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STRONGER MEASURES OF HIGHER-ORDER RISK ATTITUDES: AN EXTENSION

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Abstract

In this note, we analyze the stronger measures of higher-order risk attitudes when one compares risks with equal means that are ordered by sth degree stochastic dominance. In this way, we generalize results obtained recently for the more specific and limited case of sth degree increases in risk.

Key words and phrases: Ekern increase in risk, stochastic dominance, risk aversion, comparative risk attitude.

JEL code: D81.

1 Introduction

As is now well known, an increase in risk aversion, defined by a concavification of the utility function, does not always increase the willingness to pay for a mean preserving reduction in risk. This is why Ross (1981) proposed a stronger measure of increased risk aversion that maintains for mean preserving changes in risk the result obtained by Arrow (1965) and Pratt (1964) for risk elimination.

A mean preserving change in risk is a second order effect and it is natural to extend the analysis to higher orders. A first step in this direction was achieved by Modica and Scarsini (2005) who considered the willingness to pay for a downside risk reduction (a third order effect). Later on, using the concept of sth degree increase in risk of Ekern (1980), Li (2009) and Denuit and Eeckhoudt (2010) could generalize the previous results to any order.

The notion of sth degree increase in risk is globally elegant and it is efficient at low values for s. It includes as special cases such concepts as a mean preserving contraction in risk (Rothschild and Stiglitz, 1970), a downside risk increase (Menezes, Geiss and Tressler, 1980) or an outer risk increase (Menezes and Wang, 2005). However when s increases Ekern's notion becomes little operational. Indeed at the sth degree it requires that the first $s-1$ moments of the risks that are compared be identical. Once s exceeds the values used in the above mentioned papers (i.e., $s = 2$, 3 or 4), the equality of moments imposes very strong restrictions on the risks that can be compared so that the notion loses empirical relevance.

The purpose of the present note is to show that the stronger measures of higher-order risk aversion remain valid under much less restrictive assumptions about the risks that are compared. Indeed, we are going to show that the measures also apply when one compares random variables that have the same mean and can be ordered by sth degree stochastic dominance. As a result, no specific requirement is imposed upon the moments.

Our paper is organized as follows. In Section 2, we recall results about stochastic dominance that will be used in Section 3 to prove the main result of the paper. We briefly conclude in Section 4.

2 Stochastic dominance results

We consider random variables valued in some interval $[a, b]$ of the real line and utility functions defined on the same interval. Let us define the class $\mathcal{U}_{s-\mathrm{icv}}$, $s = 1, 2, \ldots$, of the regular s-increasing concave functions as the class containing all the functions u with derivatives $u^{(1)}, u^{(2)}, \ldots, u^{(s)}$ of degrees 1 to s such that $(-1)^{k+1}u^{(k)} \geq 0$ on $[a, b]$ for $k = 1, 2, \ldots, s$. Letting s tend to $+\infty$ gives utilities with all odd derivatives positive and all even derivatives negative. In this case, utility functions are completely monotone and express mixed risk aversion, as studied in Caballe and Pomansky (1996).

The common preferences of all the decision-makers with s-increasing concave utility functions generate the sth degree stochastic dominance rule, called here the s-increasing concave order. More precisely, given two random variables X and Y, X is said to be smaller than Y in the s-increasing concave order, denoted by $X \preceq_{s-\mathrm{icv}} Y$ when $E[u(X)] \leq E[u(Y)]$ for all the utility functions u in $\mathcal{U}_{s-\mathrm{icv}}$, provided the expectations exist.

Let F_X be the distribution function for such a random variable X. Starting from $F_X^{[1]} = F_X$, we define $F_X^{[2]}, F_X^{[3]}, \ldots$ recursively from repeated integrals:

$$F_X^{[k+1]}(x) = \int_a^x F_X^{[k]}(y)dy, \quad k = 1, 2, \ldots$$

Using integration by parts, it is easily seen that

$$F_X^{[k]}(x) = \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1} dF_X(t) = \frac{\mathbb{E}[(x-X)_+^{k-1}]}{(k-1)!}.$$

The s-increasing concave orders can be characterized as follows by means of these integrated left tails:

$$\begin{aligned} X \preceq_{s-\mathrm{icv}} Y \quad &\Leftrightarrow \quad \begin{cases} F_X^{[k]}(b) \geq F_Y^{[k]}(b) \text{ for } k = 1, 2, \ldots, s-1 \\ F_X^{[s]}(t) \geq F_Y^{[s]}(t) \text{ for } t \in [a, b] \end{cases} \\ &\Leftrightarrow \quad \begin{cases} E[(b-X)^k] \geq E[(b-Y)^k] \text{ for } k = 1, 2, \ldots, s-1 \\ E[(t-X)_+^{s-1}] \geq E[(t-Y)_+^{s-1}] \text{ for } t \in [a, b] \end{cases} \end{aligned}$$

where the interval $[a, b]$ contains the support of X and Y.

For more details, the interested readers are referred e.g. to Denuit, Lefevre and Shaked (1998) and Denuit, De Vijlder and Lefevre (1999).

3 Main result

As in Jindapon and Neilson (2007), we use the following definition: u is more sth degree Ross risk averse than v if the inequality

$$(-1)^{s+1}\frac{u^{(s)}(x)}{u^{(1)}(y)} \geq (-1)^{s+1}\frac{v^{(s)}(x)}{v^{(1)}(y)} \tag{3.1}$$

holds for all x and y. This condition has been imposed by Ross (1981) for $s = 2$, by Modica and Scarsini (2005) for $s = 3$, and by Denuit and Eeckhoudt (2010) for arbitrary $s \geq 4$. Note that this is equivalent to requiring that

$$\frac{u^{(s)}}{v^{(s)}} \geq \lambda \geq \frac{u^{(1)}}{v^{(1)}}$$

holds for some $\lambda > 0$.

Let us now extend Proposition 3.1 in Denuit and Eeckhoudt (2010) from Ekern sth degree increase in risk to sth degree stochastic dominance $\preceq_{s-\text{icv}}$ with equal means but possibly different higher moments. To this end, we need the following result of independent interest.

Property 3.1. *Consider two random variables X and Y valued in $[a, b]$. Then, for $s \geq 3$,*

$$\begin{aligned} & E[u(X)] \leq E[u(Y)] \textit{ for all } u \textit{ such that } (-1)^{k+1}u^{(k)} \geq 0 \textit{ for } k = 2, \ldots, s \\ \Leftrightarrow \; & \begin{cases} E[X] = E[Y] \\ E[(b-X)^k] \geq E[(b-Y)^k] \textit{ for } k = 2, \ldots, s-1 \\ E[(t-X)_+^{s-1}] \geq E[(t-Y)_+^{s-1}] \textit{ for all } t \in [a, b] \end{cases} \\ \Leftrightarrow \; & X \preceq_{s-icv} Y \textit{ and } E[X] = E[Y]. \end{aligned}$$

Proof. As u is s times continuously differentiable, the following expansion formula is easily obtained using integration by parts:

$$\begin{aligned} E[u(X)] &= \sum_{k=0}^{s-1}(-1)^k u^{(k)}(b)F_X^{[k+1]}(b) + (-1)^s\int_a^b u^{(s)}(x)F_X^{[s]}(x)dx \\ &= \sum_{k=0}^{s-1}(-1)^k u^{(k)}(b)\frac{E[(b-X)^k]}{k!} + (-1)^s\int_a^b u^{(s)}(x)\frac{E[(x-X)_+^{s-1}]}{(s-1)!}dx. \end{aligned} \tag{3.2}$$

Hence,

$$\begin{aligned} E[u(Y)]-E[u(X)] &= \sum_{k=2}^{s-1}(-1)^k u^{(k)}(b)\frac{E[(b-Y)^k]-E[(b-X)^k]}{k!} \\ &\quad +(-1)^s\int_a^b u^{(s)}(x)\frac{E[(x-Y)_+^{s-1}]-E[(x-X)_+^{s-1}]}{(s-1)!}dx, \end{aligned}$$

which ends the proof. □

We are now ready to state the following result which extends Ross (1981) to higher degree stochastic dominance $\preceq_{s-\text{icv}}$ keeping the mean fixed but allowing for different higher moments. We state the result for $s \geq 3$; for $s = 2$, this is the result obtained by Ross (1981).

Proposition 3.2. *Statements (i)-(iii) below are equivalent for u, v in $\mathcal{U}_{s-icv}$, $s \geq 3$:*

(i) u is more kth degree Ross risk averse than v for $k = 2, \ldots, s$;

(ii) there exists a function ϕ such that $\phi^{(1)} \leq 0$, $(-1)^{k+1}\phi^{(k)} \geq 0$ for $k = 2, \ldots, s$, and a constant $\lambda > 0$ such that $u = \lambda v + \phi$;

(iii) given X and Y such that $Y \preceq_{s-icv} X$ and $E[X] = E[Y]$,

$$\left.\begin{array}{r} E[u(Y)] = E[u(X-\pi_u)] \\ E[v(Y)] = E[v(X-\pi_v)] \end{array}\right\} \Rightarrow \pi_u \geq \pi_v.$$

Proof. To see that (i)⇒(ii), it suffices to note that

$$\phi^{(1)} \leq 0 \Leftrightarrow u^{(1)} - \lambda v^{(1)} \leq 0 \Leftrightarrow \frac{u^{(1)}}{v^{(1)}} \leq \lambda$$

and that for $k = 2, \ldots, s$,

$$(-1)^{k+1}\phi^{(k)} = (-1)^{k+1}u^{(k)} - (-1)^{k+1}\lambda v^{(k)} \geq 0 \Leftrightarrow \frac{u^{(k)}}{v^{(k)}} \geq \lambda.$$

The proof of (ii)⇒(iii) is as follows:

$$\begin{aligned} E[u(X-\pi_u)] &= \lambda E[v(Y)] + E[\phi(Y)] \\ &\leq \lambda E[v(Y)] + E[\phi(X)] \text{ by Property 3.1} \\ &= \lambda E[v(X-\pi_v)] + E[\phi(X)] \\ &\leq \lambda E[v(X-\pi_v)] + E[\phi(X-\pi_v)] \text{ as } \phi \text{ is non-increasing} \\ &= E[u(X-\pi_v)] \end{aligned}$$

which implies $\pi_u \geq \pi_v$ since u is non-decreasing. To prove (iii)⇒(i), we know from Denuit and Eeckhoudt (2010) that for $k = 2, \ldots, s$, it is possible to find lotteries $X_k(x,h)$ and $Y_k(x,h)$ ordered in the $\preceq_{k-\text{icv}}$-sense with equal $k-1$ first moments (and, thus, fulfilling the requirements of (iii)). Then, proceeding as in Denuit and Eeckhoudt (2010), we can show that this implies that u is more kth degree Ross risk averse than v. This ends the proof. $\square$

An example of function ϕ satisfying the assumptions of Proposition 3.2(ii) is $\phi(x) = -x - \exp(-x)$ defined on a subset $[a,b]$ of $(0,+\infty)$ for which $\phi^{(1)}(x) = \exp(-x) - 1 < 0$ and $\phi^{(k)}(x) = (-1)^{k+1}\exp(-x)$ for $k \geq 2$.

4 Conclusion

Many economic or financial activities lead to the reduction of existing risks while not eliminating them. To express the wilingness to pay for such reductions, the concept of a stronger measure of risk aversion and its extension to any order is indispensable. In this note, we have shown that these stronger measures are appropriate for many more risk comparisons than it was admitted so far.

For instance, while recent papers have applied the stronger measures to risk comparisons involving an sth degree risk reduction *à la* Ekern (1980), we have shown here that the measures are also useful for a much wider class or risk comparisons involving pairs of risks with equal means ordered by sth degree stochastic dominance. In his way, one avoids for large s the severe limitations imposed by the equality of the $s-1$ first moments in Ekern's ranking.

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