



Construction of cosmologically viable $f(\mathcal{G})$ gravity models

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ABSTRACT

We derive conditions under which $f(\mathcal{G})$ gravity models, whose Lagrangian densities f are written in terms of a Gauss–Bonnet term $\mathcal{G}$, are cosmologically viable. The most crucial condition to be satisfied is $d^2 f/d\mathcal{G}^2 > 0$, which is required to ensure the stability of a late-time de Sitter solution as well as the existence of standard radiation/matter dominated epochs. We present a number of explicit $f(\mathcal{G})$ models in which a cosmic acceleration is followed by the matter era. We find that the equation of state of dark energy can cross the phantom divide before reaching the present Universe. The viable models have asymptotic behavior $d^2 f/d\mathcal{G}^2 \rightarrow +0$ for $|\mathcal{G}| \rightarrow \infty$, in which case a rapid oscillation of perturbations occurs unless such an oscillating degree of freedom is suppressed relative to a homogeneous mode in the early universe. We also introduce an iterative method to avoid numerical instabilities associated with a large mass of the oscillating mode.

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1. Introduction

The late-time cosmic acceleration can, in principle, originate from a modification of gravity rather than an exotic source of matter with a negative pressure. Over the past five years, a lot of works on modified gravity have been done to identify the origin of dark energy (DE) [1]. The attractive point in modified gravity models is that they are generally more strongly constrained from cosmological observations and local gravity experiments than the models based on the exotic source of matter.

Presumably the simplest extension to Einstein gravity is the so-called $f(R)$ gravity in which f is an arbitrary function of the Ricci scalar R [2]. Even in this simple case it is not generally easy to construct viable $f(R)$ models that are consistent with cosmological and local gravity constraints. The main reason for this is that $f(R)$ gravity gives rise to a strong coupling between DE and a non-relativistic matter in the Einstein frame [3]. The models need to be carefully designed so that a scalar degree of freedom (“scalaron” [4]) is nearly frozen to suppress an effective coupling between the scalar field and matter.

The conditions for the cosmological viability of $f(R)$ models have been derived in Ref. [5]. Among those conditions the requirement, $d^2 f/dR^2 > 0$, is particularly important to give rise to a saddle matter era followed by a late-time cosmic acceleration. This is also required for the stability of cosmological perturbations [6] as well as for the consistency with local gravity experiments

[7]. The cosmologically viable $f(R)$ models need to be close to the Λ -Cold Dark Matter (Λ CDM) model in the deep matter era, but the deviation from it becomes important around the late stage of the matter era. Several examples of such viable models were presented in Refs. [8,9].

Hu and Sawicki [10] and Starobinsky [11] proposed $f(R)$ models that are consistent with local gravity constraints as well as cosmological constraints (see also Ref. [12]). In these models it is possible to find an appreciable deviation from the Λ CDM cosmology. This leaves a number of interesting observational signatures such as the peculiar evolution of the DE equation of state [13,14], the modification of the matter power spectrum [11,13,15,16] and the change of the convergence spectrum of weak lensing [17,18]. This is a welcome feature to distinguish $f(R)$ models from the Λ CDM cosmology in future observations.

There are other modified gravity DE models that are the generalizations of $f(R)$ gravity. For example Carroll et al. [19] proposed theories with the Lagrangian density $f(R, P, Q)$, where f is an arbitrary function of R , $P \equiv R_{\mu\nu}R^{\mu\nu}$ and $Q \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (here $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ are the Ricci tensor and the Riemann tensor, respectively). These theories are plagued by the appearance of spurious spin-2 ghosts unless a Gauss–Bonnet (GB) combination, i.e., $f = f(R, Q - 4P)$, is chosen [20–23]. Even in this case the graviton itself may still become a ghost in the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime, unless some no-ghost conditions are verified on the background [23].

The GB curvature invariant Lagrangian, $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, is a total derivative in the 4-dimensional FLRW background. In order to give rise to some contribution of the GB term to the Friedmann equation, we require that (i) the GB term cou-

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ples to a scalar field ϕ , i.e., $F(\phi)\mathcal{G}$, or (ii) the Lagrangian density f is a function of $\mathcal{G}$, i.e., $f(\mathcal{G})$. The GB coupling in the case (i) appears in low-energy string effective action [24] and cosmological solutions in such theory have been studied in great details [25]. It was shown by several authors that a late-time cosmic acceleration following a (scaling) matter era occurs for an exponential coupling $F(\phi) \propto e^{\lambda\phi}$ in the presence of a scalar-field potential [26–30]. Amendola et al. [31] studied local gravity constraints in such models and showed that the energy contribution coming from the GB term needs to be strongly suppressed for the compatibility with solar-system experiments. Thus, in the case (i), it is generally difficult to satisfy local gravity constraints if the GB term is responsible for DE.

In the context of $f(\mathcal{G})$ gravity there exists a de Sitter point that can be used for cosmic acceleration [32] (see also Ref. [33]). It was shown in Ref. [34] that the model with inverse powers of linear combinations of quadratic curvature invariants, i.e., $f(\mathcal{G}) = \mathcal{G}^n$ with $n < 0$, is not cosmologically viable because of the presence of separatrices between radiation and DE dominations. In Ref. [35] it was found that the model $f(\mathcal{G}) = \mathcal{G}^n$ with $n > 0$ can be consistent with solar-system tests for $n \lesssim 0.074$ if the GB term is responsible for DE. Li et al. [36] showed that it is difficult to reproduce standard expansion history of the Universe unless $f(\mathcal{G})$ is close to cosmological constant.

In this Letter we present a number of explicit models of $f(\mathcal{G})$ gravity that are cosmologically viable. These models mimic the Λ CDM cosmology in the deep matter era, but the deviation from it becomes important at late times on cosmological scales. This situation is similar to the viable $f(R)$ models proposed by Hu and Sawicki [10] and Starobinsky [11]. In $f(\mathcal{G})$ gravity, however, the GB term changes its sign during the transition from the matter era to the accelerated epoch. We need to take into account this property when we construct viable $f(\mathcal{G})$ models. For example, the $f(\mathcal{G})$ model that replaces R in the model of Hu and Sawicki or Starobinsky for $\mathcal{G}$ is not cosmologically viable.

There is another difference between $f(R)$ and $f(\mathcal{G})$ theories. The Ricci scalar R vanishes for the vacuum Schwarzschild solution, whereas the GB term has a non-vanishing value much larger than H_0^4 around the compact objects [35] (H_0 is the present Hubble parameter). In the presence of matter with a density ρ_m the Ricci scalar R is roughly of the order of ρ_m so that one has $R/H_0^2 \sim \rho_m/\rho_c$, where ρ_c is the present cosmological density. The viable $f(R)$ models [10,11] are designed to have the suppression term $(R/H_0^2)^{-n}$ ($n > 0$) in addition to the Λ CDM Lagrangian in the region of high density ($\rho_m \gg \rho_c$). In the $f(\mathcal{G})$ gravity this sort of suppression occurs even in the vacuum background because of the condition $\mathcal{G} \gg H_0^2$. Hence the $f(\mathcal{G})$ models might be less constrained by local gravity constraints relative to the $f(R)$ models. For the same reason, one could expect that for the interior star solutions these modifications of gravity could remain small corrections, provided that singularities of the kind $f_{,\mathcal{G}} \equiv d^2 f/d\mathcal{G}^2 = 0, \infty$ for finite values of $\mathcal{G}$ are not encountered. The models discussed in this Letter have exactly this feature. Note, however, that a detailed study of these issues is needed in order to further constrain these modifications, which we leave for future work.

We will show that the stability condition for a late-time de Sitter point is given by $f_{,\mathcal{G}}\mathcal{G} > 0$. This can be also derived by considering the stability of radiation and matter points. Note that the same condition has been derived in Ref. [36] by studying the evolution of cosmological perturbations. Since Li et al. [36] used the metric signature $(+, -, -, -)$ instead of $(-, +, +, +)$ that we adopt throughout this Letter, their stability condition $f_{,\mathcal{G}}\mathcal{G} < 0$ corresponds to our stability condition $f_{,\mathcal{G}}\mathcal{G} > 0$.

For the viable $f(\mathcal{G})$ models we propose in this Letter, the second derivative $f_{,\mathcal{G}}\mathcal{G}$ approaches $+0$ as $|\mathcal{G}|$ gets larger. In this case

the perturbations in the Hubble parameter H have a large mass squared proportional to $1/(H^4 f_{,\mathcal{G}}\mathcal{G})$ during radiation and matter eras. This can give rise to rapid oscillations of the Hubble parameter and matter density perturbations, as they happen for viable models in $f(R)$ gravity [11,13]. In order to avoid this, the oscillating mode needs to be suppressed relative to the homogeneous mode in the early Universe. The numerical instability we typically face in radiation and matter epochs is associated with the appearance of this oscillating mode. We shall introduce an iterative method to avoid this numerical instability in the high-redshift regime.

2. Cosmologically viable $f(\mathcal{G})$ dark energy models

We start with the following action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R + f(\mathcal{G}) \right] + S_m(g_{\mu\nu}, \Psi_m), \quad (1)$$

where R is a Ricci scalar, and f is a general differentiable function of $\mathcal{G}$. S_m is a matter action that depends on a spacetime metric $g_{\mu\nu}$ and matter fields Ψ_m . We choose units such that $\kappa^2 \equiv 8\pi G_N = 1$, where G_N is a bare gravitational constant. It should be pointed out that this action, at the classical level, can be rewritten as an auxiliary scalar field coupled to the $\mathcal{G}$ term, as shown in Ref. [23], following a trick used for the $f(R)$ theory. However there is no conformal transformation separating $\mathcal{G}$ from such a field, unlike the $f(R)$ theory in which the conformal transformation leads to an Einstein frame action with a canonical scalar field coupled to matter. An important quantity in $f(\mathcal{G})$ gravity is $1/f_{,\mathcal{G}}\mathcal{G}$, which plays the role of an effective mass for the theory (as we shall see later).

The variation of the action (1) with respect to $g_{\mu\nu}$ leads to the following equation

$$G_{\mu\nu} + 8 \left[R_{\mu\rho\nu\sigma} + R_{\rho\nu\sigma\mu} - R_{\rho\sigma\nu\mu} - R_{\mu\nu\sigma\rho} + R_{\mu\sigma\nu\rho} + \frac{R}{2} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\sigma} g_{\nu\rho}) \right] \nabla^\rho \nabla^\sigma f_{,\mathcal{G}} + (\mathcal{G} f_{,\mathcal{G}} - f) g_{\mu\nu} = T_{\mu\nu}, \quad (2)$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu}$ is the Einstein-tensor. For the energy-momentum tensor $T_{\mu\nu}$ of a matter fluid we take into account the contributions of non-relativistic matter (energy density ρ_m) and radiation (energy density ρ_{rad}). In a flat FLRW background with the metric $ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$, the 00 component of Eq. (2) gives

$$3H^2 = \mathcal{G} f_{,\mathcal{G}} - f - 24H^3 \dot{f}_{,\mathcal{G}} + \rho_m + \rho_{\text{rad}}, \quad (3)$$

where $H \equiv \dot{a}/a$, $f_{,\mathcal{G}} \equiv df/d\mathcal{G}$, and a dot represents a derivative with respect to cosmic time t . The GB term is given by

$$\mathcal{G} = 24H^2(H^2 + \dot{H}). \quad (4)$$

The energy densities ρ_m and ρ_{rad} satisfy the continuity equations $\dot{\rho}_m + 3H\rho_m = 0$ and $\dot{\rho}_{\text{rad}} + 4H\rho_{\text{rad}} = 0$, respectively.

2.1. Stability of de Sitter point

Let us first discuss the stability around a de Sitter point present in $f(\mathcal{G})$ gravity by neglecting the contribution of pressure-less matter and radiation. The Hubble parameter, $H = H_1$, at the de Sitter point satisfies

$$3H_1^2 = \mathcal{G}_1 f_{,\mathcal{G}}(\mathcal{G}_1) - f(\mathcal{G}_1), \quad (5)$$

where $\mathcal{G}_1 = 24H_1^4$. Note that we used the relations $\dot{H}_1 = 0$ and $\dot{\mathcal{G}}_1 = 0$. Considering a linear perturbation δH_1 about the de Sitter point, Eq. (3) gives

$$\delta H_1 = 4H_1^2 f_{,\mathcal{G}\mathcal{G}}(H_1) [H_1 \delta \mathcal{G}(H_1) - \delta \dot{\mathcal{G}}(H_1)]. \quad (6)$$

Substituting the relations $\delta \mathcal{G}(H_1) = 24(4H_1^3 \delta H_1 + H_1^2 \delta \dot{H}_1)$ and $\delta \dot{\mathcal{G}}(H_1) = 24H_1^2 (\delta \ddot{H}_1 + 4H_1 \delta \dot{H}_1)$ into Eq. (6), we obtain

$$\delta \ddot{H}_1 + 3H_1 \delta \dot{H}_1 + \left[\frac{1}{96H_1^6 f_{,\mathcal{G}\mathcal{G}}(H_1)} - 4 \right] H_1^2 \delta H_1 = 0. \quad (7)$$

This shows that the effective mass squared is $[(96H_1^6 f_{,\mathcal{G}\mathcal{G}}(H_1))^{-1} - 4]H_1^2$. The solution to Eq. (7) is given by

$$\delta H_1 = c_1 e^{\lambda_+ t} + c_2 e^{\lambda_- t},$$

$$\lambda_{\pm} = \frac{3H_1}{2} \left[-1 \pm \sqrt{1 - \frac{4}{9} \left(\frac{1}{96H_1^6 f_{,\mathcal{G}\mathcal{G}}(H_1)} - 4 \right)} \right], \quad (8)$$

where c_1 and c_2 are integration constants. This shows that the de Sitter point is stable under the condition

$$0 < H_1^6 f_{,\mathcal{G}\mathcal{G}}(H_1) < 1/384, \quad (9)$$

which requires that $f_{,\mathcal{G}\mathcal{G}}(H_1) > 0$. Especially when $0 < H_1^6 \times f_{,\mathcal{G}\mathcal{G}}(H_1) < 1/600$, the de Sitter point corresponds to a stable spiral (damping with oscillations).

2.2. Stabilities of radiation and matter points

We shall also study the stability of matter and radiation fixed points by using a similar method discussed above. Let us investigate the case in which the evolution of the scale factor is given by $a \propto t^p$ (p : constant) with the dominance of a fluid characterized by an energy density ρ_M (either ρ_m or ρ_{rad}). Then we have

$$3H^2 = \mathcal{G} f_{,\mathcal{G}} - f - 24H^3 f_{,\mathcal{G}\mathcal{G}} \dot{\mathcal{G}} + \rho_M. \quad (10)$$

We consider the first-order perturbations δ_H and δ_M as follows:

$$H = H^{(b)}(1 + \delta_H), \quad \rho_M = \rho_M^{(b)}(1 + \delta_M). \quad (11)$$

Here the subscript “(b)” represents background values, but in the following we omit it for simplicity. Taking the homogeneous perturbations of Eq. (10) and using the approximate relation $3H^2 \simeq \rho_M$, we find

$$\begin{aligned} \ddot{\delta}_H + \left[3 - \frac{6}{p} + \frac{96H^4 f_{,\mathcal{G}\mathcal{G}\mathcal{G}}}{f_{,\mathcal{G}\mathcal{G}}} \frac{1-p}{p^2} \right] H \dot{\delta}_H \\ + \left[\frac{21(1-p)}{p^2} + \frac{1}{96H^6 f_{,\mathcal{G}\mathcal{G}}} - 4 \right. \\ \left. + \frac{96H^4 f_{,\mathcal{G}\mathcal{G}\mathcal{G}}}{f_{,\mathcal{G}\mathcal{G}}} \left(4 - \frac{3}{p} \right) \frac{1-p}{p^2} \right] H^2 \delta_H = \frac{\delta_M}{192H^6 f_{,\mathcal{G}\mathcal{G}}} H^2. \end{aligned} \quad (12)$$

In the limit $p \rightarrow \infty$ without matter perturbations, Eq. (7) is recovered.

The solution to Eq. (12) is described by the sum of the matter-induced mode $\delta_H^{(\text{ind})}$ and the oscillating mode $\delta_H^{(\text{osc})}$ [11,13]:

$$\delta_H = \delta_H^{(\text{ind})} + \delta_H^{(\text{osc})}. \quad (13)$$

The matter-induced mode corresponds to a special solution to Eq. (12) induced by the matter perturbation δ_M . The oscillating mode is the solution of the equation with $\delta_M = 0$ in Eq. (12). For the $f(\mathcal{G})$ models whose deviation from the Λ CDM model is small during radiation and matter eras, $f_{,\mathcal{G}\mathcal{G}}$ is close to 0. In this case the mass squared

$$M^2 \equiv \frac{1}{96H^4 f_{,\mathcal{G}\mathcal{G}}}, \quad (14)$$

is the dominant contribution in front of the term δ_H in Eq. (12). In order to avoid a violent instability of perturbations we require that

$M^2 > 0$, giving the condition $f_{,\mathcal{G}\mathcal{G}} > 0$. In this case the perturbation $\delta_H^{(\text{osc})}$ oscillates with a frequency of the order of M .

During radiation and matter dominated epochs the GB term evolves as $\mathcal{G} = -24H^4$ and $\mathcal{G} = -12H^4$, respectively. Since $\mathcal{G} = 24H^2(\ddot{a}/a)$ from Eq. (4), the GB term changes its sign at the onset of the late-time acceleration. Hence the condition $f_{,\mathcal{G}\mathcal{G}} > 0$ needs to be satisfied in the region $\mathcal{G} \leq \mathcal{G}_1$. Since the term $24H^3 f_{,\mathcal{G}\mathcal{G}} \dot{\mathcal{G}}$ on the r.h.s. of Eq. (10) is of the order of $H^8 f_{,\mathcal{G}\mathcal{G}}$, this is suppressed relative to $3H^2$ under the condition $H^6 f_{,\mathcal{G}\mathcal{G}} \ll 1$. In order for this condition to hold in the radiation and matter eras, we require that the term $f_{,\mathcal{G}\mathcal{G}}$ approaches +0 with the increase of $|\mathcal{G}|$.

2.3. Viable $f(\mathcal{G})$ models

If we consider a spherically symmetric body (mass $M_\odot$ and radius $r_\odot$) with a homogeneous density, it was shown in Ref. [35] that the GB term inside and outside the body is given by $\mathcal{G} = -48(G_N M_\odot)^2/r_\odot^6$ and $\mathcal{G} = 48(G_N M_\odot)^2/r^6$, respectively (r is a distance from the center of symmetry). In the vicinity of the Sun or the Earth, $|\mathcal{G}|$ is much larger than the present cosmological GB term, $\mathcal{G}_0$. As we move from the interior to the exterior of the star, the GB term crosses 0 from negative to positive. This means that $f(\mathcal{G})$ and its derivatives with respect to $\mathcal{G}$ need to be regular for both negative and positive values of $\mathcal{G}$ whose amplitudes are much larger than $\mathcal{G}_0$.

From the above discussions the viable models need to satisfy the following conditions:

- (i) $f(\mathcal{G})$ and its derivatives $f_{,\mathcal{G}}, f_{,\mathcal{G}\mathcal{G}}, \dots$ are regular.
- (ii) $f_{,\mathcal{G}\mathcal{G}} > 0$ for $\mathcal{G} \leq \mathcal{G}_1$ and $f_{,\mathcal{G}\mathcal{G}}$ approaches +0 in the limit $|\mathcal{G}| \rightarrow \infty$.
- (iii) $0 < H_1^6 f_{,\mathcal{G}\mathcal{G}}(H_1) < 1/384$ at the de Sitter point.

A number of examples for the viable forms of $f_{,\mathcal{G}\mathcal{G}}$ are

$$\begin{aligned} \text{(a)} \quad f_{,\mathcal{G}\mathcal{G}} &= \frac{\lambda}{\mathcal{G}_*^{3/2} [1 + (\mathcal{G}^2/\mathcal{G}_*^2)^n]}, \\ \text{(b)} \quad f_{,\mathcal{G}\mathcal{G}} &= \frac{2\lambda}{\mathcal{G}_*^{3/2} (1 + \mathcal{G}^2/\mathcal{G}_*^2)^n}, \\ \text{(c)} \quad f_{,\mathcal{G}\mathcal{G}} &= \frac{\lambda}{\mathcal{G}_*^{3/2}} [1 - \tanh^2(\mathcal{G}/\mathcal{G}_*)], \end{aligned} \quad (15)$$

where λ , n and $\mathcal{G}_*$ are positive constants. These satisfy the condition $f_{,\mathcal{G}\mathcal{G}} > 0$ for all values of $\mathcal{G}$. In the following let us study the case (a) with $n = 1$, the case (b) with $n = 2$, and the case (c). Integrating $f_{,\mathcal{G}\mathcal{G}}$ with respect to $\mathcal{G}$ twice, we obtain the following models

$$\text{(A)} \quad f(\mathcal{G}) = \lambda \frac{\mathcal{G}}{\sqrt{\mathcal{G}_*}} \arctan\left(\frac{\mathcal{G}}{\mathcal{G}_*}\right) - \frac{\lambda}{2} \sqrt{\mathcal{G}_*} \ln\left(1 + \frac{\mathcal{G}^2}{\mathcal{G}_*^2}\right) - \alpha \lambda \sqrt{\mathcal{G}_*}, \quad (16)$$

$$\text{(B)} \quad f(\mathcal{G}) = \lambda \frac{\mathcal{G}}{\sqrt{\mathcal{G}_*}} \arctan\left(\frac{\mathcal{G}}{\mathcal{G}_*}\right) - \alpha \lambda \sqrt{\mathcal{G}_*}, \quad (17)$$

$$\text{(C)} \quad f(\mathcal{G}) = \lambda \sqrt{\mathcal{G}_*} \ln\left[\cosh\left(\frac{\mathcal{G}}{\mathcal{G}_*}\right)\right] - \alpha \lambda \sqrt{\mathcal{G}_*}, \quad (18)$$

where α is a constant. We have dropped the terms proportional to $\mathcal{G}$, since they do not give rise to any contribution to the evolution equations. If we demand the condition $f(\mathcal{G} = 0) = 0$ then we have $\alpha = 0$. These models are also consistent with the regularity condition (i). Note that the model $f(\mathcal{G}) = -\lambda \mathcal{G}_c^{1/2} [1 - (1 + \mathcal{G}^2/\mathcal{G}_c^2)^{-n}]$, which is a generalization of the viable $f(R)$ model proposed by Starobinsky [11], is not compatible with the condition $f_{,\mathcal{G}\mathcal{G}} > 0$ for $\mathcal{G} < 0$.

In the models (A), (B), (C) the term $\mathcal{G}f_{,\mathcal{G}} - f$ on the r.h.s. of Eq. (3) in the regime $|\mathcal{G}| \gg \mathcal{G}_*$ can be estimated by (A) $\mathcal{G}f_{,\mathcal{G}} - f \simeq (\lambda/2)\sqrt{\mathcal{G}_*}[\ln(\mathcal{G}^2/\mathcal{G}_*^2) + 2\alpha]$, (B) $\mathcal{G}f_{,\mathcal{G}} - f \simeq \lambda\sqrt{\mathcal{G}_*}(1 + \alpha)$, (C) $\mathcal{G}f_{,\mathcal{G}} - f \simeq \lambda\sqrt{\mathcal{G}_*}(\ln 2 + \alpha)$, respectively. As long as $\mathcal{G}_*$ is of the order of H_0^4 (where H_0 is the present Hubble parameter) the term $\mathcal{G}f_{,\mathcal{G}} - f$ is subdominant relative to the term $3H^2$ during radiation and matter eras (for $\alpha\lambda$ of the order of unity). Moreover we have $H^6 f_{,\mathcal{G}\mathcal{G}} \ll 1$ for $|\mathcal{G}| \gg \mathcal{G}_*$ in the above three models, which means that the condition $|24H^3 f_{,\mathcal{G}\mathcal{G}} \dot{\mathcal{G}}| \ll 3H^2$ is satisfied in this regime. The contribution coming from the $f(\mathcal{G})$ term becomes important when H decreases to the order of H_* .

Let us discuss the condition (iii) for the model (A). At the de Sitter point this model satisfies the following relation

$$\lambda = \frac{6y_1^2}{2\alpha + \ln(1 + 24^2 y_1^8)}, \quad \text{where } y_1 = \frac{H_1}{\mathcal{G}_*^{1/4}}. \quad (19)$$

When $\alpha = 0$ the r.h.s. has a minimum value $\lambda_{\min} = 0.828$ at $y_1 = 0.736$. Hence two de Sitter points exist for $\lambda > 0.828$. The stability condition for the de Sitter solution corresponds to

$$\lambda < \frac{1 + 24^2 y_1^8}{384 y_1^6}, \quad (20)$$

which gives $y_1 > 0.736$ by using Eq. (19). Hence one of the de Sitter points that exists in the region $y_1 > 0.736$ is stable.

In the case where $\alpha = 0$ we have numerically found instabilities of cosmological solutions around the region $\mathcal{G} = 0$ (which occurs during the transition from the matter era to the accelerated era). One can estimate the stability of solutions by setting $p = 1$ in Eq. (12). This gives the stability condition

$$0 < H_2^6 f_{,\mathcal{G}\mathcal{G}}(H_2) < 1/384, \quad (21)$$

where H_2 is the Hubble parameter at $\mathcal{G} = 0$. For the model (A) this translates into

$$\lambda < \frac{1}{384 y_2^6}, \quad (22)$$

where $y_2 = H_2/\mathcal{G}_*^{1/4}$. Here y_2 is slightly larger than y_1 . Since $y_1 > 0.736$ for $\alpha = 0$, the condition (22) requires that $\lambda \ll 1$. However this is not compatible with the condition $\lambda > 0.828$ for the existence of de Sitter solutions. As we will see in the next section, it is generally difficult to have a natural transition around $\mathcal{G} = 0$ for $\alpha = 0$. This anticipates that the cosmological constant term may be needed in general for the cosmological viability of the $f(\mathcal{G})$ models. If this is the case, the need of such a constant term makes the $f(\mathcal{G})$ models less attractive from a theoretical point of view.

If $\alpha \neq 0$ it is possible to make λ much smaller than 1 provided that $\alpha \gg 1$ [see Eq. (19)]. When $\alpha = 10^2$ and $y_1 = 0.1$, for example, we have $\lambda = 3.0 \times 10^{-4}$ and hence the condition (22) is satisfied even for $y_2 = 1$. In the next section we shall show that viable cosmological trajectories can be realized for α larger than the order of unity.

The stabilities of cosmological solutions for the models (B) and (C) are similar to those for the model (A) discussed above. The difference is that the models (B) and (C) have larger powers of $\mathcal{G}^2/\mathcal{G}_*^2$ in $f_{,\mathcal{G}\mathcal{G}}$ compared to the model (A). Since the mass M^2 grows rapidly toward the past in such models, it is more difficult to start solving the equations numerically from the high-redshift regime unless we use an iterative method we discuss later.

2.4. Oscillating modes in the early Universe

At the end of this section we discuss the evolution of the homogeneous perturbation δ_H during radiation and matter eras for the viable $f(\mathcal{G})$ models presented above. For the matter-induced

mode, the mass term $M^2 \delta_H$ balances with the source term on the r.h.s. of Eq. (12), giving

$$\delta_H^{(\text{ind})} \simeq \delta_M/2. \quad (23)$$

Hence the matter-induced mode grows in proportion to δ_M . In the matter era $\delta_H^{(\text{ind})} \propto t^{2/3}$ in the regime where the model is close to Λ CDM model [13].

Let us consider the model (B) for the evolution of the oscillating mode in the regime $|\mathcal{G}| \gg \mathcal{G}_*$. In this case we have $96H^4(f_{,\mathcal{G}\mathcal{G}}/f_{,\mathcal{G}\mathcal{G}})(1-p)/p^2 \simeq 16/p$ and $M^2 \simeq \mu^2 t^{-12}$, where μ is a constant. Hence the oscillating mode satisfies

$$\ddot{\delta}_H^{(\text{osc})} + \frac{3p+10}{t} \dot{\delta}_H^{(\text{osc})} + \frac{\mu^2}{t^{12}} \delta_H^{(\text{osc})} \simeq 0. \quad (24)$$

The solution to this equation can be written as the combination of Bessel differential functions:

$$\delta_H^{(\text{osc})} = \left(\frac{t_i}{t}\right)^{1+5/p} \left[\frac{\delta_{H,t_i}^{(1)}}{J_{\theta_1}(z_i)} J_{\theta_1}(z) + \frac{\delta_{H,t_i}^{(2)}}{J_{-\theta_1}(z_i)} J_{-\theta_1}(z) \right], \quad (25)$$

where $\theta_1 = (5+p)/(5p)$, $z = \mu/(5t^5)$, $z_i = z(t_i)$, and t_i is the initial time at which two modes in the square bracket of Eq. (25) have amplitudes $\delta_{H,t_i}^{(1)}$ and $\delta_{H,t_i}^{(2)}$.

As $t \rightarrow 0$, the solution reduces to

$$\delta_H^{(\text{osc})} \simeq A t^{-(10-3p)/(2p)} \left\{ C_1 \cos \left[\frac{\pi}{4} (1 + 2\theta_1) - \frac{\mu t^{-5}}{5} \right] + C_2 \cos \left[\frac{\pi}{4} (1 - 2\theta_1) - \frac{\mu t^{-5}}{5} \right] \right\}, \quad (26)$$

where $A = \mu^{-1/4} \sqrt{10/\pi}$, and C_1 and C_2 are constants. During the matter era ($p = 2/3$), the amplitude of $\delta_H^{(\text{osc})}$ is proportional to t^{-6} so that the oscillating mode tends to be negligible relative to the matter-induced mode with time. However, as we go back to the past, the amplitude grows with larger frequency of oscillations. Since one has $\text{Amp}[\delta_H^{(\text{osc})}] \propto t^{-17/2}$ for $p = 1/2$, this property also persists during the radiation-dominated epoch. As we see Section 3 the large mass term M tends to lead numerically instabilities associated with violent oscillations of δ_H , unless the oscillating mode is strongly suppressed relative to the matter-induced mode. For the model (A) the growth of the mass squared is not so strong ($M^2 \propto t^{-4}$) relative to the model (B), but still it is difficult to solve equations numerically from the high-redshift regime. This property is even severe for the model (C) because of the rapid increase of the mass squared: $M^2 \propto t^4 \exp(c/t^4)$ (c is a positive constant).

3. Cosmological dynamics

In this section we discuss cosmological dynamics for the viable models presented in the previous section. In Section 3.1 we first integrate the background equations for the model (A) directly in the low-redshift regime. It is more difficult to solve the equations numerically as we start integrating from higher redshifts. This is related to the appearance of the oscillating mode with a large frequency. In Section 3.2 we shall propose an iterative method to get approximate solutions in such a situation.

3.1. Low-redshift cosmological solutions

In order to discuss cosmological solutions in the low-redshift regime, it is convenient to introduce the following dimensionless quantities

$$x \equiv \frac{\dot{H}}{H^2}, \quad y \equiv \frac{H}{H_*}, \quad \Omega_m \equiv \frac{\rho_m}{3H^2}, \quad \Omega_{\text{rad}} \equiv \frac{\rho_{\text{rad}}}{3H^2}, \quad (27)$$

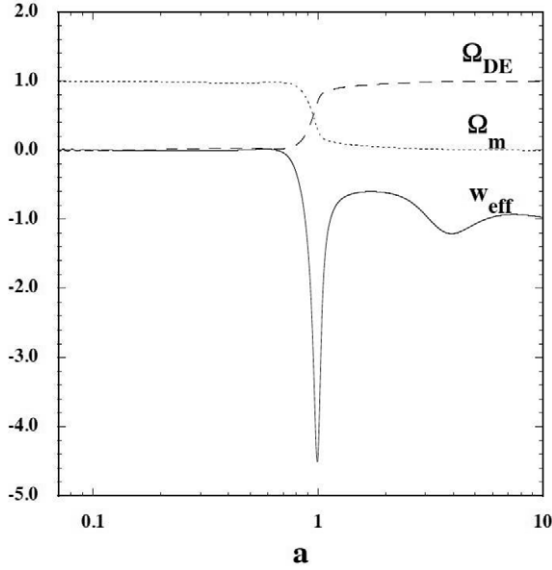


Fig. 1. The evolution of w_{eff} , Ω_{DE} and Ω_m versus the scale factor a for the model (A) with parameters $\alpha = 10.0$ and $\lambda = 7.5 \times 10^{-2}$. The initial conditions are chosen to be $x = -1.502$, $y = 20.0$, $\Omega_m = 0.9959$ and $\Omega_{\text{rad}} = 0.004$. These results are obtained by integrating Eqs. (28)–(31) forward using the lsode stiff integrator. It is clear that the matter era is followed by the accelerated epoch with the oscillation of w_{eff} around -1 .

where $H_* = G_*^{1/4}$. We then obtain the following equations of motion

$$x' = -4x^2 - 4x + \frac{1}{24^2 H^6 f_{,\mathcal{G}\mathcal{G}}} \left[\frac{\mathcal{G} f_{,\mathcal{G}} - f}{H^2} - 3(1 - \Omega_m - \Omega_{\text{rad}}) \right], \quad (28)$$

$$y' = xy, \quad (29)$$

$$\Omega'_m = -(3 + 2x)\Omega_m, \quad (30)$$

$$\Omega'_{\text{rad}} = -(4 + 2x)\Omega_{\text{rad}}, \quad (31)$$

where a prime represents a derivative with respect to $N = \ln(a)$. The quantities $H^6 f_{,\mathcal{G}\mathcal{G}}$ and $(\mathcal{G} f_{,\mathcal{G}} - f)/H^2$ can be expressed by x and y once the model is specified. The energy fraction of DE is given by $\Omega_{\text{DE}} = 1 - \Omega_m - \Omega_{\text{rad}}$. We also define the effective equation of state

$$w_{\text{eff}} \equiv -1 - \frac{2\dot{H}}{3H^2} = -1 - \frac{2}{3}x, \quad (32)$$

which changes from 0 to -1 from the matter era to the final de Sitter epoch for viable $f(\mathcal{G})$ models.

In Fig. 1 we plot the evolution of w_{eff} , Ω_{DE} and Ω_m for the model (A) with parameters $\alpha = 10.0$ and $\lambda = 7.5 \times 10^{-2}$. The present epoch corresponds to the scale factor $a = 1$ with $\Omega_{\text{DE}} = 0.72$ and $\Omega_m = 0.28$. From Eq. (19) there exists a de Sitter point at $y_1 = 0.518$ that satisfies the stability condition (20). In fact Fig. 2 shows that the quantity $y = H/H_*$ approaches this value in the asymptotic future.

In Fig. 1 the effective equation of state w_{eff} oscillates around -1 as the system enters the epoch of cosmic acceleration, which implies that the de Sitter solution is a stable spiral. It is interesting to note that w_{eff} drops down to a value less than -4 around the present epoch. We also find in Fig. 2 that the GB term switches its sign during the transition from the matter era to the accelerated epoch (which corresponds to passing through the minus infinity in logarithmic scale).

We have also tried numerical integrations by changing the model parameters α and λ . For the values of α smaller than unity

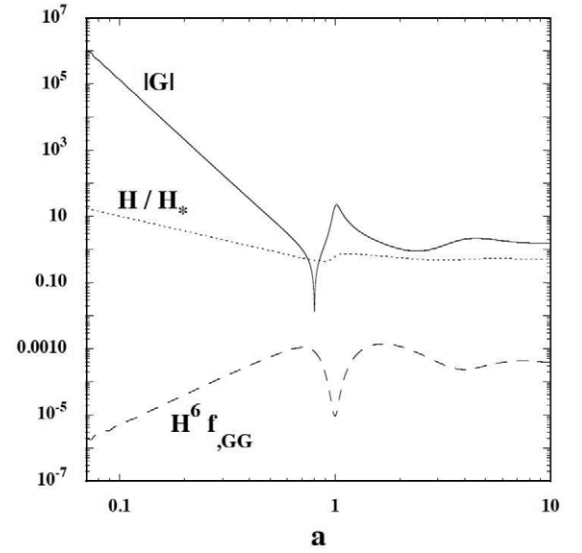


Fig. 2. The evolution of the quantities H/H_* , $|G|$ and $H^6 f_{,\mathcal{G}\mathcal{G}}$ for the same model parameters as given in Fig. 1. The GB term changes its sign from negative to positive during the transition from the matter era to the accelerated epoch. The term $H^6 f_{,\mathcal{G}\mathcal{G}}$ becomes much smaller than 1 as we go back to the past.

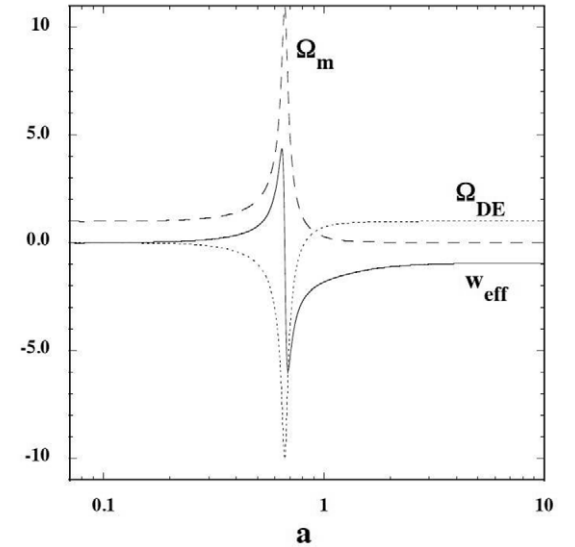


Fig. 3. The evolution of w_{eff} , Ω_{DE} and Ω_m versus the scale factor a for the model (A) with parameters $\alpha = 0$ (i.e., $f(\mathcal{G} = 0) = 0$) and $\lambda = 1$. The initial conditions are chosen to be $x = -1.502$, $y = 20.0$, $\Omega_m = 0.9959$ and $\Omega_{\text{rad}} = 0.004$. From the matter era to the accelerated epoch the solution shows an unusual transition where Ω_m exceeds the order of unity.

it is not easy to get plausible cosmological evolution. This is associated with the fact that the stability condition (22) is difficult to be satisfied around $\mathcal{G} = 0$ for smaller α . In Fig. 3 we illustrate the variation of w_{eff} , Ω_{DE} and Ω_m for the model (A) with parameters $\alpha = 0$ and $\lambda = 1$. While a stable de Sitter point exists at $y_1 = 1.075$, Ω_m becomes larger than the order of unity during the transition from the matter era to the accelerated epoch. This reflects the instability of the solutions around $\mathcal{G} = 0$. For α smaller than the order of unity, the solutions exhibit unusual behavior similar to that in Fig. 3 or they do not reach the de Sitter attractor.

For the model (A) it is difficult to integrate the equations from the redshift larger than 50 because the term $1/(96H^6 f_{,\mathcal{G}\mathcal{G}})$ in Eq. (12) gets very large as we go back to the past. The decrease of the term $H^6 f_{,\mathcal{G}\mathcal{G}}$ for larger z is in fact confirmed in Fig. 2. On

one hand, this is a good property, as the spin-2 no-ghost conditions are then satisfied [23], in addition to the fact that scalar perturbations remain stable [36]. However, this large mass leads to a rapid oscillation of the perturbations δ_H , which is hard to be dealt with numerically. The difficulty of numerical integration is even more severe for the models (B) and (C). When the equations are integrated forward, we need to choose initial conditions carefully so that the oscillating mode $\delta_H^{(\text{osc})}$ is suppressed relative to the matter-induced mode $\delta_H^{(\text{ind})}$. This property is similar to viable $f(R)$ gravity models in which the system is unstable in the high-redshift regime because of the appearance of oscillating mode [11, 13].

3.2. High-redshift approximate solutions

In the regime of high-redshifts one can use an iterative method (known as the “fixed-point” method) to find cosmological solutions approximately. We define $\bar{H}^2$ and $\bar{\mathcal{G}}$ to be $\bar{H}^2 \equiv H^2/H_0^2$ and $\bar{\mathcal{G}} \equiv \mathcal{G}/H_0^4$, where the subscript “0” represents present values (with $a_0 = 1$). The models (A), (B) and (C) can be written in the form

$$f(\mathcal{G}) = \bar{f}(\mathcal{G})H_0^2 - \bar{\Lambda}H_0^2, \quad (33)$$

where $\bar{\Lambda} = \alpha\lambda\sqrt{\mathcal{G}_*}/H_0^2$ and $\bar{f}(\mathcal{G})$ is a function of $\mathcal{G}$. The modified Friedmann equation (3) reduces to

$$\bar{H}^2 - \bar{H}_\Lambda^2 = \frac{1}{3}(\bar{f}_{,\mathcal{G}}\bar{\mathcal{G}} - \bar{f}) - 8\frac{d\bar{f}_{,\mathcal{G}}}{dN}\bar{H}^4, \quad (34)$$

where

$$\bar{H}_\Lambda^2 = \frac{\Omega_m^{(0)}}{a^3} + \frac{\Omega_{\text{rad}}^{(0)}}{a^4} + \frac{\bar{\Lambda}}{3}. \quad (35)$$

Note that H_Λ represents the Hubble parameter in the Λ CDM model. In the following we omit the tilde for simplicity.

In Eq. (34) there are derivatives of H in terms of N up to the second-order. Then we write Eq. (34) in the form

$$H^2 - H_\Lambda^2 = C(H^2, H^{2'}, H^{2''}), \quad (36)$$

where $C = (f_{,\mathcal{G}}\mathcal{G} - f)/3 - 8H^4(df_{,\mathcal{G}}/dN)$. At high redshifts ($a \lesssim 0.01$) the models (A), (B) and (C) are close to the Λ CDM model, i.e., $H^2 \simeq H_\Lambda^2$. We shall introduce an iterative method to derive approximate solutions in such a regime.

As a starting guess we set the solution to be $H_{(0)}^2 = H_\Lambda^2$. The first iteration is then

$$H_{(1)}^2 = H_\Lambda^2 + C_{(0)}, \quad (37)$$

where $C_{(0)} \equiv C(H_{(0)}^2, H_{(0)}^{2'}, H_{(0)}^{2''})$. This first iterative solution was used as an approximate solution for inverse curvature gravity in the paper of Mena et al. [37], while the authors did not pursue the idea of iterating the process again. We shall iterate the process in order to get better approximate solutions. The second iteration is

$$H_{(2)}^2 = H_\Lambda^2 + C_{(1)}, \quad (38)$$

where $C_{(1)} \equiv C(H_{(1)}^2, H_{(1)}^{2'}, H_{(1)}^{2''})$.

If the starting guess was in the basin of a fixed point, $H_{(i)}^2$ will converge to the solution of the equation after the i th iteration. For the convergence we need the following condition

$$\frac{H_{i+1}^2 - H_i^2}{H_{i+1}^2 + H_i^2} < \frac{H_i^2 - H_{i-1}^2}{H_i^2 + H_{i-1}^2}, \quad (39)$$

which means that each correction decreases for larger i . The following relation is also required to be satisfied:

$$\frac{H_{i+1}^2 - H_\Lambda^2 - C_{i+1}}{H_{i+1}^2 - H_\Lambda^2 + C_{i+1}} < \frac{H_i^2 - H_\Lambda^2 - C_i}{H_i^2 - H_\Lambda^2 + C_i}. \quad (40)$$

Once the solution begins to converge, one can stop the iteration up to the required/available level of precision.

At very high redshifts (say, the epoch of nucleosynthesis), the above method is presumably the only one that provides approximate cosmological solutions. We have checked that, for $N \gtrsim -4$ (i.e., for the redshift $z \lesssim 50$), this iterative method and the direct-forward-integration give the same results. To be more precise, the iteration is used in order to find the values of H^2 and $\mathcal{G}$ at $N = -4$. Then these values are adopted as initial conditions for the direct-forward-integrator. We integrate the equations for $-4 < N < -3$ (where the iterative method still works well) and compare H^2 as well as $\mathcal{G}$ at $N = -3$ derived by two methods. We find that these provide identical results with the precision of the order of 10^{-9} . This shows that the solution derived by direct integration remains close to the iterative one at least for all values of N at which the initial guess for the iterative method is in the basin of the fixed point.

In Fig. 4 we plot the absolute error $\log_{10}(|H_i^2 - H_\Lambda^2 - C_i|)$ as well as the relative error $\log_{10}[\frac{|H_i^2 - H_\Lambda^2 - C_i|}{|H_i^2 - H_\Lambda^2 + C_i|}]$ for the model (A) with $\alpha = 10.0$ and $\lambda = 7.5 \times 10^{-2}$ (i.e., the same model parameters as

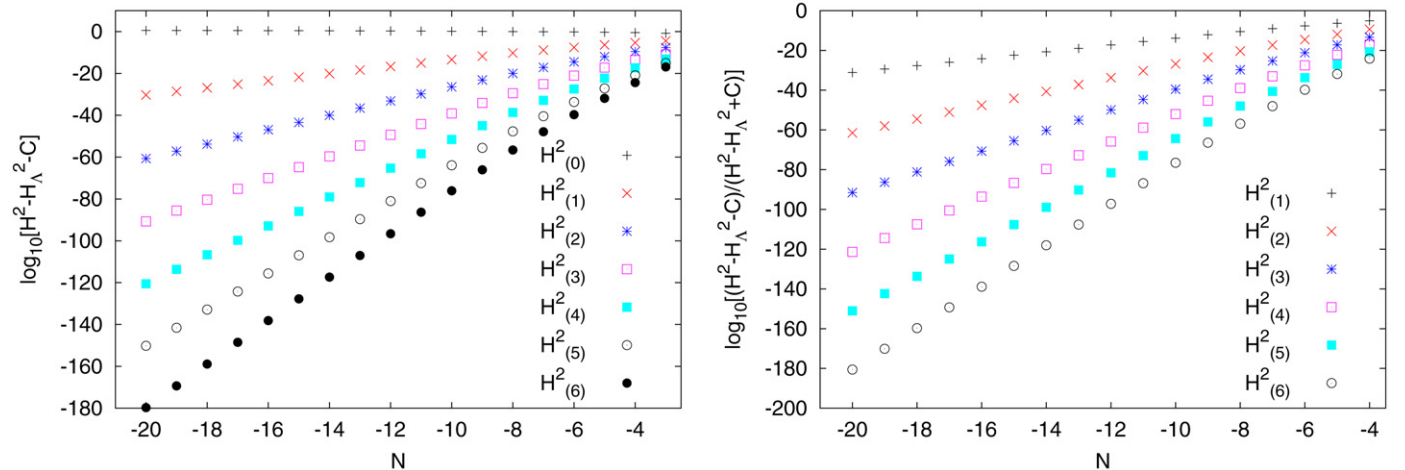


Fig. 4. The plot of the absolute errors $\log_{10}(|H_i^2 - H_\Lambda^2 - C_i|)$ (left) and $\log_{10}[\frac{|H_i^2 - H_\Lambda^2 - C_i|}{|H_i^2 - H_\Lambda^2 + C_i|}]$ (right) versus N for the model (A) with $i = 0, 1, \dots, 6$. The model parameters α and λ are the same as those in Fig. 1. The iterative method provides the solutions with high accuracy in the regime $N \lesssim -4$.

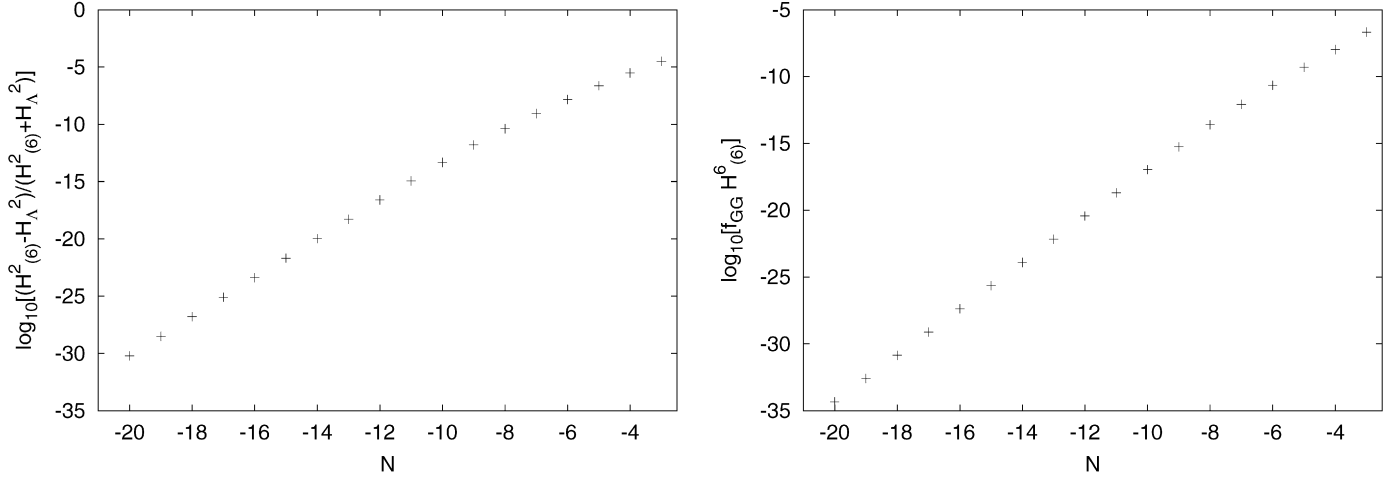


Fig. 5. The plot of the relative error $\log_{10}[\frac{|H^2_{(6)} - H^2_{\Lambda}|}{|H^2_{(6)} + H^2_{\Lambda}|}]$ for $i = 6$ (left) and the quantity $\log_{10}(f_{GG} H^6)$ (right) versus N for the model (A) with the same model parameters as used in 1.

used in Fig. 1). Note that we have carried out the iteration for 6 times. The absolute error $\log_{10}(|H^2_i - H^2_{\Lambda} - C_i|)$ is not sufficient to confirm that $H^2_i - H^2_{\Lambda}$ is really close to C_i . However the smallness of the relative error in the left panel of Fig. 5 confirms that the solution derived by the iterative method is very accurate. While this approximation tends to be worse for lower redshifts, the direct integration is well suited for $N \gtrsim -4$ as we presented in the Section 3.1.

The left panel of Fig. 5 shows that, for $N \lesssim -4$, the iterative solution is very similar to the Λ CDM solution characterized by H^2_{Λ} . Hence the Universe passes through the radiation-dominated epoch to the matter-dominated one as in the Λ CDM cosmology. From the right panel of Fig. 5 we find that the quantity $f_{GG} H^6$ is very much smaller than unity for $N \lesssim -4$, which leads to an extremely large frequency M for the perturbation δ_H . This is the main reason why we need to use the iterative method rather than the direct integration to avoid numerical instabilities in the high-redshift regime.

4. Conclusions

In this Letter we have constructed viable $f(\mathcal{G})$ gravity models that are cosmologically viable. In order to have a stable de Sitter attractor the condition (9) needs to be satisfied. For the stability of radiation and matter fixed points the mass squared M^2 given in Eq. (14) is required to be positive. These results show that the quantity f_{GG} must be positive to obtain viable cosmological evolution. Since the GB term changes its sign during the transition from the matter era to the epoch of cosmic acceleration, we need to construct models in which neither the violation of the stability conditions nor the divergence of some terms occurs in the past expansion history of the Universe.

A number of explicit $f(\mathcal{G})$ models satisfying the above requirements are given in Eqs. (16)–(18). Even if these models do not require an exotic source of matter responsible for the cosmic acceleration, it should be pointed out that in the examples presented here a cosmological constant term is still required. These models come from the integration of viable forms of f_{GG} presented in Eq. (15). In the regime where $|\mathcal{G}|$ is much larger than $\mathcal{G}_*$ (which is the same order as the present value $\mathcal{G}_0$), the model mimics the Λ CDM cosmology. The deviation from the Λ CDM cosmology tends to be important as $|\mathcal{G}|$ approaches the order of $\mathcal{G}_*$. Since the mass squared M^2 becomes very much larger than H^2 as we go back to the past, this leads to rapid oscillations of the Hubble parameter unless initial conditions are chosen such that the oscillating mode

is suppressed relative to the matter-induced mode. The direct integration of Eqs. (28)–(31) is prone to numerical instabilities in the high-redshift regime because of the very heavy mass M .

We have adopted an iterative method to derive approximate solutions in the high-redshift regime. The results of Figs. 4 and 5 show that the iterative method gives rise to the cosmic expansion history that is very close to the Λ CDM model for $z \gtrsim 50$. We have used these results as initial conditions at $z \sim 50$ for the direct forward-integration in the low-redshift region. We have found that the effective equation of state w_{eff} enters the phantom region ($w_{\text{eff}} < -1$) before reaching the de Sitter attractor with $w_{\text{eff}} = -1$ (see Fig. 1). For the models with $f(\mathcal{G} = 0) = 0$ the solutions typically exhibit unusual behavior where Ω_m grows larger than 1 during the transition from the matter era to the accelerated era (see Fig. 3). This is associated with the fact that an instability around $\mathcal{G} = 0$ is present for small $\alpha (\lesssim 1)$ for the models (16)–(18), while this is not the case for α larger than the order of unity.

Since the $f(\mathcal{G})$ models we have proposed mimic the Λ CDM model in the high-curvature regime whose energy density is much larger than the present cosmological one, it should be possible for them to satisfy local gravity constraints. We leave detailed analysis for the compatibility of our models with local gravity experiments for future work.

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