Weak Representability of Actions for categories of Leibniz algebras and Poisson algebras

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Categories and Companions Symposium

September 19–23, 2022

Joint work with Alan Cigoli (University of Turin) and Giuseppe Metere (University of Palermo)

- Let $\mathcal{C}$ be a semi-abelian category (*i.e.* $\mathcal{C}$ is pointed, Barr-exact and Bourn-protomodular with finite coproducts).
- Let $X, B \in \mathcal{C}$. A *split extension* of B by X is a diagram

$$0 \to X \xrightarrow{k} A \underset{\alpha}{\overset{\beta}{\leftrightarrows}} B \to 0$$

in $\mathcal{C}$ such that $\alpha\beta = 1_B$ and (X, k) is a kernel of α.

- For any object $X \in \mathcal{C}$, we consider the functor

$$\mathrm{SplExt}(-, \mathrm{X}) : \mathcal{C}^{op} \to \mathbf{Set}$$

where, for every $B \in \mathcal{C}$, $\mathrm{SplExt}(\mathrm{B}, \mathrm{X})$ is the set of isomorphism classes of split extensions of B by X.

- Recall that we have an an equivalence between split extensions and actions and a canonical functor isomorphism

$$\mathrm{SplExt}(-, \mathrm{X}) \cong \mathrm{Act}(-, X)$$

Action Representable Categories

Definition (F. Borceux, G. Janelidze and G.M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object $X \in \mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ is representable. This means that there exists an object $[X] \in \mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, \mathrm{X}) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X])$$

- Two examples are the category **Grp** of groups and the category $\mathbf{LieAlg}_{\mathbb{F}}$ of Lie algebras.
- In **Grp** the actor of an object H is the automorphism group $\mathrm{Aut}(H)$.
- In $\mathbf{LieAlg}_{\mathbb{F}}$ the actor of an object $\mathfrak{h}$ is the Lie algebra of derivations $\mathrm{Der}(\mathfrak{h})$.

Leibniz algebras

- Let $\mathbb{F}$ be a field with $char(\mathbb{F}) \neq 2$. A *right Leibniz algebra* over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ endowed of a bilinear map (called *commutator* or *bracket*) $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the *right Leibniz identity*

$$[[x,y],z] = [[x,z],y] + [x,[y,z]] \;\; \forall x,y,z \in \mathfrak{g}.$$

- We can define a *left Leibniz algebra*, using the left Leibniz identity

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \;\; \forall x,y,z \in \mathfrak{g}.$$

- A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*.
- Every Lie algebra is a symmetric Leibniz algebra and every Leibniz algebras with skew-symmetric commutator is a Lie algebra.

- Let $\mathfrak{g}$ be a Leibniz algebra over $\mathbb{F}$. A *derivation* of $\mathfrak{g}$ is a linear map $d\colon \mathfrak{g} \to \mathfrak{g}$ such that

$$d([x,y]) = [d(x),y] + [x,d(y)]\,,\ \forall x,y \in \mathfrak{g}.$$

- An *anti-derivation* of a right Leibniz algebra $\mathfrak{g}$ is a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ such that

$$D([x,y]) = [D(x),y] - [D(y),x],\ \ \forall x,y \in \mathfrak{g}.$$

- A *biderivation* of $\mathfrak{g}$ is a pair

$$(d,D)$$

where d is a derivation and D is an anti-derivation, such that

$$[x,d(y)] = [x,D(y)],\ \ \forall x,y \in \mathfrak{g}.$$

The Leibniz algebra of biderivations

- The set of all biderivations of $\mathfrak{g}$, denoted by $\mathrm{Bider}(\mathfrak{g})$, has a Leibniz algebra structure with the bracket

$$[(d,D),(d',D')] = (dd' - d'd, Dd' - d'D), \quad \forall (d,D),(d,D') \in \mathrm{Bider}(\mathfrak{g})$$

- The linear map $\mathfrak{g} \to \mathrm{Bider}(\mathfrak{g})$ defined by

$$x \to (-\operatorname{ad}_x, \operatorname{Ad}_x) = (-[-,x],[x,-]), \quad \forall x \in \mathfrak{g}$$

  is a Leibniz algebra morphism.

- The pair $(-\operatorname{ad}_x, \operatorname{Ad}_x)$ is called *inner biderivation* of $\mathfrak{g}$ and the set of all inner biderivations forms a Leibniz subalgebra of $\mathrm{Bider}(\mathfrak{g})$.

- Every split extension of a Leibniz algebra $\mathfrak{g}$ by another Leibniz algebra $\mathfrak{h}$ can be seen as a morphism $\mathfrak{g} \to \mathrm{Bider}(\mathfrak{h})$, but there are morphisms into $\mathrm{Bider}(\mathfrak{h})$ that do not give rise to split extensions.

Example (A.S. Cigoli, G. Metere and A. Montoli, 2013)

Let $\mathfrak{g} = \mathbb{F}$ be the abelian one-dimensional Leibniz algebra. Then the morphism $\varphi : \mathbb{F} \to \mathrm{Bider}(\mathbb{F})$ defined by

$$\varphi(a) = (d_a, D_a), \text{ where } d_a(x) = -ax, \; D_a(x) = ax, \; \forall a, x \in \mathbb{F}$$

does not define a split extension of $\mathbb{F}$ by itself. Thus $\mathbb{F}$ does not admit an actor.

- The category $\mathbf{LeibAlg}_{\mathbb{F}}$ of Leibniz algebras over $\mathbb{F}$ is not action representable.
- Other counterexamples are Associative algebras, Poisson algebras, Jordan algebras, ...

- The notion of action representable category has proven to be quite restrictive.

Theorem (X. Garcia-Martinez, M. Tsishyn, T. Van der Linden and C. Vienne, 2021)

Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ *and let $\mathcal{V}$ be a variety of non-associative algebras over $\mathbb{F}$. If $\mathcal{V}$ is action representable, then $\mathcal{V}$ must be the category* $\boldsymbol{\mathit{LieAlg}}_{\mathbb{F}}$ *of Lie algebras over $\mathbb{F}$.*

- More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly Action Representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every $X \in \mathcal{C}$, there exist an object $T \in \mathcal{C}$ and a monomorphism of functors

$$\tau \colon \mathrm{SplExt}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, T)$$

The object T is called *weak actor* of X and a morphism

$$\varphi \colon E \to T \in Im(\tau_E)$$

is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is an isomorphism.

Associative algebras (G. Janelidze, 2022)

- The category $\mathbf{Ass}_{\mathbb{F}}$ of associative algebras over $\mathbb{F}$ is weakly action representable.
- A weak actor of an object $B \in \mathbf{Ass}_{\mathbb{F}}$ is the associative algebra

  $$\mathrm{Bim}(B) = \{(f, F) \mid f(xy) = f(x)y, F(xy) = xF(y), xf(y) = F(x)y\}$$

  of *bimultipliers* of B

Proposition (A.S. Cigoli, M. M. and G. Metere, 2022)

The isomorphism classes of split extensions of an associative algebra A by B are in bijection with the class of morphisms

$$A \to \mathrm{Bim}(B),\ a \to (a * -, - * a),\ \ \forall a \in A$$

which satisfies the condition

$$a * (c * b) = (a * c) * b,\ \ \forall a, b \in A, \forall c \in B$$

i.e. the multiplier $a * -$ and the anti-multiplier $- * b$ are permutable.

Split extensions of Leibniz algebras

- Let
$$0 \to \mathfrak{h} \xrightarrow{i} \hat{\mathfrak{g}} \underset{\pi}{\overset{s}{\leftrightarrows}} \mathfrak{g} \to 0$$
be a split extension of a Leibniz algebra $\mathfrak{g}$ by another Leibniz algebra $\mathfrak{h}$.
- We have that
$$\hat{\mathfrak{g}} \cong (\mathfrak{g} \oplus \mathfrak{h}, [-,-]_{(l,r)})$$
and the Leibniz algebra structure is given by
$$[(x,a),(y,b)]_{(l,r)} = ([x,y]_{\mathfrak{g}}, [a,b]_{\mathfrak{h}} + l_x(b) + r_y(a))$$
- The left and the right components of the action
$$l \colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h} \quad r \colon \mathfrak{h} \times \mathfrak{g} \to \mathfrak{h}$$
of $\mathfrak{g}$ on $\mathfrak{h}$ are defined by
$$l_x(b) = [s(x), b]_{\hat{\mathfrak{g}}}, \ \ r_y(a) = [a, s(y)]_{\hat{\mathfrak{g}}}, \ \ \forall x, y \in \mathfrak{g}, \forall a, b \in \mathfrak{h}$$

Proposition

$(\mathfrak{g} \oplus \mathfrak{h}, [-,-]_{(l,r)})$ *is a right Leibniz algebra if and only if*

1. $r_x([a,b]) = [r_x(a), b] + [a, r_x(b)]$;
2. $l_x([a,b]) = [l_x(a), b] - [l_x(b), a]$;
3. $[a, r_x(b) + l_x(b)] = 0$;
4. $r_{[x,y]} = [r_y, r_x] = r_y \circ r_x - r_x \circ r_y$;
5. $l_{[x,y]} = [r_y, l_x] = r_y \circ l_x - l_x \circ r_y$;
6. $l_x \circ (l_y + r_y) = 0$;

for every $x, y \in \mathfrak{g}$ *and for every* $a, b \in \mathfrak{h}$.

- In this case, the resulting Leibniz algebra is the *semi-direct product* of $\mathfrak{g}$ and $\mathfrak{h}$.

- The first three equalities state that, for every $x \in \mathfrak{g}$, the pair

$$(-r_x, l_x) \in \mathrm{Bider}(\mathfrak{h})$$

- From equations *(4)-(5)*, we have that the linear map

$$\varphi\colon \mathfrak{g} \to \mathrm{Bider}(\mathfrak{h})$$

defined by

$$\varphi(x) = (-r_x, l_x), \;\; \forall x \in \mathfrak{g}$$

is a Leibniz algebra morphism.

Conversely, given a Leibniz algebra morphism

$$\varphi = (\varphi_1, \varphi_2) \colon \mathfrak{g} \to \mathrm{Bider}(\mathfrak{h})$$

satisfying the equation *(6)*

$$\varphi_2(x) \circ (\varphi_2(y) - \varphi_1(y)) = 0, \quad \forall x, y \in \mathfrak{g}$$

we can associate the split extension

$$0 \to \mathfrak{h} \xrightarrow{i} (\mathfrak{g} \oplus \mathfrak{h}, [-,-]_\varphi) \underset{\pi}{\overset{s}{\leftrightarrows}} \mathfrak{g} \to 0$$

where the Leibniz algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$ is given by

$$[(x,a),(y,b)]_\varphi = ([x,y]_\mathfrak{g}, [a,b]_\mathfrak{h} + \varphi_2(x)(b) - \varphi_1(y)(a))$$

Example (The (bi-)adjoint extension)

Let $\mathfrak{g}$ be a right Leibniz algebra and we consider the canonical action of $\mathfrak{g}$ on $\mathfrak{g}$ given by the pair of linear maps

$$r_x = \mathrm{ad}_x = [-, x], \ \ \forall x \in \mathfrak{g}$$

$$l_y = \mathrm{Ad}_y = [y, -], \ \ \forall y \in \mathfrak{g}$$

Thus we have a split extension of $\mathfrak{g}$ by itself with associated morphism

$$\mathfrak{g} \to \mathrm{Bider}(\mathfrak{g})$$

defined by

$$x \to (-\mathrm{ad}_x, \mathrm{Ad}_x), \ \ \forall x \in \mathfrak{g}$$

which obviously satisfies the condition *(6)*

$$\mathrm{Ad}_x \circ (\mathrm{Ad}_y + \mathrm{ad}_y) = 0, \ \ \forall x, y \in \mathfrak{g}$$

Theorem (A.S. Cigoli, M. M. and G. Metere, 2022)

- *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be right Leibniz algebras. The isomorphism classes of split extensions of* $\mathfrak{g}$ *by* $\mathfrak{h}$ *are in bijection with the Leibniz algebra morphisms*

$$\varphi\colon \mathfrak{g} \to \mathrm{Bider}(\mathfrak{h})$$

which satisfy the condition

$$\varphi_2(x) \circ (\varphi_2(y) - \varphi_1(y)) = 0, \ \ \forall x, y \in \mathfrak{g}$$

- *The category* $\boldsymbol{LeibAlg}_{\mathbb{F}}$ *of (right) Leibniz algebras over* $\mathbb{F}$ *is weakly action representable.*
- *A weak actor of an object* $\mathfrak{h} \in \boldsymbol{LeibAlg}_{\mathbb{F}}$ *is the Leibniz algebra* $\mathrm{Bider}(\mathfrak{h})$*.*
- $\varphi \in \mathrm{Hom}_{\boldsymbol{LeibAlg}_{\mathbb{F}}}(\mathfrak{g}, \mathrm{Bider}(\mathfrak{h}))$ *is an acting morphism if and only if it satisfies the condition*

$$\varphi_2(x) \circ (\varphi_2(y) - \varphi_1(y)) = 0, \ \ \forall x, y \in \mathfrak{g}.$$

Poisson algebras

Definition

A *Poisson algebra* over $\mathbb{F}$ is a vector space P over $\mathbb{F}$ endowed of two bilinear maps

$$\cdot\colon P \times P \to P$$

$$[-,-]\colon P \times P \to P$$

such that $(P, \cdot)$ is an associative algebra, $(P, [-,-])$ is a Lie algebra and the adjoint map $[p,-]\colon P \to P$ is a derivation of the associative algebra $(P, \cdot)$, for every $p \in P$.

Remark

Let S be a split extension of of $(P, \cdot, [-,-])$ by $(V, \cdot_V, [-,-]_V)$. Then

$$S \cong P \rtimes V = (P \times V, *, \{-,-\})$$

and the Poisson algebra structure is given by

$$(p, x) * (q, y) = (pq, x \cdot_V y + p * y + x * q)$$

$$\{(p, x), (q, y)\} = ([p, q], [x, y]_V + [\![p, y]\!] - [\![q, x]\!]),$$

where the action of P on V is defined by

$$p * y = s(p) * x, \;\; x * q = x * s(q),$$

$$[\![p, y]\!] = \{s(p), y\}, \;\; \forall p, q \in P, \forall x, y \in V$$

$P \rtimes V$ is a Poisson algebra if and only if

1. $(P \times V, *)$ is an associative algebra, *i.e.*
   - $p * (x \cdot_V y) = (p * x) \cdot_V y$;
   - $(x \cdot_V y) * p = x \cdot_V (y * p)$;
   - $x \cdot_V (p * y) = (x * p) \cdot y$;
   - $(p * x) * q = p * (x * q)$;
   - $(pq) * x = p * (q * x)$;
   - $x * (pq) = (x * p) * q$;
2. $(P \times V, \{-,-\})$ is a Lie algebra, *i.e.*
   - $[\![p, [x,y]_V]\!] = [[\![p,x]\!], y]_V + [x, [\![p,y]\!]]_V$;
   - $[\![[p,q], -]\!] = [\![p, [\![q,-]\!]]\!] - [\![q, [\![p,-]\!]]\!]$;
3. $[\![pq, x]\!] = p * [\![q,x]\!] + [\![p,x]\!] * q$;
4. $[p,q] * x = p * [\![q,x]\!] - [\![q, p * x]\!]$;
5. $x * [p,q] = [\![q,x]\!] * p - [\![q, x * p]\!]$;
6. $p * [x,y]_V = [p * x, y]_V - [\![p,y]\!] \cdot_V x$;
7. $[x,y]_V * p = [x * p, y]_V - x \cdot_V [\![p,y]\!]$;
8. $[\![p, x \cdot_V y]\!] = [\![p,x]\!] \cdot_V y + x \cdot_V [\![p,y]\!], \quad \forall p,q \in P, \forall x,y \in V$

We want to endow of a Poisson Leibniz algebra structure a reasonable subspace

$$[V] \leq Bim(V) \times Der(V)$$

such that we can associate in a natural way a Poisson algebra morphism

$$\phi\colon P \to [V],\ \phi(p) = (p * -, - * p, [\![p, -]\!]),\ \ \forall p \in P$$

with every split extension of P by V. Thus

$$\phi(pq) = \phi(p) \cdot_{[V]} \phi(q),\ \ \phi([p, q]) = [\phi(p), \phi(q)]_{[V]},\ \ \forall p, q \in P$$

and the operations in $[V]$ must satisfy

- $(p * -, - * p, [\![p, -]\!]) \cdot_{[V]} (q * -, - * q, [\![q, -]\!]) = ((pq) * -, - * (pq), p * [\![q, -]\!] + [\![p, -]\!] * q)$
- $[(p * -, - * p, [\![p, -]\!]), (q * -, - * q, [\![q, -]\!])]_{[V]} = (p * [\![q, -]\!] - [\![q, p * -]\!], [\![q, -]\!] * p - [\![q, - * p]\!], [\![p, [\![q, -]\!]]\!] - [\![q, [\![p, -]\!]]\!])$

Weak Actor for Poisson algebras

We take

$$[V] \leq \mathrm{Bim}(V) \times \mathrm{Der}(V)$$

as the subspace of all triples (f, F, D) satisfying the following equations

1. $f([x, y]_V) = [f(x), y]_V - D(y) \cdot_V x;$
2. $F([x, y]_V) = [F(x), y]_V - x \cdot_V D(y);$
3. $D(x \cdot_V y) = D(x) \cdot_V y + x \cdot_V D(y);$
4. $[f, F'] \circ D'' = 0;$
5. $f \circ D = D \circ f, \ F \circ D = D \circ F;$
6. $[f, [D', D'']] = [f', [D, D'']] + [f'', [D', D]],$
   $[F, [D', D'']] = [F', [D, D'']] + [F'', [D', D]];$
7. $[f, f'] \circ D'' = [f, D'] \circ (f'' - F''),$
   $[F, F'] \circ D'' = [F, D'](F'' - f'');$

$\forall (f', F', D'), (f'', F'', D'') \in [V], \forall x, y \in V.$

The Poisson algebra structure of $[V]$

We define the two bilinear maps

$$\cdot_{[V]} : [V] \times [V] \to [V]$$

$$[-,-]_{[V]} : [V] \times [V] \to [V]$$

by

$$(f, F, D) \cdot_{[V]} (f', F'', D'') = (f \circ f', F' \circ F, f \circ D' + F' \circ D)$$

and

$$[(f, F, D), (f', F', D')]_{[V]} = (f \circ D' - D' \circ f, F \circ D' - D' \circ F, D \circ D' - D' \circ D),$$

for every $(f, F, D), (f', F', D') \in [V]$.

Theorem (A.S. Cigoli, M. M. and G. Metere, 2022)

Let $(V, \cdot_V, [-,-]_V) \in \boldsymbol{Pois}_{\mathbb{F}}$*. Then we have the following.*

1. $([V], \cdot_{[V]}, [-,-]_{[V]})$ *is an object of* $\boldsymbol{Pois}_{\mathbb{F}}$*.*
2. *For every* $(P, \cdot, [-,-]) \in \boldsymbol{Pois}_{\mathbb{F}}$*, the isomorphism classes of split extensions of* P *by* V *are in bijection with the Poisson algebra morphisms*

$$\phi = (\phi_1, \phi_2, \phi_3) : P \to [V]$$

   such that $(\phi_1, \phi_2) : P \to \mathrm{Bim}(V)$ *is an acting morphism in the category* $\boldsymbol{Ass}_{\mathbb{F}}$ *of associative algebras over* $\mathbb{F}$*.*
3. *The category* $\boldsymbol{Pois}_{\mathbb{F}}$ *is weakly action representable. A weak actor of* V *is the Poisson algebra* $[V]$ *and the acting morphisms are precisely the ones described above.*

A counterexample: Jordan algebras

Definition

A *Jordan algebra* over a field $\mathbb{F}$ is a non-associative commutative algebra $(J, \cdot)$ over $\mathbb{F}$ which satisfies the *Jordan identity*

$$(xy)(xx) = x(y(xx)), \ \ \forall x, y \in J$$

Theorem (G. Janelidze, 2022)

Every weakly action representable category is action accessible (in the sense of D. Bourn and G. Janelidze).

Remark (A.S. Cigoli and S. Mantovani, 2012)

The category of Jordan algebras is not action accessible.

Conclusion: The category of Jordan algebras is not weakly action representable.

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Thank you for your attention!