On the combination of naive and mean-variance portfolio strategies

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Motivation

Efficient mean-variance portfolios

- Markowitz (1952) defines efficient mean-variance portfolios:

$$\boldsymbol{w}^\star = \underset{\boldsymbol{w}}{\text{argmax}} \quad \underbrace{\boldsymbol{w}^\top \boldsymbol{\mu}}_{\text{portfolio mean}} - \frac{\gamma}{2} \overbrace{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}^{\text{portfolio risk}} = \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$

  - $\boldsymbol{w}$: vector of weights on risky assets ($1 - \mathbf{1}'\boldsymbol{w}$ is invested in the risk-free asset)
  - $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$: vector of mean returns and covariance matrix of returns
  - γ: risk-aversion coefficient

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- Challenge: Investors must estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and rely on

$$\widehat{\boldsymbol{w}}^\star = \frac{1}{\gamma} \widehat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\mu}}$$

Estimation risk and poor performance

- Problem: $\widehat{\boldsymbol{w}}^{\star}$ performs poorly out-of-sample. The naive equally weighted portfolio ($\boldsymbol{w}_{ew} = \mathbf{1}/N$) often outperforms it.
- See DeMiguel et al. (2009, RFS).
- The performance problem notably stems from estimation risk

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- Plug-in: relying on $\widehat{\boldsymbol{w}}^{\star}$. It is theoretically optimal, but subject to heavy estimation risk
  - Approaches exist to limit estimation errors on parameters (e.g. LW shrinkage, norm-constrained portfolios, etc.)
- Naive: relying on $\boldsymbol{w}_{ew}$. It is insensitive to estimation risk but it is also suboptimal!
- Let's try to find the middle ground ...

Combination of naive and mean-variance portfolio strategies

- Tu and Zhou (2011, JFE) introduce a portfolio combination:

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}^{\star} + \kappa_2 \boldsymbol{w}_{ew} \quad \text{with} \quad \boldsymbol{c} := \kappa_1 + \kappa_2 = 1.$$

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$$\boldsymbol{\kappa}^{\boldsymbol{c}} = \underset{\boldsymbol{\kappa}}{\text{argmax}} \quad \mathbb{E}\underbrace{\left[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu} - \frac{\gamma}{2}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})\right]}_{\text{out-of-sample utility}} \quad \text{s.t.} \quad \boldsymbol{c}$$

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$$= \underset{\boldsymbol{\kappa}}{\text{argmax}} \quad \boldsymbol{EU} \quad \text{s.t.} \quad \boldsymbol{c}$$

- This combined rule significantly outperforms $\widehat{\boldsymbol{w}}^{\star}$ and may outperform $\boldsymbol{w}_{ew}$.

This seems all well and good, but ...

- Tu and Zhou (2011) force combination coefficients to sum to one:

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- Problem: $\hat{\boldsymbol{w}}^\star$ is not a fully invested portfolio: $\mathbf{1}'\hat{\boldsymbol{w}}^\star \neq 1$.
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- We actually invest in three portfolios: the sample tangent, the risk-free asset, and the equally-weighted portfolio...
- ... But under $\boldsymbol{c}$, we only have one coefficient to control the weight allocated to each one of them !

This paper

- We propose to relax the constraint on combination coefficients:

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}^{\star} + \kappa_2 \boldsymbol{w}_{ew} \quad \text{~~with } \boldsymbol{c}$$

- We find the unconstrained combination coefficients that maximize the expected out-of-sample utility of the portfolio combination:

$$\boldsymbol{\kappa}^{u} = \underset{\boldsymbol{\kappa}}{\text{argmax}} \quad \boldsymbol{EU} \quad \text{~~s.t. } \boldsymbol{c}$$

Contributions

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Theoretical results:

1. We derive the unconstrained combination coefficients
2. We highlight analytically the key benefits of relaxing the constraint
3. We introduce a shrinkage strategy alleviating estimation risk

Empirical results:

1. The unconstrained strategy always delivers a positive net utility
2. The constrained strategy may deliver negative net utility
3. The shrinkage strategy accomplishes its mission

Unconstrained vs. constrained combination

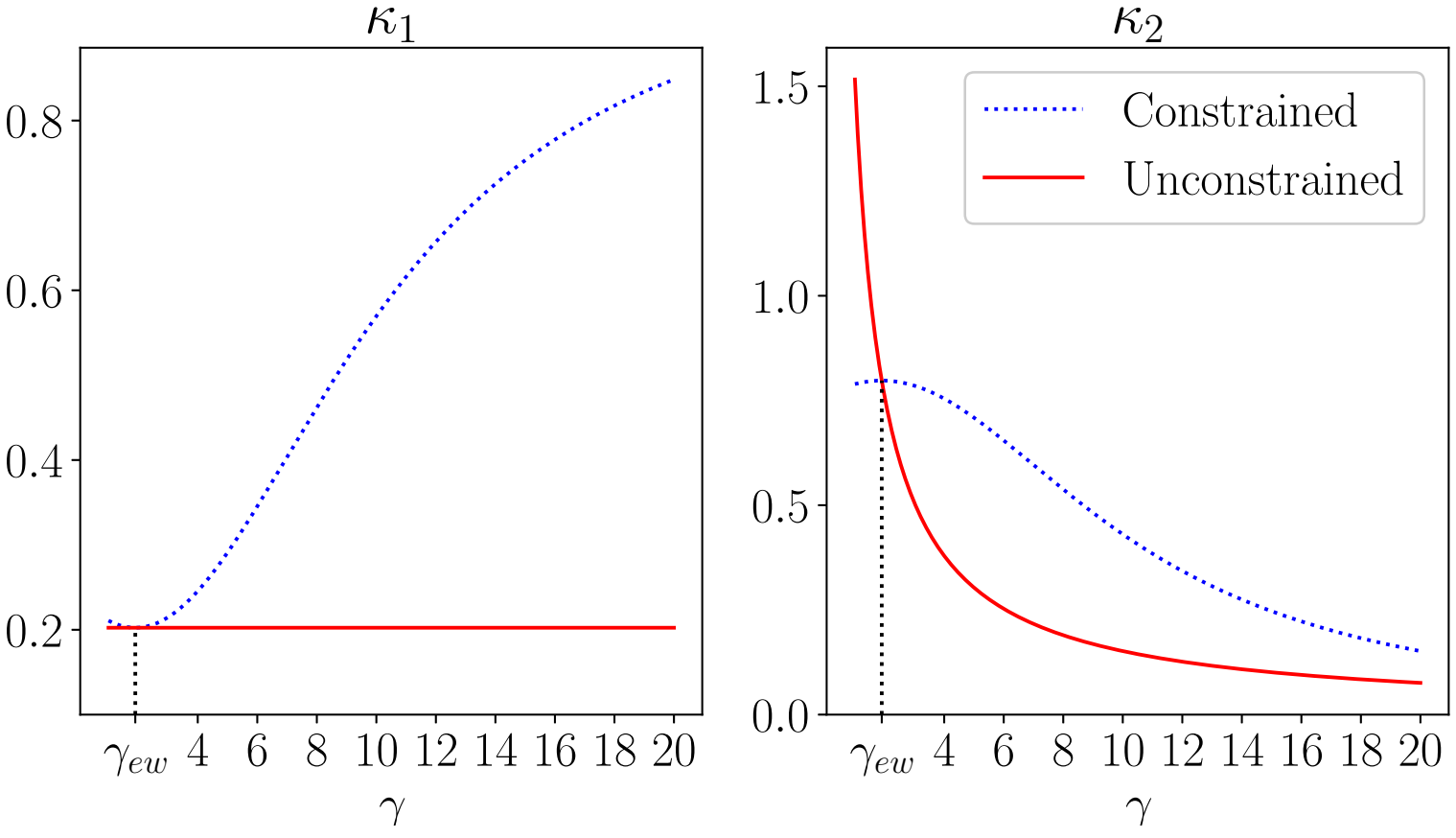
Benefits of relaxing the constraint

Key benefits of relaxing the constraint;

1. Optimal exposition to the three funds
2. Outperformance relative to the risk-free asset
3. Less extreme portfolio weights
4. Outperformance in expected out-of-sample utility (EU)
5. Outperformance relative to the two-fund rule
6. Outperformance in expected out-of-sample Sharpe Ratio

Combination coefficients

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}^{\star} + \kappa_2 \boldsymbol{w}_{ew}$$



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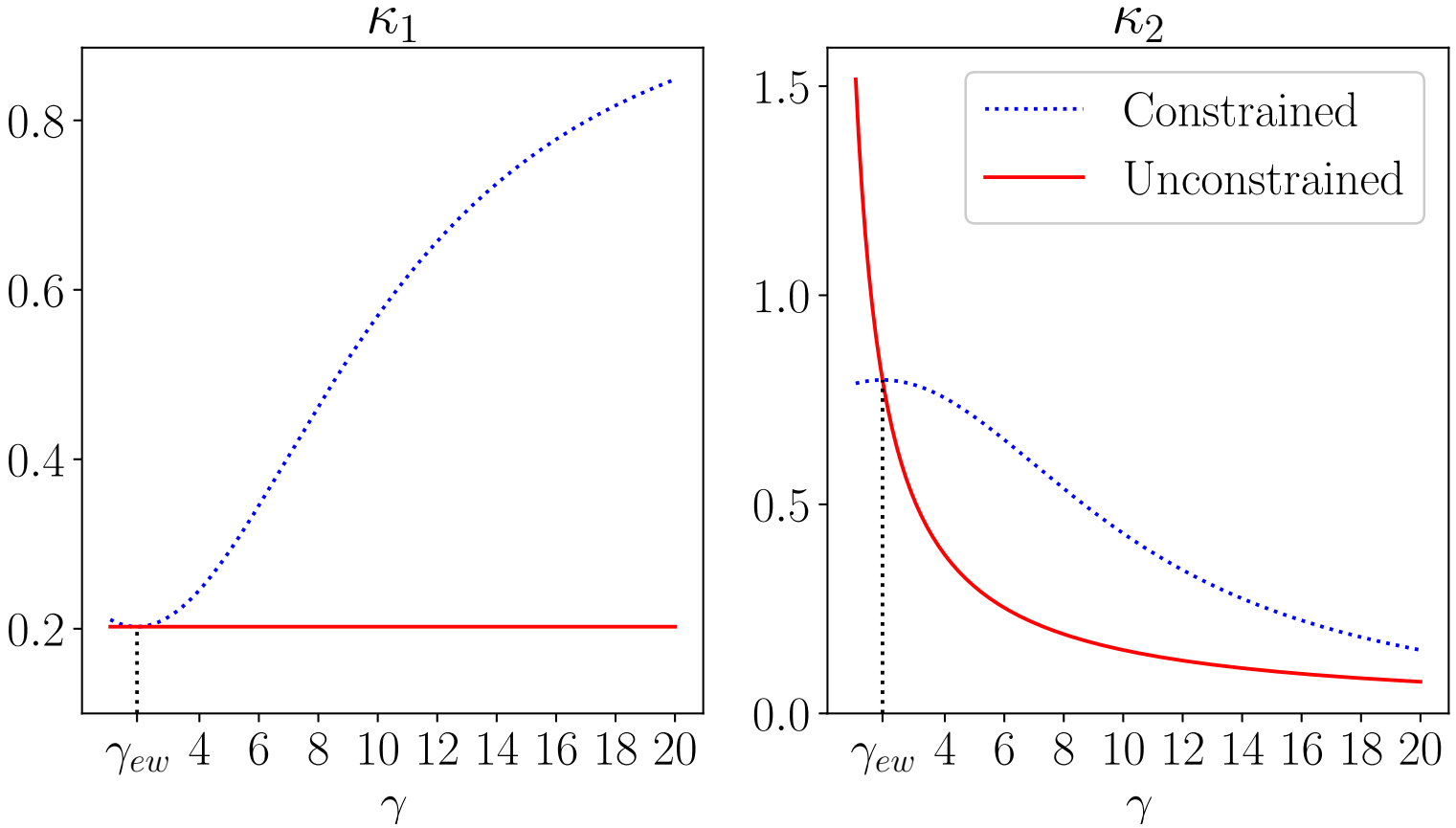
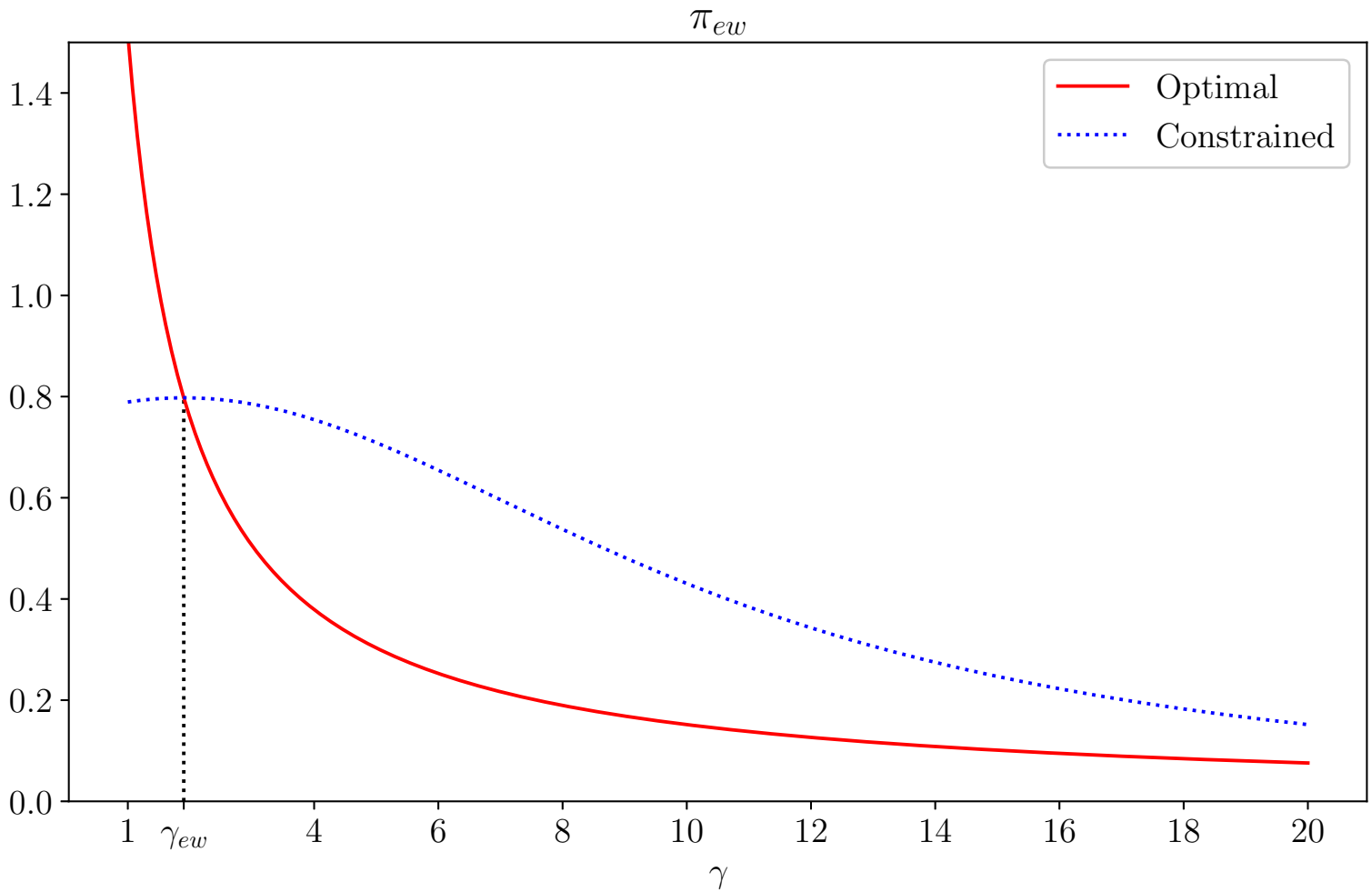


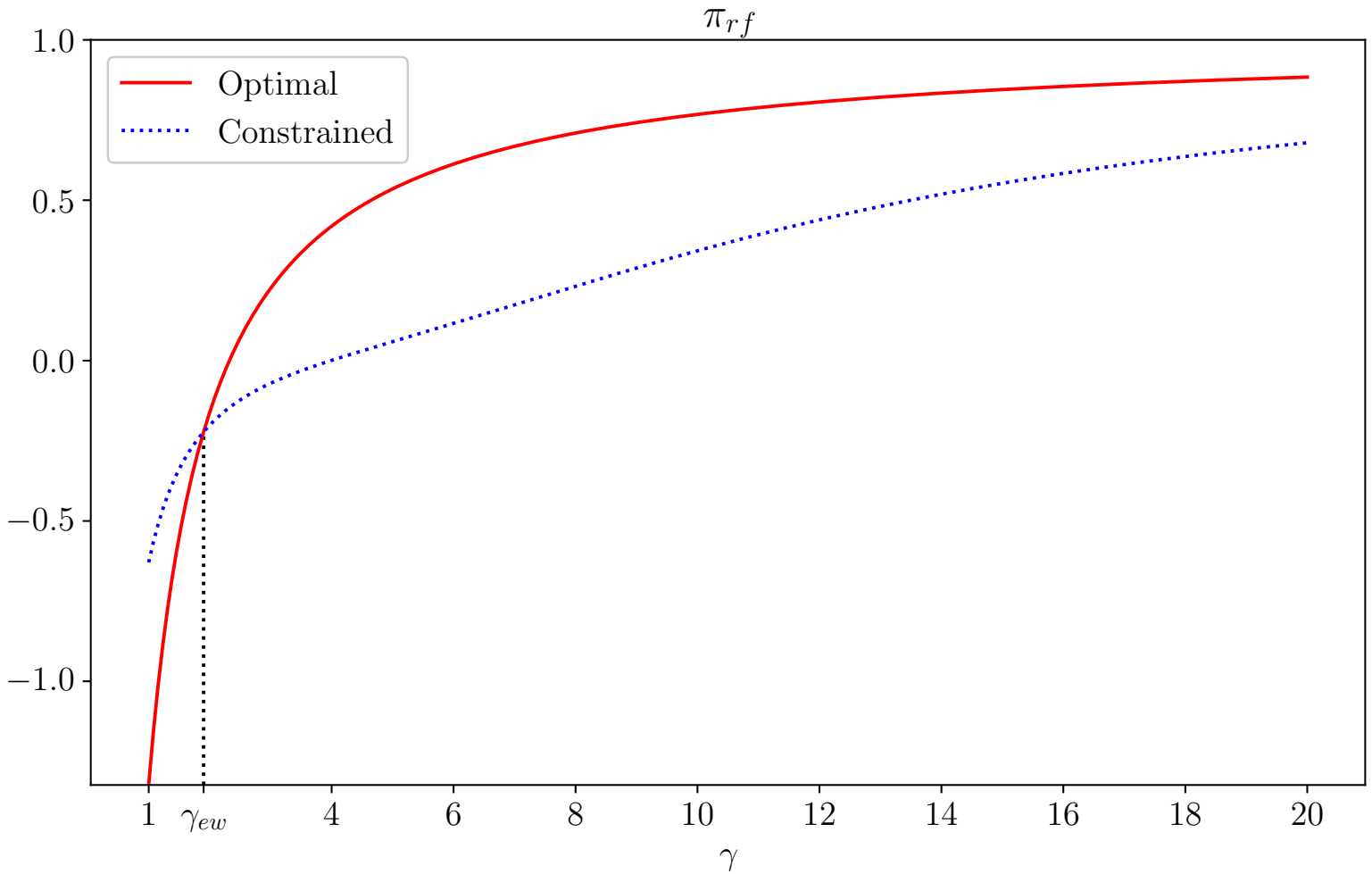
- Key facts: $\kappa_1^c \geq \kappa_1^u$ and equality iff $\gamma = \gamma_{ew} = \mu_{ew}/\sigma_{ew}^2$
- We use the 25 portfolios of stocks sorted on size and book-to-market (25SBTM) dataset spanning 07/26 to 12/21.

Benefit 1: optimal exposure to the equally-weighted portfolio



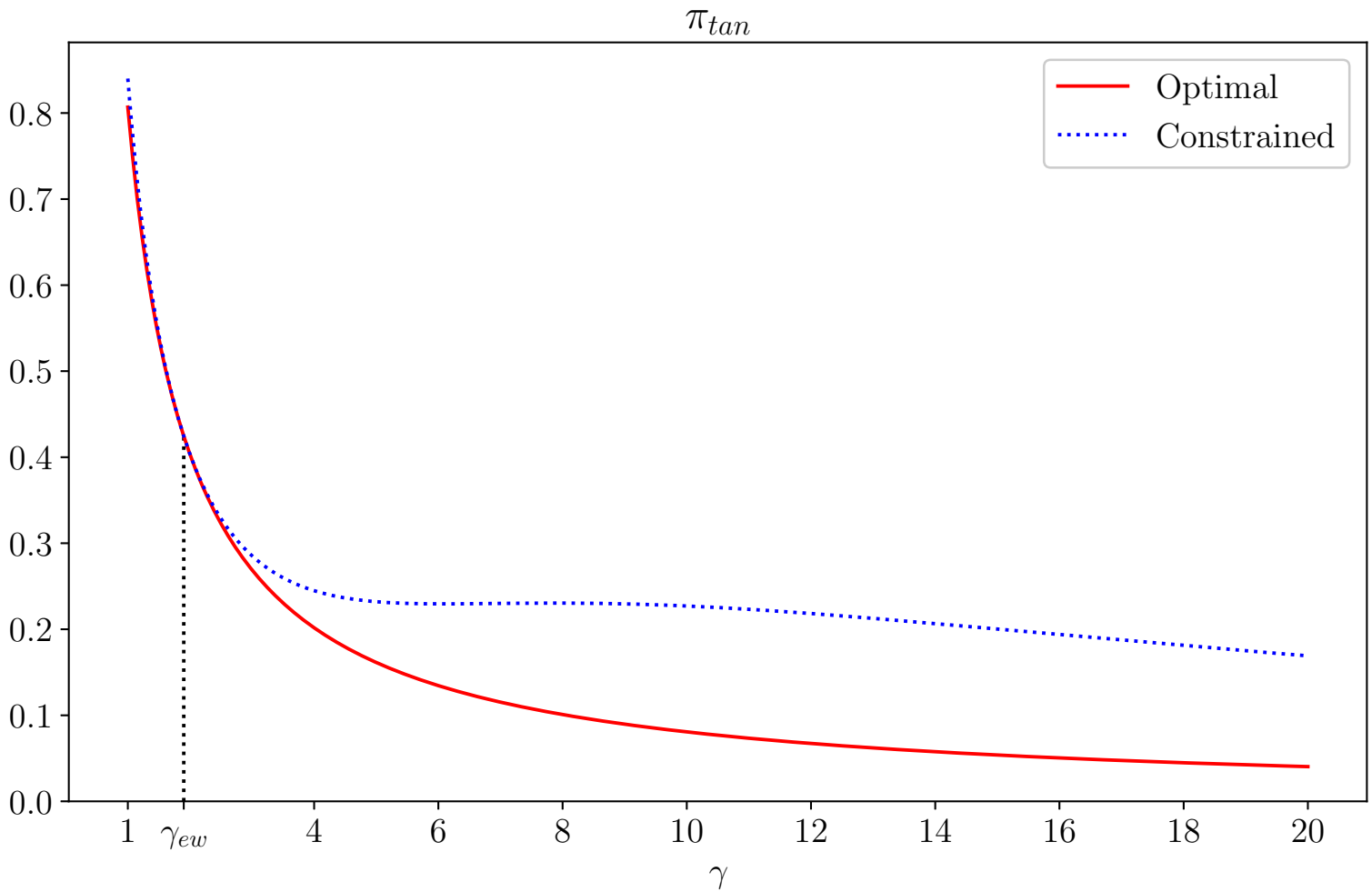
When $\gamma = 10$, $T = 120$, and $N = 25$, $\pi^c_{ew} = 43\%$ and $\pi^u_{ew} = 15\%$

Benefit 1: optimal exposure to the risk-free asset



When $\gamma = 10$, $T = 120$, and $N = 25$, $\pi^c_{rf} = 34\%$ and $\pi^u_{rf} = 77\%$

Benefit 1: optimal exposure to the tangent portfolio



When $\gamma = 10$, $T = 120$, and $N = 25$, $\pi^c_{tan} = 23\%$ and $\pi^u_{tan} = 8\%$

Benefit 2: Outperformance relative to the risk-free asset

Problem: With estimation risk, the constrained combination can underperform the risk-free asset for investors with a risk-aversion higher than a given threshold: γ_{neg}[1]

[1] $\gamma_{neg} = 5.41$ with the 25SBTM dataset.

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Problem: With estimation risk, the constrained combination can underperform the risk-free asset for investors with a risk-aversion higher than a given threshold: γ_{neg}[1]

Intuition: The combination should theoretically outperform each of its components, but it might not be possible because of the constraint.

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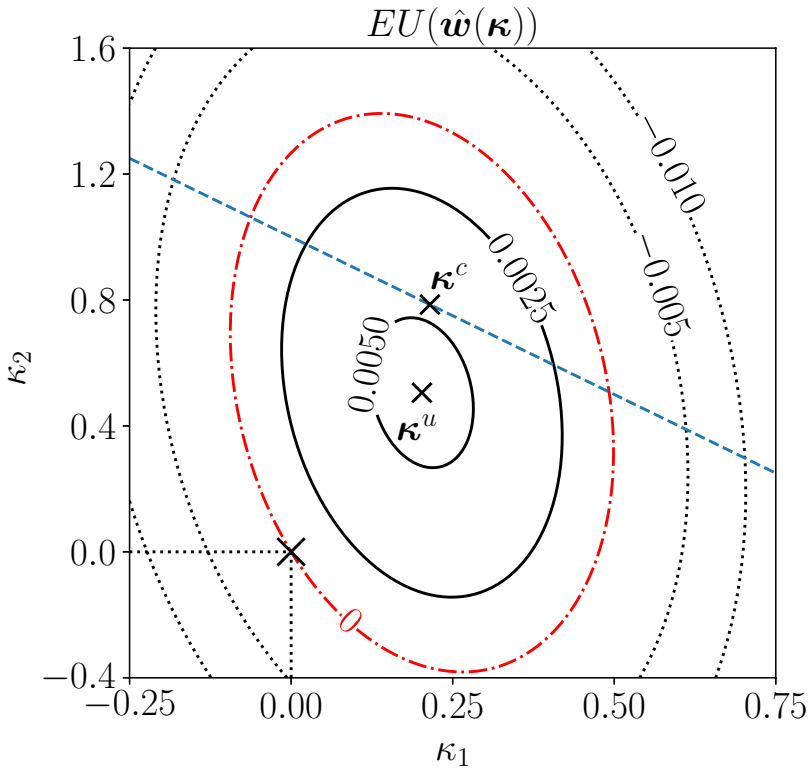
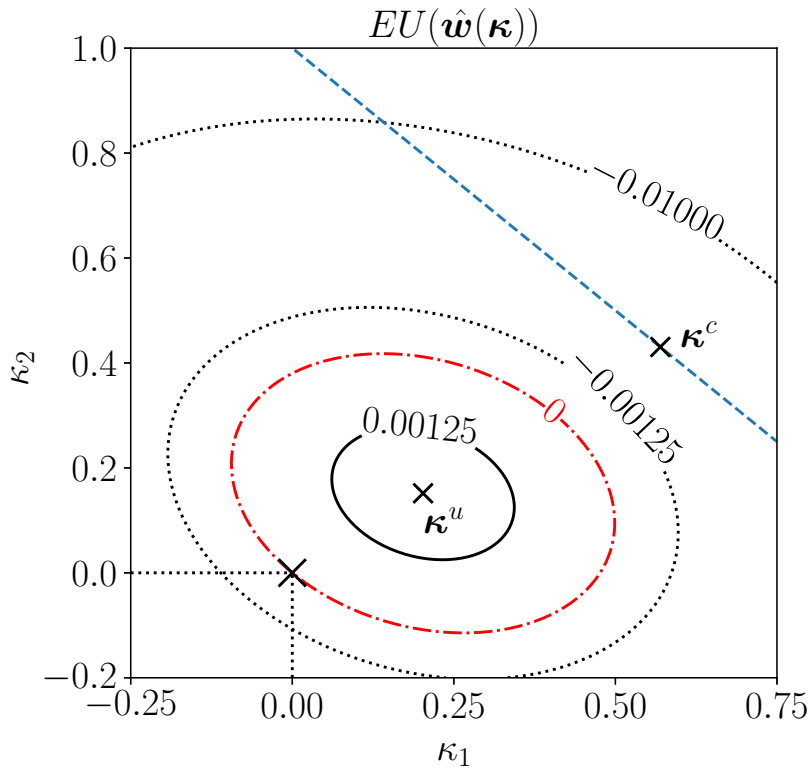
Benefit 2: Outperformance relative to the risk-free asset



(a) $\gamma = 3 < \gamma_{neg}$



(b) $\gamma = 10 > \gamma_{neg}$

Shrinkage strategy & empirical results

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- Practical concern: $\hat{\kappa}_2^u$ is unbounded and very sensitive to estimation errors in $\hat{\boldsymbol{\mu}}$

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Shrinkage strategy

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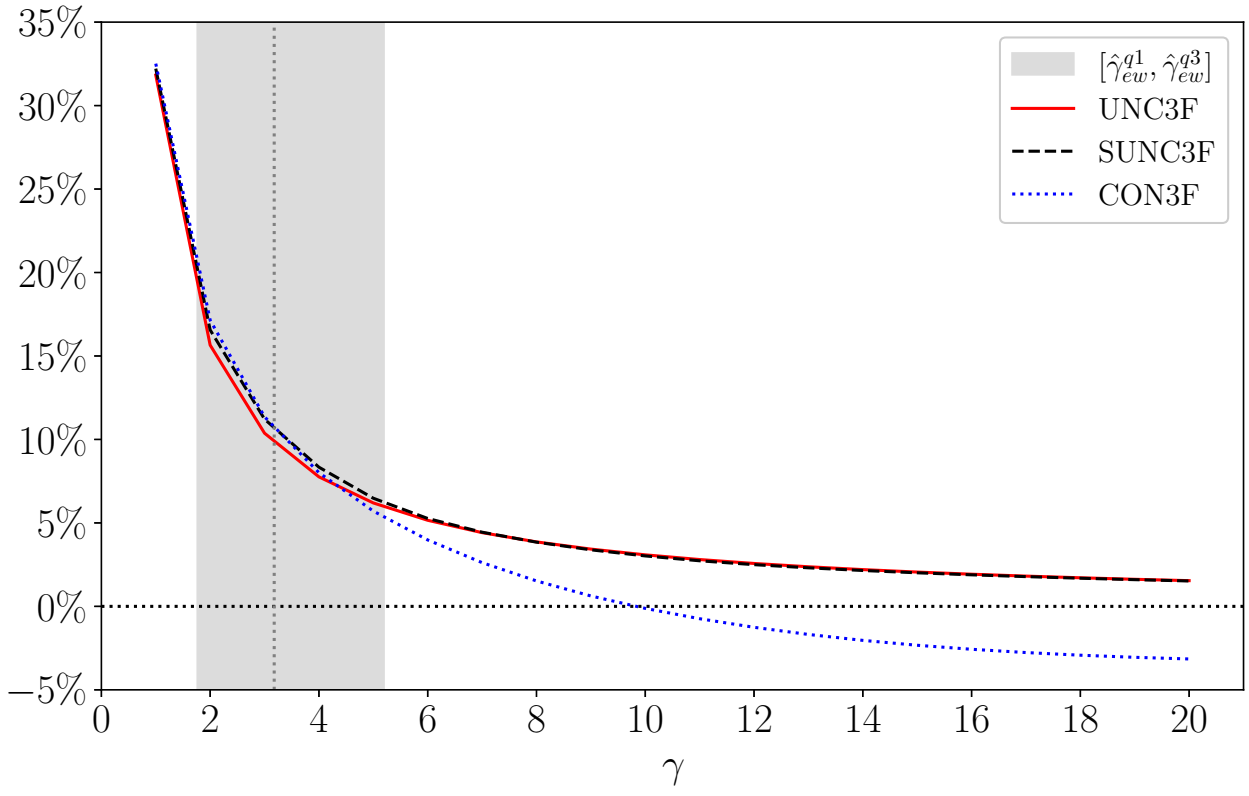
- Estimation errors have a substantial impact on out-of-sample performance. They hurt most $\boldsymbol{\kappa}^u$ relative to $\boldsymbol{\kappa}^c$ when $\gamma = \gamma_{ew}$
- $\hat{\kappa}_1^u$ is bounded in $[0, 1]$ and not very sensitive to estimation errors
- We define a shrinkage strategy in which we shrink $\hat{\kappa}_2^u$ toward the target $1 - \kappa_1^u$:

$$\hat{\kappa}_2^u(\delta) = (1 - \delta)\hat{\kappa}_2^u + \delta(1 - \kappa_1^u), \tag{1}$$

where $\delta \in [0, 1]$ is the shrinkage intensity and is calibrated to minimize the MSE of $\hat{\kappa}_2^u(\delta)$

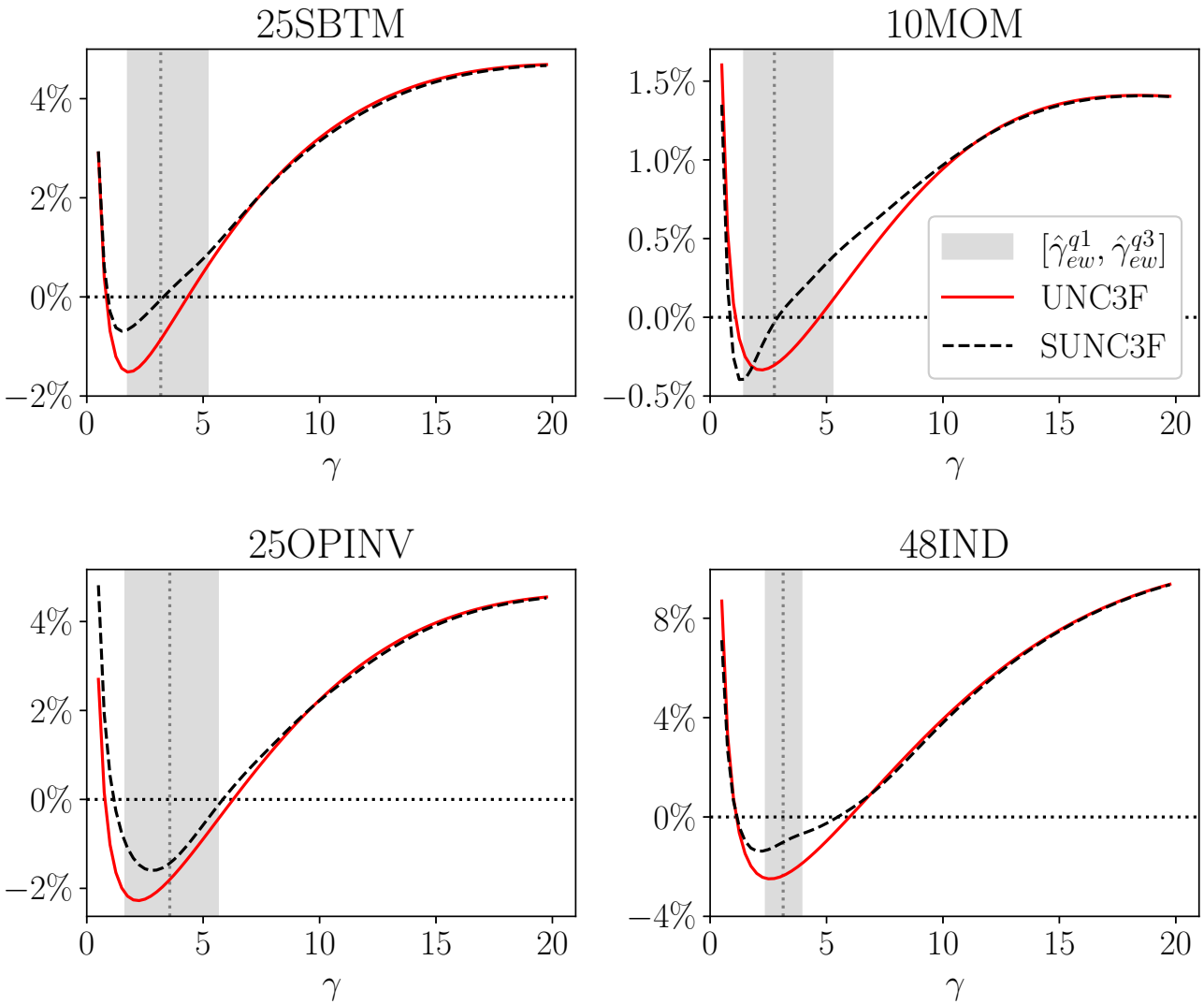
Empirical results

Net out-of-sample utility delivered by portfolio combinations (25SBTM)



Empirical results

Difference in net out-of-sample utility relative to the constrained strategy



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1. We derive the unconstrained combination coefficients
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3. We introduce a shrinkage strategy to alleviate estimation risk
4. The unconstrained strategy always delivers a positive net utility
5. The constrained strategy may deliver negative net utility
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Thank you!

Please send comments to

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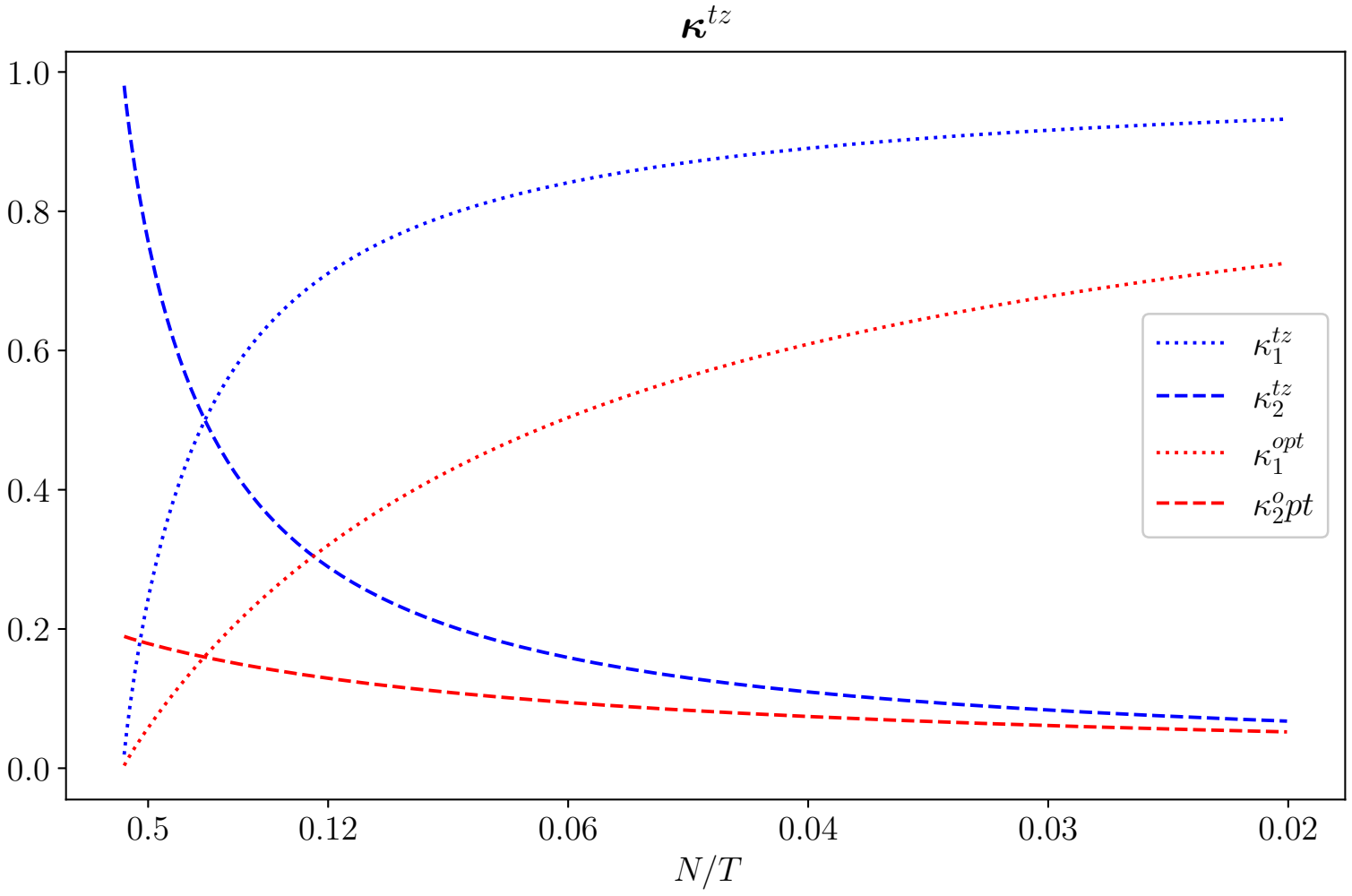
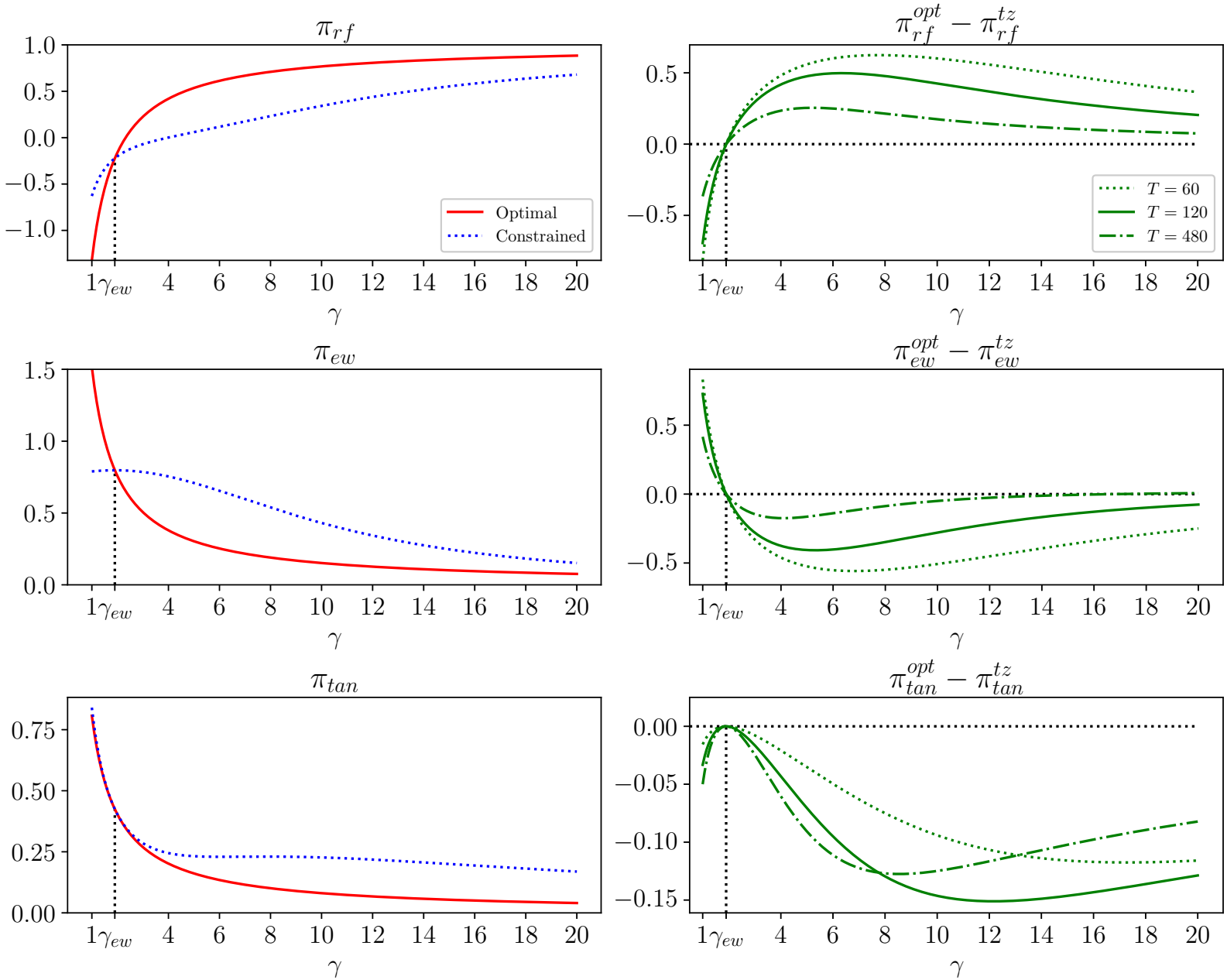
Impact of estimation risk



optimal exposure to the three funds + estimation risk



Combination coefficients

The constrained combination coefficients are

$$\kappa_1^c = \frac{\psi^2 + \sigma_{ew}^2(\gamma - \gamma_{ew})^2}{\psi^2 + \sigma_{ew}^2(\gamma - \gamma_{ew})^2 + d} \in [0,1] \quad \text{and} \quad \kappa_2^c = 1 - \kappa_1^c \in [0,1].$$

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$$\kappa_1^u = \frac{\psi^2}{\psi^2 + d} \in [0,1] \quad \text{and} \quad \kappa_2^u = \frac{\gamma_{ew}}{\gamma}(1 - \kappa_1^u) \in \mathbb{R},$$

d is a parameter that increases with estimation risk N/T

γ_{neg} threshold definition

$$\gamma > \gamma_{ew}\left(1+\sqrt{1+\frac{\theta^4/\theta_{ew}^2}{d-\theta^2}}\right) = \gamma_{neg}\,.$$

Notation

- Sample mean-variance portfolio (SMV) (from sample of size T)

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- We define $\theta^2 = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ as the maximum squared Sharpe ratio, $\theta_{ew}^2 = \mu_{ew}^2/\sigma_{ew}^2$ as the squared Sharpe ratio of the EW portfolio, and ψ^2 as the difference between the two:

$$\psi^2 = \theta^2 - \theta_{ew}^2 \geq 0.$$

Sample estimators

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{R_t}, \quad \widehat{\Sigma} = \frac{1}{T-N-2}\sum_{t=1}^{T}(\boldsymbol{R_t} - \hat{\mu})(\boldsymbol{R_t} - \hat{\mu})'.$$

Efficient frontier - Short reminder