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ON THE (INTRADAILY) SEASONALITY AND DYNAMICS OF A FINANCIAL POINT PROCESS: A SEMIPARAMETRIC APPROACH

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Abstract

A new method of estimating a component model for the analysis of financial durations is proposed. The components are long-run dynamics and seasonality. The latter is left unspecified and the former is assumed to fall within the class of a certain family of parametric functions. The proposed estimation procedure is based on a generalized profile likelihood approach and requires the assumption either of a likelihood function for the model errors or, at least, that the error density belongs to the class of exponential densities. Its main interest is twofold: first, consistent and asymptotically normal estimators for both the parameters of the long-run stochastic component and the nonparametric curve that approximates the deterministic seasonal component are provided. Hence, it is possible to derive correct inference for both parametric and nonparametric components. Second, the method is computationally very appealing since the resulting nonparametric estimator of the seasonal curve has an explicit form that turns out to be a transformation of the Nadaraya-Watson estimator. The method is applied to price and volume durations of a stock traded at the NYSE, and compared to estimation with splines and with adjustment methods. It is shown that the proposed method outperforms the other methods.

Keywords: Tick-by-tick, ACD model, Seasonal analysis, Nonparametric method.

JEL classification: C14, C41, G10.

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1 INTRODUCTION

The issue of modelling financial duration processes has been a very active area of research since Engle and Russell (1998) introduced the Autoregressive Conditional Duration (ACD) model. Their analysis is justified from an economic and a statistical point of view. On the one hand, market microstructure theory shows that time between events in a stock exchange market conveys information and thus time arrivals are worth analyzing. On the other hand, financial durations are so-called tick-by-tick data: they are none other than a one dimensional point process (with time as space). Thus arrival times are the points of the point process and hence random.

Since the above model a plethora of modifications and alternatives have been proposed. Among others, Bauwens and Giot (2000) introduce the Log-ACD model, which is an exponential version of the ACD. Grammig and Mauer (2000) used a Burr distribution in the ACD model. Zhang et al. (2001) introduce a threshold ACD (TACD). Fernandes and Grammig (2001) introduce the augmented ACD model, a very general model that covers almost all the existing ones. Drost and Werker (2001) provide a method for obtaining efficient estimators of the ACD model with no need to specify the distribution. Alternative models are the Stochastic Conditional Duration (SCD) model of Bauwens and Veredas (1999) and the Stochastic Volatility Duration (SVD) model of Ghysels et al. (1998) which are both based on latent factor models.

In most of the above studies, durations show a strong intradaily seasonality. In an explanatory graphic analysis the strong seasonal component is detected by the presence of the U (or inverted U) shape that ultra high frequency financial variables exhibit during the day. For example, Figures 1 and 2 show a measure of the intradaily behavior of price and volume durations for the Disney stock traded at the NYSE between September and November 1996.

The study of this feature when dealing with regularly spaced variables is well known, that is, when dealing with variables that are observed at equidistant periods of time. This analysis has focused mainly on the intradaily behavior of volatility either on a stock exchange market or on a foreign exchange (FX) market. Baillie and Bollerslev (1990) study the intradaily and intermarket FX volatility using a qualitative approach with hourly data. Bollerslev and Domowitz (1993) do a similar analysis but for the returns and bid-ask spread of the deutsche mark-dollar exchange rate, using data recorded at 5 minute intervals. Andersen and Bollerslev (1997) use a frequency domain approach for filtering the five-minute deutsche mark-dollar exchange rate and getting rid of the seasonal pattern. Andersen and Bollerslev (1998) use a different approach and analyze the intradaily and intraweekly seasonality using spectral analysis taking macroeconomic announcements into account. Finally, Beltratti and Morana (1999) use the half hour deutsche mark-dollar exchange rate and model it following a structural approach "*à la Harvey*".

All these previous works have used regularly spaced data sampled at different frequencies: hourly, half-hourly, every 15 minutes or every 5 minutes. For tick-by-tick data Engle and Russell (1998) introduce a method for dealing with intra-daily seasonality. It consists of decomposing the expected duration into a deterministic part, which depends on the time of the day at which the duration commences, and a stochastic part. The deterministic part deals with the seasonal effect whereas the stochastic component models the long run dynamics. If both components are assumed to belong to parametric families of functions then the two sets of parameters can be jointly estimated by Maximum Likelihood (ML) techniques.

Under some standard conditions ML estimators of the parameters of interest are consistent and asymptotically normal (see Engle and Rusell, 1998; Corollary, p. 1135). However if deterministic seasonality is approximated through nonparametric functions (mainly Fourier series, spline functions or other types of smoothers), it is well known (see Neyman and Scott, 1951) that ML estimators of finite dimensional parameters (long run dynamics) do not need to be consistent in the presence of "incidental" parameters (the nonparametric seasonal component).

The main contribution of this paper is to provide consistent and asymptotically normal estimators for both the parameters of the long-run stochastic component and the nonparametric curve that approximates the deterministic seasonal component. Hence, it is possible to derive a correct inference for both parametric and nonparametric components. The proposed estimation procedure is based on a generalized profile likelihood approach (see Severini and Wong, 1992 and Severini and Staniswalis, 1994) and requires the assumption either of a likelihood function for the model errors or, at least, that the error density belongs to the class of exponential densities.

This procedure also allows us to estimate both the deterministic and the stochastic components by a two step method (see Engle and Russell, 1995 and 1997). In the first step, the inverted U shape is removed by estimating the relationship between duration and time of the day at which this duration commences nonparametrically, through a Naradaya-Watson kernel regression estimator. In a second step, the parameters of the long-run stochastic component are estimated by (quasi)-maximum likelihood techniques using the ratio of actual to fitted values as the seasonally adjusted durations.

The proposed method is applied to Disney stock traded at the NYSE for two types of durations: price and volume durations. They describe instantaneous volatility and liquidity respectively. The basic model under which we introduce the seasonal component is the Log-ACD model of Bauwens and Giot (2000) and our estimation method is compared to the procedures proposed in Engle and Rusell (1998).

The plan of the paper is a as follows. Section two is twofold. First it develops a general ML framework for analyzing tick-by-tick financial variables, breaking the process down into the two above mentioned terms and proposing a new estimator for seasonality. Its asymptotic properties are shown. The second part of this section develops the same method but in a Generalized Linear Model (GLM) framework and hence Quasi Maximum Likelihood (QML) theory applies. It is also shown that the asymptotic properties are equivalent to those of ML. Section three is devoted to empirical application, focusing on nonparametric estimates and forecasting performance. Indeed, forecasting capability is the criteria used to evaluate the goodness of fit of all the models compared. Section four concludes. The assumptions and proofs of the main results are relegated to the Appendix.

2 ECONOMETRIC MODEL AND ESTIMATORS

In order to introduce the main contribution of the paper, a basic econometric framework is needed. Following Engle and Rusell (1998) and Engle (2000), let t_i be the time at which the i-th event occurs and let $d_i = t_i - t_{i-1}$, where $\cdots < t_i < t_{i-1} < \ldots$ be the duration between events. Notice that t_i is a sequence of times measured as accumulated seconds as from the starting date of the sample; the time gaps between market closing and opening are ignored. Consider also that at the i-th event k marks, denoted y_i, are observed. For example, if d_i are trade durations, the marks could be the price and the volume of the trade. Thus the following set of observations is available

$$\{(d_i, y_i)\}_{i=1,\cdots,n}.$$

Furthermore, assume that the i-th duration has the conditional density (depending on the past filtration)

$$d_i|I_{i-1} \sim f\left(d_i|\bar{d}_{i-1}, \bar{y}_{i-1}; \delta\right),$$

where $(\bar{d}_{i-1}, \bar{y}_{i-1}) = \bar{z}_{i-1} = (z_{i-1}, \cdots, z_1)$ is the past information of the z stochastic process and δ is a set of parameters in some possibly infinite dimensional space. Within this statistical framework, the aim is to estimate this parameter vector δ (or any nonlinear combination of its components). One way to estimate it is by using ML techniques. To this end, let us define as

$$E[d_i|\bar{d}_{i-1}, \bar{y}_{i-1}] = \psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \tag{1}$$

the expectation of the i-th duration conditionally on the past filtration. The ACD class of models consist of parameterizations of (1) and the assumption that

$$d_i = \psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1)\mu_\varepsilon(\xi)^{-1}\varepsilon_i \tag{2}$$

where ε is an *i.i.d.* random variable with known density function $p(\varepsilon_i; \xi)$ depending on a set of parameters ξ. $\mu_\varepsilon(\xi)^{-1}$ is introduced for identification reasons.

The log-likelihood function, conditionally on the past filtration and y_i, can be written as

$$L_n\left(d; \vartheta_1, \xi\right) = \sum_{i=1}^{n} \log p\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1); \xi\right). \tag{3}$$

If the conditional density is correctly specified and standard regularity conditions apply, the maximum likelihood estimators of ϑ_1 and ξ are consistent and asymptotically normal (for conditions see Engle and Rusell, 1998).

As pointed out in many recent studies, the above specification, (2), is sometimes too simple since the expected duration can vary systematically over time and can be subject to many different time effects. One way to extend the above model is to decompose the conditional mean into different effects. In standard time series literature any stochastic process can be decomposed into a combination (we adopt a multiplicative decomposition with the additive being straightforward) of cycle and trend, seasonal pattern and noise, i.e. $X_t = X_t^{CT} \cdot S_t \cdot \varepsilon_t$. This decomposition, for which a long tradition exists in time series analysis, has already been used in volatility analysis (see for example Andersen and Bollerslev, 1998) and duration analysis (see Engle and Rusell, 1998).

Thus, instead of (1), the following nonlinear decomposition is proposed:

$$E[d_i|\bar{d}_{i-1}, \bar{y}_{i-1}] = \varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_2)\right). \tag{4}$$

The function $\varphi(u, v)$ allows us to nest a great variety of models. $\varphi(u, v) = (u \times v)$ stands for an ACD representation whereas $\varphi(u, v) = \exp(u + v)$ represents a Log-ACD representation. Notice that following (4) durations, volatility, trading intensity and volume (in a tick-by-tick framework) can be modelled as a possibly nonlinear function of two components that represents long-run, $\psi(\cdot; \vartheta_1)$ and short-run, $\phi(\cdot; \vartheta_2)$, behavior respectively.

The long-run component can be considered as the core dynamics and the dynamics of the process are modelled on it. The short-run component represents the seasonal pattern. This pattern can be at any frequency, i.e. it could be hourly, daily, weekly or even quarterly. The definition of the seasonal frequency will depend on the intrinsic data features. In the current specific setting seasonality will be assumed to be daily.

Now, in what follows the choice of the different components is discussed. The long-run component is specified by means of the log of the conditional expectation of an ACD model as in Bauwens and Giot (2000). This model is useful because it avoids the positivity restrictions of the parameters of the dynamic equation and it permits the introduction of exogenous variables that are negatively correlated with the duration process. Thus the conditional mean function is specified using the following general parameterization,

$$\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1) = \omega + \sum_{j=1}^{J} \alpha_j g(d_{i-j}) + \sum_{k=1}^{K} \beta_k \psi_{i-k}, \tag{5}$$

where, for the sake of completeness of the model, an exponential form is assumed for (4), i.e. $\varphi(u, v) = \exp(u + v)$ and $g(s) = \ln(s)$.

With respect to the so called short-run component, $\phi(\cdot, \vartheta_2)$, several alternatives are available. When modelling seasonality in this type of models, it is usually assumed that the seasonal term is somehow related to the time t_i at which the i-th transaction occurs. However, in order to make this dependence more explicit, it is standard in seasonal analysis to define a re-scaled time variable, t', such that

$$t' = \begin{cases} t - \left\lfloor \frac{t - t_I}{t_F - t_I} \right\rfloor (t_F - t_I) & \text{if } t > t_F, \\ t & \text{otherwise,} \end{cases} \tag{6}$$

where $\lfloor x \rfloor$ is the integer part of x. If the seasonal period is daily, t_I and t_F stand respectively for the stock market opening and closing hour (in seconds). i.e. $t_I = 9.5 \times 60 \times 60$ and $t_F = 16 \times 60 \times 60$. Note that defined in this way, $t' \in [t_I, t_F]$ is of bounded support and is the time-of-the-day (in seconds) at which the transaction has occurred.

In the context of regularly spaced variables several functional forms have been proposed in literature. If it is assumed to fall within a known class of parametric functions then, substituting (4) into (3), we obtain the following log-likelihood function

$$L_n(d; \vartheta_1, \vartheta_3, \xi) = \sum_{i=1}^{n} \log p\left(\varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi(t'_i; \vartheta_2)\right); \xi\right), \tag{7}$$

and the parameters ϑ_1, ϑ_2 and ξ can be estimated as in the one-component case (3). If the error density is correctly specified then standard ML estimation methods apply.

Unfortunately, very little information is available about the functional form that relates seasonality and time arrivals and therefore the risk of misspecification in choosing $\phi(\cdot; \vartheta_2)$ is very high. Therefore, the use of nonparametric methods is preferred to approximate the unknown seasonal function. Let $\phi(t')$ be the deterministic seasonal component at time t'. Given the proposed specification for the components, (4) becomes

$$E\left[d_i | \bar{d}_{i-1}, \bar{y}_{i-1}\right] = \varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi(t'_i)\right), \tag{8}$$

where ϑ_1 and the function $\phi(\cdot)$, evaluated at time points $t'_1, \cdots, t'_n$, need to be estimated. Following a standard approach (as in (3) or (7)), one would be tempted to obtain estimators for ϑ_1, ξ, $\phi(t'_1)$, $\cdots$, $\phi(t'_n)$, by choosing the values that maximize the following log-likelihood function

$$L_n(d; \vartheta_1, \xi) = \sum_{i=1}^{n} \log p\left(\varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi(t'_i)\right); \xi\right). \tag{9}$$

This estimation problem is semiparametric since a nonparametric component, $\phi(\cdot)$, needs to be estimated jointly with a parametric one ϑ_1 and ξ. Unfortunately, as already remarked in Section 1, standard ML techniques do not apply directly since the estimation of the parameter vector, ϑ_1 and ξ, in the presence of infinite dimensional nuisance parameters, does not necessarily provide consistent estimators.

In order to implement a valid inference not only on the parameter estimators of the long run component but on the estimated seasonal curve as well, a generalized profile likelihood approach is proposed, introduced in an *i.i.d.* context by Severini and Wong (1992). The basic idea of this method is to estimate the nonparametric function $\phi(\cdot)$ by maximizing a local (and hence smoothed) likelihood function (see Staniswalis, 1989), and simultaneously to estimate the parameter vector ϑ_1 and ξ by maximizing the un-smoothed likelihood function. That is, for a given value of time of the day, $t_0' \in [t_I, t_F]$, and fixed values of ϑ_1 and ξ, we estimate $\phi(t_0')$ as the solution of the problem

$$\hat{\phi}_{\vartheta_1,\xi}(t_0') = \sup_{\phi\in\Phi} \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{t_0'-t_i'}{h}\right) \log p\left(\varphi\left(\psi(\bar{d}_{i-1},\bar{y}_{i-1};\vartheta_1),\phi\right);\xi\right), \tag{10}$$

where $K(\cdot)$ is a kernel function and h the corresponding bandwidth. Note also that all estimators depend on ϑ_1 and ξ. Then, $\hat{\phi}_{\vartheta_1,\xi}(t_0')$ must fulfill the following first order conditions,

$$\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{t_0'-t_i'}{h}\right) \frac{\partial}{\partial\phi}\log p\left(\varphi\left(\psi(\bar{d}_{i-1},\bar{y}_{i-1};\vartheta_1),\hat{\phi}_{\vartheta_1,\xi}(t_0')\right);\xi\right) = 0. \tag{11}$$

Given the above estimates for the nonparametric part, a simple ML estimation for ϑ_1 and ξ is performed, i.e.

$$\left(\hat{\vartheta}_{1n}\ \hat{\xi}_n\right)^T = \sup_{\vartheta_1\in\Theta}\sup_{\xi\in\Xi}\sum_{i=1}^{n}\log p\left(\varphi\left(\psi(\bar{d}_{i-1},\bar{y}_{i-1};\vartheta_1),\hat{\phi}_{\vartheta_1,\xi}(t_i')\right);\xi\right), \tag{12}$$

and $\left(\hat{\vartheta}_{1n}\ \hat{\xi}_n\right)$ must fulfill the first order conditions

$$\sum_{i=1}^{n}\frac{\partial}{\partial\left(\vartheta_1\ \xi\right)^T}\log p\left(\varphi\left(\psi(\bar{d}_{i-1},\bar{y}_{i-1};\hat{\vartheta}_{1n}),\hat{\phi}_{\vartheta_1,\xi}(t_i')\right);\hat{\xi}_n\right) = 0. \tag{13}$$

As an example of our estimation procedure, assume that $p(\cdot)$ in (3) is the generalized gamma distribution (hereafter GG). After some algebra equation (10) takes the following expression

$$\hat{\phi}_{\vartheta_1,\xi}(t_0') = \frac{1}{\gamma\nu}\log\left\{\frac{\frac{1}{nh}\sum_{i=1}^{n}K\left(\frac{t_0'-t_i'}{h}\right)\left(\frac{d_i\mu_\varepsilon}{\exp\{\psi(\bar{d}_{i-1},\bar{y}_{i-1};\vartheta_1)\}}\right)^{\gamma}}{\frac{1}{nh}\sum_{i=1}^{n}K\left(\frac{t_0'-t_i'}{h}\right)}\right\}. \tag{14}$$

This is the nonparametric estimator for the seasonal component in the Log-ACD representation. Note that when $\nu = 1$ the nonparametric seasonal estimator when $p(\cdot)$ follows a Weibull distribution is attained. When $\nu = \gamma = 1$ the estimator corresponds to the exponential distribution case.

Expression (14) is obtained as follows: the GG distribution is a four-parameter distribution. Two of them, the location and the scale parameters, are usually set at zero and one. Therefore,

$\xi = (0, 1, \gamma, \nu)$. For ease of exposition the location parameter is skipped. Notice that if $\varepsilon \sim GG(1, \gamma, \nu)$ then

$$d \sim GG\left(\mu_\varepsilon(\xi)\,\varphi\left(\psi\left(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1\right), \phi\left(t'_i\right)\right)^{-1}, \gamma, \nu\right)$$

and after some simple calculations the result in (14) follows.

In all cases the non parametric seasonal curve is estimated by a transformation of the Nadaraya-Watson nonparametric regression estimator of the duration adjusted by the long-run component, on the time values $t'_1, \cdots, t'_n$ at time t'_0. This appears as a very natural and simple idea, with a nice underlying motivation.

Within this statistical framework the results available in literature are obtained for independent observations and hence do not hold for tick-by-tick data. Nonetheless in the following theorems the equivalent statistical results to make a correct inference for the unknown parameters of the Log-ACD model are shown (the proof is given in the Appendix):

Theorem 1 *Let $\eta = (\vartheta_1\ \xi)^T$, and $\hat{\eta}_n$ be the vector of corresponding parametric estimates. Under conditions (L.1), (L.2), (A.1) to (A.3), (B.1), and (B.3) to (B.6) stated in the Appendix then,*

$$\sqrt{n}\left(\hat{\eta}_n - \eta\right) \to_d \mathbf{N}\left(0, I_\eta^{-1}(\phi, \eta)\right), \tag{15}$$

where

$$\begin{aligned} & I_\eta(\phi, \eta) \\ = {} & E\left[\frac{\partial}{\partial \eta}\log p\left(\varphi(\psi, \phi); \xi\right)\frac{\partial}{\partial \eta^T}\log p\left(\varphi(\psi, \phi); \xi\right)\right] - E\left[\frac{\partial}{\partial \eta}\log p\left(\varphi(\psi, \phi); \xi\right)\frac{\partial}{\partial \phi}\log p\left(\varphi(\psi, \phi); \xi\right)\right] \\ & \times E\left[\frac{\partial^2}{\partial \phi^2}\log p\left(\varphi(\psi, \phi); \xi\right)\right]^{-1} E\left[\frac{\partial}{\partial \phi}\log p\left(\varphi(\psi, \phi); \xi\right)\frac{\partial}{\partial \eta^T}\log p\left(\varphi(\psi, \phi); \xi\right)\right] \end{aligned}$$

and

$$\sqrt{nh}\left(\hat{\phi}_{\hat{\eta}_n}(t'_0) - \phi(t'_0)\right) \to_d \mathbf{N}\left(0, V\left(t'_0, \eta\right)\right), \tag{16}$$

where

$$V\left(t'_0, \eta\right) = \frac{\int K^2(u)du}{f(t'_0) I\left(t'_0, \eta\right)}, \tag{17}$$

$$I\left(t'_0, \eta\right) = E\left[\frac{\partial}{\partial \phi}\log p\left(\varphi(\psi, \phi); \xi\right)^2 \middle| t' = t'_0\right] \tag{18}$$

as n tends to infinity

This result is very interesting since it allows us to make an inference both on parametric and nonparametric components. However, they might be arguable since these results depend on the correct specification of the conditional density function of the error term. In order to avoid this problem, in the next subsection we weaken certain assumptions about the error density function and show that our result remains valid.

2.1 THE GLM APPROACH

As pointed out in Engle and Rusell (1998) and Engle (2000) it is of interest to have estimation techniques available that do not rely on knowledge of the functional form of the conditional density function. Two alternative approaches that allow for consistent estimation of the parameters of interest without specifying conditional density are the Quasi Maximum Likelihood technique, QML, (see Gouriéroux, Monfort and Trognon, 1984) and Generalized Linear Models, GLM, (see McCullagh and Nelder, 1989). In both approaches, it is assumed that the duration variable d, conditionally on past values of d and y, depends on a scalar parameter $\theta = h\left(\bar{d}_{i-1}, \bar{y}_{i-1}; \delta_1\right)$, and its distribution belongs to a one dimensional exponential family with conditional density

$$p\left(d_i | \bar{d}_{i-1}, \bar{y}_{i-1}; \theta\right) = \exp\left(d_i\theta - b(\theta) + c(d_i)\right),$$

where $b(\cdot)$ and $c(\cdot)$ are known functions. The main difference between the QML and the GLM approaches is simply a different parameterization of this exponential family. In this paper the GLM approach is adopted for the sake of convenience. Then it is straightforward to see that the ML estimator of θ solves the following first order conditions: $\sum_i \{d_i - b'(\theta)\} = 0$. Furthermore, by the properties of the exponential family and (4) we obtain the following relationships

$$E\left[d_i | \bar{d}_{i-1}, \bar{y}_{i-1}\right] = b'(\theta) = \varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_2)\right) \tag{19}$$

and

$$\mathrm{Var}\left[d_i | \bar{d}_{i-1}, \bar{y}_{i-1}\right] = b''(\theta) = \sigma^2 V\left\{\varphi\left(\psi(\vartheta_1), \phi(\vartheta_2)\right)\right\}, \tag{20}$$

and it is possible to show that the ML estimator of θ can also be obtained from the solution of the following equation

$$\sum_{i=1}^{n} \frac{\left(d_i - \varphi\left(\theta\right)\right) \varphi'\left(\theta\right)}{V\left(\varphi\left(\theta\right)\right)} = 0. \tag{21}$$

As can be clearly realized from equations (19), (20) and (21) the parameter of interest θ (the so called canonical parameter) can be estimated with no need to specify the whole conditional distribution function. It is only necessary to specify the functional form of the conditional mean, $\varphi(\cdot)$, and of the conditional variance $V(\cdot)$, but not the whole distribution.

The relationship between the predictors in equation (5) and the canonical parameter is given by the so called *link function.* This function depends on which member of the exponential family is used. For the exponential distribution the link function is

$$\theta = -\frac{1}{\varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_2)\right)}. \tag{22}$$

Noting that under this distribution $\mu(\theta) = -\theta^{-1}$ and $V\left(\mu\left(\theta\right)\right) = \mu^2$, then (21) are the first order conditions for the maximization of the log-likelihood function for exponentially distributed data.

In order to estimate the parameters of interest, ϑ_1 and ϑ_2, the values of the parameters that maximize the quasi-likelihood function

$$Q_n\left(d, \varphi\right) = \sum_{i=1}^{n} Q\left(\varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi(t_i; \vartheta_2)\right); d_i\right), \tag{23}$$

are taken. The (quasi-)log-likelihood $Q(\cdot)$ is obtained by integrating (21), i.e.

$$Q\left(d, g\right) = \int_g^d \frac{(s-d)}{V(s)} ds.$$

As already indicated in the likelihood context, if the seasonal component is assumed to fall within the class of a known parametric function, $\phi(t_i; \vartheta_2)$, then the properties of the QML estimators of ϑ_1 and ϑ_2 are well known (see Engle and Rusell, 1998 and Engle, 2000). Unfortunately if, instead, the seasonal component is approximated nonparametrically then standard quasi-likelihood arguments do not hold.

For the standard *i.i.d.* case, Severini and Staniswalis (1994) and Fan, Heckman and Wand (1995) propose consistent estimators of both parametric and nonparametric parts. Unfortunately, the statistical results from these papers do not apply directly in our case since they assume independent observations. Nevertheless at the end of the subsection equivalent statistical results are shown for the dependent case.

Analogously to the likelihood case let us define $\hat{\phi}_{\vartheta_1}(t'_0)$, for fixed values of ϑ_1, as the solution to the following optimization problem

$$\hat{\phi}_{\vartheta_1}(t'_0) = \arg\max_{\phi \in \Phi} \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{t'_0 - t'_i}{h}\right) Q\left(\varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi\right); d_i\right) \tag{24}$$

for $t'_0 \in [t_I, t_F]$. Then $\hat{\phi}_{\vartheta_1}(t'_0)$ must fulfill the following first order condition

$$\frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{t'_0 - t'_i}{h}\right) \frac{\partial}{\partial \eta} Q\left(\varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \phi\right); d_i\right) = 0. \tag{25}$$

The estimator of ϑ_1, $\hat{\vartheta}_{1n}$, is obtained as the solution to the following (un-smoothed) optimization problem

$$\hat{\vartheta}_{1n} = \arg\max_{\vartheta_1 \in \Theta} \sum_{i=1}^{n} Q\left(\varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \hat{\phi}_{\vartheta_1}(t'_i)\right); d_i\right),$$

and $\hat{\vartheta}_{1n}$ must fulfill the following first order condition

$$\sum_{i=1}^{n} \frac{\partial}{\partial \vartheta_1} Q\left(\varphi\left(\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1), \hat{\phi}_{\vartheta_1}(t'_i)\right); d_i\right) = 0. \tag{26}$$

As an example, set $\varphi(u, v) = (u \times v)$, the ACD representation, and $\mu = -\theta^{-1}$ and $V(\mu) = \mu^2$ (the exponential distribution). Then the quasi-likelihood function takes the form of a log-likelihood function from an exponential conditional distribution and an ACD representation, i.e.

$$-\sum_{i=1}^{n} \left[\log\left\{\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1) \times \phi(t'_i)\right\} + \frac{d_i}{\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1) \times \phi(t'_i)}\right], \tag{27}$$

and, after some derivations, the estimator (24) takes the explicit form

$$\hat{\phi}_{\vartheta_1}(t'_0) = \frac{\frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{t'_0 - t'_i}{h}\right) \frac{d_i}{\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1)}}{\frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{t'_0 - t'_i}{h}\right)}. \tag{28}$$

Since a closed expression for the parametric part is not available, an iterative algorithm must be used. Now assume instead that $\varphi(u, v) = \exp(u + v)$, then the quasi-likelihood function corresponds to the log-likelihood function from an exponential distribution and a Log-ACD representation and hence the nonparametric estimator in (24), takes the explicit form

$$\hat{\phi}_{\vartheta_1}(t'_0) = \log\left\{\frac{\frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{t'_0 - t'_i}{h}\right) \frac{d_i}{\exp\{\psi(\bar{d}_{i-1}, \bar{y}_{i-1}; \vartheta_1)\}}}{\frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{t'_0 - t'_i}{h}\right)}\right\}. \tag{29}$$

Finally, the above expressions are obtained by assuming a seasonal component for a given period (e.g. daily). However, it is also possible to extend this to several seasonal effects. For example, in the daily case, we may be interested in looking at whether the seasonal patterns for each day of the week are different and test the differences using confidence bands. In this case, for s-th day of the week, we have in the exponential and the Log-ACD representation

$$\hat{\phi}^s(t_0') = \log \left\{ \frac{\frac{1}{nh}\sum_{j=1}^{\lfloor n/5 \rfloor}\sum_{i=1}^{n} K\left(\frac{t_0'-t_i'}{h}\right) I\left(\lfloor \frac{t-t_I}{t_F-t_I}+1 \rfloor = js\right) \frac{d_i \mu_\epsilon}{\exp\{\psi(\bar{d}_{i-1},\bar{y}_{i-1};\vartheta_1)\}}}{\frac{1}{nh}\sum_{j=1}^{\lfloor n/5 \rfloor}\sum_{i=1}^{n} K\left(\frac{t_0-t_i}{h}\right) I\left(\lfloor \frac{t-t_I}{t_F-t_I}+1 \rfloor = js\right)} \right\}, \quad s = 1, \cdots, 5, \tag{30}$$

$I(\cdot)$ is the indicator function and $\lfloor x \rfloor$ is the integer part of x. The following theorem shows the equivalent statistical results for making a correct inference for the unknown parameters (the proof is given in the Appendix):

Theorem 2 *Under the conditions (A.1) to (A.5), and (B.1) to (B.6), stated in the Appendix then,*

$$\sqrt{n}\left(\hat{\vartheta}_{1n} - \vartheta_1\right) \to_d \mathbf{N}\left(0, \Sigma_{\vartheta_1}^{-1}\right), \tag{31}$$

where

$$\Sigma_{\vartheta_1} = -E\left(\frac{\partial^2}{\partial\vartheta_1\partial\vartheta_1^T} Q\left(\varphi\left(\psi(\bar{d},\bar{y};\vartheta_1), \phi(t)\right); d\right)\right),$$

and

$$\sqrt{nh}\left(\hat{\phi}_{\hat{\vartheta}_1}(t_0') - \phi(t_0')\right) \to_d \mathbf{N}\left(0, V\left(t_0'; \eta\right)\right), \tag{32}$$

where

$$V\left(t_0';\eta\right) = \frac{\int K^2(u)du}{p(t_0')} \times \frac{E\left[\left\{\frac{d-\varphi(\psi(\vartheta_1),\phi)}{V(\varphi(\psi(\vartheta_1),\phi))}\frac{\partial}{\partial\phi}\varphi\left(\psi(\vartheta_1),\phi\right)\right\}^2 | t' = t_0'\right]}{E\left[\frac{1}{V_0(\varphi(\psi(\vartheta_1),\phi))} \times \frac{\partial}{\partial\phi}\varphi\left(\psi(\vartheta_1),\phi\right)^2 | t' = t_0'\right]^2}, \tag{33}$$

as n tends to infinity

2.2 FINAL CONSIDERATIONS

Once the estimator has been proposed and its main statistical properties have been derived, it is useful to gather and highlight its main advantages, which have been outlined in the introduction and this section. The following list is just a compendium of advantages of this estimator with respect to the alternatives, mainly in comparison with piecewise polynomial splines since that is the most widely used method for this kind of processes:

- The seasonal estimator proposed here is very intuitive since it appears as the outcome of a standard optimization process. Furthermore, its closed form is just a transformation of the Naradaya Watson nonparametric kernel regression estimator.

- The statistical properties of both the parametric and nonparametric estimators are well established. This enables us to make an inference. It is possible to draw confidence bands for the proposed estimator, using (17) or (33), and hence test whether seasonality varies over time, differs in different periods, like (30), etc. Unfortunately, the same cannot be said for the estimators that result when the piecewise cubic spline is used to capture the seasonal component.

- This methodology provides a data driven method for computing the bandwidth. However for the spline there is no known method of choosing the number and location of knots and the proportionality coefficients for the end-point restrictions.
- Multivariate extensions of the seasonal estimator are straightforward. For example, considering a multivariate exponential distribution, the estimator (28) is easily adapted to the multivariate case.

3 APPLICATION TO DIFFERENT DURATION PROCESSES OF A STOCK TRADED AT THE NYSE

3.1 Data and transformations

In this section the above method is applied to several duration processes. The data, extracted from the trade and quotes (TAQ) database, are price and volume durations of the Disney stock pertaining to September, October and November 1996. These duration processes have been widely used by many researchers. See, among others, Bauwens et al. (2000) or Bauwens and Veredas (1999).

The Disney stock has been analyzed already in Bauwens et al. (2000), where it has been shown that the existing financial duration models (ACD and Log-ACD with four different distributions, TACD, SCD and the SVD) are not able to capture correctly either the density or the dynamics of this stock. In all cases the models are misspecified in mean and in distribution. These models are estimated on durations seasonally adjusted with splines.

A price duration is the minimum time interval required to witness a given cumulative price change greater than a certain threshold. As in Engle and Russell (1998), price means the average of the bid and ask prices. Price is defined as the mid quote to avoid the bid-ask bounce exhibited in the trade process. The threshold is defined as \$0.125. A volume duration is defined as the time interval needed to observe an accumulated change in volume greater than 25,000 shares.

The analysis of these duration processes provides an instantaneous overview of the state of the market: Price durations are strongly linked to the instantaneous volatility process. This is pointed out by Engle and Russell (1998) and Gerhard and Hautsch (1999). Indeed instantaneous volatility can be defined (assuming mean return of zero) as the square of the threshold over past mid price times the conditional expectation of the inverse of the price duration. Hence by computing the conditional expectation of the duration it is possible to compute the instantaneous volatility process. Finally, volume durations are appealing since they convey information on the liquidity of the market, given that they incorporate two of the three dimensions of liquidity: time and volume. See Bauwens and Giot (2001) for further implications on the use of these durations in market microstructure theory.

Prior to the analysis two transformations are performed: i) Trades and bid/ask quotes recorded before 09:30 am and after 4 pm are ignored. ii) The time gaps between market closing and opening are also ignored.

3.2 Descriptive analysis

There are 1778 observations for volume and 2160 for price (see Table 1). The first two thirds of the sample are used for estimation (in-sample) whilst the last third of the sample is used for diagnosis and prediction (out-sample). Further details are given in section 3.4. Price durations are overdispersed (the ratio of standard deviation over mean is 1.49) whilst volume durations are underdispersed (0.81). Notice also that the proportion of data below the mode is not negligible which means that very small durations are rarely observed and hence most probably the densities have a hump close to the origin. This hypothesis will be checked shortly.

In order to check for an intradaily seasonal pattern we calculate a preliminary nonparametric regression estimator of durations, d, into the modified time index, t'. For the s-th day of the week we define

$$\hat{\phi}^s(t_0') = \frac{\frac{1}{nh}\sum_{j=1}^{\lfloor n/5 \rfloor}\sum_{i=1}^{n} K\left(\frac{t_0'-t_i'}{h}\right) I\left(\lfloor \frac{t-t_I}{t_F-t_I}+1 \rfloor = js\right) d_i}{\frac{1}{nh}\sum_{j=1}^{\lfloor n/5 \rfloor}\sum_{i=1}^{n} K\left(\frac{t_0-t_i}{h}\right) I\left(\lfloor \frac{t-t_I}{t_F-t_I}+1 \rfloor = js\right)}, \quad s = 1, \cdots, 5, \tag{34}$$

$I(\cdot)$ is the indicator function and $\lfloor x \rfloor$ is the integer part of x. Figures 1 and 2 show these naive estimators of intradaily seasonality for the different days of the week. In order to know whether there are differences in the seasonal path over the days of the week they also include the pointwise confidence bands for the different curves. These confidence bands are obtained according to Bosq (1998), Theorem 3.1, p. 70.

Note that although a more formal test is needed, a preliminary analysis suggests that the hypothesis of differences between the seasonal behaviors over the days of the week is not supported. Nonetheless it can be seen that the variance of the curve estimator increases with the time of day. Figure 3 shows the mean and standard deviation in intervals of half an hour through the day. They show a daily seasonal pattern. Observe that the level at the end of the day is greater than in the rest of the day. This explains why the confidence bands are not constant in Figures 1 and 2. Moreover, notice that the variation coefficient (i.e. the ratio of standard deviation over mean) is not constant through the day either, meaning that the differences between the mean and variance are not constant.

Figures 4 and 5 are the Spearman's ρ coefficients for serial dependence and kernel density estimators, respectively, for each duration process. Figure 4 shows the need for the dynamic component in the model. Clearly the Spearman's ρ coefficients indicate the presence of serial dependencies. Notice the use of Spearman's ρ coefficients rather than autocorrelation coefficients. When dealing with time series that are irregularly spaced autocorrelation coefficients are misleading. Finally, from Figure 5 one can see that the densities have a hump, as suggested above. See Bauwens and Veredas (1999) for further discussion of the implications of the hump for the density of the duration processes.

Finally, note that (34) is indeed the first step in the two step procedure presented in the introduction. However, this naive estimator differs substantially from the curve estimator that we propose in the section above.

3.3 Estimation results

The estimation results are shown in Tables 2 and 3. As a long run component the log-ACD representation is chosen with $J = 1$ and $K = 1$ in (5). However, for the seasonal component several specifications are estimated.

Table 1: Information on Duration Data

	N	$N-in$	$N-out$	Mean	sd	mode	$\%<$ mod	min	max
Price	2160	1426	734	647	966	70.6	0.11	3	9739
Volume	1778	1173	605	801	648	304	0.24	6	4621

The original data were extracted from the TAQ database for September, October, and November 1996. Durations are measured in seconds. N denotes the total number of observations, $N-in$ the in-sample number of observations, $N-out$ the out-sample number of observations, sd the standard deviation, % <mod the proportion of observations smaller than the mode, min and max are the smallest and the largest durations.

Results are shown for different specifications:

BiNW Estimation in two steps. In the first step the seasonal component is estimated using

$$\hat{\phi}(t_0') = \frac{\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{t_0'-t_i'}{h}\right) d_i}{\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{t_0'-t_i'}{h}\right)}. \tag{35}$$

That is, no intradaily seasonality is allowed. Then in the second step the parameters of the log-ACD representation are estimated using the adjusted durations, $\tilde{d}_i = d_i/\hat{\phi}\,(t_0')$.

BiWNW The same method as in BiNW but the seasonal path is allowed to vary over the day of the week. That is, (34) is used instead of (35) in order to estimate the seasonal component.

BiSp Estimation in two steps, but the seasonal component is estimated by averaging the durations over thirty minute intervals and the resulting piece-wise constant function is smoothed via cubic splines (see for example Engle and Russell, 1997).

UniNW Joint estimation of both parametric and nonparametric components following the method proposed in Section 2.

UniWNW The same method as in UniNW but the seasonal path is allowed to vary over the days of the week, as in (30).

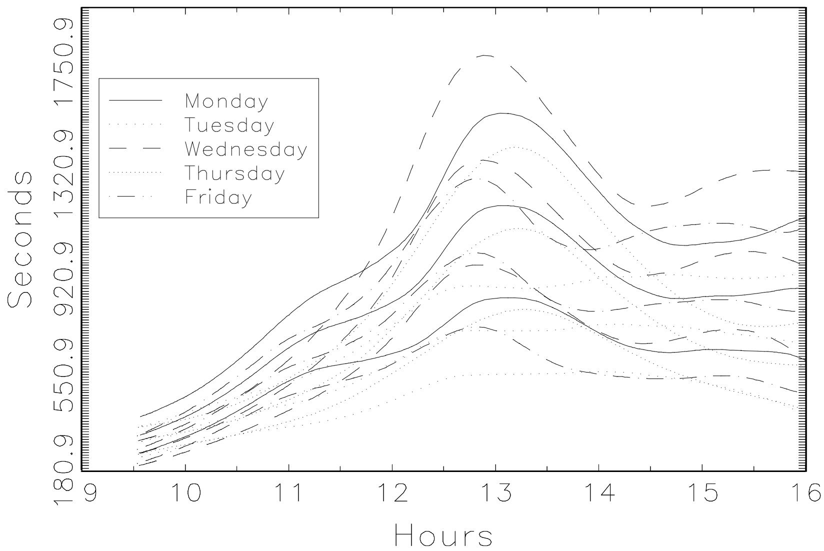
UniSp Joint estimation of both parametric and nonparametric components using the method proposed in Engle and Rusell (1998).

For the sake of computational simplicity we use a parameterization other than the one chosen in Engle and Rusell (1998, see footnote on Table 2, p 1142). It is

$$E\left[d_i|\overline{d}_{i-1}\right] = \psi_i = \exp\left\{\omega + \alpha \ln \frac{d_{i-1}}{\phi_{\theta,\delta}(t_{i-1}')} + \beta\psi_{i-1}\right\}, \tag{36}$$

$$\phi_{\theta,\delta}(t_{i-1}') = \sum_{i=1}^{4} \theta_j t_{i-1}'^{\,j-i} + \sum_{k=1}^{K} \delta_k \left(t_{i-1}' - \xi_k\right)_+^3, \tag{37}$$

where $(\cdot)_+ = \max(\cdot, 0)$, K is the number of knots and ξ_k is the kth knot. $\theta_j, j = 1, \cdots, 4$ and $\delta_k, k = 1, \cdots, K$ are the parameters of the spline to be estimated. Further details on the derivation of this form of the spline can be found in Eubank (1988, p 354).



Intradaily seasonal curves are estimators using (35) for price durations. Each curve has two associated curves representing its confidence bands.

Figure 1: Naive Diurnal estimator for price durations

To construct pointwise confidence bands for the seasonal curve in UniNW and UniWNW we need to estimate the empirical counterparts of (17) and (18). We use

$$\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}(t'_0) \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\|K\|_2^2}{\widehat{f}(t'_0)\widehat{I}(\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}, t'_0)}}, \tag{38}$$

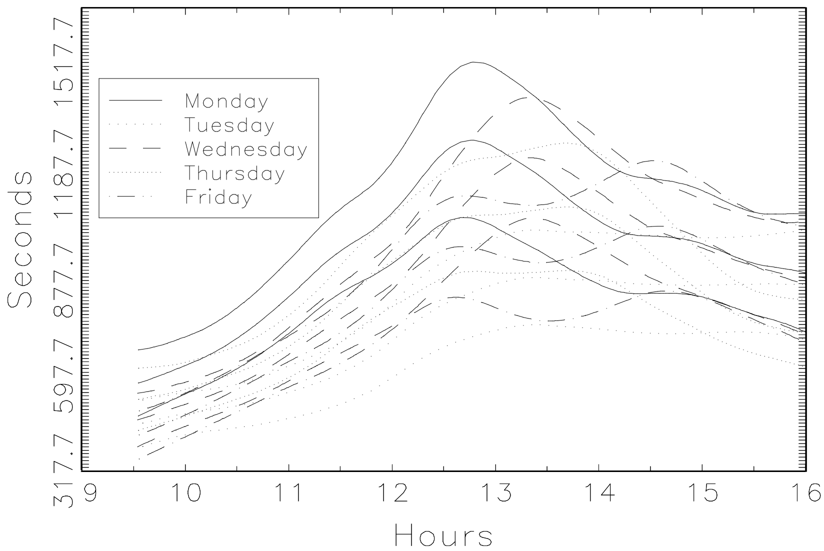
where

$$\widehat{I}(\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}, t'_0) = \frac{\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{t'_0-t'_i}{h}\right)\left[\widehat{\gamma}\left(\left(\frac{d_i\widehat{\mu}_\varepsilon}{\exp\left\{\psi(\widehat{\vartheta}_1)+\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}(t'_0)\right\}}\right)^{\widehat{\gamma}} - \widehat{\nu}\right)\right]^2}{\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{t'_0-t'_i}{h}\right)}, \tag{39}$$

$$\widehat{f}(t'_0) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{t'_0-t'_i}{h}\right), \tag{40}$$

$z_{1-\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$-quantile of the standard normal distribution and $\|K\|_2^2$ is a known constant that depends on the kernel. We choose the quartic kernel with $\|K\|_2^2 = \frac{5}{7}$.

Comparing the proposed method (UniNW and UniWNW) with the others, some basic insights that can extracted from Tables 2 and 3:



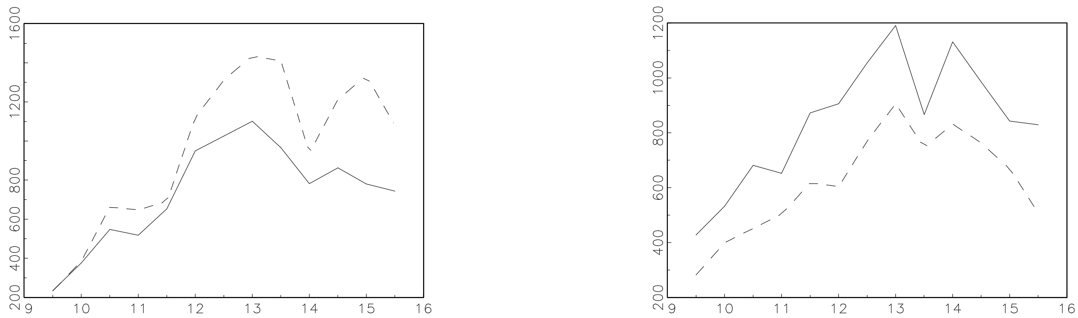
Intradaily seasonal curves are estimators using (35) for volume durations. Each curve has two associated curves representing its confidence bands.

Figure 2: Naive Diurnal estimator for volume durations

- The constant $\widehat{\omega}$ is not estimated under UniNW and UniWNW because of identification restrictions.
- Estimates using UniNW or UniWNW are very similar, meaning that differentiating between different days of the week does not affect the parameters.
- The sum of $\widehat{\alpha}+\widehat{\beta}$ is smaller (bigger) when estimating with UniNW and UniWNW than in all the other cases for price (volume) durations. This implies that the estimated unconditional mean for price (volumen) durations is smaller (bigger) when using UniNW or UniWNW than any other specification.
- The parameters of the distribution, $\widehat{\xi}$, are not statistically affected by the specification.

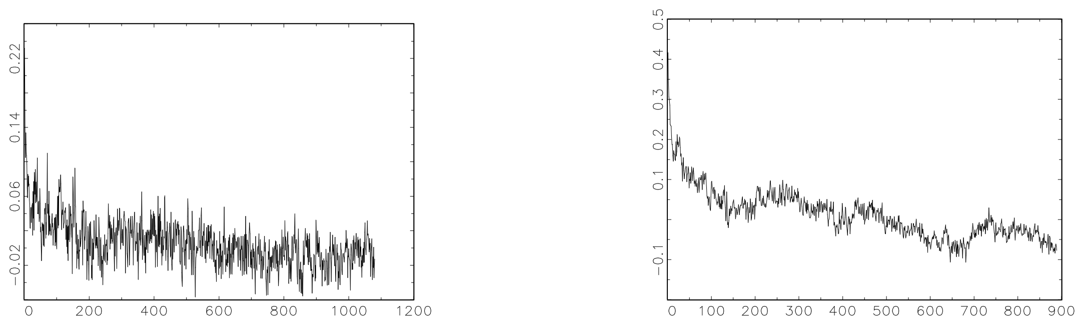
With respect to the differences in the estimated seasonal curves, Figure 6 shows the estimated curves using the specifications BiNW, BiSp, UniNW and UniSp for price and volume durations. Again, comparing UniNW with the other estimators it can be seen that:

- Under UniNW the distance between the confidence bands is constant and hence no longer increasing. It has previously being shown that the changing pattern in the confidence bands is due to the time varying variance over the day. When joint estimation is used, the confidence bands are constant. This can be explained by the fact that the variance of the estimated curves is now a function of the parameters of the distribution, see (38),



The solid line represents a rolling mean (nodes each half hour) through the day. Dashed line is a rolling standard deviation (for the same nodes). Price durations in left box and volume durations in right box.

Figure 3: Intradaily mean and standard deviations



The Y-axis shows the Spearman's ρ coefficients for serial dependence. The X-axis shows the orders of the coefficients. Price (left box) and volume (right) durations. The maximum number of orders is equal to one half of the sample.

Figure 4: Spearman's coefficients for serial dependence

which are in their turn dominating the variance of the error term. Therefore $\widehat{\xi}$ corrects for this effect.

- With respect to the differences between UniNW and UniSp, it can be seen that for both price and volume durations, UniSp generates a curve which is less smooth, with a hump at the end of the day and with a rough increase at the beginning of the day for volume durations. Under UniNW this behaviour is not present.

3.4 Predictibility

The predictibility and specification performance of the model is analyzed via a density forecast. This concept was introduced by Diebold et al. (1998) in the context of GARCH models and has been extensively used by Bauwens et al. (2000) for comparing different financial duration models. Density forecast is more accurate than point, or even interval, prediction since it relies

Table 2: Log-ACD results for Price durations

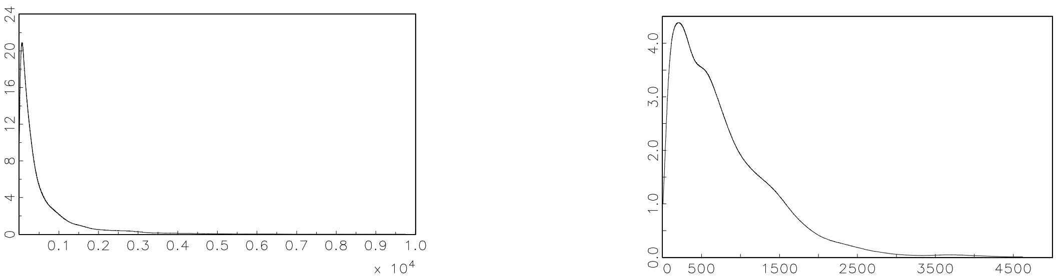
	BiNW	BiWNW	BiSp	UniNW	UniWNW	UniSp
$\widehat{\omega}$	0.0381 [0.0107]	0.0350 [0.0104]	0.1785 [0.0533]	-	-	1.2613 [0.0486]
$\widehat{\alpha}$	0.0497 [0.0138]	0.0467 [0.0137]	0.1238 [0.0259]	0.0408 [0.0171]	0.0333 [0.0169]	0.0406 [0.0086]
$\widehat{\beta}$	0.8506 [0.0419]	0.8626 [0.0415]	0.6189 [0.1193]	0.6338 [0.1095]	0.6592 [0.1099]	0.8119 [0.1124]
$\widehat{\gamma}$	0.9405 [0.0183]	0.9497 [0.0185]	0.9088 [0.0177]	0.9350 [0.0183]	0.9516 [0.0187]	0.8514 [0.0048]
$\widehat{\alpha}+\widehat{\beta}$	0.9003	0.9093	0.7427	0.6746	0.6925	0.8525
$\widehat{\omega}$	0.0501 [0.0171]	0.0442 [0.0165]	0.2115 [0.0457]	-	-	1.4121 [0.0654]
$\widehat{\alpha}$	0.0611 [0.0185]	0.0543 [0.0182]	0.1596 [0.0258]	0.0680 [0.0203]	0.0646 [0.0217]	0.0401 [0.0141]
$\widehat{\beta}$	0.8039 [0.0735]	0.8202 [0.0736]	0.5672 [0.0942]	0.6129 [0.1881]	0.5814 [0.2839]	0.7899 [0.1494]
$\widehat{\gamma}$	0.2693 [0.0605]	0.2974 [0.0618]	0.2900 [0.0579]	0.2961 [0.0599]	0.3385 [0.0624]	0.0675 [0.0014]
$\widehat{\nu}$	10.807 [4.7263]	9.0078 [3.6263]	8.7329 [3.3737]	8.7708 [3.4329]	7.9270 [2.4509]	17.85 [2.43294]
$\widehat{\alpha}+\widehat{\beta}$	0.8650	0.8745	0.7268	0.6809	0.6460	0.8300

The first part of the first column shows the parameters using a Weibull distribution and the second part using a generalized gamma distribution. Numbers in brackets are heteroskedastic-consistent standard errors.

Table 3: Log-ACD results for Volume durations

	BiNW	BiWNW	BiSp	UniNW	UniWNW	UniSp
$\widehat{\omega}$	0.0296 [0.0067]	0.0287 [0.0062]	0.0994 [0.0189]	-	-	1.5357 [0.0257]
$\widehat{\alpha}$	0.0957 [0.0173]	0.0950 [0.0166]	0.1828 [0.0248]	0.0885 [0.0189]	0.0835 [0.0242]	0.1152 [0.0227]
$\widehat{\beta}$	0.8292 [0.0378]	0.8341 [0.0344]	0.6110 [0.0675]	0.8721 [0.0423]	0.8806 [0.0552]	0.7784 [0.0488]
$\widehat{\gamma}$	1.6347 [0.0365]	1.6373 [0.0365]	1.4891 [0.0321]	1.6124 [0.0361]	1.6283 [0.0366]	1.4746 [0.0294]
$\widehat{\alpha}+\widehat{\beta}$	0.9249	0.9291	0.7938	0.9606	0.9641	0.8936
$\widehat{\omega}$	0.0306 [0.0069]	0.0294 [0.0064]	0.0896 [0.0163]	-	-	1.5361 [0.0244]
$\widehat{\alpha}$	0.1003 [0.0183]	0.0987 [0.0173]	0.1799 [0.0237]	0.0875 [0.0185]	0.0778 [0.0194]	0.1216 [0.1010]
$\widehat{\beta}$	0.8236 [0.0388]	0.8303 [0.0351]	0.6459 [0.0583]	0.8798 [0.0379]	0.8972 [0.0373]	0.7786 [0.0545]
$\widehat{\gamma}$	1.4356 [0.1328]	1.4274 [0.1312]	1.0180 [0.0972]	1.3959 [0.1328]	1.4045 [0.1361]	1.0653 [0.2376]
$\widehat{\nu}$	1.2468 [0.1912]	1.2634 [0.1930]	1.9952 [0.3373]	1.2774 [0.2020]	1.2839 [0.2067]	1.7702 [0.2453]
$\widehat{\alpha}+\widehat{\beta}$	0.9239	0.9290	0.8258	0.9673	0.9750	0.9002

See Table 2 for explanations.



Price (left box) and volume (right) durations. The nonparametric estimates are estimated by means of a gamma kernel, see Chen (2000). The optimal bandwidth is $(0.9sN^{-0.2})^2$ where N is the number of observations and s is the sample standard deviation.

Figure 5: Marginal densities

on the forecasting performance of all the moments. The behavior of the predicted density function is evaluated via the probability integral transform and if the prediction is correct the probability integral transform should be *i.i.d* and uniformly distributed.

More formally: Let $\left\{p\left(d_j|\varphi\left(\psi(\bar{d}_{j-1},\bar{y}_{j-1};\widehat{\vartheta}_1),\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}(t_j')\right);\widehat{\xi}\right)\right\}_{j=n+1}^{m}$ be a sequence of one-step-ahead density forecasts produced by the model and let $\left\{f\left(d_j|\bar{d}_{j-1},\bar{y}_{j-1}\right)\right\}_{j=n+1}^{m}$ be the sequence of densities defining the data generating process directing the duration series d_j. n and m are the number of observations in-sample and out-sample respectively. It can be shown (see Diebold et al., 1998) that the density is correctly specified if

$$\left\{p\left(d_j|\varphi\left(\psi(\bar{d}_{j-1},\bar{y}_{j-1};\widehat{\vartheta}_1),\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}(t_j')\right);\widehat{\xi}\right)\right\}_{j=n+1}^{m}=\left\{f\left(d_j|\bar{d}_{j-1},\bar{y}_{j-1}\right)\right\}_{j=n+1}^{m}.$$

To test this equality the probability integral transform,

$$z_j=\int_{-\infty}^{d_j}p\left(u|\varphi\left(\psi(\bar{d}_{j-1},\bar{y}_{j-1};\widehat{\vartheta}_1),\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}(t_j')\right);\widehat{\xi}\right)du,$$

is used. If the one-step-ahead density forecast, $p(\cdot)$, equals the density defining the data generating process, $f(\cdot)$, then z must be *i.i.d.* and uniformly distributed. Uniformity can be tested by using histograms based on the computed z sequence. If the density is correctly specified the histogram should be flat. For independence, the Spearman's ρ of various centered moments of the z sequence can reveal some dependency.

Notice that prediction is based on the observed, not the seasonally adjusted data. Therefore the seasonal component has to be introduced when computing z. Indeed assuming the GG distribution the location parameter of the density of the observed durations is $\mu_\varepsilon(\widehat{\xi})\varphi\left(\psi(\bar{d}_{i-1},\bar{y}_{i-1};\widehat{\vartheta}_1),\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}(t_j')\right)^{-1}$ when estimation is joint (using either UniNW, UniWNW or UniSp) and $\mu_\varepsilon(\widehat{\xi})\left[\psi(\bar{d}_{i-1},\bar{y}_{i-1};\widehat{\vartheta}_1)\widehat{\phi}(t_j')\right]^{-1}$ when durations are adjusted (using either BiNW, BiWNW or BiSp). In both cases the seasonal curve, $\widehat{\phi}_{\widehat{\vartheta}_1,\widehat{\xi}}(t_j')$ or $\widehat{\phi}(t_j')$, is present in the location parameter.

Figures 7 and 8 show the density forecast results for each duration process, using the GG distribution and four specifications: BiNW, BiSp, UniNW and UniSp. Results with the other specifications (because they are similar) and with the Weibull distribution (because they are uniformly worse) are not reported. The following conclusions can be drawn:

- The histograms are much better under UniNW than under any other specification. The seasonal adjustment produces misspecification in the conditional distribution of the durations. Bauwens et al. (2000) find the same results when adjusting by seasonality with the spline. Under UniSp this poor behavior is again found. The difference in the results is due to the fact that under UniNW the seasonal component adapts quickly to changes in the out-sample seasonal pattern. This is a distinctive characteristic of UniNW not present in the other specifications.

- None of the specifications captures the dynamic component correctly. This is expected as the dynamics are modelled in the same way for all specifications. However, a closer look at the Spearman's ρ coefficients reveals that the serial dependence of z is smaller under UniNW than in any other case. This is due to the differences in $\widehat{\vartheta}_1$ over specifications. This is also a confirming proof that the dynamic and seasonal components are not orthogonal.

Therefore since the only thing that changes from one specification to another is the way in which seasonality is dealt with, it seems that the method proposed in this paper is worth using since it clearly improves the results. The distributional assumption is now fulfilled. With respect to the dynamics, a more complex structure should be used. For example, Bauwens et al. (2000) show that results improve when the TACD of Zhang et al. (2001) is used. Another model could be the Augmented-ACD model of Fernandes and Grammig (2001).

4 CONCLUSIONS

This paper proposes a component model for the analysis of financial durations. The components are long-run dynamics and seasonality. The latter is left unspecified and the former is assumed to fall within the class of (Log-)ACD models. Joint estimation of the parameters of interest and the smooth curve is performed through a local (quasi-)likelihood method. For alternative specifications of the conditional density the resulting nonparametric estimator of the seasonal component shows a closed form expression that is a function of the Naradaya-Watson estimator.

A further advantage of this semiparametric component model is that under this approach any other tick-by-tick variable can be analyzed. The only requirement is that the dynamic component and the distribution be properly defined. For example tick-by-tick volatility could be successfully analyzed using this method.

The empirical application is to the price and volume duration processes of Disney stock traded at the NYSE. This stock has been shown in other articles to be difficult to model. It is now shown that using the proposed method the model is correctly specified in distribution, since poor diagnosis using previous approaches was due partly to incorrect treatment of seasonality.

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APPENDIX

Definitions and assumptions

In order to prove the results claimed in Theorems 1 and 2 we need to establish some definitions and assumptions.

(L.1) For fixed but arbitrary η_1, ϕ^+, where $\eta \in \Theta$, and $\phi^+ \in \Phi$, let

$$\rho\left(\phi^+, \eta\right) = \int \log p\left(\varphi(\psi, \phi^+); \xi\right) dF\left(\varphi(\psi, \phi^+); \xi\right),$$

$$\eta \in \Theta, \phi^+ \in \Phi$$

If $\eta \neq \eta_1$, then

$$\rho\left(\phi, \eta\right) < \rho\left(\phi^+, \eta\right)$$

Let $I_\eta\left(\phi, \eta\right)$, denote the marginal Fisher information for η in the parametric model. Then we assume that the matrix $I_\eta\left(\phi, \eta\right)$ is positive definite for all $\eta \in \Theta$ and $\phi \in \Phi$.

(L.2) Assume that for all $r, s = 0, \cdots, 4, r + s \leq 4$, the derivative

$$\frac{\partial^{r+s}}{\partial \eta^r \partial \eta^s} \log p\left(\varphi(\psi, \phi^+); \xi\right)$$

exists for almost all d and that

$$E\left\{\sup_\eta \sup_\phi \left| \frac{\partial^{r+s}}{\partial \eta^r \partial \eta^s} \log p\left(\varphi(\psi, \phi^+); \xi\right)\right|^2\right\} < \infty.$$

(A.1) The marks y take values in a compact set $\mathbf{Y} \subset R^p$.

(A.2) The observations $\{(d_i, y_i)\}_{i=1,\dots}$ are a sequence of stationary and ergodic random vectors.

(A.3) ϑ_{10} takes the values in the interior of Θ, a compact subset in R^p and ϕ takes the values in the interior of Λ, a compact subset of R.

$$\Lambda = \left\{ g \in C^2[t_I, t_F] : g(t') \in \text{int}\,(\Lambda) \quad \text{for } \forall t' \in [t_I, t_F] \right\}.$$

(A.4) Let Ξ be a compact subset of R such that $\varphi\left(\psi(\bar{d}, \bar{y}; \vartheta_1), \phi(t')\right) \in \Xi$ for all $t' \in [t_I, t_F]$, $y \in \mathrm{Y}$, $\vartheta_1 \in \Theta$ and $\phi \in \Lambda$.

(A.5) The matrix

$$\Sigma_{\vartheta_1} = E\left(\frac{\partial^2}{\partial \vartheta_1 \partial \vartheta_1^T} Q\left(\varphi\left(\psi(\bar{d}, \bar{y}; \vartheta_1), \phi(t')\right); d\right)\right)$$

is positive definite.

(B.1) The kernel function $K(\cdot)$ is of order $k > 3/2$ with support $[-1, 1]$ and it has bounded $k+2$ derivatives.

(B.2) For $r = 1, \cdots, 10+k$ the functions $\partial^r \varphi(m)/\partial m^r$ and $\partial^r V(\mu)/\partial \mu^r$ exist and they are bounded in their respective supports.

(B.3) d is a strong mixing process where the mixing coefficients must satisfy for some $p > 2$ and r being a positive integer

$$\sum_{i=1}^{\infty} i^{r-1} \alpha(i)^{1-2/p} < \infty.$$

Furthermore, for some even integer q satisfying $\frac{(k+2)(3+2k)}{(2k-3)} \leq q \leq 2r$

$$E\left\{|d|^q\right\} < \nu,$$

where ν is a constant not depending on t'.

(B.4) Let f denote the marginal density of t', and let $f(\cdot|t')$ denote the conditional density function of d given t'. f and $f(\cdot|t')$ has $k+2$ bounded derivatives uniformly in $[t_I, t_F]$.

(B.5) Let

$$M\left(\phi; \vartheta_1, t'\right) = E\left\{ \frac{\partial}{\partial \eta} Q\left(\varphi\left(\psi(\bar{d}, \bar{y}; \vartheta_1), \phi\right); d\right) \middle| t' \right\}.$$

For each fixed ϑ_1 and t', let $\phi_{\vartheta_1}(t')$ the unique solution to $M\left(\phi; \vartheta_1, t'\right) = 0$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{\vartheta_1 \in \Theta} \sup_{t' \in [t_I, t_F]} \left| \phi_{\vartheta_1}(t') - \phi(t') \right| < \epsilon$$

whenever

$$\sup_{\vartheta_1 \in \Theta} \sup_{t' \in [t_I, t_F]} \left| M\left(\phi(t'); \vartheta_1, t\right) \right| < \delta.$$

(B.6) The sequence of bandwidths must satisfy $h = O(n^{-\alpha})$ where

$$\frac{1}{4k} < \alpha < \frac{1}{4} \frac{q - (2+p)}{q + (2+p)}.$$

The following result is also needed to prove Theorems 1 and 2,

Lemma A.1 *Consider the following expression*

$$\frac{1}{nh}\sum_{i=1}^{n}\left[K\left(\frac{\tau-t_i}{h}\right)\varphi\left(\psi(\bar{d},\bar{y};\vartheta_1),\phi\right)-E\left\{K\left(\frac{\tau-t}{h}\right)\varphi\left(\psi(\bar{d},\bar{y};\vartheta_1),\phi\right)\right\}\right]$$

and define

$$W_i=\frac{1}{h}K\left(\frac{\tau-t_i}{h}\right)\varphi\left(\psi(\bar{d},\bar{y};\vartheta_1),\phi\right)-E\left\{K\left(\frac{\tau-t}{h}\right)\varphi\left(\psi(\bar{d},\bar{y};\vartheta_1),\phi\right)\right\}$$

Then, for $\epsilon > 0$

$$P\left\{\left|\frac{1}{n}\sum W_j\right|>\epsilon\right\}\leq\frac{E\left[(\sum W_i)^q\right]}{n^q\epsilon^q}\leq$$

$$\frac{1}{n^q\epsilon^q}C\left\{n^{q/2}\sum_{i=P}^{\infty}i^{q/2-1}\alpha(i)^{1-2/p}+\sum_{j=1}^{q/2}n^jP^{q-j}\nu^j\right\}$$

for any integers n *and* P *with* $0 < P < n$.

Proof of Lemma A.1

Under assumptions (A.2) and (B.3) the process $W_1,\cdots,W_n$ is strong mixing and therefore Theorem 1 from Cox and Kim (1995) applies and the sequence of inequalities holds.

Proof of Theorem 1

The proof is in two parts. First we show (15). In order to do this, we claim

$$\sup_{\vartheta_1}\sup_{t}|\hat{\phi}_{\vartheta_1}(t)-\phi(t)|=o_p\left(n^{-1/4}\right) \tag{41}$$

The proof of this expression follows the same steps as the proof of Lemma 5 from Severini and Wong (1992), p. 1784. The bias term must be treated in the same way as it is there. With respect to the variance term, an additional result must be included to account for the dependence. In fact, under assumptions (A.2) and (B.3) Lemma A.1 applies and, proceeding as for Severini and Wong (1992) in the proof of Lemma 8, the proof of (41) is completed.

Finally, the proof of (15) consists of verifying conditions I (Identification), S (Smoothness) and NP (Nuissance Parameter) from Severini and Wong (1992). Condition NP(a) is the result already shown in (41). Condition NP(b) (least favorable curve) is immediate from Lemma 6 of Severini and Wong (1992). By assuming (L.2) the smoothness condition holds. Finally, assumption (L.1) implies I. Then, under assumptions (A.1) to (A.3), a Uniform Weak Law of Large Numbers and a Central limit theorem for stationary and ergodic processes (see for example Wooldridge, 1994) holds, propositions 1 and 2 from Severini and Wong (1992) apply and the proof is completed.

The proof of (16) consists of taking a Taylor expansion around (ϑ_1,ξ). This can be done because of assumption (L.2). Then apply result (41), and the rest of the proof follows the same lines as the proof of Theorem 1 in Staniswalis (1989). Note that instead of using a standard Liapunov CLT here under assumption (A.2), we need to use one for for stationary and ergodic processes.

Proof of Theorem 2

The proof of (31) follows exactly the same lines as the proof of (15) in Theorem 1, but instead of using (L.1) for identification and (L.2) for smoothness, we use respectively (A.5) and (A.1) to (A.4). Condition NP(b) (least favorable curve) is immediate from Lemma 6 of Severini and Wong (1992). This is due to the fact that we assume that the conditional density function belongs to the exponential family.

The proof of (32) is equal to the proof of (16) except for that (L.2) is replaced by (B.2).

References

[1] Andersen, T. and Bollerslev, T. (1997). "Heterogenous Information Arrivals and Return Volatility Dynamics: Unrecovering the Long-Run in High Frequency Returns," *The Journal of Finance*, Vol. LII, n. 3, 975-1005.

[2] Andersen, T. and Bollerslev, T. (1998). "Deustche Mark-Dollar Volatility: Intraday Activity Patterns, Macroeconomic Announcements, and Longer Run Dependencies," *The Journal of Finance*, Vol. LIII, n. 1, 219-265.

[3] Baillie, R. and Bollerslev, T. (1990). "Intra-Day and Inter-Market Volatility in Foreign Exchange Rates," *Review of Economic Studies*, 58, 565-585.

[4] Bauwens, L. and Giot, P. (2001). *Econometric Modelling of Stock Market Intraday Activity.* Kluwer Academic Press.

[5] Bauwens, L. and Giot, P. (2000). "The logarithmic ACD model: an application to the bid-ask quote process of three NYSE stocks," *Annales d'Economie et de Statistique*, Vol. 60, 117-149

[6] Bauwens, L. Giot, P. Grammig, J. and Veredas, D. (2000). "A comparison of financial duration models via density forecast," CORE DP 2000/60. Université catholique de Louvain.

[7] Bauwens, L. and Veredas, D. (1999). "The Stochastic Conditional Duration Model: A latent factor model for the analysis of financial durations," CORE DP 9958. Université catholique de Louvain. *Forthcoming in Journal of Econometrics.*

[8] Beltratti, A. and Morana, C. (1999). "Computing value at risk with high frequency data," *Journal of Empirical Finance*, 6, 431-455

[9] Bollerslev, T. and Domowitz, I. (1993). "Trading Patterns and Prices in the Interbank Foreign Exchange Market," *The Journal of Finance*, Vol. XLVIII, 4, 1421-1443.

[10] Chen, S.X. (2000). "A Beta Kernel Estimation for Density Functions," *Computational Statistics and Data Analysis*, 31, 131-145.

[11] Cox, D.R. and Kim, T. Y. (1995). "Moment bounds for mixing random variables useful in nonparametric function estimation," *Stochastic processes and their applications*, 56, 151-158.

[12] Diebold, F.X., Gunther, T.A., and Tay, A.S. (1998). "Evaluating density forecasts, with applications to financial risk management," *International Economic Review* 39, 863-883.

[13] Drost,F.C. and Werker, B.J.M. (2001). "Efficient estimation in semiparametric time series: the ACD model," CentER Discussion Paper 2001-11, Tilburg University.

[14] Engle, R.F., Ito, T. and Lin, W.L. (1990). "Meteor Showers or Heat Waves? Heteroskedastic Intra-Daily Volatility in the Foreign Exchange Market," *Econometrica*, Vol. 58, n. 3, 525-42.

[15] Engle, R.F. and Russell, J.R. (1997). "Forecasting the Frequency of Changes in Quoted Foreign Exchange Prices with the ACD model," *Journal of Empirical Finance* n. 4, 187-212.

[16] Engle, R.F. and Russell, J.R. (1998). "Autoregressive conditional duration: a new approach for irregularly spaced transaction data," *Econometrica* Vol. 66, n. 5, 1127-1162.

[17] Engle, R.F. and Russell, J.R. (1995). "Autoregressive conditional duration: a new model for irregularly spaced data," unpublished manuscript , University of California, San Diego.

[18] Engle, R.F. (2000). "The Econometrics of Ultra High Frequency Data," *Econometrica* Vol. 68, n. 1, 1-22.

[19] Eubank, R.L. (1988). *Spline Smoothing and Nonparametric Regression*, New York: Marcell Decker.

[20] Fan, J., Heckman, N. E. and Wand, M. P. (1995). "Local Polynomial Kernel Regression for Generalized Linear Models and Quasi-Likelihood Functions". *Journal of the American Statistical Association*, 90, 141-150.

[21] Fernandes, M. and Grammig, J. (2001). "A family of autoregressive conditional duration models." CORE DP 2001/36. Université catholique de Louvain.

[22] Gerhard, F. and Haustch, N. (1999). "Volatility Estimation on the Basis of Price Intensities," *Journal of Empirical Finance*, 9, 57-89.

[23] Ghysels, E., Gouriéroux, C. and Jasiak, J. (1997). "Stochastic Volatility duration models," Working paper 9746. CREST Paris.

[24] Gouriéroux, C., Monfort, A. and Trognon, A. (1984). "Pseudo Maximum Likelihood Methods: Theory," *Econometrica* Vol. 52, n. 3, 681-700.

[25] Grammig, J. and Maurer K.O. (1999). "Non-monotonic hazard functions and the autoregressive conditional duration model," *The Econometrics Journal*, 3, 16-38.

[26] Harris, L. (1986). "A transaction data study of weekly and intradaily patterns in stock returns," *Journal of Financial Economics*, 16, 99-117.

[27] McCullagh, P. and Nelder, J.A. (1983). *Generalized linear models*, London :Chapman and Hall.

[28] Neyman, J. and E. L. Scott (1951). "On certain methods of estimating the linear structural relation". *Annals of Mathematical Statistics*, 22, 352-361.

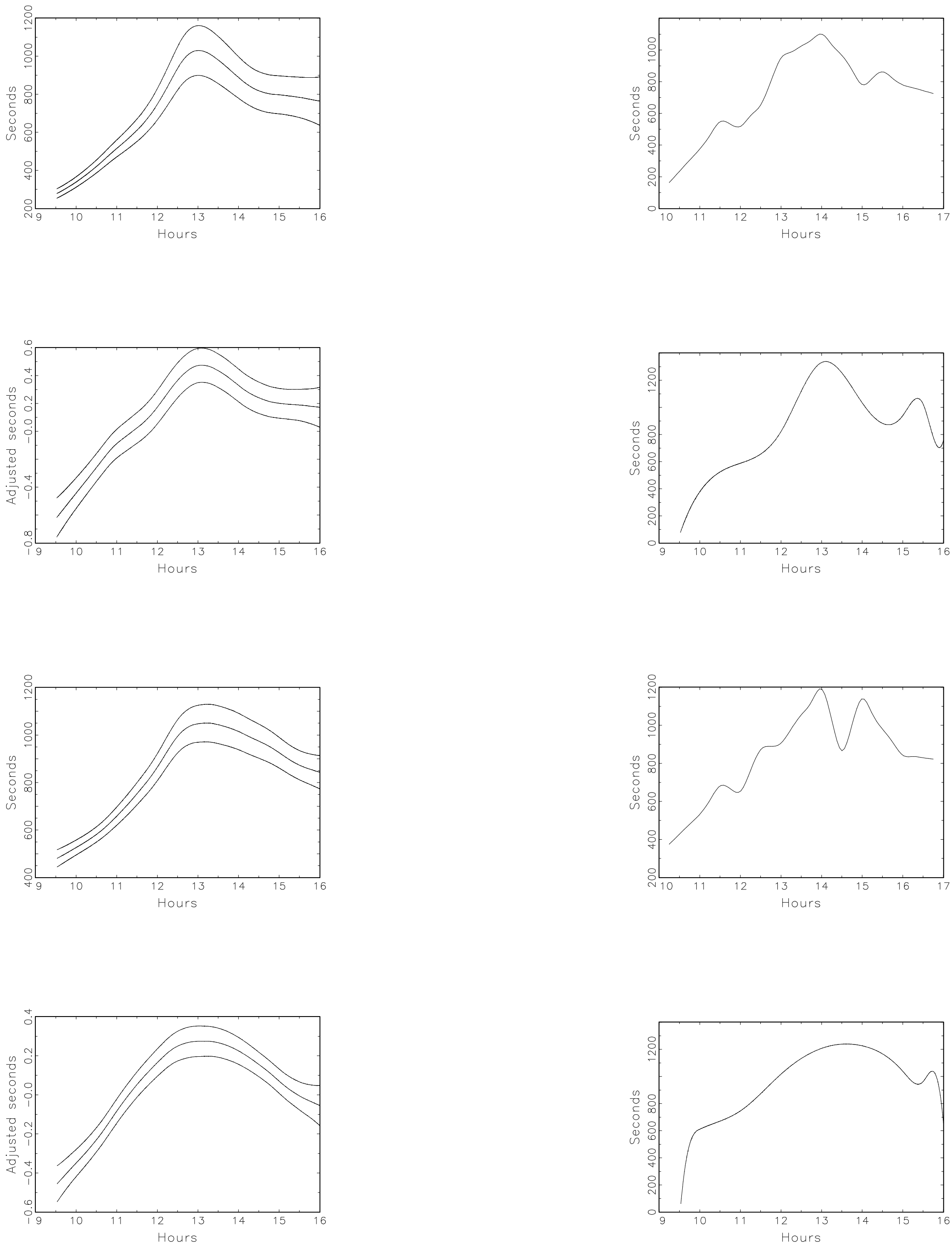
[29] Severini, T.A. and Staniswalis, J.G. (1994). "Quasi-likelihood estimation in semiparametric models," *Journal of the American Statistical Association*, 89, 501-511.

[30] Severini, T.A. and Wong, W. H. (1992). "Profile likelihood and conditionally parametric models," *Annals of Statistics*, 20, 1768-1802.

[31] Staniswalis, J.G. (1989). "On the kernel estimate of a regression function in likelihood based models," *Journal of the American Statistical Association*, 84, 276-283.

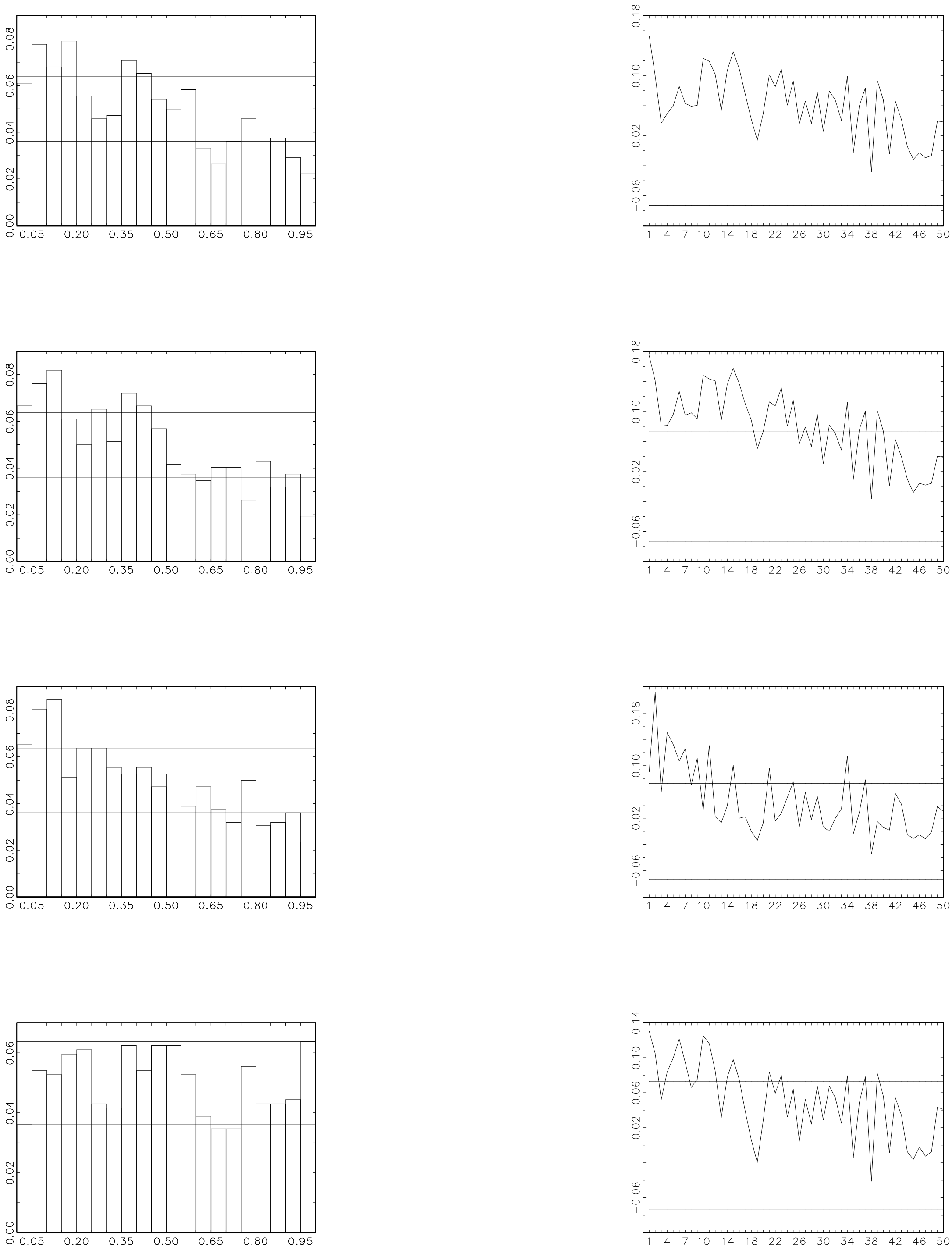
[32] Wooldridge, J.M. (1994). "Estimation and inference for dependent processes," *Handbook of Econometrics*, vol. 4. Engle, R.F. and D.L. McFadden eds. Elsevier Science. New York.

[33] Zhang, M.Y., Russell, J.R. and Tsay, R.T. (2001). "A nonlinear autoregressive conditional duration model with applications to financial transaction data," *Journal of Econometrics*, 104-1, 179-207.



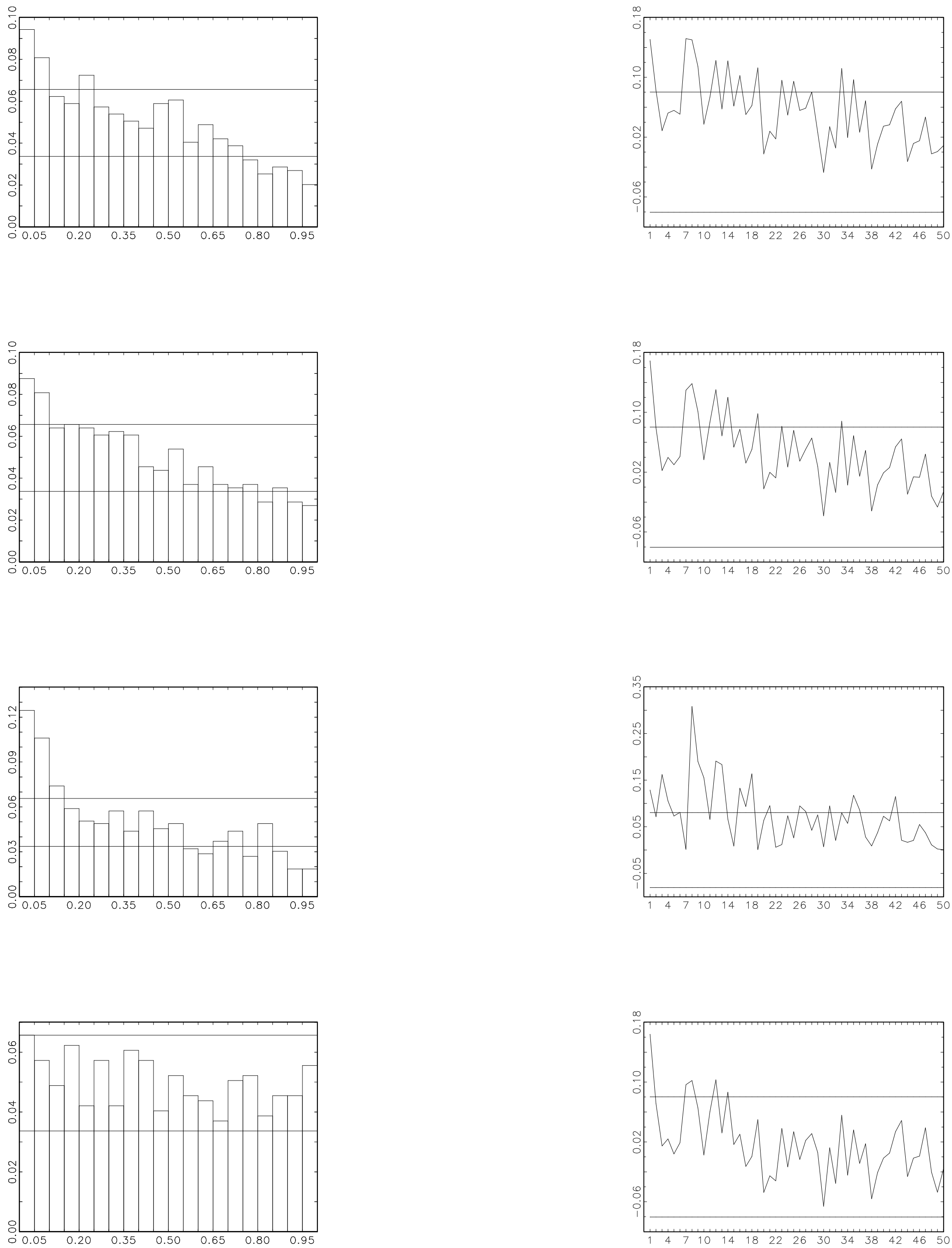
The first two rows (first block) are for price durations. The last two (second block) are for volume durations. In each block: top left BiNW, top right BiSp, bottom left UniNW (note that they are centered around the mean) and bottom right UniSp. See Section 3.3 for an explanation of the acronyms. The GG distribution is chosen.

Figure 6: Estimated seasonal curves



Histograms and autocorrelograms for z. The left column shows the histograms and in right column the Spearman's ρ coefficients. The distribution is the generalized gamma. Rows from top to bottom: BiSp, BiNW, UniSp and UniNW. See Section 3.3 for an explanation of the acronyms.

Figure 7: Out-of-sample density forecast evaluation for price durations



Histograms and autocorrelograms for z. The left column shows the histograms and the right column the Spearman's ρ coefficients. The distribution is the generalized gamma. Rows from top to bottom: BiSp, BiNW, UniSp and UniNW. See Section 3.3 for an explanation of the acronyms.

Figure 8: Out-of-sample density forecast evaluation for volume durations