

Extended Abstract of Time-changed affine models: fitting term-structures without shift

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The class of affine short-rate or intensity models are very popular in finance for tractability reasons. For instance, time-homogeneous models like Vasicek (Ornstein-Uhlenbeck), CIR (Square-Root Diffusions) and JCIR (Square-Root Diffusions with independent positive jumps) are clearly the most popular models to describe short-rate or default intensity dynamics. Consider for instance a credit-risk application where the survival probability curve $\mathbb{Q}(\tau \geq t)$ is given by a specific market-implied function $G(t) = \exp\{-\int_0^t h(s)ds\}$, extracted from defaultable bonds or CDS market quotes. Adopting CIR or JCIR dynamics for a stochastic intensity process λ yields a model-implied survival probability curve

$$\mathbb{Q}(\tau \geq t) = \mathbb{E} \left[e^{-\int_0^t \lambda_s ds} \right] ,$$

which takes the form $\exp\{A(0, t) - B(0, t)\lambda_0\}$ for some functions A, B known in closed-form. This function only depends on the few parameters involved in the corresponding stochastic differential equation with time-homogeneous coefficients (as well on the intensity's initial value λ_0). Generally speaking, it is thus not possible to make model and market-implied curves agree with time-homogeneous affine models; they just do not offer enough flexibility to allow for a perfect fit of the model curve to a specified market term-structure, thereby opening the door to arbitrage opportunities.

The standard way to circumvent this issue consists in choosing a shifted version of the above time-homogeneous intensity, i.e. choosing, as default intensity, the process λ^φ defined as :

$$\lambda_t^\varphi := \lambda_t + \varphi(t) .$$

The (deterministic) shift function φ will be chosen so as to make sure that the model and market-implied curves agree one with the other, i.e. such that :

$$\mathbb{E} \left[e^{-\int_0^t \lambda_s^\varphi ds} \right] = G(t) , \quad \forall t > 0 .$$

The function φ is available analytically in the case of affine models. This leads to the well-known Hull-White, CIR++ and JCIR++ models, respectively. Whereas this shift approach is perfectly fine in interest-rates applications, this is not always the case for the sake of modeling default intensities. The reason is that shifting a positive process with a market-implied shift may result in an intensity model taking on negative values, which is clearly inconsistent. More importantly, increasing the volatility of the underlying intensity process (keeping market- and model-implied survival probability curves equal) tend, generally speaking, to imply lower shift functions. This thus makes the problem quite significant and specially when it comes to modeling high-volatility but low-average credit spreads.

In this paper we propose an alternative tractable method to shifting time-homogeneous affine processes, based on time-change. By speeding up or slowing down the clock of an underlying time-homogeneous affine process with the help of a specific clock function Θ , we show how to build a new intensity process λ^θ defined as

$$\lambda_t^\theta := a(t)\lambda_{\Theta(t)}$$

in such a way that for any valid target survival probability curve G one has

$$\mathbb{E} \left[e^{-\int_0^t \lambda_s^\theta ds} \right] = G(t) , \quad \forall t > 0 .$$

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Because $a(t) > 0$ for all t , λ^θ is indeed an intensity (positive) process provided that so is λ . This statement does not hold for λ^φ . Moreover, the model remains fairly tractable as both the stochastic clock Θ and the scaling function a can be found by solving a simple ODE or by inverting functions.

To illustrate the power of the method, we compare the shift (Fig. 1.a.) and the time-changed (Fig. 1.b.) approaches. We use a constant hazard rate function $h(t) = h > 0$ to specify G and consider CIR dynamics for the standard affine process λ with initial value $\lambda_0 = h > 0$, speed of mean reversion κ , long-term mean μ and volatility σ . Generally speaking, there is no reason that $\mathbb{E} \left[e^{-\int_0^t \lambda_s ds} \right]$ (in red on the right-panels of Fig. 1.a and 1.b) agree with the specified G (in blue on the right panels). Choosing the φ (green curve on left panel of Fig. 1.a.) in a specific way and using λ^φ instead of λ as intensity always allows one to bring the model-implied survival probability curve (green curve on the right-panel of Fig. 1.a.) in line with the specified $G(t)$ function (blue curve on the right-panels) whatever the chosen parameters (κ, μ, σ) and the hazard rate value h . Similarly, we show that choosing a specific clock Θ (green curve on left panel of Fig. 1.b.) and using as intensity λ^θ instead of λ yields the same flexibility. However, the shift in the first approach takes on negative values. Therefore the associated default model is flawed. In particular, we cannot interpret λ^φ as a default intensity associated to a Cox process. This is in contrast with the time-changed approach which preserves the positivity of the initial CIR dynamics, so that the resulting default model is perfectly valid.

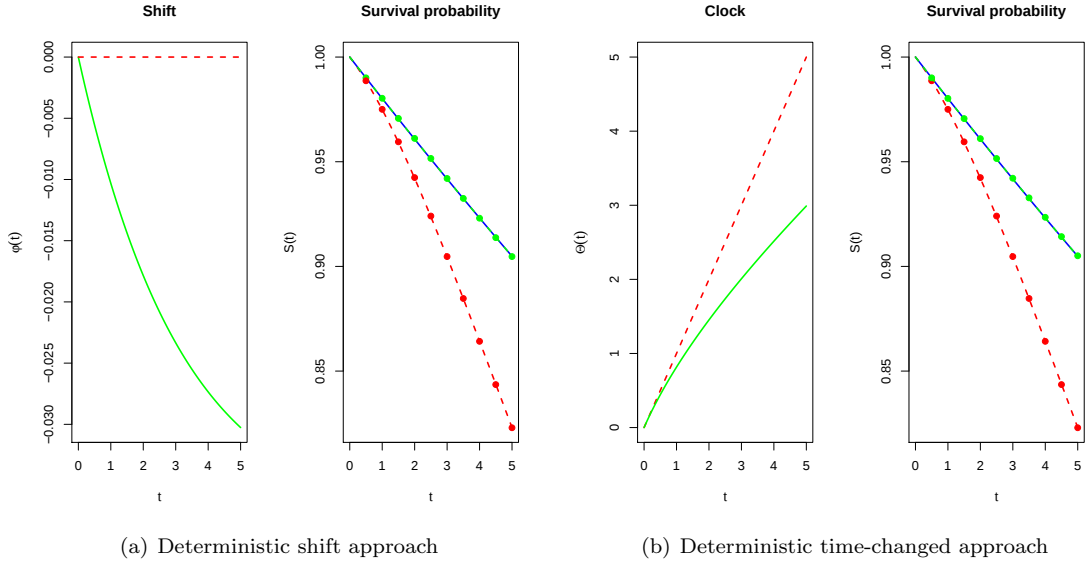


Figure 1: Example with $h(t) = h = 2\%$, $(\kappa, \mu, \sigma, \lambda_0) = (30\%, 3h, 8\%, h)$. The green curves in the left panels show the shift (Fig. a) and the clock (Fig. b) in the respective approaches. Right panels of Fig. a and b. show the specified curve G (blue) the model-implied survival probability curves associated to λ (in red) and either the shift or the time-changed curves (in green). The dots corresponds to the empirical averages from 2,000 Monte Carlo paths obtained by using Diop's scheme with time-step 0.001.

Conclusion

We introduce a new method that rules out the problem of “negative intensities” in shifted affine models. This does not affect the model’s tractability as both the time-change and scaling functions are easily computed numerically. Consequently, this method provides an appealing alternative that is expected to give room for larger intensity volatilities. This is of primary importance for various applications like pricing (e.g. CDS options), risk-management (where spread volatility matters, like Value-at-Risk on CDS) or Credit Valuation Adjustment (CVA) under wrong-way risk.