.23—coincide:

Corollary 5.2. *In an algebraically coherent semi-abelian category,*

$$[X, X, X] = [[X, X], X]$$

holds for all objects X . $\square$

Note that the commutator on the right also coincides with the Huq commutator $[[X, X]_X, X]_X$ (since one of its entries is X) and with the Smith commutator $[[\nabla_X, \nabla_X], \nabla_X]$ (by Proposition 2.2 in [28]).

6. PULLBACKS ALONG “SURJECTIONS” IN THE MAL’TSEV CONTEXT

In Section 4 we proved that coherence of *all* pullback functors in a pointed Mal’tsev context implies protomodularity (Theorem 4.1), and that in a semi-abelian context it has strong consequences such as (SH) and (NH). The present section focuses on pullbacks along “surjections” in Mal’tsev categories.

We will begin by exploring in which contexts change-of-base functors for the fibration of points are coherent for each morphism in a particular class of morphisms. We saw—this is Lemma 2.12—that the change-of-base functors for the basic fibration and hence for the fibration of points along morphisms into the terminal object in a unital category are always coherent. We will now see that also the change-of-base functors along effective descent morphisms in a Mal’tsev category are always coherent, as are the change-of-base functors along regular epimorphisms in a regular Mal’tsev category.

We give the following two lemmas without proof since they are either well known or easy to prove:

Lemma 6.1. *Let $\mathcal{C}$ be a category with pullbacks. For each commutative diagram*

$$\begin{array}{ccccc} E & \xrightarrow{r} & C & \xrightarrow{p} & A \\ \epsilon \downarrow & & \downarrow \gamma & & \downarrow \alpha \\ F & \xrightarrow{s} & D & \xrightarrow{q} & B \end{array}$$

where s is a stably strong epimorphism, if the outer rectangle and the left hand square form pullbacks, then so does the right hand square. $\square$

Lemma 6.2. *Let $\mathcal{C}$ be a category with pullbacks. For each commutative diagram*

$$\begin{array}{ccccc}
 C' & \xrightarrow{p'} & A' & & \\
 \downarrow \gamma' & \searrow g & \downarrow & \nearrow \alpha' & \\
 & C & \xrightarrow{p} & A & \\
 & \downarrow \gamma & & \downarrow \alpha & \\
 D' & \xrightarrow{q'} & B' & & \\
 \downarrow h & \searrow f & \downarrow & & \\
 & D & \xrightarrow{q} & B &
 \end{array}$$

in which the vertical faces are pullbacks and α is a monomorphism, if q' is a stably strong epimorphism, then there exists a (unique) morphism $e: A' \rightarrow A$ making the right hand side vertical face into a pullback. $\square$

The following result is a generalisation of Theorem 1.8.17 in [4].

Lemma 6.3. *Let $\mathcal{C}$ be a pointed category with finite limits. $\mathcal{C}$ is strongly unital if and only if for every morphism of split epimorphisms*

$$\begin{array}{ccc}
 C & \xrightarrow{p} & A \\
 \uparrow \delta & \downarrow \gamma & \uparrow \\
 D & \xrightarrow{q} & 0
 \end{array} \quad (\mathbf{D})$$

if p and γ are jointly monomorphic, then there exist unique morphisms $r: C \rightarrow K$, $s: K \rightarrow A$ such that $sr = p$ and $(\gamma, r): C \rightarrow D \times K$ is an isomorphism. Furthermore the morphism s is necessarily a monomorphism.

Proof. Let us begin by showing that when the morphisms r and s exist and satisfy the conditions above then s is necessarily a monomorphism. Indeed, since $(\gamma, p): C \rightarrow D \times A$ is a monomorphism which composed with an isomorphism is $1_D \times s: D \times K \rightarrow D \times A$ it follows that the composite $(1_D \times s)(0, 1_D) = (0, 1_D)s$ is a monomorphism and hence s is too. It is easy to see that $\mathcal{C}$ is strongly unital if and only if every diagram as above, where p and γ are jointly monomorphic and p is a split epimorphism, is a pullback. Therefore the “if” part follows from the fact that s as above is necessarily a monomorphism and so when p is a split epimorphism it is an isomorphism. The converse follows from [4, Theorem 1.8.17]. $\square$

As an easy corollary we obtain:

Theorem 6.4. *Let $\mathcal{C}$ be a pointed category with finite limits. $\mathcal{C}$ is strongly unital if and only if for every morphism of split epimorphisms $(\mathbf{D})$, if p is a strong epimorphism, then $(\gamma, p): C \rightarrow D \times A$ is a strong epimorphism. $\square$*

Since a category $\mathcal{C}$ is Mal'tsev if and only if its fibres $\mathbf{Pt}_B(\mathcal{C})$ are strongly unital [4, Theorem 1.8.17], as a consequence of Lemma 6.3 we obtain:

Lemma 6.5. *Let $\mathcal{C}$ be a category with finite limits. $\mathcal{C}$ is Mal'tsev if and only if for every morphism of split epimorphisms*

$$\begin{array}{ccc}
 C & \xrightarrow{p} & A \\
 \uparrow \delta & \downarrow \gamma & \uparrow \beta \\
 D & \xrightarrow{q} & B
 \end{array} \quad (\mathbf{E})$$

if q is a split epimorphism, and p and γ are jointly monomorphic, then there exist unique morphisms $r: C \rightarrow K$, $s: K \rightarrow A$ and $t: B \rightarrow K$ such that $sr = p$, $st = \beta$

and the diagram

$$\begin{array}{ccc} C & \xrightarrow{r} & K \\ \gamma \downarrow & & \downarrow \alpha s \\ D & \xrightarrow{q} & B \end{array}$$

is a pullback. Furthermore the morphism s is necessarily a monomorphism. $\square$

Lemma 6.6. *Let $\mathcal{C}$ be a Mal'tsev category. For every morphism of split epimorphisms $(\mathbf{E})$ where p and γ are jointly monomorphic, the induced morphism between the kernel pairs of the horizontal morphisms*

$$\begin{array}{ccc} C \times_A C & \xrightleftharpoons[\pi_2]{\pi_1} & C \\ \gamma \times \gamma \downarrow & & \downarrow \gamma \\ D \times_B D & \xrightleftharpoons[\pi_2]{\pi_1} & D \end{array}$$

is a discrete (op-)fibration.

Proof. The proof follows from Lemma 6.5 since π_2 is a split epimorphism and the diagram

$$\begin{array}{ccc} C \times_A C & \xrightarrow{\pi_2} & C \\ \delta \times \delta \updownarrow \gamma \times \gamma & & \delta \updownarrow \gamma \\ D \times_B D & \xrightarrow{\pi_2} & D \end{array}$$

is a morphism of split extensions where $\gamma \times \gamma$ and π_2 are jointly monomorphic, because γ and p are. $\square$

Lemma 6.7. *Let $\mathcal{C}$ be a category with finite limits. $\mathcal{C}$ is Mal'tsev if and only if every morphism of split epimorphisms $(\mathbf{E})$ where*

- p and γ are jointly monomorphic,
- p is a stably strong epimorphism or p is a strong epimorphism and q is an effective descent morphism,

is a split pullback.

Proof. For the “only if” part note that the requiring this condition for only those squares which are split epimorphisms of split epimorphisms makes each fibre $\text{Pt}_B(\mathcal{C})$ unital and hence makes $\mathcal{C}$ Mal'tsev. For the converse consider the diagram

$$\begin{array}{ccccc} C \times_A C & \xrightleftharpoons[\pi_2]{\pi_1} & C & \xrightarrow{p} & A \\ \delta \times \delta \updownarrow \gamma \times \gamma & & \delta \updownarrow \gamma & & \beta \updownarrow \alpha \\ D \times_B D & \xrightleftharpoons[\pi_2]{\pi_1} & D & \xrightarrow{q} & B. \end{array}$$

where p and γ are jointly monomorphic. According to Lemma 6.6 the left hand part of the above diagram (where the upward arrows are removed) is a discrete

fibration. Therefore when p is a stably strong epimorphism, since in the diagram

$$\begin{array}{ccccc}
 & & C & \xrightarrow{p} & A \\
 & \nearrow \pi_1 & \downarrow \gamma & \nearrow p & \downarrow \alpha \\
 C \times_A C & \xrightarrow{\pi_2} & C & & \\
 \downarrow \gamma \times \gamma & & \downarrow q & & \downarrow \gamma \\
 & \nearrow \pi_1 & D & \xrightarrow{q} & B \\
 D \times_B D & \xrightarrow{\pi_2} & D & &
 \end{array}$$

the top, bottom, left and front faces are pullbacks it follows by Lemma 6.1 that the back and right faces are too.

To complete the proof we need to now consider the case where p is a strong epimorphism and q is an effective descent morphism—see, for instance, [40] for an overview of basic results on those. In this case the coequaliser $\tilde{p}: C \rightarrow \tilde{A}$ of $\pi_1, \pi_2: C \times_A C \rightarrow C$ exists and the induced morphism $\tilde{\alpha}: \tilde{A} \rightarrow B$ makes the diagram

$$\begin{array}{ccc}
 C & \xrightarrow{\tilde{p}} & \tilde{A} \\
 \downarrow \gamma & & \downarrow \tilde{\alpha} \\
 D & \xrightarrow{q} & B
 \end{array}$$

a pullback. Now consider the diagram

$$\begin{array}{ccccc}
 C & \xrightarrow{\tilde{p}} & \tilde{A} & & \\
 \downarrow \gamma & \searrow p & \downarrow u & & \\
 D \times_B A & \xrightarrow{\pi_2} & A & & \\
 \downarrow \pi_1 & & \downarrow \alpha & & \\
 D & \xrightarrow{q} & B & &
 \end{array}$$

in which u is the unique morphism induced by the coequaliser $\tilde{p}$. Since q is an effective descent morphism it follows that π_2 is as well and hence is certainly a stably strong epimorphism. Therefore it follows by Lemma 4.3 (b) that u is a monomorphism and hence, since p is a strong epimorphism, an isomorphism. This then means that (γ, p) is an isomorphism as required. $\square$

As an immediate corollary we find the following characterisation of Mal'tsev categories, which extends the known characterisation of regular Mal'tsev categories in terms of the relative Mal'tsev axiom [25, 24] to a non-regular setting:

Theorem 6.8. *Let $\mathcal{C}$ be a category with finite limits. $\mathcal{C}$ is Mal'tsev if and only if for every morphism of split epimorphisms **(E)** where p is a stably strong epimorphism, or p is a strong epimorphism while q is an effective descent morphism, the induced morphism $(\gamma, p): C \rightarrow D \times_B A$ into the pullback is a strong epimorphism.* $\square$

Lemma 6.9. *Let $\mathcal{C}$ be a category with finite limits and let $q: D \rightarrow B$ be a stably strong epimorphism. If for every morphism of split epimorphisms **(E)** where p is a strong epimorphism, the induced morphism $(\gamma, p): C \rightarrow D \times_B A$ into the pullback is a strong epimorphism, then $q^*: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathbf{Pt}_D(\mathcal{C})$ is coherent.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
 & & S & & \\
 & i_1 \nearrow & \downarrow m & \nwarrow i_2 & \\
 D \times_B A_1 & \xrightarrow{h_1} & D \times_B A & \xleftarrow{h_2} & D \times_B A_2 \\
 \downarrow p_1 & \uparrow & \downarrow p & \uparrow & \downarrow p_2 \\
 A_1 & \xrightarrow{f_1} & A & \xleftarrow{f_2} & A_2 \\
 \downarrow \delta_1 & \uparrow & \downarrow \delta & \uparrow & \downarrow \delta_2 \\
 D & \xrightarrow{\gamma_1} & D & \xrightarrow{\gamma} & D \\
 \downarrow \beta_1 & \uparrow & \downarrow \beta & \uparrow & \downarrow \beta_2 \\
 B & \xrightarrow{\alpha_1} & B & \xrightarrow{\alpha} & B \\
 & \nwarrow q & \nwarrow q & \nwarrow q & \\
 & & B & &
 \end{array}$$

in which f_1 and f_2 are jointly strongly epimorphic, $h_i = q^*(f_i)$ for i in $\{1, 2\}$, and m is monomorphism. Since the morphisms p_i are strongly epimorphic it follows that $f_1 p_1 = p m i_1$ and $f_2 p_2 = p m i_2$ are jointly strongly epimorphic and hence that pm is an extremal epimorphism. Therefore since the diagram

$$\begin{array}{ccc}
 S & \xrightarrow{pm} & A \\
 \delta_1 i_1 \uparrow & \gamma m & \downarrow \beta \\
 D & \xrightarrow{q} & B
 \end{array}$$

is a morphism of split epimorphisms where pm is a strong epimorphism, it follows that the induced morphisms into the pullback $m: S \rightarrow D \times_B A$ is a strong epimorphism and hence an isomorphism. $\square$

From Theorem 6.8 and Lemma 6.9 together with Lemma 2.12, we obtain:

Theorem 6.10. *Let $\mathcal{C}$ be a Mal'tsev category. If at least one of the conditions*

- (a) $q: D \rightarrow B$ is a effective descent morphism in $\mathcal{C}$;
- (b) strong epimorphisms are pullback-stable in $\mathcal{C}$ and q is a strong epimorphism;
- (c) $\mathcal{C}$ is regular and q is a regular epimorphism.

holds, then the functor $q^: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathbf{Pt}_D(\mathcal{C})$ is coherent.* $\square$

Combined with Theorem 4.5, this gives us:

Corollary 6.11. *Let $\mathcal{C}$ be a Mal'tsev category satisfying at least one of the following:*

- (a) *for every D in $\mathcal{C}$ the unique morphism to the terminal object is an effective descent morphism;*
- (b) *strong epimorphisms are pullback-stable in $\mathcal{C}$ and $\mathbf{Inh}(\mathcal{C}) = \mathcal{C}$;*
- (c) *$\mathcal{C}$ is regular and $\mathbf{Inh}(\mathcal{C}) = \mathcal{C}$.*

Then, $\mathcal{C}$ is protomodular and algebraically coherent if and only if for every D in $\mathcal{C}$ the fibre $\mathbf{Pt}_D(\mathcal{C})$ is algebraically coherent.

Proof. The “only if” part follows from Proposition 2.22. The “if” part follows from Theorem 4.5 and Theorem 6.10, under conditions (a), (b) or (c) respectively, using the factorisation in the proof of Lemma 4.4. $\square$

7. SUMMARY OF RESULTS IN THE SEMI-ABELIAN CONTEXT

In this section we give several short summaries. We begin with a summary of conditions that follow from algebraic coherence for a semi-abelian category $\mathcal{C}$:

- (a) preservation of Higgins and Huq commutators, normal closures and co-kernels by change-of-base functors with respect to the fibration of points, see 4.14;
- (b) (SH) and (NH), see 4.15;
- (c) as a consequence, the category $\mathcal{C}$ is necessarily peri-abelian [11, 31] and thus satisfies the universal central extension condition (UCE) [19, 31];
- (d) (SSH), see 4.16;
- (e) strong protomodularity, see 4.18;
- (f) fibre-wise algebraic cartesian closedness (FWACC), if $\mathcal{C}$ is a variety, see 4.20 and [14, 29];
- (g) $[K, L, M] = [[K, L], M] \vee [[M, K], L]$ for $K, L, M \triangleleft X$, see 5.1.

We also give a summary of semi-abelian categories which are algebraically coherent. These include all abelian categories; all *categories of interest*: (all subvarieties of) groups, the varieties of Lie algebras, Leibniz algebras, rings, associative algebras, Poisson algebras; cocommutative Hopf algebras over a field of characteristic zero; n -nilpotent or n -solvable groups, rings Lie algebras; internal reflexive graphs, categories and (pre)crossed modules in such; compact Hausdorff algebras over such; arrows, extensions and central extensions in such—note, however, that the latter two categories are only homological in general.

Finally we give a summary of semi-abelian categories which are not algebraically coherent. These include (commutative) loops, digroups, non-associative rings, Jordan algebras.

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