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Carathéodory, Helly and Radon Numbers for Sublattice
Convexities

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DISCUSSION PAPER

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Largest minimal inversion-complete and pair-complete sets of permutations¹

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Abstract

The Carathéodory, Helly, and Radon numbers are three main invariants in convexity theory. They relate, respectively, to minimal representations of points in a convex hull; to the size of minimal infeasible inequality systems; and to VC-dimensions and the existence of centerpoints (generalized medians). These invariants have been determined, exactly or approximately, for a number of different convexity structures. We consider convexity structures defined by the sublattices and by the convex sublattices of finite-dimensional Euclidian, integer and Boolean spaces. Such sublattices arise as feasible sets in submodular optimization (lattice programming) and in monotone comparative statics of optimization and fixed-point problems. We present new results on the exact Carathéodory numbers for these sublattice convexities. Our results imply, for example, that if a subset S of a finite set D can be obtained with unions and intersections from a given family G of subsets of D , then S can be obtained with unions and intersections from a small subfamily of G . Convex sublattice and integral L-natural convexities are induced by polyhedra defined by dual generalized network flow constraint systems. We reduce the problem of finding the Carathéodory number for the integral L-natural convexity to an extremal problem in the theory of permutations, namely, finding the maximum size of a minimal cover of all ordered pairs of elements from a finite set using permutations of that set; this extremal problem is solved in a companion paper co-authored with Eric Balandraud. We also find very close upper and lower bounds for the other Carathéodory numbers, and the exact Helly and Radon numbers of most of these convexities. We leave as open problems the determination of the Helly and Radon numbers of the integer convex sublattice convexity.

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1 Introduction.

Submodular optimization and lattice programming (Fujishige [29], Veinott [65, 64], Topkis [60]) have emerged as fundamental methodologies in several areas in discrete mathematics, economics and operations research. Both are concerned with the optimization (minimization or maximization) of a submodular or supermodular function over a lattice or sublattice, with a focus on submodular (or supermodular) *set* functions for submodular optimization, and on more general product spaces for lattice programming. These studies have led to important results, both algorithmic and structural. Algorithmic results include fast (i.e., polynomial time) versions of dynamic programming when certain submodularity and lattice properties hold (see, e.g., Chapter 3 in Topkis [60]); fast algorithms for minimizing a submodular function over a finite distributive lattice (even when the function is only accessible through a value oracle; see, e.g., Fujishige [29] and McCormick’s survey [41]); and approximation guarantees for related NP-hard problems, such as submodular set cover (e.g., Fujito [31]) or maximizing submodular set functions (e.g., Lee et al. [38]). Structural results concern the existence and properties of optimal solutions and equilibria; in particular, qualitative postoptimality analysis (Granot and Veinott [34]), also known as monotone comparative statics (Milgrom and Shannon [42]), studies directions of change in optimal solutions and equilibria as parameters change, see, e.g., Topkis [60] for a comprehensive treatment, and Granot et al. [33] for the special case of parametric minimum *s-t*-cuts.

Analogies between submodular functions and convex functions have long been recognized, in both discrete and continuous settings (e.g., Topkis [59], Lovász [40], Murota [43], Fujishige [29]). These analogies extend to (sub)lattices, the feasible sets of lattice programming, which may be viewed as analogues of convex sets. Indeed, the framework of abstract convexity (e.g., Danzer et al. [20], Duchet [24], Eckhoff [26], van de Vel [61]), allows us to view (standard) convex sets (say, in the Euclidian space $\mathbb{R}^d$) and sublattices, as well as many structures arising in geometry, graph theory, and other areas of mathematics, as particular instantiations of convex structures that share many common properties. A *convex structure* (or *convexity*) $(X, \mathcal{F})$ consists of a (ground) set X and a family $\mathcal{F} \subseteq 2^X$ of subsets of X that (i) contains the full and empty sets, X and $\emptyset$; (ii) is stable (closed) under arbitrary intersections; and (iii) is stable under nested union. The first two properties (which define a Moore family, or closure system) imply that every subset $A \subseteq X$ has a “convex” hull $\text{co}_{\mathcal{F}}(A)$, which is simply the smallest subset in $\mathcal{F}$ that contains A (i.e., the intersection of all $F \in \mathcal{F}$ that satisfy $A \subseteq F$). The third property implies that for every $x \in \text{co}_{\mathcal{F}}(A)$ there is a *finite* subset $B \subseteq A$ such that $x \in \text{co}_{\mathcal{F}}(B)$, a fundamental result in abstract convexity (see, e.g., [61, Theorem 1.3]). For the standard convexity $(\mathbb{R}^d, \mathcal{C})$, where $\mathcal{C}$ is the set of all (standard) convex subsets in $\mathbb{R}^d$, i.e., of all subsets $C \subseteq \mathbb{R}^d$ that contain every convex combination of every two points in C , this is equivalent to stating that the convex hull of A is the set of all convex combinations of finite subsets of A .

This finiteness has led to the study of *convexity invariants*, most notably the

Carathéodory, Helly and Radon numbers of convex structures. These invariants are named after corresponding seminal results about the standard convexity $(\mathbb{R}^d, \mathcal{C})$. Informally (precise definitions will be given in Section 2 below), for a convex structure $(X, \mathcal{F})$:

- The Carathéodory number $c(X, \mathcal{F})$ is the smallest integer c which guarantees that, for every $A \subseteq X$ and every $x \in \text{co}_{\mathcal{F}}(A)$, there is a subset B of at most c elements of A such that $x \in \text{co}_{\mathcal{F}}(B)$. Carathéodory's classic result [18] is that $c = d + 1$ for the standard convexity $(\mathbb{R}^d, \mathcal{C})$.
- The Helly number $h(X, \mathcal{F})$ is the smallest integer h which guarantees that the intersection of any finite collection $\mathcal{I} \subseteq \mathcal{F}$ of convex sets is nonempty when the intersection of each of its subcollections $\mathcal{J} \subseteq \mathcal{I}$ of size $|\mathcal{J}| \leq h$ is nonempty. Helly's classic result [35] is that $h = d + 1$ for the standard convexity $(\mathbb{R}^d, \mathcal{C})$.
- The Radon number $r(X, \mathcal{F})$ is the smallest integer r which guarantees that every set $A \subseteq X$ with cardinality $|A| \geq r$ admits a partition $A = (B, A \setminus B)$ into two subsets whose convex hulls intersect (i.e., $\text{co}_{\mathcal{F}}(B) \cap \text{co}_{\mathcal{F}}(A \setminus B) \neq \emptyset$). Such a partition $(B, A \setminus B)$ is called a *Radon partition* of A , and any point in the intersection $\text{co}_{\mathcal{F}}(B) \cap \text{co}_{\mathcal{F}}(A \setminus B)$ a *Radon point* of A . Radon's classic result [50] is that $r = d + 2$ for the standard convexity $(\mathbb{R}^d, \mathcal{C})$.

These invariants capture important structural properties of a convexity structure $(X, \mathcal{F})$. These structural properties often have algorithmic or complexity implications (see Section 2.2).

In this paper we study these three invariants for convexities defined by sublattices in the continuous space $\mathbb{R}^d$, in the discrete space $\mathbb{Z}^d$ of integer d -vectors, and in the Boolean space $\mathbb{B}^d$ of binary d -vectors. These three spaces are the natural settings for most of the current applications of lattice programming: $\mathbb{R}^d$ for economics and most game-theoretic applications, $\mathbb{Z}^d$ for integer programming, and $\mathbb{B}^d$ for combinatorial optimization. We study three types of sublattice convexities in $\mathbb{R}^d$ and $\mathbb{Z}^d$:

- the sublattice convexity $\mathcal{L}$ that consists of all sublattices of $\mathbb{R}^d$ or $\mathbb{Z}^d$, i.e., all subsets thereof that are closed under componentwise maximum and minimum (join and meet);
- the convex sublattice convexity $\mathcal{L} \cap \mathcal{C}$ of $\mathbb{R}^d$ that consists of all sublattices that are also (standard) convex sets; and the collection $\mathcal{L} \cap \mathcal{C}|\mathbb{Z}^d$ of all integer convex sublattices, i.e., of the intersections $F \cap \mathbb{Z}^d$ for all $F \in \mathcal{L} \cap \mathcal{C}$;
- the integral $L^{\natural}$ convexity $\mathcal{L}^{\natural}$ that consists of all integral $L^{\natural}$ polyhedra, i.e., of all convex polyhedra in $\mathbb{R}^d$ defined by linear inequalities of the type $x_i \leq u_i$, $x_j \geq l_j$ and $x_i - x_j \leq b_{ij}$ where all right-hand sides u_i , l_j and b_{ij} are integral; and the corresponding discrete convexity $\mathcal{L}^{\natural}|\mathbb{Z}^d$ in $\mathbb{Z}^d$.

The restrictions to $\mathbb{B}^d$ of these 3 convexities coincide, and give the Boolean sublattice convexity, which is isomorphic to the collections of all rings of subsets of $\{1, \dots, d\}$, i.e., of all collections of subsets that are closed under union and intersections.

Our contributions are as follows.

- We introduce a new field of research, connecting abstract convexity with lattice and related continuous and discrete convexity structures in vector spaces. This leads to structural questions, such as the determination of convexity invariants addressed in this paper, as well as algorithmic issues, such as determining minimum-size membership (and non-membership) certificates. It also relates to other fields, such as graph theory, network optimization, and the theory of permutations.
- We determine exact values for several of these invariants and, for several other cases, very close lower and upper bounds that differ only by lower order terms.
- We use a variety of methods in studying these invariants. We present efficient algorithms for the membership problem in the sublattice hull and convex sublattice hull of a given finite set of points. These algorithms return membership and non-membership certificates, from which we derive upper bounds on Carathéodory numbers. We also present matching, or very close, lower bounds on these Carathéodory numbers.
- We show that the Carathéodory number for the integral $L^\natural$ convexities equals the optimum value, which we determine in a companion paper [3], of an extremal problem in the theory of permutations, namely, to find the maximum size of a minimal cover of all ordered pairs of elements from a finite set using permutations of that set, see Section 4.4 for details.
- We present preliminary results for small dimensions for the remaining cases, and aim to motivate further research on these open problems.

The results of this paper are summarized, and compared with related results, in Table 1.

The contents of this paper are as follows. In Section 2 we present more formal definitions and basic properties of lattices and sublattices, and of abstract convexity. In particular we define the seven convexities introduced above (Section 2.1), and the convexity invariants that are the main subject of this work (Section 2.2). We also mention (Section 2.3) related convexities, in addition to the standard and integer convexities already described: box convexities, and the recently introduced Max-plus convexities. We discuss Helly and Radon numbers in Section 3. We first consider general sublattice and subsemilattice convexities in Section 3.1, and then in Section 3.2 turn to the seven convexities introduced above. Section 4 is devoted to Carathéodory numbers. We first determine the Carathéodory number of the Boolean sublattice convexity in Section 4.1. We then present in Section 4.2 a sublattice hull membership algorithm

Convexity	Invariant		
	Helly number $h(d)$	Radon number $r(d)$	Carathéodory number $c(d)$
Standard ($\mathbb{R}^d, \mathcal{C}$)	$d + 1$ Helly (1923)	$d + 2$ Radon (1921)	$d + 1$ Carathéodory (1907)
Integer ($\mathbb{Z}^d, \mathcal{C} \mathbb{Z}^d$)	2^d Doignon (1973)	$r(d) \leq d(2^d - 1) + 3$ Sierksma (1977) $r(d) \geq 5 \cdot 2^{d-2} + 1$, $r(2) = 6$: Onn (1991) $r(3) \leq 17$: Bezdek and Blokhuis (2003)	$d + 1$ Doignon (1973)
Box ($\mathbb{R}^d, \mathcal{B}$), ($\mathbb{Z}^d, \mathcal{B}$)	2 Soltan (1976)	$\min \left\{ m : \binom{m}{\lfloor m/2 \rfloor} > 2d \right\}$ Eckhoff (1969)	d Reay (1970)
Max-plus ($\mathbb{R}^d, \mathcal{M}^+$) ($\mathbb{R}_{\max}^d, \mathcal{M}^+$)	$d + 1$ Briec and Horvath (2004) Gaubert and Sergeev (2007)	$d + 2$ Butkovič (2003) Gaubert and Meunier (2010)	$d + 1$ Develin and Sturmfels (2004) Develin and Sturmfels (2004)
Sublattice ($\mathbb{R}^d, \mathcal{L}$), ($\mathbb{Z}^d, \mathcal{L}$)	$+\infty$ Theorem 1(iii)	$+\infty$ Theorem 1(iii)	$\left\lfloor \frac{d^2}{4} \right\rfloor + d$ Theorem 4
Convex sublattice ($\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}$) ($\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} \mathbb{Z}^d$)	$d + 1$ Theorem 1(i) $d + 1 \leq h(d) \leq 2^d$ Corollary 3 2^d for $d \leq 3$ Proposition 5	$d + 2$ Theorem 1(ii) $d + 2 \leq r(d) \leq r(\mathbb{Z}^d, \mathcal{C} \mathbb{Z}^d)$ Corollary 3 $r(2) = 6$ Proposition 5	$2 \left\lfloor \frac{d^2}{4} \right\rfloor \leq c(d) \leq 2 \left\lfloor \frac{d^2}{4} \right\rfloor + d + 1$ Theorem 5 $c(2) = 3$ Proposition 9
Integral L -natural ($\mathbb{R}^d, \mathcal{L}^\natural$), ($\mathbb{Z}^d, \mathcal{L}^\natural$)	$d + 1$ Theorem 1(i)	$d + 2$ Theorem 1(ii)	$\max \left\{ \left\lfloor \frac{(d+1)^2}{4} \right\rfloor, d + 1 \right\}$ Corollary ??
Boolean sublattice ($\mathbb{B}^d, \mathcal{L}$)	$d + 1$ Theorem 1(i)	$d + 2$ Theorem 1(ii)	$\max \left\{ \left\lfloor \frac{d^2}{4} \right\rfloor, d \right\}$ Theorem 3

Table 1: Invariants of sublattice and related convexities in $\mathbb{R}^d$ and related spaces. Top part: known results (see Section 2.3 for details); bottom: new results in this paper. (For simplicity and readability, subscripts are omitted from the convexities.)

and use it to determine the Carathéodory numbers of the sublattice convexities $\mathcal{L}$ in $\mathbb{R}^d$ and $\mathbb{Z}^d$. In Section 4.3 we consider the Carathéodory numbers of the convex sublattice convexities $\mathcal{L} \cap \mathcal{C}$ in $\mathbb{R}^d$ and $\mathbb{Z}^d$. After reviewing structural results on convex sublattices, we present a convex sublattice hull membership algorithm and use it to determine very close lower and upper bounds on these Carathéodory numbers. Finally in Section 4.4 we consider the Carathéodory numbers of the integral L^1 convexities $\mathcal{L}^1$ in $\mathbb{R}^d$ and $\mathbb{Z}^d$. We show that it is equal to the maximum size, determined in [3], of a minimal cover of all ordered pairs of elements from a set of size $d + 1$ using permutations of that set. We conclude with some open questions.

2 Definitions, Basic Properties, and Related Results.

2.1 Lattices and semilattices.

We first recall definitions and basic properties of lattices and semilattices (see, e.g., [10, 21]). A *meet semilattice* is a poset (partially ordered set) $(S, \leq)$ in which every pair of elements x, y has a greatest common lower bound $x \wedge y$, called their *meet*. Dually, $(S, \leq)$ is a *join semilattice* if every pair of elements x, y has a least common upper bound $x \vee y$, called their *join*. A *lattice* is a poset that is both a meet and a join semilattice. A subset of a meet semilattice is a *meet subsemilattice* if it is closed for the meet operation; *join subsemilattices* are defined dually. A subset of a lattice is a *sublattice* if it is both a meet and a join subsemilattice, or, equivalently, if it is closed for the meet and join operations. We denote by $\mathcal{M}_S$ the family of all meet subsemilattices of a meet semilattice $(S, \leq)$, and by $\mathcal{L}_S$ the family of all sublattices of a lattice $(S, \leq)$. A lattice $(L, \leq)$ is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$ (or dually and equivalently, if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$).

We consider the vector space $\mathbb{R}^d$ equipped with the usual componentwise partial order $\leq$ (whereby $x \leq y$ iff $x_i \leq y_i$ for all $i = 1, \dots, d$). Thus, the join $x \vee y$ and meet $x \wedge y$ of two points $x, y \in \mathbb{R}^d$ have components

$$(x \vee y)_i = \max\{x_i, y_i\} \quad \text{and} \quad (x \wedge y)_i = \min\{x_i, y_i\}$$

for all $i \in D = \{1, \dots, d\}$, and $\mathbb{R}^d$ is a distributive lattice for these join and meet operations. Furthermore, the set $\mathbb{Z}^d$ of all integer d -vectors and the set $\mathbb{B}^d = \{0, 1\}^d$ are sublattices of $\mathbb{R}^d$, and thus are themselves lattices for the same (componentwise) join and meet operations. Recall that $(\mathbb{B}^d, \leq)$ is isomorphic to the Boolean lattice $(2^D, \subseteq)$ of all subsets of D . In this paper we are particularly interested in the families $\mathcal{L}_{\mathbb{R}^d}$, $\mathcal{L}_{\mathbb{Z}^d}$ and $\mathcal{L}_{\mathbb{B}^d}$ of all sublattices of $\mathbb{R}^d$, $\mathbb{Z}^d$ and $\mathbb{B}^d$. For simplicity, we suppress the subscript and let $\mathcal{L}$ denote either of these sets when the meaning is clear from the context.

We recall some fundamental results on representations of sublattices of $\mathbb{R}^d$ and $\mathbb{B}^d$. For every coordinate $i \in D$ the *i -th coordinate projection* of a subset $T \subseteq \mathbb{R}^d$ is $\pi_i T = \{v \in \mathbb{R} : \exists x \in T \text{ with } x_i = v\}$. The Cartesian product

$U = \bigotimes_{i=1}^d \pi_i T$ of these coordinate projections is a sublattice of $\mathbb{R}^d$ (in fact, a rectangular box therein), and $T \subseteq U$. Similarly, for $i, j \in D$ ($i \neq j$) the (two-dimensional) ij -th coordinate projection of T is $\pi_{ij}T = \{(v, w) \in \mathbb{R}^2 : \exists x \in T \text{ with } x_i = v \text{ and } x_j = w\}$. Topkis's 2D-Projections Theorem [58, Theorem 1] implies that any sublattice S of $\mathbb{R}^d$ is entirely defined by its two-dimensional coordinate projections $\pi_{ij}S$, that is, $x \in S$ iff $\pi_{ij}x \in \pi_{ij}S$ for all $i, j \in D$ ($i \neq j$).

We now turn to the descriptions of convex sublattices of $\mathbb{R}^d$. A linear inequality $ax \leq b$ is *bimonotone* [58] if it is of the form

$$a_i x_i - a_j x_j \leq b \quad \text{with } a_i, a_j \geq 0$$

for some $i, j \in D$ (with possibly $i = j$). Veinott [63] showed that a polyhedron is a sublattice if and only if it can be described as the solution set of a finite system of bimonotone linear inequalities. This result has been extended by the authors [47] to closed convex sublattices and infinite (countable) systems of bimonotone linear inequalities. A well-known representation theorem by Birkhoff [9, 10] (see also [47]) implies that a sublattice of $\mathbb{B}^d$ can be represented as the set of integer solutions to a system of bimonotone linear inequalities of the form

$$a_i x_i - a_j x_j \leq b \quad \text{with } a_i, a_j \in \mathbb{B}, \quad (1)$$

and $|b| \in \mathbb{B}$. Note that any (finite) system of inequalities (1) has a totally unimodular constraint matrix.

Given a subset X of $\mathbb{R}^d$ we let $\mathcal{L}_X^{\natural}$ denote the family of all subsets of X defined by a (possibly empty) system of inequalities of the form (1) with $b \in \mathbb{Z}$. (Note that we may restrict attention to finite systems of such inequalities.) If X is a sublattice of $\mathbb{R}^d$, then each member of $\mathcal{L}_X^{\natural}$ is a sublattice of X . It follows from Birkhoff's Theorem that $\mathcal{L}_{\mathbb{B}^d}^{\natural} = \mathcal{L}_{\mathbb{B}^d}$. The members of the family $\mathcal{L}_{\mathbb{Z}^d}^{\natural}$ are called *$L^{\natural}$ -convex sets* (Fujishige and Murota [30]), and the members of $\mathcal{L}_{\mathbb{R}^d}^{\natural}$ are called *integral $L^{\natural}$ -convex polyhedra* (Murota and Shioura [44]); see also Murota [43] for a comprehensive treatment of $L^{\natural}$ -convexity and of related notions of discrete convexity. As above, we suppress the subscript and let $\mathcal{L}^{\natural}$ denote either of these convexities when the meaning is clear from the context.

2.2 Abstract Convexity and Invariants.

We now recall definitions and basic results from abstract convexity theory, see, e.g., Danzer et al. [20], Eckhoff [26], van de Vel [61]. A *convexity* (or *alignment*) on a set X is a family $\mathcal{F}$ of subsets of X (called *convex sets*) that contains $\emptyset$ and X , is stable for intersections (i.e., the intersection of any collection of convex sets is a convex set), and is stable for nested unions (i.e., if $F_i \in \mathcal{F}$ for $i \in I$ and $\{F_i\}_{i \in I}$ is totally ordered by inclusion, then $\bigcup_{i \in I} F_i \in \mathcal{F}$). A *convexity structure* is a pair $(X, \mathcal{F})$, where $\mathcal{F}$ is a convexity on X .

A convexity $\mathcal{F}$ determines a *convex hull operator* $\text{co}_{\mathcal{F}}$ on the power set 2^X of X that associates to any $S \subseteq X$ the smallest subset, denoted $\text{co}_{\mathcal{F}}(S)$, in $\mathcal{F}$ containing S ; i.e., $\text{co}_{\mathcal{F}}(S)$ is the intersection of all subsets in $\mathcal{F}$ containing S .

Several results and notions of the classical convexity theory have been extended to this abstract convexity setting (see van de Vel [61] and references therein for a very detailed survey). In particular, a topic of considerable interest (e.g., [26, 37, 54, 55, 56] and [61, Chapter 2]) is the extension to various convexity structures of the main *convexity invariants* of classical convexity theory, namely, the Carathéodory, Helly and Radon numbers.

The *Carathéodory number* $c(X, \mathcal{F})$ of a convexity structure $(X, \mathcal{F})$ is the least number k such that for every subset T of X and every point $z \in \text{co}_{\mathcal{F}}(T)$ there exists a subset U of T such that $|U| \leq k$ and $z \in \text{co}_{\mathcal{F}}(U)$. We define $c(X, \mathcal{F}) = +\infty$ when such a least number does not exist.

The *Helly number* $h(X, \mathcal{F})$ is the least number k such that the intersection of any finite collection of convex sets is nonempty when the intersection of each of its subcollections consisting of at most k convex sets is nonempty (with $h(X, \mathcal{F}) = +\infty$ when there is no such least number).

In a convexity structure $(X, \mathcal{F})$, a partition $\{T_1, T_2\}$ of a subset $T \subseteq X$ (i.e., with $T = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$) which satisfies $\text{co}_{\mathcal{F}}(T_1) \cap \text{co}_{\mathcal{F}}(T_2) \neq \emptyset$ is called a *Radon partition* of T . The *Radon number* $r(X, \mathcal{F})$ is the least number k with the property that every set T in X with $|T| \geq k$ admits a Radon partition (with $r(X, \mathcal{F}) = +\infty$ when there is no such least number).¹

We recall that for the family $\mathcal{C}$ of all (standard) convex subsets of $\mathbb{R}^d$, the classical results of Carathéodory, Helly and Radon state that $c(\mathbb{R}^d, \mathcal{C}) = h(\mathbb{R}^d, \mathcal{C}) = d + 1$ and $r(\mathbb{R}^d, \mathcal{C}) = d + 2$. These invariants capture properties of the convexity structure $(\mathbb{R}^d, \mathcal{C})$ (e.g., Barvinok [6]). These properties also have important algorithmic or complexity implications, for example:

- The Carathéodory number is routinely used (e.g., when minimizing submodular functions, see [29, 41]) to guarantee the existence of a concise (i.e., polynomial sized) *membership certificate*: if $x \in \text{co}_{\mathcal{F}}(A)$ then this can be verified by giving a subset B of at most $c(X, \mathcal{F})$ elements of A and verifying that $x \in \text{co}_{\mathcal{F}}(B)$. For example, in the standard convexity $(\mathbb{R}^d, \mathcal{C})$, a certificate that a point x is in the convex hull of a set A consists of a subset of at most $d + 1$ points in A and a (usually simple) certificate that x is in the convex hull of these points. Bárány [4] gives a stronger, “colourful” version of Carathéodory’s Theorem.
- The Helly number is used to guarantee the existence of a concise *infeasibility certificate*: if an intersection $\bigcap_{i \in I} F_i$ is empty (i.e., if the constraint system $[x \in F_i \text{ for all } i \in I]$ is infeasible), then there is an infeasible subsystem $J \subseteq I$ of size $|J| \leq h(X, \mathcal{F})$. For example, let $\mathcal{C}|\mathbb{Z}^d = \{C \cap \mathbb{Z}^d : C \in \mathcal{C}\}$ denote the collection of all “integer convex sets”, i.e., of all sets of integer points in (standard) convex sets in $\mathbb{R}^d$. A classic result of Doignon [23]

¹M. van de Vel defines the Radon number as the maximum cardinality of a subset T that admits no Radon partition, i.e., $r(X, \mathcal{F}) - 1$. While we agree that this definition is more consistent with the definitions of the Carathéodory and Helly numbers and leads to simpler and more uniform results, we choose to conform with the traditional definition of the Radon number.

(independently re-discovered by Bell [7] and by Scarf [53]) states that the Helly number of this integer convexity structure $(\mathbb{Z}^d, \mathcal{C}|\mathbb{Z}^d)$, is equal to 2^d . Thus an infeasible system of linear (or convex) inequalities in d integer variables must contain an infeasible subsystem of size at most 2^d . Since 2^d is best possible, it is not possible to guarantee, in general, the existence of a *concise* infeasibility certificate of that form. Averkov and Weismantel [2] extend this result to the mixed-integer case. Helly’s Theorem has also been extended to oriented matroids by Edmonds [28]. Bárány and Matoušek [5] show that the *fractional* Helly number of $(\mathbb{Z}^d, \mathcal{C}|\mathbb{Z}^d)$ is equal to $d + 1$.

- Helly’s and Radon’s theorems are also used in robust statistics (e.g., Amenta et al. [1]) and in computational geometry (e.g., Clarkson et al. [19]), where they can be used to prove the existence of “centerpoints”, i.e., higher-dimensional generalizations of medians. In particular, Rado [49] uses Helly’s Theorem to prove a result now known as the “Centerpoint Theorem”: given any set A of n points in $\mathbb{R}^d$, there exists a point c such that any closed halfspace containing c contains at least $n/(d + 1)$ points of A .
- The Radon number $r(\mathbb{R}^d, \mathcal{C})$ is also used in statistical (or computational, or machine) learning, because it implies that halfspaces cannot shatter sets of $d + 2$ or more points in $\mathbb{R}^d$, where a family $\mathcal{H}$ of subsets of a set X *shatters* a subset $S \subseteq X$ iff for every subset $A \subseteq S$ there is an $H \in \mathcal{H}$ such that $A = S \cap H$. The *VC-dimension* (Vapnik and Chervonenkis [62]) of a subset family $\mathcal{H}$ is the largest size of a subset of X it can shatter. The VC-dimension is a measure of the capacity of $\mathcal{H}$ as a classification model for X . Thus Radon’s theorem implies that the VC-dimension of the family of halfspaces in $\mathbb{R}^d$ is at most $d + 1$.

The meet subsemilattice family $\mathcal{M}_S$ of a meet semilattice S is a convexity on S , which we call its *subsemilattice convexity* (or *semilattice alignment*, [36]). Similarly, the *sublattice convexity* (or *lattice alignment*, *ibid.*) of a lattice S is its sublattice family $\mathcal{L}_S$. The Carathéodory number of a sublattice convexity is called the *breadth* of the lattice by Birkhoff [10, p. 99]. The sublattice families $\mathcal{L}_{\mathbb{R}^d}$, $\mathcal{L}_{\mathbb{Z}^d}$, and $\mathcal{L}_{\mathbb{B}^d}$ are sublattice convexities on the spaces $\mathbb{R}^d$, $\mathbb{Z}^d$, and $\mathbb{B}^d$. Furthermore, the family $\mathcal{L}_{\mathbb{Z}^d}^{\natural}$ of $L^{\natural}$ -convex sets is a convexity in $\mathbb{Z}^d$, and the family $\mathcal{L}_{\mathbb{R}^d}^{\natural}$ of integral $L^{\natural}$ -convex polyhedra is a in $\mathbb{R}^d$ (see Section 4.4 for details). In sections 3 and 4 we provide exact or approximate values for the convexity invariants of these convexities and of the convexity $\mathcal{L} \cap \mathcal{C}$ of all sublattices of $\mathbb{R}^d$ that are also convex sets in the standard convexity.

2.3 Related Convexities.

Before concluding this Section, we briefly mention several related convexities that have been extensively studied. We use the following definition: given a convexity structure $(X, \mathcal{F})$ and a subset $Y \subseteq X$, the *relative convexity* $\mathcal{F}|Y$ is

the family of all $F \cap Y$ for $F \in \mathcal{F}$. (For example, the $L^{\natural}$ -convexity $\mathcal{L}_{\mathbb{Z}^d}^{\natural}$ defined above is just the relative convexity $\mathcal{L}_{\mathbb{R}^d}^{\natural}|\mathbb{Z}^d$.)

The *order convexity* (or *interval convexity*, or *segment convexity*) on a lattice $(L, \leq)$ is the set of all (order) *intervals* (or *segments*) $I \subseteq L$ defined by the property that $x \leq y \leq z$ and $x, z \in I$ imply $y \in I$; see examples I§§5.2–5.4 in [61] and the references therein. In particular, when restricted to closed subsets, the order convexity in $\mathbb{R}^d$ is called the *box convexity* $\mathcal{B}$. Thus box convex sets are rectangular boxes (i.e., parallelepipeds) in $\mathbb{R}^d$, and the convex hull of a set is the smallest box containing it. They are also special types of $L^{\natural}$ -convex polyhedra, that are only defined by the bound constraints $l_j \leq x_j \leq u_j$. In the relative convexity $(\mathbb{Z}^d, \mathcal{B}|\mathbb{Z}^d)$ the convex sets are the sets of integer points in rectangular boxes; without loss of generality we may assume that these boxes have integer extreme points; the resulting integral box-convex sets are thus also special types of $L^{\natural}$ -convex sets; see also Section II§2 in [61]. The convexity invariants for box convexities were determined by Eckhoff [25], Reay [51] and Soltan [57]. Eckhoff [27, Section 3] shows that the Radon number $r(\mathbb{R}^d, \mathcal{B})$ given in Table 1 may be written as $\log_2 d + \theta(\log \log d)$. See also [14] for related work.

The *integer convexity* in $\mathbb{Z}^d$ is the relative convexity $\mathcal{C}_{\mathbb{R}^d}|\mathbb{Z}^d$, i.e., its (“integer”) convex sets are the sets of integer points in the (standard) convex sets of $\mathbb{R}^d$. Doignon [23] determined its Carathéodory and Helly numbers; the determination of its Radon number remains an open problem for $d \geq 3$ (Onn [46], Bezdek and Blokhuis [8]).

Let H be a subset of the unit sphere S^{d-1} in $\mathbb{R}^d$ that is not contained in any closed hemisphere of S^{d-1} . Then a subset of $\mathbb{R}^d$ is called *H-convex* [13] (see also [12]) if it can be obtained as the intersection of halfspaces of the form $ax \leq b$, where $a \in H$ and $b \in \mathbb{R}$. The integral $L^{\natural}$ -convex polyhedra and closed convex sublattices are *H-convex* subsets for $H = \{(a_i e^i - a_j e^j) / \|a_i e^i - a_j e^j\|_2 : a_i, a_j \in \mathbb{B}; i, j = 1, \dots, d\}$ and $H = \{(a_i e^i - a_j e^j) / \|a_i e^i - a_j e^j\|_2 : a_i, a_j \in \mathbb{R}_+; i, j = 1, \dots, d\}$, respectively, where e^i denote the i -th unit vector in $\mathbb{R}^d$. Boltyanski [13] determined the Helly number for *H-convexities*; his result implies that $h(\mathbb{R}^d, \mathcal{L}^{\natural}) = h(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) = d + 1$, as also shown in Theorem 1 herein. Furthermore, Boltyanski and Martini [11] proved that for $d \geq 3$ the Carathéodory number for any *H-convexity* is bounded below by $d - 1$. In contrast we obtain in Section 4 very tight *quadratic* upper and lower bounds for the Carathéodory numbers of the special *H-convexities* formed by integral $L^{\natural}$ convex polyhedra and by closed convex sublattices.

Finally, the *max-plus convexity* $\mathcal{M}^+$ in $\mathbb{R}^d$ and $\mathbb{R}_{\max}^d$, where $\mathbb{R}_{\max}^d = \mathbb{R} \cup \{-\infty\}$, arising from studies of discrete dynamic systems, is defined by the convex hull $\text{co}_{\mathcal{M}^+}(\{x^1, \dots, x^d\}) = \{\bigvee_{i=1}^n (\lambda_i + x^i) : \lambda \in \mathbb{R}^n \text{ and } \bigvee \{\lambda^1, \dots, \lambda^n\} = 0\}$ where $\lambda_i + x^i$ is the d -vector with components $(\lambda_i + x^i)_j = \lambda_i + x_j^i$ for all $j = 1, \dots, d$. Note that the resulting max-plus convex sets are join subsemilattices of $(\mathbb{R}^d, \leq)$. The convexity invariants for these max-plus convexities were determined by Butković [16], Briec and Horvath [15], Develin and Sturmfels [22], and Gaubert and Meunier [32]; see also the latter reference for further details and related results.

These convexities are related to, but different from those studied in the present paper. The top half of Table 1 summarizes the known values of these convexities in $\mathbb{R}^d$, $\mathbb{Z}^d$ and $\mathbb{B}^d$, spanning over one century of mathematical research.

3 Helly and Radon Numbers.

To derive the Helly and Radon numbers we use the following definitions and known properties, mostly quoted from the comprehensive monograph by M. van de Vel on abstract convexity [61].

A convexity $\mathcal{F}$ is *finer* than a convexity $\mathcal{G}$ on the same set X if $\mathcal{F} \supseteq \mathcal{G}$. A direct argument (or Theorem II§1.10 in [61, p.170] applied to the identity function f from X into itself) establishes the first result we shall need:

Proposition 1. *If convexity $\mathcal{F}$ is finer than convexity $\mathcal{G}$ on a set X , then their Helly and Radon numbers satisfy $h(X, \mathcal{F}) \geq h(X, \mathcal{G})$ and $r(X, \mathcal{F}) \geq r(X, \mathcal{G})$.*

Recall that, given a convexity structure $(X, \mathcal{F})$ and a subset $Y \subseteq X$ the *relative convexity* $\mathcal{F}|Y$, is the family of all $F \cap Y$ for $F \in \mathcal{F}$. The *relative hull formula* (I§1.9.1 in [61]) states that $\text{co}_{\mathcal{F}|Y}(T) = \text{co}_{\mathcal{F}}(T) \cap Y$ for all $T \subseteq Y$. The Helly and Radon numbers of a relative convexity satisfy:

Proposition 2 (Theorem II§1.11(2) in [61]). *If $(X, \mathcal{F})$ is a convexity structure and $Y \in \mathcal{F}$, then $h(Y, \mathcal{F}|Y) \leq h(X, \mathcal{F})$ and $r(Y, \mathcal{F}|Y) \leq r(X, \mathcal{F})$.*

Calder [17] gave the following useful characterization of the Helly number (see also Theorem II§1.7 in [61]):

Proposition 3 (Calder [17]). *For any integer $n \geq 1$, the Helly number of a convexity structure $(X, \mathcal{F})$ satisfies $h(X, \mathcal{F}) \leq n$ iff*

$$\bigcap_{a \in T} \text{co}_{\mathcal{F}}(T \setminus \{a\}) \neq \emptyset \text{ for all } T \subseteq X \text{ with } |T| > n.$$

Using the terminology of [61], we say that a subset T of X is *H-independent* if $\bigcap_{a \in T} \text{co}_{\mathcal{F}}(T \setminus \{a\}) = \emptyset$. Thus Proposition 3 implies that the Helly number $h(X, \mathcal{F})$ is the maximum cardinality of an H-independent subset of X . Note that a subset of an H-independent set is itself H-independent, i.e., H-independence is a hereditary property [61, II§1.16.1 p.173].

3.1 Helly and Radon Numbers for Semilattice Convexities.

Recall that a *chain* in a poset S is a totally ordered subset of S . The *depth* $\text{depth}(S)$ of a poset $(S, \leq)$ is the maximum cardinality of a chain in S (with $\text{depth}(S) = +\infty$ if S contains arbitrarily long chains). The following result is stated without proof in [61, II§1.23 p.177]. We include a proof for completeness.

Proposition 4. *The Helly number $h(S, \mathcal{M}_S)$ of the subsemilattice convexity on a (meet) semilattice S is equal to $\text{depth}(S)$.*

Proof. (i) We first show that $h(S, \mathcal{M}_S) \geq \text{depth}(S)$. Let C be a finite chain in S . Then C is a subsemilattice, and we may consider the relative subsemilattice convexity $\mathcal{M}_C = \mathcal{M}_S|_C$ on C . For every $a \in C$ the set $C \setminus \{a\}$ is itself a chain in C . Therefore $\text{co}_{\mathcal{M}_C}(C \setminus \{a\}) = C \setminus \{a\}$ and $\bigcap_{a \in C} \text{co}_{\mathcal{M}_C}(C \setminus \{a\}) = \emptyset$. By Proposition 3 this implies $h(C, \mathcal{M}_C) > \text{depth}(S) - 1$, and by Proposition 2, $h(S, \mathcal{M}_S) \geq h(C, \mathcal{M}_C) \geq \text{depth}(S)$.

(ii) We now show the reverse inequality, i.e., $h(S, \mathcal{M}_S) \leq \text{depth}(S)$. Note that every singleton subset $\{x\}$ of S is H-independent. Let F be any nonempty finite H-independent subset of S . Let $F_1 = F$. For $i = 1, \dots, |F|$ inductively define $x_i = \bigwedge F_i$, so $x_i \notin \text{co}_{\mathcal{M}_S}(F_i \setminus \{a_i\})$ for some $a_i \in F_i$, and define $F_{i+1} = F_i \setminus \{a_i\}$. For every $i < |F|$, $F_{i+1} \subset F_i$ and, by heredity, F_{i+1} is H-independent. Thus $x_{i+1} = \bigwedge F_{i+1}$ is well defined and satisfies $x_{i+1} \geq x_i$. But then $x_{i+1} \in \text{co}_{\mathcal{M}_S}(F_{i+1}) = \text{co}_{\mathcal{M}_S}(F_i \setminus \{a_i\})$ while $x_i \notin \text{co}_{\mathcal{M}_S}(F_i \setminus \{a_i\})$, and we must have $x_i < x_{i+1}$. Thus $x_1 < x_2 < \dots < x_{|F|}$, forming a chain of cardinality $|F|$ in S . This shows that $\text{depth}(S) \geq |F|$ for every finite H-independent subset of S , and therefore $\text{depth}(S) \geq h(S, \mathcal{M}_S)$. The proof is complete. $\square$

Remark 1. Part (i) in the proof of Proposition 4 also follows from the observation that the relative subsemilattice convexity of a chain C coincides with that of the power set 2^C , i.e., the *free convexity* on C , for which it is known (statement II§1.4.4 (iii) in [61]) that $h(C, 2^C) = |C|$.

Corollary 1. *The Helly number $h(L, \mathcal{L})$ of the sublattice convexity $\mathcal{L}$ on a lattice L is equal to $\text{depth}(L)$.*

Proof. The inequality $h(L, \mathcal{L}) \geq \text{depth}(L)$ follows from considering any finite chain in L as in part (i) of the proof of Proposition 4. Since the meet subsemilattice convexity $\mathcal{M}$ defined by the meet operation of L is finer than its sublattice convexity, Propositions 1 and 4 imply $h(L, \mathcal{L}) \leq h(L, \mathcal{M}) = \text{depth}(L)$. $\square$

We now turn to the Radon number of subsemilattice and sublattice convexities. The *Levi inequality* [39] (see also Theorem II§1.9(1) in [61]) states that $r(X, \mathcal{F}) \geq h(X, \mathcal{F}) + 1$ for any convexity structure $(X, \mathcal{F})$. Combining this inequality with Proposition 2 and Corollary 1 we get:

Corollary 2. (i) *The Radon number of the subsemilattice convexity $\mathcal{M}$ of a semilattice $(S, \leq)$ satisfies $r(S, \mathcal{M}) \geq \text{depth}(S) + 1$.*

(ii) *The Radon number of the sublattice convexity $\mathcal{L}$ of a lattice $(L, \leq)$ satisfies $r(L, \mathcal{L}) \geq \text{depth}(L) + 1$.*

The following example shows that, in contrast with Proposition 4, the inequality in Corollary 2(i) may be strict.

Example 1. Consider the meet semilattice S with 16 elements: $i = 0, \dots, 5$ and (i, j) with $1 \leq i < j \leq 5$; and the intransitive partial order relations $0 < (i, j) < i$ and $(i, j) < j$ for all i, j ($1 \leq i < j \leq 5$). Its depth is 3. Since

every $\text{co}_{\mathcal{M}}(\{i\}) = \{i\} \notin \text{co}_{\mathcal{M}}(T \setminus \{i\})$ and every $\text{co}_{\mathcal{M}}(\{i, j\}) = \{i, j, (i, j)\}$ with $(i, j) \notin \text{co}_{\mathcal{M}}(T \setminus \{i, j\})$, the subset $T = \{1, 2, 3, 4, 5\}$ admits no Radon partition. Therefore, $r(S, \mathcal{M}) \geq 6 = \text{depth}(S) + 3$.

3.2 Helly and Radon Numbers for Sublattice Convexities in $\mathbb{B}^d$, $\mathbb{Z}^d$ and $\mathbb{R}^d$.

We now determine the Helly and Radon number of several of the sublattice convexities defined above. For improved readability we write $\text{co}_{(\mathbb{Z}^d, \mathcal{L}^{\natural})}(T)$ instead of $\text{co}_{\mathcal{L}_{\mathbb{Z}^d}^{\natural}}(T)$, and similarly for other convexities.

Theorem 1. *For all integers $d \geq 1$ the following equalities hold:*

- (i) $h(\mathbb{B}^d, \mathcal{L}_{\mathbb{B}^d}) = h(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}^{\natural}) = h(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}^{\natural}) = h(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d} \cap \mathcal{C}) = h(\mathbb{R}^d, \mathcal{C}) = d+1$;
- (ii) $r(\mathbb{B}^d, \mathcal{L}_{\mathbb{B}^d}) = r(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}^{\natural}) = r(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}^{\natural}) = r(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d} \cap \mathcal{C}) = r(\mathbb{R}^d, \mathcal{C}) = d+2$;
- (iii) $h(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}) = r(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}) = h(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}) = r(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}) = +\infty$.

Proof. (i) We prove the following chain of inequalities:

$$d+1 \leq h(\mathbb{B}^d, \mathcal{L}_{\mathbb{B}^d}) \leq h(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}^{\natural}) \leq h(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}^{\natural}) \leq h(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d} \cap \mathcal{C}) \leq h(\mathbb{R}^d, \mathcal{C}) = d+1.$$

The first inequality follows from Corollary 1 by observing that $(\mathbb{B}^d, \leq)$ contains a chain of length $d+1$. The second inequality follows from Proposition 2 by noting that $\mathbb{B}^d$ is a convex set in $(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}^{\natural})$ and that $\mathcal{L}_{\mathbb{B}^d} = \mathcal{L}_{\mathbb{Z}^d}^{\natural} \upharpoonright \mathbb{B}^d$, i.e., the intersections with $\mathbb{B}^d$ of all the $L^{\natural}$ -convex sets of $\mathbb{Z}^d$ define precisely all sublattices of $\mathbb{B}^d$.

To prove the third inequality, note that for every $T \subseteq \mathbb{Z}^d$ we have $\text{co}_{(\mathbb{Z}^d, \mathcal{L}^{\natural})}(T) = (\text{co}_{(\mathbb{R}^d, \mathcal{L}^{\natural})}(T)) \cap \mathbb{Z}^d$ (since an $L^{\natural}$ -convex set is the set of all integer points in the corresponding integral $L^{\natural}$ -convex polyhedron). Then observe that every nonempty $\mathcal{L}_{\mathbb{R}^d}^{\natural}$ -convex set (i.e., integral $L^{\natural}$ -convex polyhedron) contains an integer point. Therefore if $T \subseteq \mathbb{Z}^d$ satisfies $|T| > h(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}^{\natural})$ then, by Proposition 3, $\bigcap_{a \in T} \text{co}_{(\mathbb{R}^d, \mathcal{L}^{\natural})}(T \setminus \{a\})$ is nonempty, and therefore contains an integer point. Thus $\bigcap_{a \in T} \text{co}_{(\mathbb{Z}^d, \mathcal{L}^{\natural})}(T \setminus \{a\}) \neq \emptyset$, implying $h(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}^{\natural}) \leq h(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}^{\natural})$.

The fourth inequality follows from Proposition 1 since the convexity $\mathcal{L} \cap \mathcal{C}$ on $\mathbb{R}^d$ is finer than the convexity $\mathcal{L}^{\natural}$ (every integral $L^{\natural}$ -convex polyhedron is both a sublattice and a convex set). The last inequality follows from Proposition 1 since the standard convexity $\mathcal{C}$ on $\mathbb{R}^d$ is finer than the convexity $\mathcal{L} \cap \mathcal{C}$. The final equality is Helly's original theorem.

(ii) Similarly we show that

$$d+2 \leq r(\mathbb{B}^d, \mathcal{L}_{\mathbb{B}^d}) \leq r(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}^{\natural}) \leq r(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}^{\natural}) \leq r(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d} \cap \mathcal{C}) \leq r(\mathbb{R}^d, \mathcal{C}) = d+2.$$

The third inequality follows from the fact that if $T \subseteq \mathbb{Z}^d$ admits a Radon partition $\{T_1, T_2\}$ in $(\mathbb{R}^d, \mathcal{L}^{\natural})$ then the intersection $\text{co}_{(\mathbb{R}^d, \mathcal{L}^{\natural})}(T_1) \cap \text{co}_{(\mathbb{R}^d, \mathcal{L}^{\natural})}(T_2)$ is non-empty and thus contains an integer point z . For $i = 1, 2$ we have $z \in$

$\text{co}_{(\mathbb{R}^d, \mathcal{L}^\natural)}(T_i) \cap \mathbb{Z}^d = \text{co}_{(\mathbb{Z}^d, \mathcal{L}^\natural)}(T_i)$, so $\text{co}_{(\mathbb{Z}^d, \mathcal{L}^\natural)}(T_1) \cap \text{co}_{(\mathbb{Z}^d, \mathcal{L}^\natural)}(T_2) \neq \emptyset$ and $\{T_1, T_2\}$ is also a Radon partition of T in $(\mathbb{Z}^d, \mathcal{L}^\natural)$. The other inequalities follow from a similar argument as used for part (i). The final equality is Radon's original theorem.

(iii) This follows from Corollaries 1 and 2(ii) by noting that the lattices $\mathcal{L}_{\mathbb{Z}^d}$ and $\mathcal{L}_{\mathbb{R}^d}$ contain arbitrarily long chains. $\square$

Corollary 3. *For all integers $d \geq 1$ the following inequalities hold:*

- (i) $d + 1 \leq h(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) \leq 2^d$;
- (ii) $d + 2 \leq r(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) \leq r(\mathbb{Z}^d, \mathcal{C})$;

Proof. The lower bounds in (i) and (ii) follow from Proposition 1 and Theorem 1, since the convexity $\mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d$ is finer than the $\mathcal{L}^\natural$ -convexity $\mathcal{L}_{\mathbb{Z}^d}^\natural$ (every integer $\mathcal{L}^\natural$ -convex subset of $\mathbb{Z}^d$ is both a sublattice of $\mathbb{Z}^d$ and an integer convex set). The upper bounds follow from Proposition 1 and, for (i), from $h(\mathbb{Z}^d, \mathcal{C} | \mathbb{Z}^d) = 2^d$ (Doignon 1973 [23]), since the integer convexity $\mathcal{C} | \mathbb{Z}^d$ is finer than $\mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d$. $\square$

The following small-dimension results suggest that the Helly and Radon numbers of $(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d)$ might be closer to the upper bounds than to the lower bounds of Corollary 3:

Proposition 5. (i) $h(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) = 2^d$ for $d \leq 3$; and

(ii) $r(\mathbb{Z}^2, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^2) = 6$.

In the proof of Proposition 5 we will use the following observations. Recall that a linear function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is *bimonotone* if it has the form $f(x) = a_i x_i - a_j x_j$ with $a_i, a_j \geq 0$. It follows from the Bimonotone Representation Theorem in [47] that for any nonempty finite subsets A and B of $\mathbb{R}^d$, $A \cap \text{co}_{\mathcal{L} \cap \mathcal{C}}(B) = \emptyset$ if (and only if) there exists a bimonotone function $f : \mathbb{R}^d \mapsto \mathbb{R}$ such that $\min_{x \in A} f(x) > b := \max_{x \in B} f(x)$; in such case we say that A is *separated from* $\text{co}_{\mathcal{L} \cap \mathcal{C}}(B)$ by the bimonotone inequality $f(x) \leq b$. Note that this also implies that $\text{co}_{\mathcal{L} \cap \mathcal{C}}(A)$ is also separated from $\text{co}_{\mathcal{L} \cap \mathcal{C}}(B)$ by that same bimonotone inequality.

Proof of Proposition 5. Recall that for any finite $T \subset \mathbb{Z}^d$ its integer convex sublattice hull $\text{co}_{\mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d}(T)$ is finite (since $\text{co}_{\mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d}(T) \subseteq \{x \in \mathbb{Z}^d : \bigwedge T \leq x \leq \bigvee T\}$) and is thus closed.

(i) The case $d = 1$ follows directly from Corollary 3. For $d = 2$, the set $T = \{(0, 0), (1, 0), (1, 1), (2, 1)\}$ is H-independent since each $t \in T$ is separated from $\text{co}_{\mathcal{L} \cap \mathcal{C} | \mathbb{Z}^2}(T \setminus \{t\})$ by the bimonotone inequality $-x_1 \leq -1$; $x_1 - 2x_2 \leq 0$; $-x_1 + 2x_2 \leq 0$ and $x_1 \leq 1$, respectively, and no integer $\bar{x} \in \mathbb{Z}^2$ satisfies all these four inequalities (for they imply $2x_2 = x_1 = 1$, that has no integral solution). Therefore, T is H-independent and $4 = |T| \leq h(\mathbb{Z}^2, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^2) \leq 4$ (by Corollary 3). For $d = 3$ consider the set $T = \{t^0, t^1, \dots, t^7\}$ where

$$t_j^i = \left\lfloor \frac{i}{2^j} + \frac{1}{2} \right\rfloor \text{ for all } j = 1, \dots, d \quad (2)$$

for all $i = 0, \dots, 7$. Each t^i is separated from $\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^3 \setminus \{t^i\})$ by the following bimonotone inequalities:

$$\begin{array}{rclcl} 1 & \leq & x_1 & \leq & 3 & \text{for } t^0 \text{ and } t^7, \text{ resp.;} \\ 0 & \leq & -x_1 + 3x_2 & \leq & 2 & \text{for } t^1 \text{ and } t^6, \text{ resp.;} \\ 0 & \leq & 2x_1 - 3x_2 & \leq & 2 & \text{for } t^2 \text{ and } t^5, \text{ resp.;} \\ -1 & \leq & -x_1 + 3x_3 & \leq & 0 & \text{for } t^3 \text{ and } t^4, \text{ resp.} \end{array}$$

We now show that no $x \in \mathbb{Z}^3$ satisfies all these 8 inequalities, implying that $\bigcap_{i=0}^7 \text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^3 \setminus \{t^i\}) = \emptyset$, i.e., that T is H-independent. Indeed, by contradiction assume that some $x \in \mathbb{Z}^3$ does satisfy all these 8 inequalities. Then, adding the first two pairs of inequalities (those for t^0, t^7, t^1 and t^6) implies $1/3 \leq x_2 \leq 5/3$, and thus $x_2 = 1$. The third pair of inequalities then imply $3/2 \leq x_1 \leq 5/2$, and thus $x_1 = 2$. Finally, the last pair of inequalities then imply $1/3 \leq x_3 \leq 2/3$, that has no integral solution, a contradiction. Therefore, T is H-independent and $8 = |T| \leq h(\mathbb{Z}^3, \mathcal{L} \cap \mathcal{C}|\mathbb{Z}^3) \leq 8$ (by Corollary 3).

(ii) Consider the chain $T = \{(0, 0), (2, 0), (4, 1), (5, 2), (5, 4)\}$. First, note that each $t \in T$ is separated from $\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^2 \setminus \{t\})$ by a bimonotone inequality. Therefore there is no Radon Partition $(T_1, T \setminus T_1)$ of T with either $|T_1|$ or $|T \setminus T_1|$ equal to 1. On the other hand, when $|T_1| = 2$ we have the following three cases:

- $T_1 = \{(0, 0), (2, 0)\}$ is separated from $\text{co}_{\mathcal{L} \cap \mathcal{C}}(T \setminus T_1)$ by the bimonotone inequality $-x_2 \leq -1$, and thus

$$(\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^2(T_1)) \cap (\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^2(T \setminus T_1))) \subseteq (\text{co}_{\mathcal{L} \cap \mathcal{C}}(T_1)) \cap (\text{co}_{\mathcal{L} \cap \mathcal{C}}(T \setminus T_1)) = \emptyset.$$
- $T_1 = \{(5, 2), (5, 4)\}$ is separated from $\text{co}_{\mathcal{L} \cap \mathcal{C}}(T \setminus T_1)$ by the bimonotone inequality $x_1 \leq 4$, and thus $(\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^2(T_1)) \cap (\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^2(T \setminus T_1))) = \emptyset$.
- Otherwise, let $T_1 = \{t', t''\}$ with $t' < t''$. Then $\text{co}_{\mathcal{L} \cap \mathcal{C}}(T_1)$ is the line segment $\text{co}_{\mathcal{C}}(T_1)$ connecting t' and t'' . Since no such line segment contains an integer point, $\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^2(T_1)) = \{t', t''\}$. But each of t' and t'' is also separated from $\text{co}_{\mathcal{L} \cap \mathcal{C}}(T \setminus T_1)$ by a bimonotone inequality, and we have

$$(\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^2(T_1)) \cap (\text{co}_{\mathcal{L} \cap \mathcal{C}}(\mathbb{Z}^2(T \setminus T_1))) \subseteq \{t', t''\} \cap (\text{co}_{\mathcal{L} \cap \mathcal{C}}(T \setminus T_1)) = \emptyset.$$

This shows that T cannot have a Radon partition, and thus $6 = |T| + 1 \leq r(\mathbb{Z}^2, \mathcal{L} \cap \mathcal{C}|\mathbb{Z}^2) \leq 6$ (by Corollary 3(ii) and [46]). $\square$

Remark 2. For $d = 4$ the set T consisting of the 12 points t^i defined by (2) for $i \in \{0, \dots, 15\} \setminus \{5, 6, 9, 10\}$ is H-independent, hence $12 \leq h(\mathbb{Z}^4, \mathcal{L} \cap \mathcal{C}|\mathbb{Z}^4) \leq 16$. It would be interesting to close the gap between these bounds, and more generally between the very different upper and lower bounds in Corollary 3 on the Helly and Radon numbers for the integer convex-sublattice convexity $(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C}|\mathbb{Z}^d)$.

3.3 The Radon Number of the Sublattice Convexity of a Finite Distributive Lattice.

We conclude Section 3 by deriving from Theorem 1(ii) and Birkhoff's representation theorem [9, 10] the Radon number of the sublattice convexity of a finite distributive lattice. An element x of a lattice $(L, \leq)$ is *join irreducible* if (i) $x \neq \bigwedge L$ when L has a least element, and (ii) x cannot be expressed as the join of two other elements in the lattice (i.e., $x = y \vee z$ implies $y = x$ or $z = x$). Let $J(L)$ denote the set of join irreducible elements in the lattice $(L, \leq)$. Birkhoff's representation theorem states that every finite distributive lattice is isomorphic to a sublattice of the Boolean lattice $(2^{J(L)}, \subseteq)$.

Theorem 2. *The Radon number $r(L, \mathcal{L})$ of the sublattice convexity $\mathcal{L}$ of a finite distributive lattice $(L, \leq)$ is $r(L, \mathcal{L}) = \text{depth}(L) + 1$.*

Proof. By Birkhoff's representation theorem, a finite distributive lattice $(L, \leq)$ is isomorphic to a sublattice $(L', \subseteq)$ of $(2^{J(L)}, \subseteq)$. Let $\mathcal{L}'$ denote the sublattice convexity of $(L', \subseteq)$. We have

$$\begin{aligned} \text{depth}(L) + 1 &\leq r(L, \mathcal{L}) = r(L', \mathcal{L}') \leq r(2^{J(L)}, \mathcal{L}_{(2^{J(L)}, \subseteq)}) \\ &= r(\mathbb{B}^{|J(L)|}, \mathcal{L}_{\mathbb{B}^{|J(L)|}}) = |J(L)| + 2 \leq \text{depth}(L) + 1, \end{aligned}$$

where the first inequality follows from Corollary 2(ii); the next equality from the invariance of the Radon number under lattice isomorphism; the next inequality from Proposition 1. The equality at the beginning of the second line also follows from lattice isomorphism; the next equality from Theorem 1(ii); and the final inequality from Lemma 2 in [10, p.58]. $\square$

4 Carathéodory Numbers.

Given a convexity structure $(X, \mathcal{F})$, a subset $T \subseteq X$ is *C-dependent* if $\text{co}_{\mathcal{F}}(T) \subseteq \bigcup_{a \in T} \text{co}_{\mathcal{F}}(T \setminus \{a\})$, and it is *C-independent* otherwise. Thus the Carathéodory number $c(X, \mathcal{F})$ equals the largest cardinality of a C-independent subset. We will use the following general result on relative convexities:

Proposition 6 (Theorem II§1.11(1) in [61]). *If $(X, \mathcal{F})$ is a convexity structure and $Y \subseteq X$, then $c(Y, \mathcal{F}|_Y) \leq c(X, \mathcal{F})$.*

As mentioned in the Introduction, the Carathéodory number $c(X, \mathcal{L})$ of a lattice $\mathcal{L}$ is also called its breadth. As in [47, 48], we denote the sublattice hull $\text{co}_{\mathcal{L}}(Q)$ of a subset $Q \subseteq L$ in a lattice $(L, \leq)$ with the shorter notation $\mathcal{L}Q$. Assume that we can express $\bar{x} \in \mathcal{L}Q$ as $\bar{x} = \bigvee_{i \in I} \bigwedge_{j \in J(i)} q^{i,j}$ for some index sets I and $J(i)$ ($i \in I$) and points $q^{i,j} \in Q$: we then say that the points $q^{i,j}$ generate $\bar{x}$. Let $\mathbf{0}$ denote the d -vector in $\mathbb{R}^d$ with all its components equal to 0, and $\mathbf{1}$ that with all its components equal to 1.

4.1 Carathéodory Numbers for the Boolean Sublattice Convexity.

Theorem 3. *For any positive integer d the Carathéodory number of the convexity structure $(\mathbb{B}^d, \mathcal{L})$ is*

$$c(\mathbb{B}^d, \mathcal{L}) = \max \left\{ \left\lfloor \frac{d^2}{4} \right\rfloor, d \right\}. \quad (3)$$

Proof. Let $\gamma(d)$ denote the right-hand side of equation (3). Consider $Q \subseteq \mathbb{B}^d$ and $\bar{x} \in \mathcal{L}Q$. If $\bar{x} = \mathbf{0}$ then for all $i \in D$ there exists $q^i \in Q$ such that $q^i_i = 0$. Therefore, $\bar{x} = \bigwedge_{i \in D} q^i$ and at most d elements from Q suffice to generate $\bar{x}$ in this case. A dual argument shows that at most d elements from Q also suffice to generate $\bar{x}$ if $\bar{x} = \mathbf{1}$. Otherwise, the set $Z = \{i \in D : \bar{x}_i = 0\}$ satisfies $Z \neq \emptyset \neq D \setminus Z$. For all $i \in Z$ and $j \in D \setminus Z$ there exists $q^{i,j} \in Q$ (not necessarily all distinct) with $q^{i,j}_i = 0$ and $q^{i,j}_j = 1$, for otherwise Q would be contained in the sublattice $\{y \in \mathbb{B}^d : y_i \geq y_j\}$ whereas $\bar{x}$ is not, contradicting $\bar{x} \in \mathcal{L}Q$. Therefore

$$\bar{x} = \bigvee_{i \in D} \bigwedge_{\substack{j \in D \\ i \neq j}} q^{i,j}$$

and at most $|Z| \cdot |D \setminus Z| \leq \lfloor \frac{d}{2} \rfloor \cdot \lceil \frac{d}{2} \rceil = \lfloor \frac{d^2}{4} \rfloor$ elements from Q suffice to generate $\bar{x}$. Thus in all cases $\gamma(d)$ elements from Q suffice to generate any point $\bar{x} \in \mathcal{L}Q$. That is, we have shown that $c(\mathbb{B}^d, \mathcal{L}) \leq \gamma(d)$ for all d .

To show that this upper bound $\gamma(d)$ is attained, consider the following instances. If $d \leq 4$ then $\gamma(d) = d$. For these values of d let $Q = \{e^i : i \in D\}$ where each e^i is the i -th unit vector. Then $\mathcal{L}Q = \mathbb{B}^d$ and, for every $i \in D$, $\mathcal{L}(Q \setminus \{e^i\}) = \{y \in \mathbb{B}^d : y_i = 0\}$. Therefore $\mathbf{0} \in \mathcal{L}Q \setminus \bigcup_{q \in Q} \mathcal{L}(Q \setminus \{e^i\})$, so Q is C-independent, implying that $c(\mathbb{B}^d, \mathcal{L}) \geq |Q| = \gamma(d)$ for all $d \leq 4$.

If $d \geq 4$ then let $Z = \{i \in D : i \leq d/2\}$, so $2 \leq |Z| \leq d-2$ and $|Z| \cdot |D \setminus Z| = \lfloor \frac{d}{2} \rfloor \cdot \lceil \frac{d}{2} \rceil = \gamma(d)$. Let

$$Q = \{q^{i,j} : i \in Z \text{ and } j \in D \setminus Z\} \quad \text{where} \quad q^{i,j} = e^i + \sum_{k \in (D \setminus Z) \setminus \{j\}} e^k.$$

Define $\bar{x} \in \mathbb{B}^d$ with components $\bar{x}_i = 1$ if $i \in Z$ and 0 otherwise, and consider its 2D-projections $\pi_{h,k}\bar{x}$, where $h < k \in D$. If both $h, k \in Z$ then $\pi_{h,k}\bar{x} = (1, 1) = \pi_{h,k}(q^{h,j} \vee q^{k,j})$ for any $j \in D \setminus Z$, and thus $\pi_{h,k}\bar{x} \in \pi_{h,k}\mathcal{L}Q$. Dually, if both $h, k \in D \setminus Z$ then, for any $i \in Z$, $\pi_{h,k}\bar{x} = (0, 0) = \pi_{h,k}(q^{i,h} \wedge q^{i,k}) \in \pi_{h,k}\mathcal{L}Q$. Finally, if $h \in Z$ and $k \in D \setminus Z$ then $\pi_{h,k}\bar{x} = (1, 0) = \pi_{h,k}q^{h,k} \in \pi_{h,k}\mathcal{L}Q$. By the 2D-Projections Theorem [58, Theorem 1], it follows that $\bar{x} \in \mathcal{L}Q$. On the other hand, for every $i \in D$ and $j \in D \setminus Z$, $Q \setminus \{q^{i,j}\}$ is contained in the sublattice $\{y \in \mathbb{B}^d : y_i \leq y_j\}$ whereas $\bar{x}$ is not, and thus $\bar{x} \notin \mathcal{L}(Q \setminus \{q^{i,j}\})$. Therefore Q is C-independent and $c(\mathbb{B}^d, \mathcal{L}) \geq |Q| = \gamma(d)$ for all $d \geq 4$. $\square$

Remark 3. Since $(\mathbb{B}^d, \leq)$ is isomorphic to the Boolean lattice $(2^D, \subseteq)$ with union and intersection as join and meet operations, we have the following interesting set-theoretic interpretation of Theorem 3 mentioned in the abstract: if a subset S of D can be obtained with unions and intersections from a family $\mathcal{G}$ of subsets of D , then S can be obtained with unions and intersections from a subfamily $\mathcal{G}'$ of at most $\max\left\{\left\lfloor \frac{d^2}{4} \right\rfloor, d\right\}$ elements of $\mathcal{G}$.

4.2 Carathéodory Numbers for the Sublattice Convexities in $\mathbb{R}^d$ and $\mathbb{Z}^d$.

We next turn to the evaluation of the Carathéodory numbers of the sublattice convexities in $\mathbb{R}^d$ and $\mathbb{Z}^d$, i.e., of the breadth of the lattices $(\mathbb{R}^d, \leq)$ and $(\mathbb{Z}^d, \leq)$. We first show that these two numbers are equal:

Lemma 4. $c(\mathbb{Z}^d, \mathcal{L}) = c(\mathbb{R}^d, \mathcal{L})$.

Proof. Let T be a finite C-independent subset of $(\mathbb{Z}^d, \mathcal{L})$. Since the join and meet (pointwise minimum and maximum) of integer points are integer points, $\text{co}_{(\mathbb{Z}^d, \mathcal{L})}(T) = \text{co}_{(\mathbb{R}^d, \mathcal{L})}(T)$. Therefore T is C-independent in $(\mathbb{R}^d, \mathcal{L})$, implying that $c(\mathbb{Z}^d, \mathcal{L}) \leq c(\mathbb{R}^d, \mathcal{L})$. Conversely, let T be a finite C-independent subset of $(\mathbb{R}^d, \mathcal{L})$. For every $i \in D$ write the i -th coordinate projection of T as $\pi_i T = \{v_{i,1}, \dots, v_{i,k(i)}\}$ with $v_{i,1} < v_{i,2} < \dots < v_{i,k(i)}$. To every point x in the Cartesian product $U = \bigotimes_{i=1}^d \pi_i T$ associate an integer point $\alpha(x) \in K = \bigotimes_{i=1}^d \{1, 2, \dots, k(i)\} \subseteq \mathbb{Z}^d$ defined by $\alpha(x)_i = j$ iff $x_i = v_{i,j}$ for all $i \in D$. Note that $\alpha : U \mapsto K$ is a lattice isomorphism. It follows that $\alpha(T)$ is a C-independent set in $(\mathbb{Z}^d, \mathcal{L})$ with $|\alpha(T)| = |T|$, implying that $c(\mathbb{R}^d, \mathcal{L}) \leq c(\mathbb{Z}^d, \mathcal{L})$. $\square$

To evaluate the Carathéodory number $c(\mathbb{R}^d, \mathcal{L})$, we first present an algorithm which solves the following *sublattice hull membership* problem [48] in $\mathbb{R}^d$: given a subset $Q \subseteq \mathbb{R}^d$ and a point $\bar{x} \in \mathbb{R}^d$, decide whether $\bar{x} \in \mathcal{L}Q$. If $\bar{x} \in \mathcal{L}Q$ then the algorithm returns as a *certificate* a “small” subset $R \subseteq Q$ such that $\bar{x} \in \mathcal{L}R$. Here, “small” means that $|R| \leq \tau(d)$, where

$$\tau(d) = \left\lfloor \frac{d^2}{4} \right\rfloor + d. \quad (4)$$

This yields an upper bound on the Carathéodory number $c(\mathbb{R}^d, \mathcal{L})$.

The sublattice hull membership algorithm that is given in [48] provides a certificate R (i.e., a subset $R \subseteq Q$ such that $\bar{x} \in \mathcal{L}R$) of size $|R| \leq d^2$. In order to reduce this upper bound from d^2 to $\tau(d)$ we refine that algorithm. For this, we use the following notions, where $\bar{x} \in \mathbb{R}^d$ is fixed. With any subset $R \subseteq \mathbb{R}^d$ we associate the non-reflexive binary relation $\prec^R$ on $D = \{1, \dots, d\}$ defined by:

$$i \prec^R j \quad \text{if and only if} \quad i \neq j \text{ and for all } r \in R \text{ either } r_i < \bar{x}_i \text{ or } r_j > \bar{x}_j. \quad (5)$$

The close connection with the sublattice hull membership problem is revealed by the following lemma. We say that a binary relation $\prec$ is *vacuous* if $i \prec j$ does not hold for any $i, j \in D$.

Lemma 5. *Let $\bar{x} \in \mathbb{R}^d$. For all $R \subseteq \mathbb{R}^d$ such that $\bar{x} \in \bigotimes_{i \in D} \pi_i R$, the binary relation $\prec^R$ is vacuous if and only if $\bar{x} \in \mathcal{LR}$.*

Proof. If $\bar{x} \in \mathcal{LR}$ then the relation $\prec^R$ must be vacuous, for $i \prec^R j$ would imply that R is contained in the sublattice $\{x \in \mathbb{R}^d : x_i < \bar{x}_i \text{ or } x_j > \bar{x}_j\}$ whereas $\bar{x}$ is not. If, on the other hand, $\bar{x} \notin \mathcal{LR}$ then, by the Sublattice Hull Representation Theorem [48], there exist indices $i, j \in D$, $i \neq j$ such that $\phi_{ij}^R(\bar{x}_i) > \bar{x}_j$, where $\phi_{ij}^R(\bar{x}_i) = \inf \{r_j : r \in R \text{ and } r_i \geq \bar{x}_i\}$. This implies that $i \prec^R j$. $\square$

A binary relation $\prec$ on D is a *strict partial order* if it is non-reflexive, antisymmetric (that is, if $i \prec j$ then we cannot have $j \prec i$) and transitive (that is, if $i \prec j$ and $j \prec k$ then we must have $i \prec k$).

Lemma 6. *Let $\bar{x} \in \mathbb{R}^d$. For all $R \subseteq \mathbb{R}^d$ such that $\bar{x} \in \bigotimes_{i \in D} \pi_i R$, the binary relation $\prec^R$ is a strict partial order on D .*

Proof. If $i \prec^R j$ and $j \prec^R i$ then for all $r \in R$ we have (i) $r_i < \bar{x}_i$ or $r_j > \bar{x}_j$ and (ii) $r_j < \bar{x}_j$ or $r_i > \bar{x}_i$, a contradiction with the fact that $r_i = \bar{x}_i$ for some $r \in R$ (since $\bar{x} \in \bigotimes_{i \in D} \pi_i R$); this shows that $\prec^R$ is antisymmetric. On the other hand, if $i \prec^R j$ and $j \prec^R k$ then $k \neq i$ and for all $r \in R$ we have (i) $r_i < \bar{x}_i$ or $r_j > \bar{x}_j$ and (ii) $r_j < \bar{x}_j$ or $r_k > \bar{x}_k$; this implies $r_i < \bar{x}_i$ or $r_k > \bar{x}_k$, that is, $i \prec^R k$; this shows that $\prec^R$ is transitive. $\square$

An element $s \in D$ is a *proper source* for the binary relation $\prec$ on D if $i \prec s$ does not hold for any $i \in D$ and $s \prec j$ holds for some $j \in D$. Dually, t is a *proper sink* for the binary relation $\prec$ on D if $t \prec j$ does not hold for any $j \in D$ and $i \prec t$ holds for some $i \in D$. The proof of the following lemma follows from a standard argument and is omitted.

Lemma 7. *Let $\prec$ be a non-vacuous strict partial order on a finite set D . Then there exist a proper source s and a proper sink t for $\prec$ such that $s \prec t$.*

As mentioned above, if $\bar{x} \in \mathcal{LR}$ then the following algorithm returns a certificate R , i.e., a subset $R \subseteq Q$ such that $\bar{x} \in \mathcal{LR}$. If, on the other hand, $\bar{x} \notin \mathcal{LQ}$ then the algorithm returns as a certificate (that $\bar{x} \notin \mathcal{LQ}$) either an index $i \in D$ such that $\bar{x}_i \notin \pi_i Q$ or two indices $i \neq k \in D$ such that $\bar{x}_{ik} \notin \pi_{ik} Q$. Recall that d is an integer, $d \geq 2$ and $D = \{1, \dots, d\}$. In lines 7 and 13 of the algorithm, S^R and T^R denote the sets of all proper sources and proper sinks, respectively, for $\prec^R$. The **while**-loop counter n in steps 6 and 9 is introduced to facilitate the discussion of the algorithm.

SUBLATTICE HULL MEMBERSHIP($d, \bar{x}, Q$)

- 1 $R \leftarrow \emptyset$
- 2 **for** $i \leftarrow 1$ **to** d
- 3 **do if** there exists $q^i \in Q$ such that $q^i_i = \bar{x}_i$

```

4         then  $R \leftarrow R \cup \{q^i\}$ 
5         else return ( "NO",  $i$ )
6    $n \leftarrow 0$ 
7   define the strict partial order  $\prec^R$  on  $D$  and the subsets  $S^R$  and  $T^R$ 
8   while the partial order  $\prec^R$  is not vacuous
9     do  $n \leftarrow n + 1$ 
10      choose  $s \in S^R$  and  $t \in T^R$  such that  $s \prec^R t$ 
11      if there exists  $q^{s,t} \in Q$  with  $q^{s,t}_s \geq \bar{x}_s$  and  $q^{s,t}_t \leq \bar{x}_t$ 
12        then  $R \leftarrow R \cup \{q^{s,t}\}$ 
13        update  $\prec^R$  and the subsets  $S^R$  and  $T^R$ 
14      else return ( "NO",  $s, t$ )
15 return ( "YES",  $R$ )

```

Proposition 7. *For any integer $d \geq 2$, subset $Q \subseteq \mathbb{R}^d$ and point $\bar{x} \in \mathbb{R}^d$ the algorithm SUBLATTICE HULL MEMBERSHIP terminates after at most $\left\lfloor \frac{d^2}{4} \right\rfloor$ iterations of its **while** loop 8–14 and decides whether or not $\bar{x} \in \mathcal{L}Q$. Furthermore,*

- (i) *if $\bar{x} \in \mathcal{L}Q$ then it returns in line 15 a subset $R \subseteq Q$ such that $\bar{x} \in \mathcal{L}R$ and $|R| \leq \tau(d)$;*
- (ii) *if $\bar{x} \notin \mathcal{L}Q$ then it returns in line 5 an index $i \in D$ such that $\bar{x}_i \notin \pi_i \mathcal{L}Q$, or in line 14 two indices $s \neq t \in D$ such that $\bar{x}_{st} \notin \pi_{st} \mathcal{L}Q$.*

Proof. First, recall that $\bigotimes_{i \in D} \pi_i Q$ is a sublattice containing Q , and therefore also containing $\mathcal{L}Q$. Thus if $\bar{x} \in \mathcal{L}Q$ then for every $i \in D$ we have $\bar{x}_i \in \pi_i Q$ and therefore there exists $q^i \in Q$ with $q^i_i = \bar{x}_i$. This justifies steps 1–5 of the algorithm.

Assume the algorithm did not terminate in steps 1–5 and let R^0 denote the resulting subset R . To simplify notations let $\prec^n$ denote the strict partial order associated with the subset $R^n = R$ just before the counter n is incremented in step 9. Let S^n and T^n be the corresponding sets of proper sources and proper sinks, respectively.

If $\bar{x} \in \mathcal{L}R^0$ then by Lemma 5 the relation $\prec^0$ is vacuous and the algorithm correctly returns in step 15 the subset $R = R^0$ satisfying $|R^0| \leq d < \tau(d)$. Else, it constructs a strictly increasing sequence $R^0 \subset R^1 \subset \dots$ of subsets of Q . Indeed, at each iteration $n = 0, 1, \dots$ such that $\bar{x} \notin \mathcal{L}R^n$, Lemmata 6 and 7 imply that we may choose in step 10 a proper source $s \in S^n$ and sink $t \in T^n$ such that $s \prec^n t$. If $\bar{x} \in \mathcal{L}Q$ then there exists $q^{s,t} \in Q$ with $q^{s,t}_s \geq \bar{x}_s$ and $q^{s,t}_t \leq \bar{x}_t$, for otherwise Q would be contained in the sublattice $\{x \in \mathbb{R}^d : x_s < \bar{x}_s \text{ or } x_t > \bar{x}_t\}$ whereas $\bar{x}$ is not. Thus we may add $q^{s,t}$ to R and proceed to the next iteration. This justifies steps 11–14 of the algorithm. Finally, Lemma 5 justifies step 15 for any $n \geq 1$.

We now show that the algorithm terminates after a number N of iterations satisfying $N \leq \left\lfloor \frac{d^2}{4} \right\rfloor$. First, note that, since $R^{n+1} \supset R^n$ every $i, j \in D$ such that $i \prec^{n+1} j$ also satisfy $i \prec^n j$; that is, the relation $\prec^{n+1}$ is *weaker* than $\prec^n$.

It is *strictly* weaker since, letting $R^{n+1} = R^n \cup \{q^{s,t}\}$, we have $s \prec^n t$ but not $s \prec^{n+1} t$. (Although this does not affect the analysis, notice that if $s \prec^n j \prec^n t$ then either $q_j^{s,t} \leq \bar{x}_j$ and thus $s \not\prec^{n+1} j$, or else $q_j^{s,t} > \bar{x}_j$ and $j \not\prec^{n+1} t$; so we may be deleting more than one pair of related elements when moving from $\prec^n$ to $\prec^{n+1}$ at an iteration.)

Let $\tilde{S}^n$ be the set of all *sources* of $\prec^n$, that is, of all $j \in D$ such no $i \in D$ satisfies $i \prec^n j$; and let $\tilde{T}^n$ be the set of all *sinks* of $\prec^n$, that is, of all $j \in D$ such no $k \in D$ satisfies $j \prec^n k$. Then $U^n = \tilde{S}^n \cap \tilde{T}^n$ is the set of *isolated elements* for $\prec^n$. Note that the set of proper sources is $S^n = \tilde{S}^n \setminus U^n$ and that of proper sinks is $T^n = \tilde{T}^n \setminus U^n$. Since $\prec^n$ is strictly weaker than $\prec^{n-1}$, (i) every isolated element in $\prec^{n-1}$ remains isolated in $\prec^n$; (ii) no source in $\prec^{n-1}$ can become a proper sink in $\prec^n$, that is, $j \in \tilde{S}^{n-1}$ implies $j \notin T^n$; and (iii) similarly, no sink in $\prec^{n-1}$ can become a proper source in $\prec^n$. Thus every proper source $s \in S^{n-1}$ remains a proper source or becomes (and remains) isolated in $\prec^n, \dots, \prec^N$; and similarly every proper sink $t \in T^{n-1}$ remains a proper sink or becomes (and remains) isolated in $\prec^n, \dots, \prec^N$.

As a result, the sets $S^* = \bigcup_{n=0}^N S^n$ and $T^* = \bigcup_{n=0}^N T^n$ are disjoint. Since there can be at most a total of $|S^*| \cdot |T^*|$ pairs $(i, j) \in S^* \times T^*$ such that $i \prec^0 j$, and since each iteration removes at least one of these pairs from the current strict partial order, the algorithm must terminate after a number N of iterations satisfying $N \leq |S^*| \cdot |T^*| \leq \lfloor \frac{d}{2} \rfloor \cdot \lceil \frac{d}{2} \rceil = \lfloor \frac{d^2}{4} \rfloor$. This completes the justification of the algorithm. Since $|R^N| = |R^0| + N \leq \tau(d)$, the proof of Proposition 7 is complete. $\square$

Theorem 4. *For every positive integer d the Carathéodory numbers of the convexity structures $(\mathbb{R}^d, \mathcal{L})$ and $(\mathbb{Z}^d, \mathcal{L})$ are*

$$c(\mathbb{R}^d, \mathcal{L}) = c(\mathbb{Z}^d, \mathcal{L}) = \left\lfloor \frac{d^2}{4} \right\rfloor + d. \quad (6)$$

Proof. In view of Lemma 4 it suffices to prove the theorem for $(\mathbb{R}^d, \mathcal{L})$. If $d = 1$ then every subset $Q \subset \mathbb{R}$ is a chain and thus satisfies $\mathcal{L}Q = Q$; therefore $c(\mathbb{R}^1, \mathcal{L}) = 1$ and equation (6) holds. Hence assume $d \geq 2$ and recall that $\tau(d)$ denotes the right-hand side of equation (6). Proposition 7 implies that $c(\mathbb{R}^d, \mathcal{L}) \leq \tau(d)$. To show that this upper bound $\tau(d)$ is attained, consider the following instance defined for every $d \geq 2$. For all $i \in D$ define $q^i \in \{-1, 0, 1\}^d$ with components

$$q_k^i = \begin{cases} -1 & \text{if } k < i; \\ 0 & \text{if } k = i; \\ 1 & \text{if } k > i, \end{cases}$$

Let $Z = \{i \in D : i \leq d/2\}$ and for all $i \in Z$ and $j \in D \setminus Z$ define $q^{i,j} \in \{-1, 0, 1\}^d$ with components

$$q_k^{i,j} = \begin{cases} -1 & \text{if } k = j \text{ or } k \in Z \setminus \{i\}; \\ 1 & \text{if } k = i \text{ or } k \in (D \setminus Z) \setminus \{j\}. \end{cases}$$

The set

$$Q = \{q^i : i \in D\} \cup \{q^{i,j} : i \in Z \text{ and } j \in D \setminus Z\}$$

has size $|Q| = \tau(d)$. Finally let $\bar{x} = \mathbf{0}$.

Define subset $R \subseteq \{-1, 0 + 1\}^d$ to be **0-critical** if $\mathbf{0} \in \mathcal{L}R$ and $\mathbf{0} \notin \mathcal{L}(R \setminus \{r\})$ for any $r \in R$. Thus the size of any **0-critical** set R is a lower bound on the Carathéodory number of the convexity structures $(\{-1, 0 + 1\}^d, \mathcal{L})$ and $(\mathbb{R}^d, \mathcal{L})$, that is, $c(\mathbb{R}^d, \mathcal{L}) \geq c(\{-1, 0 + 1\}^d, \mathcal{L}) \geq |R|$.

For all $i \in Z$ let $y^i = q^i \vee (\bigvee_{h < i} q^{h,d}) \in \mathcal{L}Q$, and we have $y^i = \mathbf{1} - e^i$. On the other hand, for all $j \in D \setminus Z$ let $y^j = q^j \vee (\bigvee_{i \in Z} q^{i,j}) \in \mathcal{L}Q$, and we also have $y^j = \mathbf{1} - e^j$. Therefore $\mathbf{0} = \bigwedge_{i \in D} y^i \in \mathcal{L}Q$. To complete the proof that Q is **0-critical**, first note that, for all $i \in D$, q^i is the only element $q \in Q$ with $q_i = 0$; therefore $\mathbf{0} \notin \mathcal{L}(Q \setminus \{q^i\})$. Next, for $i \in Z$ and $j \in D \setminus Z$, note that $q^{i,j}$ is the unique element $q \in Q$ with $q_i \geq 0$ and $q_j \leq 0$; therefore $Q \setminus \{q^{i,j}\}$ is contained in the sublattice $\{x \in \{-1, 0 + 1\}^d : x_i = -1 \text{ or } x_j = 1\}$ whereas $\mathbf{0}$ is not. Thus Q is **0-critical** and, since $|Q| = \tau(d)$, this completes the proof that $c(\mathbb{R}^d, \mathcal{L}) = \tau(d)$. $\square$

Remark 4. Note that the values of the Carathéodory numbers in Theorems 3 and 4 depend only on the dimension d of the product space and the cardinality of its component chains, and not on the fact that these chains are specifically $\mathbb{B}$, $\mathbb{Z}$ or $\mathbb{R}$. Indeed, consider any d -dimensional product $\bigotimes_{i=1}^d T_i$ of chains, with all $|T_i| \geq 2$. The lower bound from Theorem 3 and Proposition 7 imply

$$\max \left\{ \left\lfloor \frac{d^2}{4} \right\rfloor, d \right\} \leq c \left(\bigotimes_{i=1}^d T_i, \mathcal{L} \right) \leq \left\lfloor \frac{d^2}{4} \right\rfloor + d.$$

The upper bound from Theorem 3 implies that the leftmost of these two inequalities holds as an equality if all $|T_i| = 2$; and the lower bound from Theorem 4 implies that the rightmost inequality holds as an equality if all $|T_i| \geq 3$. We leave to the interested reader the exact determination of the Carathéodory number when some, but not all, $|T_i| = 2$.

4.3 Carathéodory Numbers for the Convex Sublattice Convexities.

In this section we consider the Carathéodory numbers of the convexity structures $(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C})$ and $(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C}|\mathbb{Z}^d)$. For easier reference to other papers whose results are used in the proofs, we replace the notation $\text{co}_{\mathcal{L} \cap \mathcal{C}}(Q)$ for the convex hull operator with respect to this convexity, with the notation $\mathcal{L} \text{conv } Q$. This notation is justified since, as shown in [48], $\mathcal{L} \text{conv } Q$ is the lattice hull of the (standard) convex hull $\text{conv } Q$.

We start with the case $d = 2$. Given two points $a, b \in \mathbb{R}^2$, let

$$\text{box}(a, b) = \{x \in \mathbb{R}^2 : a \wedge b \leq x \leq a \vee b\}.$$

The following proposition characterizes the convex sublattice hull of a compact subset in $\mathbb{R}^2$:

Proposition 8. *Let Q be a compact subset of $\mathbb{R}^2$ and, for $i = 1, 2$, let*

$$u^i \in \arg \min \{x_i : x \in Q\} \quad \text{and} \quad v^i \in \arg \max \{x_i : x \in Q\}.$$

Then the convex sublattice hull of Q is

$$\mathcal{L} \operatorname{conv} Q = (\operatorname{box}(u^1, u^2)) \cup (\operatorname{conv} Q) \cup (\operatorname{box}(v^1, v^2)).$$

Proof. Since Q is compact, so is its convex hull Q' . Then, by Theorem 3.7 in [48], the lattice convex hull $L = \mathcal{L} \operatorname{conv} Q = \mathcal{L} Q'$ is closed. Since $L \subseteq \bigotimes_{i=1}^2 \pi_i Q'$, L is a compact sublattice. Note that, for $i = 1, 2$, $\pi_i L = \pi_i Q' = [u_i^i, v_i^i]$. Since Q' is compact, the following functions $f_{ij} : \pi_i Q' \mapsto \pi_j Q'$, are well-defined, where

$$f_{ij}(h) = \min \{x_j : x \in Q' \text{ such that } x_i = h\} \quad \text{for all } h \in \pi_i Q'.$$

Recall that, when B is a poset, the *epigraph* $\mathcal{E}g$ of a function $g : A \mapsto B$ is

$$\mathcal{E}g = \{(h, k) \in A \times B : k \geq g(h)\}.$$

Let

$$\tilde{\mathcal{E}}g = \{(k, h) \in B \times A : (h, k) \in \mathcal{E}g\}.$$

Since Q' is a compact convex subset of $\mathbb{R}^2$, we have

$$Q' = \mathcal{E}f_{12} \cap \tilde{\mathcal{E}}f_{21}.$$

As in [48], define for $i \neq j \in \{1, 2\}$ the function $\phi_{ij} : \pi_i Q' \mapsto \pi_j Q'$ by

$$\phi_{ij}(h) = \min \{x_j : x \in Q' \text{ such that } x_i \geq h\}.$$

Since Q' is a compact convex set, we have

$$\phi_{ij}(h) = \begin{cases} u_j^j & \text{if } h \leq u_i^i; \\ f_{ij}(h) & \text{otherwise;} \end{cases}$$

and thus

$$\mathcal{E}\phi_{12} = ([u_1^1, u_1^2] \times \pi_2 Q') \cup \mathcal{E}f_{12}$$

and

$$\tilde{\mathcal{E}}\phi_{21} = (\pi_1 Q' \times [u_2^2, u_2^1]) \cup \tilde{\mathcal{E}}f_{21}.$$

Applying Theorem 3.4 in [48], we have

$$\begin{aligned} \mathcal{L} Q' &= \mathcal{E}\phi_{12} \cap \tilde{\mathcal{E}}\phi_{21} \\ &= \operatorname{box}(u^1, u^2) \cup (\mathcal{E}f_{12} \cap \tilde{\mathcal{E}}f_{21}) \cup \operatorname{box}(v^1, v^2) \end{aligned}$$

and the result follows. $\square$

The following corollary will be used for determining the Carathéodory numbers for $d = 2$, and also later in our analysis of the case $d \geq 3$.

Corollary 8. *Let $Q \subset \mathbb{R}^2$ be a compact subset, $q \in Q$ and $\bar{x} \in \mathcal{L} \text{conv } Q$. Then there exist $a, b \in Q$ (not necessarily distinct) such that $\bar{x} \in \mathcal{L} \text{conv } \{q, a, b\}$.*

Proof. Using the notations of Proposition 7, either $\bar{x} \in \text{box}(u^1, u^2)$, or $\bar{x} \in \text{box}(v^1, v^2)$, or else $\bar{x} \in \text{conv } Q$. The corollary follows in the first two cases by letting $\{a, b\} = \{u^1, u^2\}$, or $\{a, b\} = \{v^1, v^2\}$. It follows similarly in the third case if $\text{conv } Q$ is not full-dimensional; otherwise it follows from noting that, in $\mathbb{R}^2$, the convex hull of a compact subset Q can be covered with triangles having vertices in Q , each triangle having as vertex a same arbitrary given point $q \in Q$. $\square$

Proposition 9. *When $d = 2$, the Carathéodory numbers of the convex sublattice convexity structures $(\mathbb{R}^2, \mathcal{L} \cap \mathcal{C})$ and $(\mathbb{Z}^2, \mathcal{L} \cap \mathcal{C}|_{\mathbb{Z}^2})$ are $c(\mathbb{R}^2, \mathcal{L} \cap \mathcal{C}) = c(\mathbb{Z}^2, \mathcal{L} \cap \mathcal{C}|_{\mathbb{Z}^2}) = 3$.*

Proof. We prove the chain of inequalities

$$3 \leq c(\mathbb{Z}^2, \mathcal{L} \cap \mathcal{C}|_{\mathbb{Z}^2}) \leq c(\mathbb{R}^2, \mathcal{L} \cap \mathcal{C}) \leq 3.$$

The first inequality follows from the fact that the set $Q = \{(0, 0), (1, 0), (2, 3)\}$ is $\mathcal{C}$ -independent, since the integer convex sublattice hulls $\text{co}_{\mathcal{L} \cap \mathcal{C}|_{\mathbb{Z}^2}}(Q \setminus \{q\}) = Q \setminus \{q\}$ for all $q \in Q$, whereas $(1, 1) \in \text{co}_{\mathcal{L} \cap \mathcal{C}|_{\mathbb{Z}^2}}(Q) \setminus Q$. The second inequality follows from Proposition 6. To prove the last inequality, let S be an arbitrary subset of $\mathbb{R}^2$ and $\bar{x} \in \mathcal{L} \text{conv } S$. By a fundamental result in abstract convexity mentioned in the Introduction, there exists a *finite* set of points $Q \subseteq S$ such that $\bar{x} \in \mathcal{L} \text{conv } S$. Since Q is compact, the last inequality now follows from Corollary 8. $\square$

For the case $d \geq 3$ we now present an algorithm which solves the *convex sublattice hull membership* problem in $\mathbb{R}^d$: given a subset $Q \subseteq \mathbb{R}^d$ and a point $\bar{x} \in \mathbb{R}^d$, decide whether or not $\bar{x} \in \mathcal{L} \text{conv } Q$. If $\bar{x} \in \mathcal{L} \text{conv } Q$ then the algorithm returns as a *certificate* a subset $R \subseteq Q$ such that $\bar{x} \in \mathcal{L} \text{conv } R$ and with size $|R| \leq \tau(d) + 1$ where, as defined in equation (4), $\tau(d) = \left\lfloor \frac{d^2}{4} \right\rfloor + d$. This implies $c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) \leq \tau(d) + 1$. Our algorithm and developments will be similar to those in Section 4.2 but with some important differences.

For any $i \neq j$ in D , we call a linear function $f : \mathbb{R}^d \mapsto \mathbb{R}$ *ij-bimonotone* if it has the form $f(x) = a_i x_i - a_j x_j$ with $a_i > 0$ and $a_j > 0$. Recall that a real-valued linear function on $\mathbb{R}^d$ *separates* a point $\bar{x} \in \mathbb{R}^d$ from a subset $R \subseteq \mathbb{R}^d$ if $f(\bar{x}) < \inf \{f(r) : r \in R\}$. Given $\bar{x} \in \mathbb{R}^d$, we associate with any subset $R \subseteq \mathbb{R}^d$ the non-reflexive binary relation $\propto^R$ on D defined by:

$$i \propto^R j \quad \text{if and only if} \quad i \neq j \quad \text{and} \quad \text{there exists a linear } ij\text{-bimonotone} \\ \text{function which separates } \bar{x} \text{ from } R. \quad (7)$$

The connection with the convex sublattice hull membership problem is revealed by the following lemma.

Lemma 9. *Let $\bar{x} \in \mathbb{R}^d$. For all $R \subseteq \mathbb{R}^d$ such that $\mathcal{L} \text{ conv } R$ is closed and $\bar{x} \in \bigotimes_{i \in D} \pi_i \text{ conv } R$, the binary relation $\propto^R$ is vacuous if and only if $\bar{x} \in \mathcal{L} \text{ conv } R$.*

Proof. Let $\bar{x}$ and R be as stated in the Lemma. If $\bar{x} \in \mathcal{L} \text{ conv } R$ then the relation $\propto^R$ must be vacuous, for $i \propto^R j$ would imply that R is contained in a closed convex sublattice $\{x \in \mathbb{R}^d : a_i x_i - a_j x_j \geq a_i \bar{x}_i - a_j \bar{x}_j + \varepsilon\}$ for some $\varepsilon > 0$, whereas $\bar{x}$ is not. If, on the other hand, $\bar{x} \notin \mathcal{L} \text{ conv } R$ then, by the Linear Bimonotone Representation Theorem [47], there exists an ij -bimonotone linear function f which separates $\bar{x}$ from $\mathcal{L} \text{ conv } R$. Since $R \subseteq \mathcal{L} \text{ conv } R$, f also separates $\bar{x}$ from R . Let $f(x) = a_i x_i - a_j x_j$ with $a_i \geq 0$ and $a_j \geq 0$. Since $\bar{x} \in \bigotimes_{k \in D} \pi_k \text{ conv } R$, we cannot have $a_i = 0$ or $a_j = 0$. Therefore f must be ij -bimonotone for some $i \neq j \in D$. These two indices thus satisfy $i \propto^R j$, implying that $\propto^R$ is not vacuous. $\square$

Lemma 10. *Let $\bar{x} \in \mathbb{R}^d$. For all $R \subseteq \mathbb{R}^d$ such that $\bar{x} \in \bigotimes_{i \in D} \pi_i \text{ conv } R$, the binary relation $\propto^R$ is a strict partial order on D .*

Proof. If $i \propto^R j$ and $j \propto^R i$ then an ij -bimonotone function $f(x) = a_i x_i - a_j x_j$ satisfies $f(\bar{x}) < \inf \{f(r) : r \in R\}$, and a ji -bimonotone function $f'(x) = a'_j x_j - a'_i x_i$ satisfies $f'(\bar{x}) < \inf \{f'(r) : r \in R\}$. Therefore

$$(a_i a'_j - a'_i a_j) \bar{x}_j = a_i f'(\bar{x}) + a'_i f(\bar{x}) > a_i f'(r) + a'_i f(r) = (a_i a'_j - a'_i a_j) r_j$$

for all $r \in R$, a contradiction with $\bar{x} \in \bigotimes_{i \in D} \pi_i \text{ conv } R$; this shows that $\propto^R$ is antisymmetric.

If $i \propto^R j$ and $j \propto^R k$ then $k \neq i$ and an ij -bimonotone function $f(x) = a_i x_i - a_j x_j$ satisfies $f(\bar{x}) < \inf \{f(r) : r \in R\}$, and a jk -bimonotone function $f'(x) = a'_j x_j - a'_k x_k$ satisfies $f'(\bar{x}) < \inf \{f'(r) : r \in R\}$. Since all four coefficients a_i, a_j, a'_j and a'_k are positive, the function $f'' = a'_j f + a_j f'$ is ik -bimonotone and separates $\bar{x}$ from R . Therefore $i \prec^R k$; this shows that $\propto^R$ is transitive. $\square$

In steps 8 and 11 of the algorithm below, S^R and T^R denote the sets of all proper sources and proper sinks, respectively, for $\propto^R$.

CONVEX SUBLATTICE HULL MEMBERSHIP($d, \bar{x}, Q$)

```

1   $R \leftarrow \emptyset$ 
2  choose  $q^0 \in Q$ 
3  for all  $i \in D$  such that  $\bar{x}_i \neq q_i^0$ 
4      do if there exists  $q^i \in Q$  such that  $(q_i^i \leq \bar{x}_i < q_i^0)$  or  $(q_i^i \geq \bar{x}_i > q_i^0)$ 
5          then  $R \leftarrow R \cup \{q^i\}$ 
6          else return ( "NO",  $i$ )
7   $n \leftarrow 0$ 
8  define the strict partial order  $\propto^R$  on  $D$  and the subsets  $S^R$  and  $T^R$ 
9  while the partial order  $\propto^R$  is not vacuous
10     do  $n \leftarrow n + 1$ 
11         choose  $s \in S^R$  and  $t \in T^R$  such that  $s \propto^R t$ 
12         if there exist  $q^{s,t}, q^{t,s} \in Q$  s.t.  $\pi_{st} \bar{x} \in \mathcal{L} \text{ conv } \{\pi_{st} q^0, \pi_{st} q^{s,t}, \pi_{st} q^{t,s}\}$ 
```

```

13         then  $R \leftarrow R \cup \{q^{s,t}, q^{t,s}\}$ 
14             update  $\alpha^R$  and the subsets  $S^R$  and  $T^R$ 
15         else return ( "NO",  $s, t$ )
16 return ( "YES",  $R$ )

```

Proposition 10. *For any integer $d \geq 3$, subset $Q \subseteq \mathbb{R}^d$ and point $\bar{x} \in \mathbb{R}^d$ the algorithm CONVEX SUBLATTICE HULL MEMBERSHIP terminates after at most $\left\lfloor \frac{d^2}{4} \right\rfloor$ iterations of its **while** loop 9–15 and decides whether or not $\bar{x} \in \mathcal{L}Q$. Furthermore,*

- (i) *if $\bar{x} \in \mathcal{L}Q$ then it returns in line 16 a subset $R \subseteq Q$ such that $\bar{x} \in \mathcal{L} \text{conv } R$ and $|R| \leq 2 \left\lfloor \frac{d^2}{4} \right\rfloor + d + 1$;*
- (ii) *if $\bar{x} \notin \mathcal{L}Q$ then it returns in line 6 an index $i \in D$ such that $\bar{x}_i \notin \pi_i \text{conv } Q$, or in line 15 two indices $s \neq t \in D$ such that $\bar{x}_{st} \notin \pi_{st} \mathcal{L} \text{conv } Q$.*

Proof. The proof is similar to that of Proposition 7. The existence of $q^{s,t}$ and $q^{t,s}$ in line 12, when $\bar{x} \in \mathcal{L} \text{conv } Q$, follows from Corollary 8. $\square$

Propositions 6 and 10 imply that

$$c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) \leq c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) \leq 2 \left\lfloor \frac{d^2}{4} \right\rfloor + d + 1. \quad (8)$$

and therefore these Carathéodory numbers are finite. Thus we will be interested in the structure of the convex sublattice hull of a finite set of points.

Proposition 11. *If $Q \subset \mathbb{R}^d$ is nonempty and finite then its convex sublattice hull $\mathcal{L} \text{conv } Q$ is a polytope defined by a system of bimonotone inequalities.*

Proof. Since Q is finite, its standard convex hull $\text{conv } Q$ is a polytope. By Corollary 12 in [48], the convex sublattice hull $\mathcal{L} \text{conv } Q = \mathcal{L}(\text{conv } Q)$ is a polyhedron. Since Q is contained in $\text{box}(\bigwedge Q, \bigvee Q)$, which is a convex sublattice of $\mathbb{R}^d$, $\mathcal{L} \text{conv } Q$ is also contained in that box, and is thus a polytope. Finally, from Veinott [63] it follows that this polytope is defined by a system of bimonotone inequalities. $\square$

Theorem 5. *For every positive integer $d \geq 3$ the Carathéodory number of the convexity structure $(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C})$ satisfies*

$$2 \left\lfloor \frac{d^2}{4} \right\rfloor \leq c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) = c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) \leq 2 \left\lfloor \frac{d^2}{4} \right\rfloor + d + 1. \quad (9)$$

Proof. In view of (8) it suffices to prove (i) $c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) \geq c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C})$, and (ii) $c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) \geq 2 \lfloor d^2/4 \rfloor$.

(i) We prove the inequality $c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) \geq c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C})$ by induction on d . The base case $d = 2$ follows from Proposition 9. Thus for any $d \geq 3$ assume

that $c(\mathbb{R}^{d-1}, \mathcal{L} \cap \mathcal{C}) = c(\mathbb{Z}^{d-1}, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^{d-1})$. Let Q be a $\mathcal{C}$ -independent set in $\mathbb{R}^d$ with $|C| = c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C})$. By Proposition 11, $P := \mathcal{L} \text{ conv } Q$ is a polytope defined by a system of bimonotone inequalities.

First, consider the case where P is not full dimensional. Hence (Theorem 3.5.b in [45]), P is contained in the hyperplane induced by one of the bimonotone inequalities defining P . W.l.o.g., we may thus assume that P is contained in the (bimonotone) hyperplane defined by an equation of the form $a_i x_i - a_d x_d = b$ with $a_i a_d \geq 0$ and $a_d \neq 0$. For every $y \in \mathbb{R}^d$ let $y' \in \mathbb{R}^{d-1}$ denote its projection onto $\mathbb{R}^{d-1}$ (defined by $y'_j = y_j$ for all $j = 1, \dots, d-1$), and thus let $Y' = \{y' : y \in Y\}$ denote the projection of subset $Y \subseteq \mathbb{R}^d$. Since Q is contained in this hyperplane, different points $q \in Q$ have different projections q' and thus $|Q'| = |Q|$. We claim that Q' is $\mathcal{C}$ -independent in $\mathbb{R}^{d-1}$. Since Q is $\mathcal{C}$ -independent there exists $\bar{x} \in P \setminus \bigcup_{q \in Q} \mathcal{L} \text{ conv}(Q \setminus \{q\})$. Say that a vector $\alpha \in \mathbb{R}^n$ is *bimonotone* if $\alpha \neq 0$ and at most two components, say, α_i and α_j of α are nonzero and satisfy $\alpha_i \alpha_j \leq 0$. Thus if $\alpha' \in \mathbb{R}^{d-1}$ is bimonotone then its “trivial lifting” $\alpha \in \mathbb{R}^d$, defined by $\alpha_j = \alpha'_j$ for all $j = 1, \dots, d-1$ and $\alpha_d = 0$, is bimonotone; and conversely, if $\alpha \in \mathbb{R}^d$ is bimonotone then so is its projection $\alpha' \in \mathbb{R}^{d-1}$. Since $\bar{x} \in \mathcal{L} \text{ conv } Q$, for every bimonotone vector $\alpha \in \mathbb{R}^d$ there exist r and $s \in Q$ such that $\alpha r \leq \alpha \bar{x} \leq \alpha s$. Therefore, for every bimonotone vector $\alpha' \in \mathbb{R}^{d-1}$ there exist r and $s \in Q$ such that $\alpha r \leq \alpha \bar{x} \leq \alpha s$ for the lifting α of α' ; but since $\alpha_d = 0$ we have $\alpha' r' \leq \alpha' \bar{x}' \leq \alpha' s'$. Since this is true for every bimonotone vector $\alpha' \in \mathbb{R}^{d-1}$, it implies that $\bar{x}' \in \mathcal{L} \text{ conv } Q'$. On the other hand, since $\bar{x} \notin \bigcup_{q \in Q} \mathcal{L} \text{ conv}(Q \setminus \{q\})$, for every $q \in Q$ there exists a bimonotone vector $\alpha \in \mathbb{R}^d$ which “separates” $\bar{x}$ from $\mathcal{L} \text{ conv}(Q \setminus \{q\})$, i.e., such that (w.l.o.g.) $\alpha \bar{x} < \alpha r$ for all $r \in Q \setminus \{q\}$. Fix $q \in Q$ and consider such an associated separating bimonotone α . If $\alpha_d = 0$ then, letting $\beta = \alpha'$, for every $x \in \mathbb{R}^d$ we have $\beta x' = \alpha x$, implying $\beta \bar{x}' < \beta r'$ for all $r' \in (Q \setminus \{q\})' = Q' \setminus \{q'\}$. Else $\alpha_d \neq 0$ and at most one α_k , $1 \leq k \leq d-1$, is nonzero, and thus of sign opposite to that of α_d . Since every $x \in P = \mathcal{L} \text{ conv } Q$ satisfies $a_i x_i - a_d x_d = b$ and $a_d \neq 0$, we have $x_d = (a_i x_i - b)/a_d$, and thus $\alpha x = \beta x' - \alpha_d b/a_d$, where $\beta \in \mathbb{R}^{d-1}$ is defined by $\beta_i = \alpha_i + \alpha_d a_i/a_d$ and $\beta_j = \alpha_j$ for all $j \neq i$. Therefore $\beta \bar{x}' < \beta r'$ for all $r \in Q \setminus \{q\}$, and thus for all $r' \in Q' \setminus \{q'\}$. This implies that $\beta \neq 0$. If $\alpha_i \neq 0$ then β has just one nonzero component, and is thus bimonotone. Else β has at most two nonzero components, $\beta_k = \alpha_k$ and $\beta_i = \alpha_d a_i/a_d$. Since $a_i/a_d \geq 0$, β_i has the same sign as α_d , i.e., opposite to that of α_k . This implies that β is bimonotone. In each case we have shown that for every $q' \in Q'$ there exists a bimonotone vector $\beta \in \mathbb{R}^{d-1}$ which separates $\bar{x}'$ from $Q' \setminus \{q'\}$. This implies that $\bar{x}' \notin \bigcup_{q' \in Q'} \mathcal{L} \text{ conv}(Q' \setminus \{q'\})$, and therefore that Q' is $\mathcal{C}$ -independent, as claimed.

Conversely, if R' is $\mathcal{C}$ -independent in $\mathbb{Z}^{d-1}$ then its trivial lifting $R = \{(r', 0)^T \in \mathbb{Z}^d : r' \in R'\}$ is $\mathcal{C}$ -independent in $\mathbb{Z}^d$, for the bimonotone hyperplane defined by $x_d = 0$ is a convex sublattice of $\mathbb{Z}^d$, and thus $\mathcal{L} \text{ conv } R$ is contained in this hyperplane. This implies

$$c(\mathbb{Z}^{d-1}, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^{d-1}) \leq c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d). \quad (10)$$

As a consequence,

$$\begin{aligned}
c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) &= |Q| = |Q'| \leq c(\mathbb{R}^{d-1}, \mathcal{L} \cap \mathcal{C}) \quad (\text{since } Q' \text{ is } C\text{-independent}) \\
&= c(\mathbb{Z}^{d-1}, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^{d-1}) \quad (\text{by the inductive assumption}) \\
&\leq c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d) \quad (\text{by (10)}) \\
&\leq c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) \quad \text{by (8)}
\end{aligned}$$

hence equality must hold throughout, and thus $c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) = c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d)$, as claimed.

Now consider the case where $P = \mathcal{L} \text{ conv } Q$ is full dimensional. Hence (Theorem 6.3 in [52]) P is the (topological) closure of its interior, i.e., $P = \text{cl}(\text{int } P)$. Let $R = \bigcup_{q \in Q} \mathcal{L} \text{ conv}(Q \setminus \{q\})$, so R is a finite union of polytopes, hence a closed set, and since Q is C -independent, R is strictly contained in P . The open set $\bar{R} = \mathbb{R}^d \setminus R$ intersects P hence (Corollary 6.3.2 in [52]) this open set $\bar{R}$ intersects $\text{int } P$, and thus $P \setminus R = P \cap \bar{R}$ has a nonempty interior. W.l.o.g. (or after an affine change of variables), we may assume that $0 \in \text{int}(P \setminus R)$. Thus there exists $\epsilon > 0$ such that the ball $B(0, \epsilon)$ centered at 0 and with radius ϵ is contained in $P \setminus R$. There exist rational vectors $\tilde{q} \in \mathbb{Q}^d$, $q \in Q$, all distinct and such that $\|q - \tilde{q}\| \leq \epsilon/2$. Let $\tilde{Q} = \{\tilde{q} : q \in Q\}$. Note that the convex sublattice hull $\mathcal{L} \text{ conv } Y$ of any finite set $Y \subset \mathbb{R}^d$ is a polytope which may be defined by *normalized* bimonotone inequalities $ax \leq \max_{y \in Y} ay$, i.e., such that $\|a\| = 1$. Let $A = \{a \in \mathbb{R}^d : a \text{ is bimonotone and } \|a\| = 1\}$. By the Cauchy-Schwarz inequality, for every $a \in A$ and $q \in Q$, $a\tilde{q} = aq + a(\tilde{q} - q) \leq aq + \|a\| \cdot \|\tilde{q} - q\| \leq aq + \epsilon/2$, and, similarly, $a\tilde{q} \geq aq - \epsilon/2$. Since $B(0, \epsilon) \subseteq P$ and the vectors ϵa and $-\epsilon a$ are in $B(0, \epsilon)$, for every $a \in A$ we have $\min_{q \in Q} aq \leq \min_{x \in B(0, \epsilon)} ax \leq -\epsilon$ and $\max_{q \in Q} aq \geq \max_{x \in B(0, \epsilon)} ax \geq \epsilon$. Therefore $\min_{q \in Q} a\tilde{q} \leq \min_{q \in Q} (aq + \epsilon/2) \leq -\epsilon/2$ and $\max_{q \in Q} a\tilde{q} \geq \max_{q \in Q} (aq - \epsilon/2) \geq \epsilon/2$. This implies that $0 \in \mathcal{L} \text{ conv } \tilde{Q}$. On the other hand, since $B(0, \epsilon) \subseteq \bar{R}$, for every $q \in Q$ there exists $a \in A$ that separates $B(0, \epsilon)$ from $\mathcal{L} \text{ conv}(Q \setminus \{q\})$, i.e., w.l.o.g., such that $ax \leq \min_{r \in Q \setminus \{q\}} ar$ for all $x \in B(0, \epsilon)$. Therefore $\min_{\tilde{r} \in \tilde{Q} \setminus \{\tilde{q}\}} ar \geq \min_{r \in Q \setminus \{q\}} ar \geq \epsilon/2$ for all $x \in B(0, \epsilon)$, and thus $0 \notin \mathcal{L} \text{ conv}(\tilde{Q} \setminus \{\tilde{q}\})$. Thus we have shown that $0 \in (\mathcal{L} \text{ conv } \tilde{Q}) \setminus \left(\bigcup_{\tilde{q} \in \tilde{Q}} \mathcal{L} \text{ conv}(\tilde{Q} \setminus \{\tilde{q}\}) \right)$, implying that $\tilde{Q}$ is C -independent in $\mathbb{Q}^d$. Then we may scale all $\tilde{q}$ by an integer M (say, the least common multiple of all denominators in all $\tilde{q} \in \tilde{Q}$) so that $\hat{Q} = \{M\tilde{q} : q \in Q\}$ is C -independent in $\mathbb{Z}^d$. Therefore

$$c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) = |Q| = |\hat{Q}| \leq c(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C} | \mathbb{Z}^d).$$

This complete our inductive proof of (i).

(ii) To prove the lower bound $2 \left\lfloor \frac{d^2}{4} \right\rfloor$, consider the following instance Q derived from that in the proof of Theorem 4: use the $\left\lfloor \frac{d^2}{4} \right\rfloor$ points $\tilde{q}^{i,j} = q^{i,j} + \mathbf{1}$ where each $q^{i,j}$ is as defined in that proof, and add the $\left\lfloor \frac{d^2}{4} \right\rfloor$ points $p^{i,j} = q^{i,j} - \mathbf{1}$. Note that these $2 \left\lfloor \frac{d^2}{4} \right\rfloor$ points are distinct, since they are all nonzero, all $\tilde{q}^{i,j} \geq 0$

and all $p^{i,j} \leq 0$. Recall that $Z = \{k \in D : k \leq d/2\}$ and let

$$\bar{x} = \frac{1}{2} \left(\bigvee_{i \in Z} \bigwedge_{j \in D \setminus Z} \tilde{q}^{i,j} \right) + \frac{1}{2} \left(\bigvee_{i \in Z} \bigwedge_{j \in D \setminus Z} p^{i,j} \right).$$

So, $\bar{x} \in \text{conv } \mathcal{L}Q \subseteq \mathcal{L} \text{conv } Q$, with $\bar{x}_k = 1$ if $k \in Z$, and $\bar{x}_k = -1$ otherwise. For $i \in Z$ and $j \in D \setminus Z$, we have

$$x_j - 2x_i > -3 = \bar{x}_j - 2\bar{x}_i$$

for all $x \in Q \setminus \{\tilde{q}^{i,j}\}$; hence each $\tilde{q}_{ij}$ is $\bar{x}$ -critical. Similarly, we have

$$2x_j - x_i > -3 = 2\bar{x}_j - \bar{x}_i$$

for all $x \in Q \setminus \{p^{i,j}\}$; hence each p_{ij} is $\bar{x}$ -critical. This instance thus implies the lower bound $c(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C}) \geq 2 \left\lfloor \frac{d^2}{4} \right\rfloor$. $\square$

4.4 Carathéodory Numbers for the Integral $L^\natural$ Convexities.

In this section we consider the Carathéodory numbers of the integral $L^\natural$ convexity structures $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}^\natural)$ and $(\mathbb{Z}^d, \mathcal{L}_{\mathbb{Z}^d}^\natural)$. As before, we write $\mathcal{L}^\natural$ for $\mathcal{L}_{\mathbb{R}^d}^\natural$ or $\mathcal{L}_{\mathbb{Z}^d}^\natural$, whenever the meaning is clear from the context.

We first introduce some notations that will be used in the sequel, and we also use them to verify that these structures are indeed convexities. Let $V = \{0, 1, \dots, d\}$, where the index 0 will be associated with lower and upper bound constraints. Given $X \in \{\mathbb{R}^d, \mathbb{Z}^d\}$, define the functions $\delta_{i,j} : X \mapsto \mathbb{R}$ for all pairs $(i, j) \in A_V$ where $\delta_{i,0}(x) = x_i$, $\delta_{0,j}(x) = -x_j$, and $\delta_{i,j}(x) = x_i - x_j$. Letting $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$, the integral $\mathcal{L}^\natural$ convexity on X is $\mathcal{L}^\natural X = \{P_X(b) : b \in \bar{\mathbb{Z}}^{A_V}\}$ where

$$P_X(b) = \{x \in X : \delta_{i,j}(x) \leq b_{i,j} \text{ for all } (i, j) \in A_V\}.$$

This is indeed a convexity since (i) it includes the empty set (i.e., $P_X(b)$ for any b for which it is empty) and the full set (i.e., $P_X(b^\infty)$ where b^∞ has all its components $b_{i,j}^\infty = +\infty$); (ii) the intersection of any family $(P_X(b^k))_{k \in K}$ is $\bigcap_{k \in K} P_X(b^k) = P_X(\bigwedge_{k \in K} b^k)$ if the meet $\bigwedge_{k \in K} b^k$ is finite, and $\emptyset$ otherwise; and (iii) such a family is nested if K is totally ordered and $b^k \leq b^l$ whenever $k \leq l$, and thus the union $\bigcup_{k \in K} P_X(b^k) = P_X(\bigvee_{k \in K} b^k)$ (where the join $\bigvee_{k \in K} b^k$ has components $\sup_{k \in K} b_{i,j}^k \in \bar{\mathbb{Z}}$)². As a consequence, when $X = \mathbb{R}^d$, the integral $\mathcal{L}^\natural$ convex hull of a nonempty, finite subset $Q \subset X$ is the convex polytope $\text{co}_{\mathcal{L}^\natural} Q = P_X(b(Q))$ where $b(Q)$ has components $b(Q)_{i,j} = \max_{q \in Q} \delta_{i,j}(q)$; and when $X = \mathbb{Z}^d$ it is the set of integer points in this convex polytope.

²Note, however, that when $X = \mathbb{R}^d$ the restriction to *integral* right-hand sides b is essential here, for with noninteger b^k 's, $P_X(\bigvee_{k \in K} b^k)$ might be defined by *strict* inequalities $\delta_{i,j}(x) < \sup_{k \in K} b_{i,j}^k$ when the supremum is not attained.

We now relate the Carathéodory numbers $c(\mathbb{R}^d, \mathcal{L}^\natural)$ and $c(\mathbb{Z}^d, \mathcal{L}^\natural)$ with the optimum value of an extremal problem in the theory of permutations. Given a finite set V , let S_V denote the *symmetric group* of V , i.e., the set of all permutations of V (so $|S_V| = |V|!$), and $A_V = \{(i, j) : i, j \in V, i \neq j\}$ the set of all (ordered) pairs from V (so $|A_V| = |V|(|V| - 1)$). A permutation $\sigma = (\sigma(1), \dots, \sigma(|V|))$ covers the pair $(i, j) \in A_V$ if $(i, j) = (\sigma(k), \sigma(l))$ with $k < l$, i.e., if i appears before j in σ . A set $T \subseteq S_V$ of permutations is a *cover* of A_V , or simply a *pair cover*, if every pair in A_V is covered by at least one permutation in T .³ A cover Q of A_V is a *minimal* cover of A_V if no proper subset of Q is a cover of A_V .

Pair covers arise as follows from the study of integral $\mathcal{L}^\natural$ convex hulls. First, we extend any vector $x \in \mathbb{R}^d$ to $x^0 \in \mathbb{R}^V$ by adding a zero component $x_0^0 = 0$. Given any subset $Q \subseteq X$ and $\bar{x} \in X$ (and thus $\bar{x}^0 \in X^0 = \{y^0 : y \in X\}$), let for every $q \in Q$, $S(q)$ denote the set of all permutations $\sigma \in S_V$ that sort the components of $\bar{x}^0 - q^0$ in nondecreasing order (i.e., $\bar{x}_{\sigma(0)}^0 - q_{\sigma(0)}^0 \leq \bar{x}_{\sigma(1)}^0 - q_{\sigma(1)}^0 \leq \dots \leq \bar{x}_{\sigma(d)}^0 - q_{\sigma(d)}^0$; note that one of these differences, where $\sigma(i) = 0$, is zero). Then observe that $\bar{x} \in \text{co}_{\mathcal{L}^\natural} Q$ iff for every $(i, j) \in A_V$ there exists $q \in Q$ such that $\delta_{i,j}(\bar{x}) \leq \delta_{i,j}(q)$, i.e., iff $\bar{x}_{\sigma(i)}^0 - q_{\sigma(i)}^0 \leq \bar{x}_{\sigma(j)}^0 - q_{\sigma(j)}^0$. Therefore, $\bar{x} \in \text{co}_{\mathcal{L}^\natural} Q$ iff for every $(i, j) \in A_V$ there is some $q \in Q$ and some permutation $\sigma \in S(q)$ that covers the pair (i, j) . That is, we have shown:

Lemma 11. *Given a nonempty, finite subset $Q \subset X \in \{\mathbb{R}^d, \mathbb{Z}^d\}$ (with $d \geq 2$), a point $\bar{x} \in X$ is in the integral $\mathcal{L}^\natural$ convex hull of Q iff the set $\bigcup_{q \in Q} S(q)$ defined above is a pair cover.*

By eliminating superfluous permutations, we may restrict attention to minimal pair covers. For any integer $n \geq 2$ let $\gamma(n)$ denote the largest cardinality of a minimal pair cover⁴ of A_V for any n -element set V . (Indeed, this number only depends on the cardinality of V and not on the identity its elements). In a companion paper [3] co-authored with Eric Balandraud we show

Theorem 6. *For every integer $n \geq 2$, $\gamma(n) = \max \left\{ n, \lfloor \frac{n^2}{4} \rfloor \right\}$.*

We now prove:

Theorem 7. *For every $d \geq 1$, the Carathéodory numbers of the integral $\mathcal{L}^\natural$ convexities satisfy $c(\mathbb{R}^d, \mathcal{L}^\natural) = c(\mathbb{Z}^d, \mathcal{L}^\natural) = \gamma(d + 1)$.*

Proof. The case $d = 1$ is straightforward. Indeed, for $X \in \{\mathbb{R}, \mathbb{Z}\}$ and $d = 1$, $X^d = X$ is a chain and the integral $\mathcal{L}^\natural$ convex hull of any nonempty, finite $Q \subset X$ is the interval $\text{co}_{\mathcal{L}^\natural} Q = \{x \in X : \bigwedge_{q \in Q} q \leq x \leq \bigvee_{q \in Q} q\}$, which is generated by at most $2 = \gamma(d + 1)$ endpoints.

³In analogy with “inversion-complete sets” in the theory of permutations, pair covers are called in [3] *pair-complete sets* of permutations. We use here the term “pair cover” for its connection with set cover problems in discrete optimization.

⁴The qualifier “minimal” is essential here, for the whole symmetric group S_V is itself a pair cover, and thus trivially of maximum possible cardinality given V .

For $d \geq 2$ we prove the following chain of inequalities:

$$\gamma(d+1) \leq c(\mathbb{Z}^d, \mathcal{L}^\natural) \leq c(\mathbb{R}^d, \mathcal{L}^\natural) \leq \gamma(d+1). \quad (11)$$

To prove the first inequality, let $P \subseteq S_V$ be a maximum-cardinality minimal cover of A_V , so $|P| = \gamma(d+1)$. For every $\sigma \in P$ let m_σ be the position such that $\sigma(m_\sigma) = 0$, and define $q^\sigma \in \mathbb{Z}^d$ with components $q_{\sigma(l)}^\sigma = l - m_\sigma$ for all $l \neq m_\sigma$. Let $Q := \{q^\sigma : \sigma \in P\}$ and consider the point $\bar{x} = \mathbf{0} \in \mathbb{Z}^d$. Since P is a pair cover, for every pair $(j, i) \in A_V$ there exists a permutation $\sigma \in P$ and positions $0 \leq k < l \leq d$ such that $\sigma(k) = j$ and $\sigma(l) = i$. If $i \neq 0 \neq j$ then

$$\delta_{i,j}(\mathbf{0}) = 0 < l - k = q_{\sigma(l)}^\sigma - q_{\sigma(k)}^\sigma = \delta_{i,j}(q^\sigma) \leq b(Q)_{i,j}.$$

Similarly, $\delta_{0,j}(\mathbf{0}) < \delta_{0,j}(q^\sigma) \leq b(Q)_{0,j}$ if $i = 0$, and $\delta_{i,0}(\mathbf{0}) < \delta_{i,0}(q^\sigma) \leq b(Q)_{i,0}$ if $j = 0$. This implies that $\mathbf{0} \in P_{\mathbb{Z}^d}(b(Q)) = \text{co}_{\mathcal{L}^\natural} Q$. On the other hand, since P is a *minimal* pair cover, for every $\sigma \in P$ there is a “critical” pair $(j, i) \in A_V$ which is covered by σ and by no other $\tau \in P$. Thus fix $\sigma \in P$ and the corresponding critical pair (j, i) . For every $\tau \in P \setminus \{\sigma\}$, $\tau(k) = j$ and $\tau(l) = i$ imply $k > l$. If $i \neq 0 \neq j$, then

$$\delta_{i,j}(q^\sigma) > \delta_{i,j}(\mathbf{0}) = 0 > l - k = q_{\tau(l)}^\tau - q_{\tau(k)}^\tau = \delta_{i,j}(q^\tau).$$

Similarly, $\delta_{0,j}(q^\sigma) > \delta_{0,j}(\mathbf{0}) > \delta_{0,j}(q^\tau)$ if $i = 0$, and $\delta_{i,0}(q^\sigma) > \delta_{i,0}(\mathbf{0}) > \delta_{i,0}(q^\tau)$ if $j = 0$. It follows that

$$\delta_{i,j}(q^\sigma) > \delta_{i,j}(\mathbf{0}) > \max_{q \in Q \setminus \{q^\sigma\}} \delta_{i,j}(q) = b(Q \setminus \{q^\sigma\})_{i,j}$$

i.e., q^σ and $\mathbf{0}$ each violates the inequality $\delta_{i,j}(x) \leq b(Q \setminus \{q^\sigma\})_{i,j}$, which is valid for $P_{\mathbb{Z}^d}(b(Q \setminus \{q^\sigma\})) = \text{co}_{\mathcal{L}^\natural}(Q \setminus \{q^\sigma\})$. This implies that $q^\sigma \notin \text{co}_{\mathcal{L}^\natural}(Q \setminus \{q^\sigma\})$, and therefore all q^σ in Q are distinct and $|Q| = |P| = \gamma(d+1)$. It also implies that for every $q \in Q$, $\mathbf{0} \notin \text{co}_{\mathcal{L}^\natural}(Q \setminus \{q\})$. Therefore Q is C-independent and $\gamma(d+1) = |Q| \leq c(\mathbb{Z}^d, \mathcal{L}^\natural)$, as claimed.

The middle inequality in (11) follows from proposition 6 since $(\mathbb{Z}^d, \mathcal{L}^\natural)$ is identical to the relative convexity $(\mathbb{R}^d, \mathcal{L}^\natural | \mathbb{Z}^d)$.

We now prove the last inequality in (11) by induction on $d \geq 2$. The base case $d = 1$ was established at the beginning of this proof. Thus assume that $c(\mathbb{R}^{d-1}, \mathcal{L}^\natural) \leq \gamma(d)$ and consider any C-independent subset Q in $(\mathbb{R}^d, \mathcal{L}^\natural)$. We need to show that $|Q| \leq \gamma(d+1)$. First consider the case where $\text{co}_{\mathcal{L}^\natural} Q$ is not full-dimensional. Then all $x \in \text{co}_{\mathcal{L}^\natural} Q$ satisfy one the defining linear inequalities $\delta_{i,j}(x) \leq b(Q)_{i,j}$ as an equality, that is, $\text{co}_{\mathcal{L}^\natural} Q$ is contained in the affine subspace $L_{i,j} = \{x \in \mathbb{R}^d : \delta_{i,j}(x) = b(Q)_{i,j}\}$. Note that $L_{i,j}$ is also a sublattice of $(\mathbb{R}^d, \leq)$ and it is isomorphic to $\mathbb{R}^{d-1}$ by projection onto the coordinate subspace $\mathbb{R}^{D'}$ where $D' = D \setminus \{h\}$ and $h = \max\{i, j\}$. (This projection is indeed a lattice homomorphism, and it is a bijection since every $y \in \mathbb{R}^{D'}$ uniquely determines $x \in L_{i,j}$ such that $y = \pi_{D'} x$ by letting $x_k = y_k$ for all $k \neq h$ and then using the equation $\delta_{i,j}(x) = b(Q)_{i,j}$ to define x_h .) Thus, by lattice isomorphism, $\pi_{D'} Q$ is

C-independent in $(\mathbb{R}^{D'}, \mathcal{L}^{\natural})$ and, by the inductive assumption, $|Q| = |\pi_{D'} Q| \leq c(\mathbb{R}^{d-1}, \mathcal{L}^{\natural}) \leq \gamma(d) < \gamma(d+1)$.

Otherwise, $\text{co}_{\mathcal{L}^{\natural}} Q$ is full-dimensional. Since $\text{co}_{\mathcal{L}^{\natural}} Q$ and all $\text{co}_{\mathcal{L}^{\natural}}(Q \setminus \{q\})$ are convex polytopes and the set difference $\Gamma = \text{co}_{\mathcal{L}^{\natural}} Q \setminus \bigcup_{q \in Q} \text{co}_{\mathcal{L}^{\natural}}(Q \setminus \{q\})$ is nonempty, it contains a (full-dimensional) open set. Furthermore, since the union $\bigcup_{(i,j) \in A_V, q \in Q} \Delta_{i,j}(q)$ of the $d(d+1)|Q|$ affine subspaces $\Delta_{i,j}(q) = \{x \in \mathbb{R}^d : \delta_{i,j}(x) = \delta_{i,j}(q)\}$ is not full-dimensional, there exists a point $\bar{x} \in \Gamma$ with $\delta_{i,j}(\bar{x}) \neq \delta_{i,j}(q)$ for all $(i,j) \in A_V$ and all $q \in Q$. Thus for each $q \in Q$ there is a unique permutation in $S(q)$. By Lemma 11, $\bigcup_{q \in Q} S(q)$ is a pair cover of A_V . Since $\bar{x} \notin \text{co}_{\mathcal{L}^{\natural}}(Q \setminus \{q\})$, $\bar{x}$ violates one of the inequalities $\delta_{i,j}(x) \leq b(Q \setminus \{q\})_{i,j}$ defining that polytope, and thus $\max_{r \in Q \setminus \{q\}} \delta_{i,j}(r) < \delta_{i,j}(\bar{x}) \leq \delta_{i,j}(q)$. Therefore $\bigcup_{r \in Q \setminus \{q\}} S(r)$ does not cover the pair $(i,j) \in A_V$. This implies that $\bigcup_{q \in Q} S(q)$ is a minimal pair cover of A_V , and thus $|Q| = |\bigcup_{q \in Q} S(q)| \leq \gamma(d+1)$. Hence we have shown that every C-independent subsets $Q \subset \mathbb{R}^d$, whether of full dimension or not, must have cardinality $|Q| \leq \gamma(d+1)$. Therefore $c(\mathbb{R}^d, \mathcal{L}^{\natural}) \leq \gamma(d+1)$, completing our inductive proof of the last inequality in (11). $\square$

5 Conclusion and Open Questions

In this paper we have introduced the study of convexity invariants for several convexity structures induced by various classes of sublattices in the Euclidian, integer and Boolean spaces $\mathbb{R}^d$, $\mathbb{Z}^d$ and $\mathbb{B}^d$. We have determined the exact values of these invariants for most of these convexity structures. A look at Table 1 indicates that the remaining open questions concern the convex sublattice convexities $(\mathbb{R}^d, \mathcal{L} \cap \mathcal{C})$ and $(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C}|\mathbb{Z}^d)$. For these convexities we have close bounds (with asymptotically vanishing relative difference) for the Carathéodory numbers, but no exact values for $d \geq 3$.

On the other hand, we have large gaps between linear lower bounds and exponential upper bounds for the Helly and Radon numbers for the latter, integer convexity. We also gave exact values for very small dimensions $d = 2$ and, in one case, for $d = 3$. Interesting open problems are to narrow these gaps, and if possible to determine the exact values of these invariants. Advances on the Radon number for the integer convex sublattice convexity $(\mathbb{Z}^d, \mathcal{L} \cap \mathcal{C}|\mathbb{Z}^d)$ are likely to be related with advances for the Radon number for the integer (standard) convexity $(\mathbb{Z}^d, \mathcal{C}|\mathbb{Z}^d)$, which has been an outstanding open question for about 40 years.

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