Hierarchical implementation of the analytical singularity evaluation technique

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***Abstract*—Using the Method of Moments, the computation of the near-field interactions requires to evaluate 4D singular integrals. Recently, a method has been proposed to compute these integrals fully analytically, rapidly providing accurate results. In this paper, we reformulate the analytical technique to highlight the different intermediate results that are involved in the full procedure. We show that, by storing and reusing intermediate results, the evaluation of the singular integrals is accelerated by several orders of magnitudes.**

***Index Terms*—antenna, propagation, measurement.**

I. Introduction

In computational electromagnetics, the methods based on Surface Integral Equation (SIE), and more particularly the Method of Moments (MoM), play an important role for many reasons [1]. One is the reduced number of unknowns one has to deal with due to the fact that only the surfaces between different media have to be meshed. The MoM also implicitly satisfies the radiation boundary conditions, so that no absorbing layer is required to model open problems. But the greatest advantage it offers is its high modularity and flexibility, which have been extensively used in the literature to devise acceleration techniques: the Macro-Basis Functions (MBF) for complex structures [2], the Multi-Level Fast Multipole Algorithm (MLFMA) for large-scale problems [3] or the use of specialized Green's functions, *e.g.* for stratified media [4].

However, the fast and accurate computation of the near-field interactions is still a challenge since the Green's function and its gradient must be convolved with the Basis Functions (BF) and integrated over the Testing Functions (TF). The computation of near-field interactions therefore requires to routinely evaluate 4D integrals whose integrand is singular. To deal with this special kind of integrals, many techniques have been proposed, which can be classified in to two categories.

The singularity cancellation techniques [5], [6] rely on the use of specialized quadrature and irregular sampling of the Green's function over the BF and TF to evaluate the integral in a purely numerical way. The positions of the sampling points are concentrated around the singular part, so that the weighting coefficient applied to each sample is compensating for the singularity of the Green's function.

The singularity extraction techniques [7], [8], [9] aim at simplifying the problem by integrating analytically the Green's function or its singular part over a limited number of dimensions. The analytical integration of the Green's function provides a result that gets less singular as one integrates over every successive dimensions. Therefore, after a few analytical integrations, the rest of the integral can be computed numerically. Recently, a method has been proposed to perform the full 4D integral in a fully analytical way.

A more detailed review of the literature can be found in [9], to which should be added the recent work of Botha *et al.* on the evaluation of the error made when evaluating integrals numerically [10], the recent extensions [11] proposed by Bleszynski *et al.* to their work [12] and the new evaluation technique proposed by Wilton *et al.* [13], [14].

The goal of this paper is to propose an object-oriented framework for the fully analytical evaluation of the near-field interactions proposed in [9]. Using this new framework, the singular integrals can be implemented using a bookkeeping strategy. Reusing intermediate results leads to a huge time saving with respect to a simple brute-force recursive implementation.

In Section 2, the 4D integrals involved in the MoM and the method of [9] are summarized. In Section 3, the whole analytical integration procedure is rapidly reminded using a new framework. This new framework clearly highlights the intermediate results involved in the procedure and the interactions between them. Using this framework, the analytical evaluation of the integrals has been implemented using a recursive-based and a bookkeeping based strategies. Both methods are compared in Section 4 in terms of time and number of terms computed. Using both metrics, the bookkeeping strategy outperforms the recursive one by several orders of magnitude.

II. Geometrical Method of Moments

Using the equivalence principle, the scattering of incident fields by a structure is reformulated in terms of equivalent currents on the surface of the structure [15]. These equivalent currents can be found by imposing proper boundary conditions, which depend on the formulation used [16]. Using the MoM, the equivalent currents on the interface between different media are discretized using a set of BFs and the boundary conditions are imposed on average along a set of TFs. Then, the equivalent currents can be found by solving a linear system of equations:

$$\underline{\underline{Z}}_{\text{tot}} \cdot \mathbf{x} = -\mathbf{b}, \tag{1}$$

with $\underline{\underline{Z}}_{\text{tot}}$ the impedance matrix, whose (m, n) entry is related to the fields generated by unit currents of the n^{th} basis function along the m^{th} testing function through one or several media. If the BF and TF are div-conforming, the fields generated by a given BF inside a given medium of permittivity ε and permeability μ on a chosen TF can be computed by evaluating the following integrals:

$$\begin{aligned} Z^{EJ} &= \eta^2 Z^{HM} && \text{(2a)} \\ &= -\frac{j\eta}{4\pi k}\iint\iint \frac{\exp(-jkR)}{R}\Big(k^2\mathbf{F}_B(\mathbf{r})\cdot\mathbf{F}_T(\mathbf{r}') \\ &\qquad -\big(\boldsymbol{\nabla}\cdot\mathbf{F}_B(\mathbf{r})\big)\big(\boldsymbol{\nabla}'\cdot\mathbf{F}_T(\mathbf{r}')\big)\Big)dS(\mathbf{r})dS'(\mathbf{r}'), \\ Z^{EM} &= -Z^{HJ} && \text{(2b)} \\ &= \frac{1}{4\pi}\iint\iint \boldsymbol{\nabla}'\Big(\frac{\exp(-jkR)}{R}\Big) \\ &\qquad \times \mathbf{F}_B(\mathbf{r})\cdot\mathbf{F}_T(\mathbf{r}')dS(\mathbf{r})dS'(\mathbf{r}'), \end{aligned}$$

with $\eta = \sqrt{\mu/\varepsilon}$ the impedance of the medium, $k = \omega\sqrt{\varepsilon\mu}$ the wavenumber of the excitation, ω the angular frequency of the excitation, $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}'$ the derivative operator over the $\mathbf{r}$ and $\mathbf{r}'$ coordinates, respectively, $R = |\mathbf{r}-\mathbf{r}'|$ the distance between the points $\mathbf{r}$ and $\mathbf{r}'$ and $\mathbf{F}_B(\mathbf{r})$ and $\mathbf{F}_T(\mathbf{r}')$ the value of the BF and TF at positions $\mathbf{r}$ and $\mathbf{r}'$, respectively. Z^{EJ} (resp. Z^{EM}) is the value of the electric field generated on the TF by electric (resp. magnetic) currents on the BF, while Z^{HJ} (resp. Z^{HM}) is the value of the magnetic field generated on the TF by these same currents.

To express the impedance matrix as a weighted sum of purely geometrical terms, one can apply a Taylor expansion of the R dependency of the phase term [17], [9]. In order to improve the convergence of the series, the Taylor expansion is expanded around the value $R = R_0$, R_0 corresponding to the mean distance between the BF and TF. It leads to the rapidly converging series [17], [9]:

$$Z^{EJ} = -\frac{j\eta_0\exp(-jkR_0)}{4\pi k}\sum_{i=0}^{N} K_i^N(k,R_0)(k^2A_i - B_i) \tag{3a}$$

$$Z^{EM} = \frac{\exp(-jkR_0)}{4\pi}\sum_{i=0}^{N} K_i^N(k,R_0)C_i \tag{3b}$$

with

$$A_i = \iint_{S'_T}\iint_{S_B} R^{(i-1)}\big(\mathbf{F}_B\cdot\mathbf{F}_T\big)dSdS' \tag{4a}$$

$$B_i = \iint_{S'_T}\iint_{S_B} R^{(i-1)}\big(\boldsymbol{\nabla}\cdot\mathbf{F}_B\big)\big(\boldsymbol{\nabla}\cdot\mathbf{F}_T\big)dSdS' \tag{4b}$$

$$C_i = (i-1)\iint_{S'_T}\iint_{S_B} R^{(i-3)}\mathbf{R}\times\mathbf{F}_B\cdot\mathbf{F}_T dSdS' \tag{4c}$$

$$K_i^N(k,R_0) = \sum_{\alpha=i}^{N}\frac{(-jk)^\alpha}{(\alpha-i)!\,i!}(-R_0)^{(\alpha-i)} \tag{4d}$$

and S'_T and S_B the domains over which the TF and BF are not vanishing, respectively. This results is convenient for two reasons. First, the purely geometrical integrals involved in (4) can be computed analytically. A second advantage is that every of these 4D integrals is independent from the frequency and material parameters. Once the geometrical terms have been evaluated, the impedance matrix can be computed for many different frequencies or for many different material parameters at nearly no cost.

III. Analytical solution of the geometrical integrals

Considering polynomial basis and testing functions on polygonal supports and using a specialized system of coordinates in which $\hat{\mathbf{x}}$ is parallel to both the BF and TF, $\hat{z}$ is perpendicular to the BF and the latter is located in the plane $z = 0$, the A_i term can be expressed as a weighted sum of integrals of the type:

$$\iint_{S'_T}\iint_{S_B} x^{k_1}y^{k_2}x'^{k_3}y'^{k_4}z'^{k_5}R^{k_0}dSdS' \tag{5}$$

with

$$R = \sqrt{(x-x')^2+(y-y')^2+(z')^2}. \tag{6}$$

Due to the lack of space, the case of parallel basis and testing functions will not be considered in this work.

The integrand of (5) is easy to integrate analytically except for the factor R^{k_0}, which is depending on all the coordinates. To circumvent this problem, a change of variables is used:

$$\begin{aligned} u_1 &= \frac{x-x'}{\sqrt{2}} \quad & u_4 &= \frac{x+x'}{\sqrt{2}} \\ u_2 &= \frac{y-y'}{\sqrt{2}} \quad & u_5 &= \frac{y+y'}{\sqrt{2}} \\ u_3 &= z' \end{aligned} \tag{7}$$

The integrand of (5) can be reformulated in the new basis as a sum of terms of the shape:

$$I_1\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\k_4\\k_5\end{pmatrix} = \int_{\Omega^4} u_1^{k_1}u_2^{k_2}u_3^{k_3}u_4^{k_4}u_5^{k_5}R^{k_0}dV_4^5 \tag{8}$$

where we introduced the shorthand notation I_1 whose parameters correspond to the exponents of the different factors in (8), Ω^4 the 4D polyhedron corresponding to the support of the BF and TF in the 5D space parameterized by the u_i coordinates and dV_i^j indicating that a iD integral is performed in jD space. It should be noted that all the exponents are integers. Moreover, except for k_0, they are all positive.

Following the same reasoning with the B_i and C_i terms, one finds generic integrals of the same form as (8). To solve these integrals, we follow the method described in [9].

First, the u_5 dependency of the integrand is removed using the fact that within a 4D volume within a 5D space, provided that the BF and TF are not parallel, the u_5 coordinate is linearly depending on the other coordinates, so that

$$\begin{aligned} I_1\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\k_4\\k_5\end{pmatrix} =& \frac{d_{v,1}}{\hat{\mathbf{n}}_{v,1}\cdot\hat{\mathbf{u}}_5} I_1\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\k_4\\k_5-1\end{pmatrix} \\ &-\sum_{j=1}^{4}\frac{\hat{\mathbf{n}}_{v,1}\cdot\hat{\mathbf{u}}_j}{\hat{\mathbf{n}}_{v,1}\cdot\hat{\mathbf{u}}_5} I_1\begin{pmatrix}k_0\\k_1+\delta_{1j}\\k_2+\delta_{2j}\\k_3+\delta_{3j}\\k_4+\delta_{4j}\\k_5-1\end{pmatrix} \end{aligned} \tag{9}$$

with $\hat{\mathbf{n}}_{v,1}$ the normal to the 4D volume in the 5D space, $d_{v,1} = \hat{\mathbf{n}}_{v,1}\cdot\mathbf{u}$ for $\mathbf{u}\in\Omega^4$ and δ_{ij} the Kronecker delta, whose value is 1 if $i=j$ and 0 otherwise.

Applying iteratively identity (9), one can reach $k_5 = 0$, which means that the integrand does not depend upon variable u_5 anymore. If $k_5 = 0$, one can do the orthogonal projection of the volume into the $u_5 = 0$ space and apply the divergence theorem on the 4D volume of integration, leading to

$$I_1\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\k_4\\0\end{pmatrix} = \frac{1}{|\hat{\mathbf{n}}_{v,1}\cdot\hat{\mathbf{u}}_5|}\sum_{\{p|\hat{\mathbf{n}}_p\cdot\hat{\mathbf{u}}_4\neq 0\}}\frac{(\hat{\mathbf{n}}_p\cdot\hat{\mathbf{u}}_4)}{(k_4+1)} I_2^p\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\k_4+1\end{pmatrix} \tag{10}$$

with

$$I_2^p\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\k_4\end{pmatrix} = \iiint_{\Omega_p^3} u_1^{k_1}u_2^{k_2}u_3^{k_3}u_4^{k_4}R^{k_0}dV_3^4, \tag{11}$$

Ω_p^3 the p^{th} 3D volume that is part of the boundary of the projected 4D volume and $\hat{\mathbf{n}}_p$ the outer normal to this 3D volume in the 4D space parameterized by $\{u_1,u_2,u_3,u_4\}$.

Once this step has been performed, once again, the u_4 dependency of the integrand can be removed by recursively applying

$$\begin{aligned} I_2^p\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\k_4\end{pmatrix} =& \frac{d_p}{\hat{\mathbf{n}}_p\cdot\hat{\mathbf{u}}_4} I_2^p\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\k_4-1\end{pmatrix} \\ &-\sum_{j=1}^{3}\frac{\hat{\mathbf{n}}_p\cdot\hat{\mathbf{u}}_j}{\hat{\mathbf{n}}_p\cdot\hat{\mathbf{u}}_4} I_2^p\begin{pmatrix}k_0\\k_1+\delta_{1j}\\k_2+\delta_{2j}\\k_3+\delta_{3j}\\k_4-1\end{pmatrix}. \end{aligned} \tag{12}$$

with $d_p = \hat{\mathbf{n}}_p\cdot\mathbf{u}$ for $\mathbf{u}\in\Omega_p^3$. Once the integrand is made independent from u_4, one can project the 3D volume of integration into the subspace $u_4 = 0$ and then apply the divergence theorem once again to obtain the 3D integral in terms of 2D integrals.

To do so, two recursive relations can be used. If $k_1 = k_2 = k_3 = 0$, one can use

$$I_2^p\begin{pmatrix}k_0\\0\\0\\0\\0\end{pmatrix} = \frac{1}{|\hat{\mathbf{n}}_p\cdot\hat{\mathbf{u}}_4|}\sum_{\{q|S_q\in\partial V_p\}}\frac{d_q}{k_0+3} I_3^q\begin{pmatrix}k_0\\0\\0\\0\end{pmatrix}. \tag{13}$$

Otherwise, one should use recursively the following identity:

$$\begin{aligned} I_2^p\begin{pmatrix}k_0\\k_1\\k_2\\k_3\\0\end{pmatrix} =& \frac{1}{|\hat{\mathbf{n}}_p\cdot\hat{\mathbf{u}}_4|}\sum_{\{q|S_q\in\partial V_p\}}\frac{(\hat{\mathbf{n}}_q\cdot\hat{\mathbf{u}}_\alpha)}{k_0+2} I_3^q\begin{pmatrix}k_0+2\\k_1-\delta_{\alpha 1}\\k_2-\delta_{\alpha 2}\\k_3-\delta_{\alpha 3}\end{pmatrix} \\ &-\frac{k_\alpha-1}{k_0+2} I_2^p\begin{pmatrix}k_0+2\\k_1-2\delta_{1\alpha}\\k_2-2\delta_{1\alpha}\\k_3-2\delta_{1\alpha}\\0\end{pmatrix}, \end{aligned} \tag{14}$$

with S_q the q^{th} 2D surface of the original 4D volume of integration that has been projected in the space $u_4 = u_5 = 0$, V_p the 3D volume Ω_p^3 that has been projected into the space $u_4 = 0$, ∂V_p its boundary, $\hat{\mathbf{n}}_q$ the outer normal to the 2D surface q, $d_q = \hat{\mathbf{n}}_q\cdot\mathbf{u}$ for $\mathbf{u}\in S_q$ and

$$I_3^q\begin{pmatrix}k_0\\k_1\\k_2\\k_3\end{pmatrix} = \int_{S_q} u_1^{k_1}u_2^{k_2}u_3^{k_3}R^{k_0}dS. \tag{15}$$

In order to transform these 2D integrals into 1D integrals, one needs to apply the surface divergence theorem. In order to simplify the formulation, the $\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2 - \hat{\mathbf{u}}_3$ system of coordinates can be rotated into the $\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2 - \hat{\mathbf{v}}_3$ system of coordinates, such that $\hat{\mathbf{v}}_3$ is perpendicular to the surface ($\hat{\mathbf{v}}_3 = \hat{\mathbf{n}}_q$). In this case, the polynomial involved in (15) can be reformulated in the new basis:

$$I_3^q\begin{pmatrix}k_0\\k_1\\k_2\\k_3\end{pmatrix} = \sum_{j_1=0}^{k_{\text{tot}}}\sum_{j_2=0}^{k_{\text{tot}}-j_1} A_{j_1,j_2}^{k_{\text{tot}}} d_q^{k_{\text{tot}}-j_1-j_2} I_4^q\begin{pmatrix}k_0\\j_1\\j_2\end{pmatrix} \tag{16}$$

with $A_{j_1,j_2}^{k_{\text{tot}}}$ the coefficients corresponding to the change of basis, $k_{\text{tot}} = k_1+k_2+k_3$ and

$$I_4^q\begin{pmatrix}k_0\\k_1\\k_2\end{pmatrix} = \iint_{S_q} v_1^{k_1}v_2^{k_2}R^{k_0}dS. \tag{17}$$

The 2D integrals of (17) can be solved recursively using the identities

$$
\begin{aligned}
I_4^q \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} = \sum_{\{r|l_r \in \partial S_q\}} \frac{(\hat{\mathbf{n}}_r \cdot \hat{\mathbf{v}}_\alpha)}{k_0+2} I_5^{q,r} \begin{pmatrix} k_0+2 \\ k_1-\delta_{\alpha 1} \\ k_2-\delta_{\alpha 2} \end{pmatrix} \\
- \frac{k_\alpha - 1}{k_0+2} I_4^q \begin{pmatrix} k_0+2 \\ k_1-2\delta_{\alpha 1} \\ k_2-2\delta_{\alpha 2} \end{pmatrix}
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
I_4^q \begin{pmatrix} k_0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{k_0+2} I_6^q \left(k_0+2\right) - \sum_{\{r|l_r \in \partial S_q\}} \frac{|d_q|^{k_0+2}}{k_0+2} \\
\times \left[\arctan\left(\frac{l}{d_r}\right)\right]_{\mathbf{u}_r^-}^{\mathbf{u}_r^+}
\end{aligned}
\tag{19}
$$

with $\hat{\mathbf{n}}_r$ the outer normal to the edge r relative to surface q, l_r the support of edge r, $d_r = \hat{\mathbf{n}}_r \cdot \mathbf{u}$ for $\mathbf{u} \in l_r$, $\mathbf{u}_r^-$ and $\mathbf{u}_r^+$ the position of the two extreme points of l_r, $\hat{\mathbf{l}} = \mathbf{u}_r^+ - \mathbf{u}_r^- / |\mathbf{u}_r^+ - \mathbf{u}_r^-|$, $l = \mathbf{u} \cdot \hat{\mathbf{l}}$ and the new shorthand notations

$$
I_5^{q,r} \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} = \int_{l_r} v_1^{k_1} v_2^{k_2} R^{k_0} dl, \tag{20}
$$

$$
I_6^q \left(k_0\right) = \int_{\partial S_q} \frac{R^{k_0}}{P^2} \mathbf{P} \cdot \hat{n}\, dS, \tag{21}
$$

$$
[f(\mathbf{u})]_{\mathbf{u}_1}^{\mathbf{u}_2} = f(\mathbf{u}_2) - f(\mathbf{u}_1), \tag{22}
$$

with $\mathbf{P} = l\hat{\mathbf{l}} + d_r \hat{\mathbf{n}}_r$ and $P^2 = \mathbf{P} \cdot \mathbf{P}$. The second term of (19) corresponds to the singular term described in [7].

To solve I_6^q, one can recursively use the identity:

$$
I_6^q \left(k_0\right) = d_q^2 I_6^q \left(k_0 - 2\right) + I_7^q \left(k_0 - 2\right) \tag{23}
$$

$$
I_6^q \left(k_0\right) = \frac{1}{d_q^2} I_6^q \left(k_0 + 2\right) - \frac{1}{d_q^2} I_7^q \left(k_0\right) \tag{24}
$$

$$
I_6^q \left(-1\right) = \frac{1}{d_q} \sum_{\{r|l_r \in \partial S_q\}} \left[\arctan\left(\frac{l\, d_q}{|d_r||\mathbf{u}|}\right)\right]_{\mathbf{u}_r^-}^{\mathbf{u}_r^+} \tag{25}
$$

$$
I_6^q \left(0\right) = \sum_{\{r|l_r \in \partial S_q\}} \left[\arctan\left(\frac{l}{d_r}\right)\right]_{\mathbf{u}_r^-}^{\mathbf{u}_r^+}, \tag{26}
$$

with

$$
I_7^q \left(k_0\right) = \iint_{S_q} \boldsymbol{\nabla} \cdot \left(R^{k_0} \mathbf{P}\right) dS. \tag{27}
$$

The I_7^q term can be simplified by considering that

$$
I_7^q \left(k_0\right) = \sum_{\{r|l_r \in S_q\}} d_r I_8^r \begin{pmatrix} k_0 \\ 0 \end{pmatrix} \tag{28}
$$

with

$$
I_8^r \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = \int_{l_r} l^{k_1} R^{k_0} dl. \tag{29}
$$

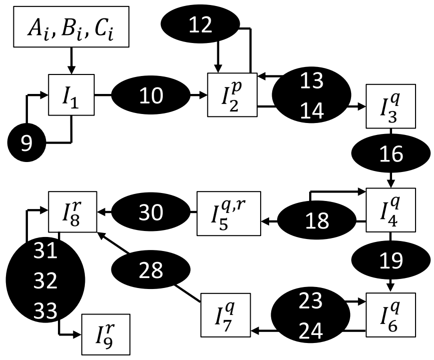


Fig. 1. Schematic of the sequence of operations required to solve analytically the A_i, B_i and C_i terms. The square boxes represent intermediate results. The rounded boxes represent operations whose corresponding equations' numbers are indicated.

To solve the $I_5^{q,r}$ term involved in (18), the $\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2 - \hat{\mathbf{v}}_3$ basis is rotated to match the $\hat{\mathbf{l}} - \hat{\mathbf{n}}_r - \hat{\mathbf{n}}_q$ directions. The polynomial factor in the integrand should be changed accordingly:

$$
I_5^{q,r} \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} = \sum_{j=0}^{k_1+k_2} A_j^{(k_1+k_2)} I_8^r \begin{pmatrix} k_0 \\ j \end{pmatrix} \tag{30}
$$

with A_j^k the coefficients corresponding to the change of basis.

Last, the remaining 1D integrals can be solved recursively using the following identities

$$
\begin{aligned}
I_8^r \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = \frac{1}{k_0+2} I_9^r \begin{pmatrix} k_0+2 \\ k_1-1 \end{pmatrix} \\
- \frac{k_1-1}{k_0+2} I_8^r \begin{pmatrix} k_0+2 \\ k_1-2 \end{pmatrix}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
I_8^r \begin{pmatrix} k_0 \\ 0 \end{pmatrix} = \frac{k_0}{k_0+1} (d_q^2 + d_r^2) I_8^r \begin{pmatrix} k_0-2 \\ 0 \end{pmatrix} \\
+ \frac{1}{k_0+1} I_9^r \begin{pmatrix} k_0 \\ 1 \end{pmatrix}
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
I_8^r \begin{pmatrix} k_0 \\ 0 \end{pmatrix} = \frac{k_0+3}{(d_q^2+d_r^2)(k_0+2)} I_8^r \begin{pmatrix} k_0+2 \\ 0 \end{pmatrix} \\
- \frac{1}{(d_q^2+d_r^2)(k_0+2)} I_9^r \begin{pmatrix} k_0+2 \\ 1 \end{pmatrix}
\end{aligned}
\tag{33}
$$

$$
I_8^r \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{d_q^2 + d_r^2}} \left[\arctan\left(\frac{l}{\sqrt{d_r^2 + d_q^2}}\right)\right]_{\mathbf{u}_r^-}^{\mathbf{u}_r^+} \tag{34}
$$

$$
I_8^r \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \left[\ln\left(|\mathbf{u}| + l\right)\right]_{\mathbf{u}_r^-}^{\mathbf{u}_r^+} \tag{35}
$$

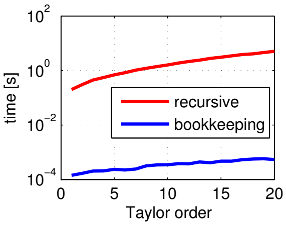
with

$$
I_9^r \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = \left[l^{k_1} |\mathbf{u}|^{k_0}\right]_{\mathbf{u}_r^-}^{\mathbf{u}_r^+} \tag{36}
$$

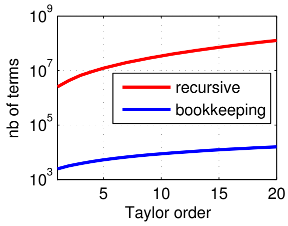
The whole procedure is summarized in Figure 1.

IV. Bookkeeping implementation

The framework used previously has two advantages. First, it highlights the different intermediate results that are involved in



(a)



(b)

Fig. 2. Comparison of the efficiency of recursive-based and bookkeeping-based implementations of the analytical integral evaluation method. The efficiency is assessed for different Taylor orders in Equation (3) and for a pair of linear triangular basis and testing functions that are not parallel. (a) Mean time required to evaluate (3) for one pair of basis and testing function. (b) Number of intermediate results that need to be evaluated for each pair.

the evaluation of the 4D integral. Indeed, for each edge, there is a finite set of I_9^r and I_8^r values that need to be evaluated. Once these have been evaluated, one can evaluate the terms associated to surfaces, etc. Building a structure that can store these intermediate results and then reusing that same structure for every successive pair of (non-parallel) BF and TF, a much faster procedure is reached.

To quantify the efficiency of the bookkeeping strategy, we compared it with a brute-force recursive implementation of the method proposed in [9]. In the latter, the different operations involved in the method are called recursively, so that intermediate results need to be computed several time.

We first compared the mean computation time for a pair of basis and testing functions as a function of the Taylor order involved in Equation (3). The basis and testing functions consist of half Rao-Wilton-Glisson basis functions [18], which correspond to linear functions on triangular support. The position, size and orientation of the basis and testing functions have no impact on the speed of either method (provided that they are not parallel [9]). The computation time is averaged over a huge number of interactions to provide more reliable results, which are displayed in Figure 2(a). Since the time required for computation may strongly depend on the implementation, we also compared the total number of intermediate results computed using both strategies and displayed the results in Figure 2(b). It can be clearly seen that bookkeeping strategy outperforms the brute-force implementation by several orders of magnitude.

V. Conclusion

In this paper, we provide an object-oriented framework for the evaluation of the singular integrals involved in the computation of the MoM impedance matrix. This framework highlights the different intermediate results that are required, so that the integral evaluation can be implemented using a bookkeeping strategy. It is shown that, storing and reusing the intermediate results, the computation of the whole 4D integral can be accelerated by three orders of magnitude.

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