Supplementary material to the paper

Fracture mechanisms in Ti and Co-Cr growing rods and impact on clinical practice

Arnaud Ribesse - Karim Ismail - Maïté Croonenborghs - Nadia Irda – Lotfi Miladi – Pascal J. Jacques –Maryline Mousny –Thomas Pardoen

Derivation of the beam model

Figure S.1 indicates the sign convention followed for the derivation of the beam model.

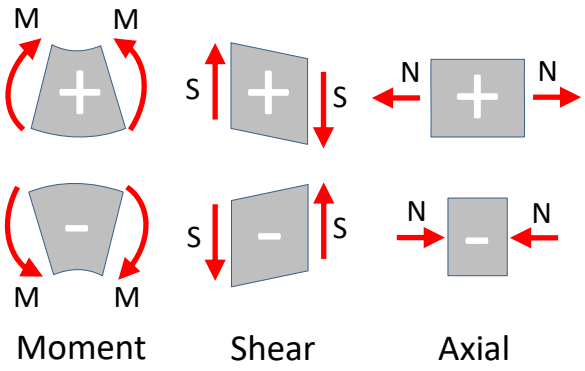


Figure S.1. Sign convention

Figure S.2 presents the boundary conditions and loading parameters of the beam bending model.

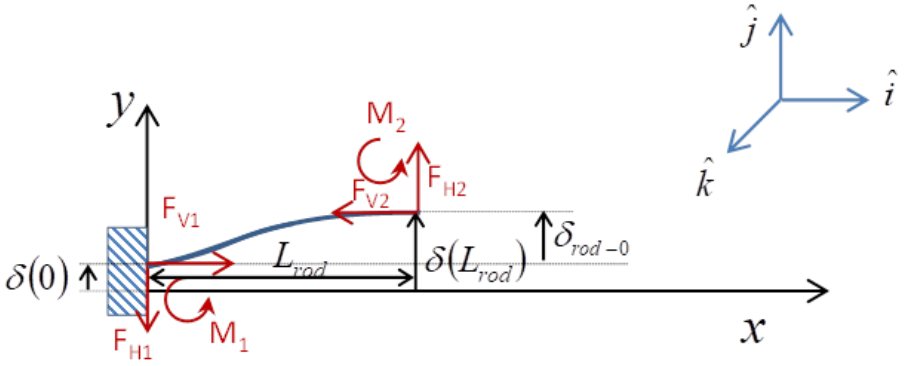


Figure S.2. Boundary and loading conditions

The different vectors entering the problem are defined as

$$\vec{F}_{v1} = F_{v1}\hat{i} \qquad \vec{N}(x) = N(x)\hat{i} \qquad \vec{\delta}_{rod-0} = \delta_{rod-0}\, j = \left[\delta(L_{rod}) - \delta(0)\right] j$$

$$\vec{F}_{h1} = -F_{h1}\hat{j} \qquad \vec{S}(x) = -S(x)\, j \qquad \vec{\delta}_{x-0} = \delta_{x-0}\, j = \left[\delta(x) - \delta(0)\right] j$$

$$\vec{F}_{v2} = -F_{v2}\hat{i} \qquad \vec{M}(x) = M(x)k$$

$$\vec{F}_{h2} = F_{h2}\,\hat{j} \qquad \vec{\delta}(x) = \delta(x)\, j$$

$$\vec{M}_1 = M_1\,\hat{k}$$

Force and moment equilibrium is expressed as

$$\Sigma F_x = 0 \Leftrightarrow F_{v1} = F_{v2} \ , \tag{1}$$

$$\Sigma F_y = 0 \Leftrightarrow F_{h1} = F_{h2} \ , \tag{2}$$

$$\Sigma M_{x=0} = 0$$

$$\Leftrightarrow \vec{M}_1 + \vec{M}_2 + \left(\vec{F}_{v2} \times \vec{\delta}_{rod-0}\right) + \vec{F}_{h2} \times L_{rod}\,\hat{i} = 0$$

$$\Leftrightarrow M_1 = -\left(M_2 + F_{v2}\,\delta_{rod-0} + F_{h2}\,L_{rod}\right) \ . \tag{3}$$

Figure S.3. shows the imaginary cut made in the beam to determine the forces and moment along its length. This leads to the following expression for the forces and moment

$$\Sigma F_{x=x} = 0 \Leftrightarrow \begin{cases} N(x) = -F_{v1} \\ S(x) = -F_{h1} \end{cases} \tag{4}$$

$$\Sigma M_{x=x} = 0$$

$$\Leftrightarrow \vec{M}(x) + \vec{M}_1 + \left(\vec{F}_{v1} \times \vec{\delta}_{x-0}\right) + \left(\vec{F}_{h1} \times x\hat{i}\right) = 0$$

$$\Leftrightarrow M(x) = -M_1 - F_{v1}\delta_{x-0} - F_{h1}x$$

$$\Leftrightarrow M(x) = M_2 + F_{v1}\delta_{rod-0} + F_{h1}L_{rod} - F_{v1}\delta_{x-0} - F_{h1}x$$

$$\Leftrightarrow M(x) = M_2 + F_{v1}(\delta_{rod-0} - \delta_{x-0}) + F_{h1}(L_{rod} - x)$$

$$\Leftrightarrow M(x) = M_2 + F_{v2}(\delta(L_{rod}) - \delta(x)) + F_{h2}(L_{rod} - x) \ . \tag{5}$$

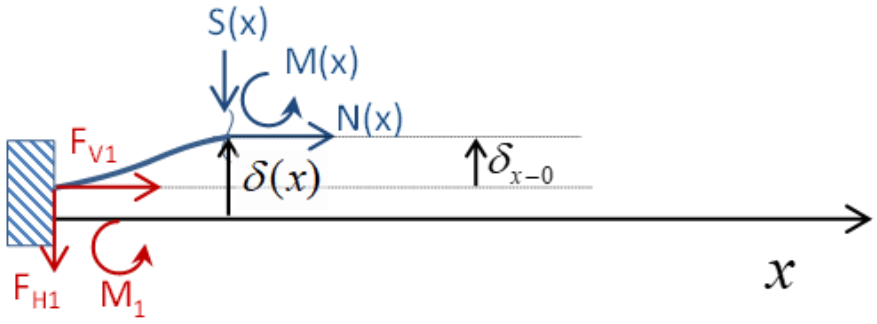


Figure S.3 Imaginary cut in the beam.

The stress is determined as

$$\sigma(x) = -\frac{y\,M(x)}{I_{rod}} + \frac{N(x)}{A_{rod}} \ . \tag{6}$$

Corresponding for the dorsal part, $y = -R$, to

$$\sigma_{surf}(x) = \frac{R_{rod}\,M(x)}{I_{rod}} + \frac{N(x)}{A_{rod}}$$

$$\sigma_{surf}(x) = \frac{R_{rod}}{I_{rod}}\left(M_2 + F_{v2}(\delta(L_{rod}) - \delta(x)) + F_{h2}(L_{rod} - x)\right) - \frac{F_{v2}}{A_{rod}}$$

$$\sigma_{surf}(x) = \frac{F_{v2}\,R_{rod}}{I_{rod}}\left(\frac{M_2}{F_{v2}} + \delta(L_{rod}) - \delta(x) + \frac{F_{h2}}{F_{v2}}(L_{rod} - x)\right) - \frac{F_{v2}}{A_{rod}} \tag{7}$$

The maximum tensile stress localization ($x_{\sigma\,max}$) is obtained by differentiating σ_{surf} with respect to *x* and setting the result to zero:

$$\frac{\partial \sigma_{surf}(x)}{\partial x} = 0$$

$$\Leftrightarrow \frac{F_{v2} R_{rod}}{I_{rod}} \left(-\frac{\partial \delta_x(x)}{\partial x} - \frac{F_{h2}}{F_{v2}} \right) = 0$$

$$\Leftrightarrow \frac{F_{h2}}{F_{v2}} = -\frac{\partial \delta_x(x)}{\partial x}$$

$$\Leftrightarrow \frac{F_{h2}}{F_{v2}} = \frac{\delta_{spine}}{2} \frac{2\pi}{L_{spine}} \sin\left(\frac{2\pi}{L_{spine}} \left(x_{\sigma\,max} + X_1 \right) \right) . \quad (8)$$

By assuming that fracture occurs at the position corresponding to the maximum tensile stress ($x_{fract} = x_{\sigma\,max}$), one gets

$$\frac{F_{h2}}{F_{v2}} = \frac{\delta_{spine}\,\pi}{L_{spine}} \sin\left(\frac{2\pi}{L_{spine}} \left(x_{fract} + X_1 \right) \right) ,$$

$$\sigma_{max} = \frac{F_{v2} R_{rod}}{I_{rod}} \left(\frac{M_2}{F_{v2}} + \delta\left(L_{rod}\right) - \delta\left(x_{fract}\right) + \frac{F_{h2}}{F_{v2}} \left(L_{rod} - x_{fract} \right) \right) - \frac{F_{v2}}{A_{rod}} . \quad (9)$$

Considering that fatigue cracking occurs as soon as the tensile stress in the dorsal part of the rod reaches the fatigue endurance limit of the material of the rod ($\sigma_{max} = \sigma_{end}$)

$$F_{v2} = \frac{\sigma_{end}}{\frac{R_{rod}}{I_{rod}} \left(\frac{M_2}{F_{v2}} + \delta\left(L_{rod}\right) - \delta\left(x_{fract}\right) + \frac{F_{h2}}{F_{v2}} \left(L_{rod} - x_{fract} \right) \right) - \frac{1}{A_{rod}}} . \quad (10)$$

The quantities σ_{max} and $x_{\sigma\,max}$ can be determined in the following way

$$\frac{F_{h2}}{F_{v2}}\frac{L_{spine}}{\delta_{spine}}\frac{1}{\pi}=\sin\left(\frac{2\pi}{L_{spine}}\left(x_{\sigma\,max}+X_1\right)\right)$$

$$\Leftrightarrow \frac{2\pi}{L_{spine}}\left(x_{\sigma\,max}+X_1\right)=(-1)^n\arcsin\left(\frac{F_{h2}}{F_{v2}}\frac{L_{spine}}{\delta_{spine}}\frac{1}{\pi}\right)+n\pi$$

$$\text{for}\left[-\frac{\pi}{2}+n\pi\le\frac{2\pi}{L_{spine}}\left(x_{\sigma\,max}+X_1\right)\le\frac{\pi}{2}+n\pi\right]$$

$$\Leftrightarrow x_{\sigma\,max}=\frac{L_{spine}}{2\pi}(-1)^n\arcsin\left(\frac{F_{h2}}{F_{v2}}\frac{L_{spine}}{\delta_{spine}}\frac{1}{\pi}\right)+\frac{L_{spine}}{2}n-X_1$$

$$\text{for}\left[L_{spine}\frac{2n-1}{4}-X_1\le x_{\sigma\,max}\le L_{spine}\frac{2n+1}{4}-X_1\right].$$

(11)

Among these solutions, only four of them are comprised between 0 and L_{spine}

$$\Leftrightarrow x_{\sigma_{\max}}=\begin{cases}x_{\sigma_{\max}0}=0 & \text{for}\quad\left[1<\frac{F_{h2}}{F_{v2}}\frac{L_{spine}}{\delta_{spine}}\frac{1}{\pi}\le-1\right]\\ x_{\sigma_{\max}1}=\frac{L_{spine}}{2\pi}\arcsin\left(\frac{F_{h2}}{F_{v2}}\frac{L_{spine}}{\delta_{spine}}\frac{1}{\pi}\right)-X_1 & \text{for}\quad\left[-\frac{L_{spine}}{4}-X_1<x\le\frac{L_{spine}}{4}-X_1\right]\\ x_{\sigma_{\max}2}=-\frac{L_{spine}}{2\pi}\arcsin\left(\frac{F_{h2}}{F_{v2}}\frac{L_{spine}}{\delta_{spine}}\frac{1}{\pi}\right)+\frac{L_{spine}}{2}-X_1 & \text{for}\quad\left[\frac{L_{spine}}{4}-X_1<x\le\frac{3L_{spine}}{4}-X_1\right]\\ x_{\sigma_{\max}3}=\frac{L_{spine}}{2\pi}\arcsin\left(\frac{F_{h2}}{F_{v2}}\frac{L_{spine}}{\delta_{spine}}\frac{1}{\pi}\right)+L_{spine}-X_1 & \text{for}\quad\left[\frac{3L_{spine}}{4}-X_1<x\le\frac{5L_{spine}}{4}-X_1\right]\end{cases}.(12)$$

In order to find the final value, a two-step procedure is followed. First, the solutions in terms of $x_{\sigma\,max}$ are estimated. The solutions lying outside the $0 \le x \le L_{rod}$ interval are not valid and are rejected. Finally, the remaining solutions are inserted into $\sigma_{surf}(x)$ expression to calculate the level of stress. The final solution for $x_{\sigma\,max}$ is the one leading to the largest stress on the dorsal surface of the rod:

$$\Leftrightarrow \forall \left(0 \le x_{\sigma_{\max} i} \le L_{rod}\right), \quad \sigma_{\max} = \max \begin{cases} \left|\sigma_{surf}\left(x_{\sigma_{\max} 0}\right)\right| \\ \left|\sigma_{surf}\left(x_{\sigma_{\max} 1}\right)\right| \\ \left|\sigma_{surf}\left(x_{\sigma_{\max} 2}\right)\right| \\ \left|\sigma_{surf}\left(x_{\sigma_{\max} 3}\right)\right| \end{cases} . \tag{13}$$