A note on the estimation of orders of magnitude in marine sciences

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Fluid mechanics and reactive transport equations are non linear. In addition, they encompass a large number of terms involving, for most of them, partial derivatives. Therefore, analytical solutions hardly ever exist. This does not imply, however, that one is completely helpless. There are many methods for obtaining qualitative and quantitative information as to the solutions of these equations. In this respect, estimating orders of magnitude often plays a crucial role.

Clearly, identifying terms that are negligible in a given situation allows deriving simplified equations, which, presumably, will be easier to deal with than the complete ones. However, the question as to how to estimate the order of magnitude of each term without explicitly knowing the solution of the equations is not an easy one. Physical intuition, experience and a little luck are necessary in this matter. Nonetheless, a few basic notions are worth bearing in mind.

When dealing with orders of magnitude, specific notations are usually resorted to. The symbol "≈" means "of the same order of magnitude as", whilst "<<" (">>") means "much smaller (larger) than" or, equivalently, "at least one order of magnitude smaller (larger) than". For instance, 1=2 obviously is not correct, but $1 \approx 2$ is. In general, a quantity is considered to be one order of magnitude larger than another if their ratio is of the order of 10. Occasionally, symbol "$\tilde{<}$" ("$\tilde{>}$") is employed, whose signification is "smaller (larger) than or of the same order of magnitude as".

Let $f(t,\mathbf{x})$ represent a function of time t and position $\mathbf{x}$. To estimate its order of magnitude and that of its derivatives, it is appropriate to write this function as follows:

$$f(t,\mathbf{x}) = F + \Delta F\, \tilde{f}(t,\mathbf{x}) \quad , \tag{1}$$

where F and ΔF are constants, whose physical dimension is the same as that of $f(t,\mathbf{x})$. As a consequence, $\tilde{f}(t,\mathbf{x})$ is a dimensionless function. As will be seen, for a fruitful scaling of the function under consideration to be obtained, it is highly desirable that $\tilde{f}(t,\mathbf{x})$ be of order unity, i.e.

$$\tilde{f}(t,\mathbf{x}) \approx 1 \quad . \tag{2}$$

This expression does not imply that $\tilde{f}(t,\mathbf{x})$ is positive definite: when estimating orders of magnitude, it is customary to ignore the sign of the quantities under consideration — unless it is absolutely necessary to do otherwise. Accordingly, in the vast majority of cases, (2) is equivalent to $\left|\tilde{f}(t,\mathbf{x})\right| \approx 1$, though the latter expression is somewhat less ambiguous.

Two markedly different cases must be distinguished. If $F >> \Delta F$, then F may be viewed as the order of magnitude of the function under study, i.e. $f(t,\mathbf{x}) \approx F$, whilst ΔF denotes the order of magnitude of its variations. If, on the other hand, $F \tilde{<} \Delta F$, expression (1) may be simplified to $f(t,\mathbf{x}) = \Delta F\, \tilde{f}(t,\mathbf{x})$. Therefore, ΔF represents both the order of magnitude of $f(t,\mathbf{x})$ and that of its variations. For notational simplicity, the latter expression is often transformed to $f(t,\mathbf{x}) = F\, \tilde{f}(t,\mathbf{x})$. This type of notation is a very common one, but is not entirely consistent with (1).

To illustrate these points, consider first $u(t,\mathbf{x})$, a horizontal component of the water velocity in the sea, which usually satisfies $-1\,\mathrm{m\,s^{-1}} \lesssim u(t,\mathbf{x}) \lesssim 1\,\mathrm{m\,s^{-1}}$. Therefore, a suitable scaling of this function reads $u(t,\mathbf{x}) = U\,\tilde{u}(t,\mathbf{x})$, with $U \approx 1\,\mathrm{m\,s^{-1}}$, implying that $\tilde{u}(t,\mathbf{x}) \approx 1$ as expected. In this case, U refers to both the order of magnitude of the velocity component under study and that of its variations. On the other hand, so simple a scaling cannot apply to the atmospheric pressure at sea level $p_a^s(t,\mathbf{x})$. This is because the order of magnitude of the pressure, $P_a^s \approx 101\times10^3$ Pa, is much larger than that of pressure variations, $\Delta P_a^s \approx 10^3$ Pa. This requires the surface pressure to be formulated as $p_a^s(t,\mathbf{x}) = P_a^s + \Delta P_a^s\,\tilde{p}_a^s(t,\mathbf{x})$, which is directly inspired by (1). Accordingly, dimensionless function $\tilde{p}_a^s(t,\mathbf{x})$ is expected to be of order unity.

It is now necessary to introduce the concept of timescale, which is also termed characteristic time. For a function defined in accordance with (1), the characteristic time (of variation) is the time needed for its value to vary by about ΔF. For instance, for exponentially decreasing function $f(t) = f_0 e^{-t/\tau}$, where f_0 and τ are positive constants, it seems quite natural to set $F \approx f_0 \approx \Delta F$ and take the e-folding timescale τ as the relevant timescale. For periodic function $f(t) = f_0\cos(\omega t)$, where constant ω denotes the angular frequency, it is customary to regard ω^{-1} as the relevant characteristic time. One could argue that period $2\pi\omega^{-1}$ would be an equally acceptable timescale. This is not quite so, as will be seen below.

Using timescale T, the dimensionless time may defined:

$$\tilde{t} = \frac{t}{T} \quad . \tag{3}$$

Then, combining (1) and (3) leads to the following expression of the time derivative of $f(t,\mathbf{x})$:

$$\frac{\partial f}{\partial t} = \frac{\Delta F}{T}\frac{\partial \tilde{f}}{\partial \tilde{t}} \quad . \tag{4}$$

Clearly, $\partial \tilde{f} / \partial \tilde{t}$ is dimensionless, whilst the physical dimension of ratio $\Delta F / T$ is equivalent to that of $\partial f / \partial t$. In addition, if the scaling is well designed, i.e. suitable values of ΔF and T are selected, then $\partial \tilde{f} / \partial \tilde{t}$ is of order unity so that $\Delta F / T$ provides the order of magnitude of time derivative $\partial f / \partial t$. In other words, it is possible to evaluate the order of magnitude of the time derivative of a function without explicitly knowing its value at any time: it suffices to estimate the order of magnitude of its variations and the associated characteristic time.

By way of illustration, consider again exponentially decreasing function $f(t) = f_0 e^{-t/\tau}$. As suggested above, one sets $\Delta F = f_0$ and $T = \tau$, yielding

$$\frac{df}{dt} = \frac{f_0}{\tau}\underbrace{e^{-\tilde{t}}}_{\approx 1} \quad \Rightarrow \quad \frac{df}{dt} \approx \frac{f_0}{\tau} \quad , \tag{5}$$

as expected. Turning one's attention to periodic function $f(t) = f_0\cos(\omega t)$, the following result is readily obtained:

$$\frac{df}{dt} = -\frac{f_0}{\omega^{-1}}\underbrace{\sin\tilde{t}}_{\approx 1} \quad \Rightarrow \quad \frac{df}{dt} \approx \frac{f_0}{\omega^{-1}} \quad . \tag{6}$$

This suggests that the inverse of the angular frequency presumably is a more suitable timescale than the period. This is not to say that the taking the period as the characteristic time is completely irrelevant, but one must bear in mind that the ratio of the period to the inverse of the angular frequency is equal to 2π. Thus, the previous timescale is almost one order of magnitude larger than the latter. This difference is too large to go unnoticed, but too small to regard the period as a completely incorrect timescale.

The developments concerning timescales and time derivatives may be transposed easily to length scales and space derivatives. It must be realised, however, that space variations are not necessarily isotropic, which may render it inappropriate to rely on a single length scale. For most marine phenomena, the horizontal length scale, L_h, is much larger than the vertical one, L_v. Let x and y denote Cartesian horizontal coordinates and let z be the vertical coordinate. For illustrative purposes, consider a function representing a travelling wave:

$$f(t,\mathbf{x}) = f_0 \cos[\omega t - (kx + ly + mz)] \quad , \tag{7}$$

where f_0, ω, k, l and m are constants. Then, expressions

$$F = f_0 = \Delta F \ , \ \ T = 1/\omega \ , \ \ L_h = 1/\sqrt{k^2 + l^2} \ , \ \ L_v = 1/m \quad , \tag{8}$$

are in line with elementary physical intuition.

The difference between the horizontal length scale and the vertical one must be taken into account when introducing dimensionless space coordinates and derivatives. The dimensionless space coordinates then are $(\tilde{x},\tilde{y}) = (x,y)/L_h$ and $\tilde{z} = z/L_v$. Next, the del operator, ∇, must be split into its horizontal part, ∇_h, and it vertical one, $\mathbf{e}_z \, \partial/\partial z$, i.e.

$$\nabla = \nabla_h + \mathbf{e}_z \frac{\partial}{\partial z} \quad . \tag{9}$$

The dimensionless counterpart thereof is readily seen to be

$$\nabla = \frac{1}{L_h} \tilde{\nabla}_h + \mathbf{e}_z \frac{1}{L_v} \frac{\partial}{\partial \tilde{z}} \quad . \tag{10}$$

The above considerations suggest that when estimating orders of magnitude vagueness and ambiguity are unavoidable. However, this tool has proven countless times to be an invaluable one. Indeed, it has been used successfully in fluid mechanics for over a century and will continue to be resorted to for many years to come. No viable alternative is likely to be worked out in the foreseeable future. However, the limitations of this technique must be borne in mind at all times in order to avoid drawing erroneous conclusions. This is particularly true when a multi-scale problem is being tackled, i.e. a problem in which multiple characteristic times or length scales must be taken into account, possibly involving boundary layers.
