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Abstract

New dynamic models for realized covariance matrices are proposed. The expected value of the realized covariance matrix is specified in two steps: one for each realized variance, and one for the realized correlation matrix. The realized correlation model is a scalar dynamic conditional correlation model. Estimation can be done in two steps as well, and a QML interpretation is given to each step, by assuming a Wishart conditional distribution. The model is applicable to large matrices since estimation can be done by the composite likelihood method.

Keywords: realized covariance, dynamic conditional correlations, covariance targeting, Wishart distribution, composite likelihood.

JEL Classification: C13, C32, C58

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1 Introduction

The availability of measures of daily variances of financial returns, and covariances between these, allows researchers to model time series of covariance matrices. One interest of these models is that they can be used for forecasting future values, which is typically of use in financial applications such as hedging, option pricing, risk management, and portfolio allocation. Another potential interest of models for realized covariance matrices is that they allow researchers to study the macroeconomic and financial determinants of the changes in multivariate volatility. GARCH models can be used for the same purposes - see for example Engle and Rangel (2008) - but since they rely on daily observed returns, in principle they provide less precise estimates and forecasts of variances and covariances than measures based on intraday data.

Models have firstly been proposed for realized variances alone, such as ARFIMA models, see e.g. Andersen, Bollerslev, Diebold, and Labys (2003), and the HAR model of Corsi (2009). The development of multivariate models has come afterwards. Modeling a covariance matrix is challenging since the dimension of the object to be modeled is proportional to the square of the number of assets (denoted by n), and thus the number of parameters is likely to be large even for a handful of assets. Another challenge is that the model should be congruent with the property that covariance matrices are positive definite. Several dynamic models for realized covariance matrices use the Wishart distribution. We also use this distribution, which is a natural choice since the support of the Wishart is the set of positive definite symmetric matrices.

Our basic idea is to combine the Wishart assumption with DCC-type specifications[1] of the GARCH literature to specify the dynamics of realized correlation matrices, and we show that this has several important advantages:

1. The model can be specified in steps, one for each realized variance, and one for the realized correlation matrix, without departing from the Wishart assumption;

[1] See Tse and Tsui (2002), Engle (2002a), and Aielli (2008).

2. Correspondingly, maximum likelihood (ML) estimation can be split in two steps, one for the parameters of each realized variance dynamic process and one for those of the realized correlation process;

3. The estimators at each step have a quasi-ML interpretation;

4. Due to the properties of the Wishart distribution and of the scalar DCC specifications we propose, the estimation of the second step can be done by a composite likelihood (CL) approach.

5. The scalar DCC specifications also allow us to use correlation targeting of the matrix constant term of the dynamic equation for the correlation matrix, i.e. preliminary estimation of this matrix by a method of moment estimator. The number of remaining parameters in the dynamic correlation process is then fixed and independent of the dimension of the matrices being modeled.

The last two elements allow us to estimate the models for a relatively large order of the realized covariance matrices (up to fifty in the empirical illustration of this paper and to one hundred in simulations). Our ability to model matrices of large dimensions contrasts with the existing models that also rely on the Wishart assumption,[2] and also with other modeling approaches for realized covariance matrices.[3]

Golosnoy, Gribisch, and Liesenfeld (2012) and Noureldin, Shephard, and Sheppard (2012) also combine the idea of a time-varying Wishart distribution and a dynamic matrix process inspired by a multivariate GARCH model (the BEKK process). In these papers, the number of parameters of the dynamic equation of the scale matrix is thus proportional to n^2.

The paper is structured as follows. Realized dynamic conditional correlation models are presented in Section 2. In Section 3 we present the two-step QML estimation procedure

[2]Gouriéroux, Jasiak, and Sufana (2009) work with three assets, Bonato, Caporin, and Ranaldo (2009) four, Golosnoy, Gribisch, and Liesenfeld (2012) and Jin and Maheu (2010) five, Noureldin, Shephard, and Sheppard (2012) ten, and Bonato, Caporin, and Ranaldo (2011) twelve.

[3]Chiriac and Voev (2011) work with six assets and Bauer and Vorkink (2011) with five.

applicable to these models. Correlation targeting is discussed in Section 4, and estimation by the composite maximum likelihood (CML) method in Section 5. In Section 6 we report the results of a simulation study comparing the QML and CML estimators. In Section 7, we apply the methods to real data sets, and we offer some conclusions in the last section.

2 Model specifications

Let C_t be a sequence of PDS realized covariance matrices of order n, for $t = 1, \ldots, T$. We assume that conditional on past information I_{t-1} consisting of C_τ for $\tau \leq t-1$, and for all t, C_t follows a n-dimensional central Wishart distribution and denote this assumption by

$$C_t | I_{t-1} \sim W_n(\nu, S_t/\nu), \tag{1}$$

where ν ($> n-1$) is the degrees of freedom parameter and S_t/ν is a PDS scale matrix of order n. Equation (1) defines a generic conditional autoregressive Wishart (CAW) model, as proposed by Golosnoy, Gribisch, and Liesenfeld (2012). From the properties of the Wishart distribution - see e.g. Anderson (1984)- it follows that

$$E(C_t | I_{t-1}) := E_{t-1}(C_t) = S_t, \tag{2}$$

so that the i, j-th element of S_t is defined as the conditional covariance between returns on assets i and j, $cov(r_{i,t}, r_{j,t} | I_{t-1})$, for $i, j = 1, \ldots, n$, $r_{i,t}$ denoting the logarithmic return on asset i between the ends of periods $t-1$ and t.

Several choices are available for specifying the dynamics of S_t. Golosnoy, Gribisch, and Liesenfeld (2012) use the BEKK formulation of the multivariate GARCH literature. Assuming only one lag, this corresponds to

$$S_t = GG' + AC_{t-1}A' + BS_{t-1}B', \tag{3}$$

where A and B are square matrices of order n, and G is a lower triangular matrix such that GG' is PDS. This choice ensures that S_t is PDS for all t if S_0 is itself PDS. For large n, this choice renders the estimation infeasible due to the high number of parameters. Golosnoy, Gribisch, and Liesenfeld (2012) are able to estimate this model for five assets and two lags, for a total of one hundred and sixteen parameters. This is a remarkable performance that is probably difficult to be improved on unless imposing strong parameter restrictions, as for instance, common dynamics for all the elements of S_t. However, they do not consider covariance targeting. This is easy to implement, since the unconditional expectation of C_t and of S_t is known analytically, see Corollary 1 in their paper. Nevertheless the number of parameters in (3) remains of order n^2. This holds even with covariance targeting, unless the matrices A and B are restricted to be diagonal, to have their rank equal to one, or are replaced by scalars.

The scalar Re-BEKK (Re for realized) model imposes that A and B are diagonal matrices, and their diagonal elements are all equal to $\sqrt{a}$ and $\sqrt{b}$, respectively. Hence equation (3) can be written

$$S_t = (1 - a - b)\bar{S} + aC_{t-1} + bS_{t-1}, \tag{4}$$

where a and b are positive scalars, restricted by $a+b < 1$, so that $\bar{S} = E(C_t) = E(S_t)$. The latter results allows us to target $\bar{S}$, i.e. estimate it consistently by the sample average of the C_t matrices. When this estimator is substituted for $\bar{S}$, the likelihood function depends on a and b, which enables their ML estimation for large n (with an efficiency loss).

This scalar model implies that the conditional variances and covariances all follow the same dynamic pattern. This is restrictive but reduces the number of parameters enormously. To enlarge the class of possible models, we use the representation of the covariance matrix in terms of the corresponding diagonal matrix of standard deviations and correlation matrix. Thus, we express S_t in equation (1) as

$$S_t = D_t R_t D_t, \tag{5}$$

where R_t is the conditional correlation matrix of the return vector $r_t = (r_{1,t}, \ldots, r_{n,t})'$ and $D_t = \{\mathrm{diag}(S_t)\}^{1/2}$ is the diagonal matrix whose i-th diagonal entry is given by the conditional standard deviation $\sqrt{S_{ii,t}}$ of asset i.

This decomposition, introduced in a similar context by Engle (2002a) and Tse and Tsui (2002), enables us to separately specify the dynamic equation of each conditional variance and of the conditional correlation matrix R_t. For the conditional variances, we can choose among available univariate specifications, such as a GARCH-type equation, the HAR equation of Corsi (2009), an ARFIMA model as in Andersen, Bollerslev, Diebold, and Labys (2003), or any other suitable model. Each univariate model for $C_{ii,t}$ depends on lags of $C_{ii,t}$ and in some cases of $S_{ii,t}$, but cannot depend on lags of other realized variances or conditional realized variances (spillover terms) to allow for the two-step estimation developed in Section 3.

In the following subsections we propose and discuss two scalar realized dynamic conditional correlation specifications for R_t. Non-scalar specifications are possible but not developed in this paper.

2.1 The scalar Realized-DCC model

The first scalar specification that we propose imposes a scalar dynamic equation on the conditional correlation matrix. A dynamic updating equation for R_t, inspired by that of Tse and Tsui (2002) for multivariate GARCH models, is given by

$$R_t = (1 - \alpha - \beta)\bar{R} + \alpha P_{t-1} + \beta R_{t-1}, \tag{6}$$

where

$$P_t = \{\text{diag}(C_t)\}^{-1/2} C_t \{\text{diag}(C_t)\}^{-1/2} \tag{7}$$

is the realized correlation matrix at time t. The parameters α and β, and their sum, are constrained to lie between zero and one. The matrix $\bar{R}$ is a parameter that must satisfy the constraints of a correlation matrix, i.e. positive definite symmetric with unit diagonal elements. Since P_t has unit diagonal elements, R_t is a well defined correlation matrix for all t if the initial matrix R_0 is a correlation matrix.[4] A drawback of this specification is

[4]The matrix $\bar{R}$ can be parameterized by using the representation $\bar{R} = \{\text{diag}(CC')\}^{-1/2} CC' \{\text{diag}(CC')\}^{-1/2}$ where C is a lower triangular matrix of parameters. Notice that, although C is uniquely identifiable only if its diagonal elements are constrained to be positive (i.e. if it is obtained as the result of a Cholesky type decomposition) and CC' is identifiable only up to a multiplicative constant, $\bar{R}$ remains uniquely identifiable.

that it does not imply that $\bar{R}$ is the unconditional expectation of P_t and of R_t, which has some consequences discussed in Section 4. We label this model 'scalar Re-DCC'.

2.2 The scalar consistent Re-DCC model

A different specification is in spirit close to the cDCC model of Aielli (2008), which is itself a modification of Engle (2002a). Thus, the representation in (5) is complemented by

$$R_t = \{\text{diag}(Q_t)\}^{-1/2} Q_t \{\text{diag}(Q_t)\}^{-1/2}, \tag{8}$$

The correlation driving process Q_t is defined by

$$Q_t = (1 - \alpha - \beta)\bar{Q} + \alpha P^*_{t-1} + \beta Q_{t-1}, \tag{9}$$

where

$$P^*_t = \{\text{diag}(Q_t)\}^{1/2} D_t^{-1} C_t D_t^{-1} \{\text{diag}(Q_t)\}^{1/2}. \tag{10}$$

We label this model 'scalar Re-cDCC' (c for consistent).

By taking expectations on both sides of (9), one obtains, assuming $\alpha + \beta < 1$, that $E(Q_t) = \bar{Q}$ if $E(P^*_t) = E(Q_t)$. The latter result holds using (2), (5) and (8) since

$$\begin{aligned} E\big(P^*_t\big) &= E\big[\{\text{diag}(Q_t)\}^{1/2} D_t^{-1} E_{t-1}\big(C_t\big) D_t^{-1} \{\text{diag}(Q_t)\}^{1/2}\big] \\ &= E\big[\{\text{diag}(Q_t)\}^{1/2} D_t^{-1} D_t R_t D_t D_t^{-1} \{\text{diag}(Q_t)\}^{1/2}\big] \\ &= E\big[\{\text{diag}(Q_t)\}^{1/2} R_t \{\text{diag}(Q_t)\}^{1/2}\big] = E(Q_t). \end{aligned} \tag{11}$$

In both scalar models, the number of parameters is $O(n^2)$ due to the matrix $\bar{R}$ or $\bar{Q}$. We discuss targeting, i.e. estimation of these matrices before ML estimation of the remaining parameters, in Section 4. Targeting enables us to use the models for large dimensions. In Section 4 we show that an advantage of (6) is that the targeting is easier than in (9) and does not depend on unknown parameters and thus is robust to specification errors in the variance equations. Its drawback is that the targeting is not consistent, but simulation results in Section 6 show that this inconsistency does not create worrying finite sample biases in the estimation of α and β. In practice the specifications (6) and (9) provide close empirical results, as illustrated in Section 7.

3 Two-step QML estimation

In this section we focus on the estimation by the ML method of the parameters of the Re-(c)DCC models defined in the previous section. We treat the degrees of freedom ν as a nuisance parameter. We do not focus in this section on the targeting of the constant matrix term ($\bar{R}$ or $\bar{Q}$), this issue being discussed in Section 4. In sub-section 3.1, we show that, based on the Wishart distribution assumption of equation (1), the estimation can be split in two steps, one for the parameters of the conditional variance equations if they do not include spillover terms and are variation free, and one for the parameters of the conditional correlation equation. In sub-section 3.2, we show that the estimators of the variance and correlation equations have in each step a quasi-ML (QML) interpretation and can be obtained without estimating the degrees of freedom, which justifies our treatment of the latter as a nuisance parameter. Moreover, in some applications of multivariate volatility modeling, such as optimal portfolio choice and hedging, the interest of the modeler is in the estimation of the conditional covariance matrix rather than in its distributional properties. In these contexts, the parameters of interest would not include the degrees of freedom.

The QML interpretation is interesting for the estimation of the realized variance parameters, since it has been reported that the distribution of realized variances is often very close empirically to being lognormal, see e.g. Andersen, Bollerslev, Diebold, and Ebens (2001) and Andersen, Bollerslev, Diebold, and Labys (2001). The Wishart assumption implies that the distribution of a realized variance is marginally gamma, but since the estimation method we propose has a QML interpretation, the estimator is consistent even if the true distribution is lognormal and the conditional mean is correctly specified.

Our estimation method is different from that of Golosnoy, Gribisch, and Liesenfeld (2012), who deal with the BEKK formulation and extensions of it. They use ML estimation, but do not give a QML interpretation to it. Our results partly differ also from those of Noureldin, Shephard, and Sheppard (2012), who deal with a model that includes a realized covariance matrix equation in addition to a modified multivariate GARCH equation. They do provide a QML interpretation to ML estimators based on the Wishart assumption in the context of BEKK formulations, but the latter also prevents two-step estimation.

3.1 The Wishart likelihood function

The vector of unknown parameters to be estimated, denoted by θ, can be partitioned into $(\theta_c', \theta_v')'$ where θ_v and θ_v are the vectors containing the conditional variance and correlation parameters, respectively. If targeting is used, θ_c does not include the parameters of the matrix $\bar{R}$ of (6) and $\bar{Q}$ of (9), otherwise it does. We also partition θ_v into $\theta_v = (\theta_v^{(1)'}, \theta_v^{(2)'}, \dots, \theta_v^{(n)'})'$, where $\theta_v^{(i)}$ is the vector containing the parameters of the conditional variance equation specific to asset i.

Using the expression of a Wishart density function, and of S_t in (5), we obtain the log-likelihood contribution $\ell(C_t; \theta | I_{t-1})$ of observation t, denoted by $\ell_t(\theta)$:

$$\begin{aligned} \ell_t(\theta) &= \frac{\nu n}{2} \log \frac{\nu}{2} + \frac{\nu - n - 1}{2} \log |C_t| - \sum_{i=1}^{n} \log \Gamma[(\nu + 1 - i)/2] \\ &\quad - \frac{\nu}{2} \log |D_t R_t D_t| - \frac{\nu}{2} \mathrm{tr}\{(D_t R_t D_t)^{-1} C_t\}. \end{aligned} \tag{12}$$

Proposition 1. *The likelihood contribution $\ell_t(\theta)$ in (12) can be written as*

$$\ell_t(\theta) = \ell_{1t}(\nu, \theta_v) + \ell_{2t}(\nu, \theta_c, \theta_v) + \ell_{0t}(\nu), \tag{13}$$

where

$$\ell_{1t}(\nu, \theta_v) = -\nu \log(D_t) - \frac{\nu}{2} tr\{D_t^{-1} C_t D_t^{-1}\}, \tag{14}$$

$$\ell_{2t}(\nu, \theta_c, \theta_v) = -\frac{\nu}{2} \log |R_t| - \frac{\nu}{2} tr\{(R_t^{-1} - I_n) D_t^{-1} C_t D_t^{-1}\}, \tag{15}$$

and $\ell_{0t}(\nu) = \frac{\nu n}{2} \log \frac{\nu}{2} + \frac{\nu-n-1}{2} \log |C_t| - \sum_{i=1}^{n} \log[\Gamma((\nu + 1 - i)/2]$. Moreover, assuming that the univariate equations for the conditional variances do not include spillover terms and that their parameters are variation-free, ℓ_{1t} can be written as the sum of n univariate functions:

$$\ell_{1t}(\nu, \theta_v) = \frac{\nu}{2} \left[-\sum_{i=1}^{n} \log S_{ii,t} - \sum_{i=1}^{n} S_{ii,t}^{-1} C_{ii,t} \right] = \frac{\nu}{2} \sum_{i=1}^{n} \ell_{1t}^{(i)}(\theta_v^{(i)}). \tag{16}$$

Proof. The proof relies on the following results:
(i) $\log|D_t R_t D_t| = 2\log|D_t| + \log|R_t|$.
(ii) $\mathrm{tr}\{(D_t R_t D_t)^{-1}C_t\} = \mathrm{tr}\{R_t^{-1}D_t^{-1}C_t D_t^{-1}\} = \mathrm{tr}\{(R_t^{-1} - I_n)D_t^{-1}C_t D_t^{-1}\} + \mathrm{tr}\{D_t^{-1}C_t D_t^{-1}\}$.
(iii) $2\log|D_t| = \sum_{i=1}^{n}\log S_{ii,t}$.
(iv) $\mathrm{tr}\{D_t^{-1}C_t D_t^{-1}\} = \sum_{i=1}^{n} S_{ii,t}^{-1}C_{ii,t}$. □

At this stage there are three important considerations to make:

1. The ℓ_{1t} part of the log-likelihood is proportional to the shape parameter ν. This implies that it can be maximized with respect to the elements of θ_v independently of the value of ν which is not affecting the first order conditions for θ_v.

2. Each function $\ell_{1t}^{(i)}$, defined as the terms between square brackets in (16), only depends on the parameters $\theta_v^{(i)}$ specific to the conditional variance equation of asset i. It follows that maximization of ℓ_{1t} can be achieved through n separate optimizations (under the assumptions stated in the proposition). Notice that $\ell_{1t}^{(i)}$ corresponds to the log-likelihood of an exponential distribution.

3. The ℓ_{2t} part of the log-likelihood depends on the whole set of parameters θ. Since it is linear in ν, it can be maximized with respect to these parameters independently of the value of ν.

The main interest of these results is that we can adopt a two-step procedure to estimate the model parameters:

1. In step 1, the conditional realized variance parameters are estimated by maximizing $\ell_{1t}^{(i)}$ with respect to $\theta_v^{(i)}$ for $i = 1, \ldots, n$.

2. In step 2, the correlation equation parameters are estimated by maximizing the ℓ_{2t} function with respect to θ_c, after fixing θ_v to the estimate provided by step 1.

3.2 QML interpretations

By equation (16), we can just consider the estimation of the parameter $\theta_v^{(i)}$ for a given i. Using the law of iterated expectation it is easy to show that at the true parameter value $\theta_{v,0}^{(i)}$, the expected first step score for observation t is equal to 0:

$$E[\partial\ell_{1t}^{(i)}/\partial\theta_v^{(i)}] = E\left\{E_{t-1}\left[(-1 + C_{ii,t}/S_{ii,t})(1/S_{ii,t})\partial S_{ii,t}/\partial\theta_v^{(i)}\right]\right\} = 0 \tag{17}$$

since $E_{t-1}(C_{ii,t}) = S_{ii,t}$, see equation (2). This implies that (16) is a quasi-likelihood (QL) function and its maximizer $\hat{\theta}_v$ is a QML estimator (QMLE). Hence, by the results in Bollerslev and Wooldridge (1992), under the stated regularity conditions, consistency and asymptotic normality hold. The result in (17), and the implied interpretation of $\hat{\theta}_v$ as a QMLE, makes our approach robust to misspecification of the distribution of univariate realized volatilities. Even if the gamma assumption (implied by the Wishart) is not satisfied, we still obtain consistent estimates of the elements of θ_v. In the literature, a similar estimation problem has been considered by Engle and Russell (1998), Engle (2002b), and Engle and Gallo (2006) in the estimation of ACD and multiplicative error models with a gamma conditional distribution.

Consistency of the first step estimator $\hat{\theta}_v$ implies consistency of the second step estimators of θ_c obtained as

$$\hat{\theta}_c = \underset{\theta_c}{\operatorname{argmax}} \sum_{t=1}^{T} \ell_{2t}(\nu, \theta_c, \hat{\theta}_v). \tag{18}$$

This result directly follows from the application of Theorem 3.10 in White (1994), under the regularity conditions that are stated there. Notice that if consistent targeting is used in the second step, the consistency of the second step estimators of α and β is kept. Furthermore, the consistency of the estimator of θ_c still holds if the first step parameters in θ_v are consistently estimated by optimizing an objective function different from $\sum_{t=1}^{T} \ell_{1t}$. For example, following the mainstream literature on univariate modeling of realized variances, we could adopt a maximum likelihood estimator based on the maximization of a lognormal likelihood.

For the estimation of θ_c in the second step, we can show that the score vector related to θ_c, for observation t, has its expected value equal to zero when $\theta_c = \theta_{c,0}$, where $\theta_{c,0}$ is the value of θ_c in the data generating process. Namely, by applying standard results on the differentiation of matrix functions, and taking expectations conditional on past information I_{t-1}, we obtain

$$E_{t-1}\left(\partial \ell_{2t}/\partial \theta_c\right) = \frac{\nu}{2}\left\{\mathrm{tr}\left(R_t^{-1}\frac{\partial R_t}{\partial \theta_c}\right) - \mathrm{tr}\left(D_t^{-1}E_{t-1}(C_t)D_t^{-1}R_t^{-1}\frac{\partial R_t}{\partial \theta_c}R_t^{-1}\right)\right\} = 0,$$

since at the true parameter value $\theta_c = \theta_{c,0}$, $E_{t-1}(C_t) = D_t R_t D_t$ by equation (2).

This result has great practical relevance since it implies that, under the usual regularity conditions - see e.g. Newey and McFadden (1994), Wooldridge (1994) - an estimator based on the moment conditions $\partial \ell_{2t}/\partial \theta_c = 0$ is consistent for θ_c. In other words, even if the 'true' distribution of C_t is not Wishart, we can consider $\hat{\theta}_c$, defined by (18), as a QMLE.

Concerning the asymptotic distribution of the second step estimator $\hat{\theta}_c$, we can rely in principle on Theorem 6.1 in Newey and McFadden (1994) to invoke its asymptotic normality, at least if targeting is not used. If consistent targeting is used, the asymptotic distribution has to be derived, and will include an adjustment to account for the efficiency loss due to the preliminary estimation of the constant term.

4 Correlation targeting and profiling

In the scalar Re-(c)DCC models presented in Section 2, the dynamic equations depend on constant matrices $\bar{R}$ or $\bar{Q}$, see (6) and (9). To avoid having a large number of parameters (of order n^2) in the numerical maximization of the QL function of the second step of the estimation of the models, which renders the computations impossible in practice for large values of n, we can use 'targeting'. This means usually a preliminary estimation of these constant matrices by a method of moment estimator. If this estimator is substituted for the corresponding parameter matrix in the QL function (of the second step), the numerical burden is much reduced since the number of parameters is independent of n. It is desirable that the targeting estimator of a parameter is consistent, even if it is inefficient. Indeed, since the QML estimators of the remaining parameters depend on the targeting estimator, they cannot be consistent if the targeting estimator is not consistent.

4.1 Re-DCC

We have mentioned in Section 2.1 that the specification of the Re-DCC model does not imply that $\bar{R}$ is the unconditional expectation of P_t and of R_t. Indeed, although $E_{t-1}(C_t) = S_t$, by assumption, see (2), and thus $E(C_t) = E(S_t)$, this does not imply that $E(P_t)$ is equal to the unconditional correlation matrix $\{\text{diag}[E(S_t)]\}^{-1/2} E(S_t) \{\text{diag}[E(S_t)]\}^{-1/2}$ deduced from the unconditional covariance $E(S_t)$, due to the non-linearity of the transformation from covariances to correlations. Thus a consistent estimator of the unconditional correlation matrix, given by

$$\bar{P}_T = \sum_{t=1}^{T} P_t / T \tag{19}$$

is not consistent for $\bar{R}$ (because $\bar{R}$ is not the unconditional correlation matrix), and targeting $\bar{R}$ by $\bar{P}_T$ is inconsistent. However the finite sample bias of doing that may not be important if C_t is constructed consistently from a large enough number (H) of high-frequency returns $r_{t,h}$, assumed to be independently distributed and $N_n(0, S_t/H)$ for $h = 1, 2, \ldots, H$. Indeed this assumption implies that $C_t := C_{t,H} = \sum_{h=1}^{H} r_{t,h} r'_{t,h} \sim W_n(H, S_t/H)$. Then $C_{t,H} \xrightarrow{p} S_t$ as $H \to \infty$, and

$$P_{t,H} = \{\text{diag}(C_{t,H})\}^{-1/2} C_{t,H} \{\text{diag}(C_{t,H})\}^{-1/2} \xrightarrow{p} R_t = \{\text{diag}(S_t)\}^{-1/2} S_t \{\text{diag}(S_t)\}^{-1/2}.$$

Thus for large H, $E_{t-1}(P_{t,H})$ should be close to R_t, hence $E(P_{t,H})$ should be close to $E(R_t)$. Then $\bar{P}_T$ is estimating $E(R_t)$ consistently. If $E(P_t)$ were equal to $E(R_t)$, then $\bar{R}$ would be equal to $E(R_t)$. Since this holds approximately for large enough H, we expect that targeting $\bar{R}$ by $\bar{P}_T$ should not lead to a strong bias if the observed matrices are obtained form high frequency data and free from contamination by microstructure noise.

4.2 Re-cDCC

For the Re-cDCC models, we have shown in Section 2.2 that $\bar{Q} = E(Q_t) = E(P_t^*)$. Hence,

$$\bar{P}_T^* = \sum_{t=1}^{T} P_t^*/T \xrightarrow{p} \bar{Q}. \tag{20}$$

In practice, this estimator is not feasible since it depends on the unknown parameters of the conditional variance equations through D_t. We have explained in Section 3 that the parameters of the variance equations can be estimated consistently in the first step of estimation. If D_t is thus replaced by a consistent estimator $\hat{D}_t$ and $\hat{P}_t^*$ stands for (10) with $\hat{D}_t$ replacing D_t, then

$$\bar{\hat{P}}_T^* = \sum_{t=1}^{T} \hat{P}_t^*/T \xrightarrow{p} \bar{Q}. \tag{21}$$

Nevertheless $\bar{\hat{P}}_T^*$ cannot be used for targeting $\bar{Q}$ since it depends also on the parameters α and β of (9) through the diagonal elements of Q_t itself. We can obviously estimate the Re-cDCC model by maximizing a QL function with respect to $\bar{Q}$ in addition to α and β. This approach limits the use of the model to small dimensions since the number of parameters is then $O(n^2)$.

To circumvent this problem, we maximize a profile QL function, see Severini (1998), that is, we substitute $\bar{\hat{P}}_T^*(\alpha, \beta)$ (making the dependence of $\bar{\hat{P}}_T^*$ on α and β explicit) for $\bar{Q}$ in (9) in evaluating the QL function for any value of α and β. If $\hat{\alpha}$ and $\hat{\beta}$ are the values that maximize the QL function, we finally estimate consistently $\bar{Q}$ by $\bar{\hat{P}}_T^*(\hat{\alpha}, \hat{\beta})$. This procedure makes it possible to estimate the scalar Re-cDCC model for a large number of assets. The same idea is used in the context of the cDCC MGARCH model by Aielli (2011).

In practice this requires n additional univariate recursions for calculating the diagonal elements of Q_t, on which the value of P_t^* depends. Letting $\delta_{A,t}$ denote the vector that stacks the diagonal elements of the matrix A_t, we have

$$\delta_{Q,t} = (1 - \alpha - \beta)\delta_{\bar{Q}} + \alpha \delta_{P^*,t-1} + \beta \delta_{Q,t-1} \tag{22}$$

where we impose $\delta_Q = \mathbf{1}_n$. An analogous restriction is imposed by Aielli (2008) in order to guarantee the identifiability of the cDCC MGARCH model. In our case, this restriction is not necessary, but we impose it to save parameters and ease estimation.

5 Composite likelihood estimation

The second step of the estimation method presented in Section 3 may not be practicable for very large dimensions. This is due to the need to invert the matrix R_t appearing in the log-likelihood function for each observation. This operation is time consuming for the sample sizes of typical empirical applications. The same issue arises in the estimation of the DCC version of a multivariate GARCH model and has motivated Engle, Shephard, and Sheppard (2008) to use the composite likelihood (CL) method based on the conditional normal distribution for the return vector. It turns out that for the scalar Re-(c)DCC and Re-BEKK models, the Wishart assumption also enables us to use the CL method explained below.

The method is based on three results for which we need the following notations. For any square matrix M_t of order n, we denote by $M_{AA,t}$ a square matrix of order n_A extracted from M_t, which has its main diagonal elements on the main diagonal of M_t. Namely, if A stands for a subset of n_A different indices of $\{1, 2, \ldots, n\}$, $M_{AA,t}$ is the matrix that consists of the intersection of the rows and columns of M_t corresponding to the selection of indices denoted by A. The three results are:

R1: If $C_t \sim W_n(\nu, S_t/\nu)$, $C_{AA,t} \sim W_{n_A}(\nu, S_{AA,t}/\nu)$ for any selection of n_A indices.

R2: If $S_t = D_t R_t D_t$, $S_{AA,t} = D_{AA,t} R_{AA,t} D_{AA,t}$.

R3 (Re-DCC): if $R_t = (1-\alpha-\beta)\bar{R} + \alpha P_{t-1} + \beta R_{t-1}$, $R_{AA,t} = (1-\alpha-\beta)\bar{R}_{AA} + \alpha P_{AA,t-1} + \beta R_{AA,t-1}$.

R3 (Re-cDCC): $R_{AA,t} = \mathrm{diag}(Q_{AA,t})^{-1/2} Q_{AA,t} \mathrm{diag}(Q_{AA,t})^{-1/2}$, and if $Q_t = (1-\alpha-\beta)\bar{Q} + \alpha P^*_{t-1} + \beta Q_{t-1}$, $Q_{AA,t} = (1-\alpha-\beta)\bar{Q}_{AA} + \alpha P^*_{AA,t-1} + \beta Q_{AA,t-1}$.

Result 1 is a property of the Wishart distribution already mentioned at the end of Section 3. Notice that applied with $n_A = 1$, it corresponds to the result that the marginal distribution of a diagonal element of a Wishart matrix is a gamma, a result that is used in

Section 3 to form the log-likelihood for the first step of the estimation procedure. Results 2, given that D_t is diagonal, and 3 are obvious algebraic results.

A CL second step estimator of the parameters α and β is then defined as the maximizer of the sum of a number of Wishart marginal log-likelihoods for sub-matrices $P_{AA,t}$ corresponding to different choices of indices A. The most obvious choice is to select all the log-likelihoods corresponding to sub-matrices of order 2, i.e. to all the $n(n-1)/2$ correlation coefficients or pairs of assets. In each bivariate Wishart term, the parameters of the conditional variances are fixed at the estimates of the first step, and the matrix $\bar{R}_{AA}$ is set to the corresponding matrix extracted from $\bar{P}_T$. Notice that in these bivariate Wishart log-likelihoods, only matrices of order 2 must be inverted, which can be efficiently programmed. Such a CL is denoted $CL2_t$ for the contribution of observation t. Formally,

$$CL2_t(\nu, \alpha, \beta, \hat{\bar{R}}, \hat{\theta}_v) = \sum_{h=2}^{n} \sum_{k<h} \ell_{hk,t}(\nu, \alpha, \beta, \bar{P}_T^{(hk)}, \hat{\theta}_v^{(h)}, \hat{\theta}_v^{(k)}) \tag{23}$$

with

$$\begin{aligned} \ell_{hk,t}(.) \quad = \quad & \nu \log\left(\frac{\nu}{2}\right) + \frac{\nu-3}{2} \log|C_t^{(hk)}| - \sum_{i=1}^{2} \log\Gamma[(\nu+1-i)/2] \\ & -\frac{\nu}{2} \log|D_t^{(hk)} \bar{P}_T^{(hk)} D_t^{(hk)}| - \frac{\nu}{2} \mathrm{tr}\{(D_t^{(hk)} \bar{P}_T^{(hk)} D_t^{(hk)})^{-1} C_t^{(hk)}\}, \end{aligned} \tag{24}$$

where for any matrix M_t, $M_t^{(hk)}$ is the matrix of order 2 extracted at the intersection of rows and columns h and k of M_t. One can use less terms (e.g. consecutive pairs) than the $n(n-1)/2$ terms in (23) especially if the number of terms is very large. One can also use marginal log-likelihoods of sub-matrices of higher dimension, e.g. all or a subset of triplets of indices of $\{1, 2, \ldots, n\}$, to form a CL of order three, denoted $CL3$.

The application of CL estimation to the scalar Re-BEKK model is straightforward and estimation is in one step. Notice that for this model, R1 is used together with the fact that the model defined in (4) implies that $S_{AA,t} = (1-a-b)\bar{S}_{AA} + aC_{AA,t-1} + bS_{AA,t-1}$.

In order to derive the asymptotic properties of CML estimators it is useful to consider composite likelihoods as misspecified likelihoods where the misspecification derives from neglecting the dependence between the low-dimensional blocks of observations used for building the overall CL function, see e.g. Varin, Reid, and Firth (2011). In particular, consistency immediately follows from observing that $\nabla^1_\theta CL = \sum_{t=1}^T \nabla^1_\theta CL_t = 0$ is an unbiased estimating equation, where $\nabla^h_\theta X$ denotes the h-order gradient of X with respect to θ and $\theta = (\nu, \alpha, \beta)'$. Furthermore, under standard regularity assumptions (see e.g. Ng, Joe, Karlis, and Liu (2011); Engle, Shephard, and Sheppard (2008)), it can be proven that the CML estimator is asymptotically normal with asymptotic covariance matrix given by $G(\theta)^{-1} = H(\theta)^{-1}J(\theta)H(\theta)^{-1}$, where $G(.)$ denotes the Godambe information matrix, $H(\theta) = E(\nabla^2_\theta CL)$ and $J(\theta) = var(\nabla_\theta CL)$.

Thus we conjecture that the CL estimators of Re-DCC and Re-cDCC models can be shown to be consistent and asymptotically normal if a consistent targeting estimator is used.

As anyway we do not have a consistent estimator of $\bar{R}$ in the Re-DCC model, this issue is not much relevant and we rely on a simulation study to get insights on the finite sample bias of some CL estimators, and their efficiency with respect to the ML estimator. The simulation study will also give information on the finite sample properties of the estimators for the Re-cDCC model.

6 Simulation study

This section presents the results of a Monte Carlo simulation study aimed at comparing the finite sample properties of the maximum likelihood (L) and composite maximum likelihood estimators of the parameters of the conditional correlation process. For CL, we use estimators based on pairs (CL2) and on triplets (CL3). We are interested in the bias of the estimators, for the Re-DCC model with targeting since we know that the targeting we use is not consistent, and for the Re-cDCC because of the profiling method we use. We are also interested in the relative efficiencies of L, CL2, and CL3.

6.1 Simulation design

We consider as data generating processes (DGP) a scalar Re-DCC model and a scalar Re-cDCC model, where $\alpha = \alpha_0$ and $\beta = \beta_0$, α_0 and β_0 being positive scalars such that $\alpha_0 + \beta_0 < 1$. Both $\bar{R}$ and $\bar{Q}$ are equicorrelated matrices, i.e. matrices having diagonal elements equal to one and off-diagonal elements equal to ρ. The value of ρ is fixed to 0.6 because this is in the range of plausible values for stock markets. Non reported results for different values of ρ show that the specific value of ρ does not change the conclusions drawn from the simulations.

In the simulations for each of the two DGP considered, we have generated 1000 time series of length $T = 1000$ and $T = 2500$ with three different choices of α_0 and β_0 and six different values of the cross sectional dimension n (5, 15, 25, 50, 75, 100). In all the cases, the value of the degrees of freedom parameter (ν) has been set equal to $3n$. The DGP for the realized variances associated to the Re-DCC and Re-cDCC correlation models are GARCH-type recursions defined by

$$S_{ii,t} = (1 - \gamma_i - \delta_i) + \gamma_i C_{ii,t-1} + \delta_i S_{ii,t-1} \qquad i = 1, \ldots, n. \tag{25}$$

In order to allow for some variation in the volatility dynamics, for each $i = 1, \ldots, n$, we draw γ_i and δ_i from dependent uniform distributions defined as

$$\gamma_i \sim U(\gamma_0 - 0.02, \gamma_0 + 0.02), \qquad \delta_i | \gamma_i \sim U(2\delta_0 + \gamma_i - 1 + \epsilon, 1 - \gamma_i - \epsilon),$$

with γ_0=0.05 and δ_0=0.90. This ensures that $E(\gamma_i) = \gamma_0$ (set to 0.05), $E(\delta_i) = \delta_0$ (set to 0.90) and $\gamma_i + \delta_i < 1 - \epsilon$ (= 0.99).

In both cases (Re-DCC and Re-cDCC), the estimated model corresponds to the model class to which the DGP belongs, so that the estimated model is correctly specified. Estimation is performed in two steps (by each method – L, CL2, and CL3) with correlation targeting for Re-DCC, and with profiling for Re-cDCC. For the latter we use the approach described in Section 2.2 for the estimation of $\bar{Q}$. The first step of the estimation of the likelihood and composite likelihood methods being identical, we do not report the corresponding simulation results.

For CL2, we use all pairs of assets, and for CL3 we use all of them for $n \leq 25$, while we use 5000 randomly selected triplets for $n > 25$ since the number of triplets is then so large that the Monte Carlo study would require too much time.

In order to assess the statistical properties of the estimates, we have computed from the simulated values the percentage relative bias (RB) and root mean squared error (RMSE):

$$RB(\theta) = 100 \times \frac{1}{1000} \sum_{i=1}^{1000} \frac{(\hat{\theta}_i - \theta)}{\theta},$$

$$RMSE(\theta) = 100 \times \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta}_i - \theta)^2},$$

where $\hat{\theta}_i$ is either $\hat{\alpha}_i$ (when θ is α_0) or $\hat{\beta}_i$ (when θ is β_0), with $(\hat{\alpha}_i, \hat{\beta}_i)$ denoting the estimated parameter values for the i-th simulated series.

6.2 Bias results

The simulation results for the scalar Re-DCC and Re-cDCC processes are reported in Tables 1 and 2, respectively. A few conclusions arise from these results:

1) The biases for β_0 are negative and in most cases very small, being smaller than one per cent in absolute value, except for the Re-cDCC model, $n = 5$, when β_0 is equal to 0.90 and 0.85, where the largest bias is 1.22 per cent.

2) For α_0, the biases are positive (except for $\alpha_0 = 0.03$ by CL2 and CL3). The largest biases occur for $n = 5$ and the Re-DCC model, with the maximum being 6.35 per cent (4.61 for the Re-cDCC). They decrease as n increases, being smaller than one per cent for $n \geq 50$ for both models. This effect (bias decrease) is not visible for β_0 (except comparing $n = 5$ and 15) since the biases are very small.

3) The biases are smaller in the Re-cDCC model than in the Re-DCC, but the differences are far from impressive. Thus, in the Re-DCC case, the targeting of the constant matrix of the correlation process by a (presumably hardly) inconsistent estimator does not seem to create a serious bias problem in the estimation of the dynamic parameters. The same conclusion holds for the profiling method in the Re-cDCC model.

4) The biases tend to decrease when T increases from 1000 to 2500; exceptions happen only for α, and the increases are minor.

We did similar simulations with the Re-BEKK model but do not report the tables to save space. The biases are smaller than one percent, even for α_0 and small n, and they are smaller than for the DCC versions. This may be due to to the fact that targeting is consistent.

6.3 Efficiency results

The simulation results for the scalar Re-DCC and Re-cDCC processes are reported in Tables 3 and 4, respectively. What is reported in the tables is the ratio of the RMSE of each CL estimator and the L estimator, and of CL2 with respect to CL3. Several conclusions emerge from these results:

1) As expected, the efficiency of the CL2 and CL3 estimators is smaller than that of the L estimator. The efficiency loss is very large especially for α_0 and large n. Remember that for $n = 50$ or more, not all triplets are used, implying that the efficiency ratios are not really comparable to those for $n \leq 25$. Actually for $n = 100$, 5000 pairs only correspond to 3 per cent of all pairs.

2) The efficiency of CL2 is smaller than that of CL3, but the gap is not so high as with respect to L (at most 33 per cent, and in most cases much less).

3) The efficiency gap in favor of the L estimator increases with n. This is due to the increase of information brought by a larger cross-sectional dimension, given the scalar nature of the models.

4) Efficiency ratios for α_0 tend to be larger for $T = 1000$ than for $T = 2500$. For β_0, the reverse occurs.

5) The efficiency ratios for the two types of models are generally very close to each other.

Results for the Re-BEKK model are of the same type as in the tables above. The main difference is that the efficiency ratios for CL2/L and CL3/L are a little larger, and for CL2/CL3 a little lower. This may be due to the fact that the Re-BEKK model imposes the same parameters on the variances and covariances, hence there is more information

Table 1: Relative biases of estimators of α and β, scalar Re-DCC model

n	L	CL2	CL3	L	CL2	CL3	L	CL2	CL3
T = 1000		α_0=0.03			α_0=0.05			α_0=0.10	
5	5.39	4.65	4.75	6.35	6.30	6.27	5.99	5.98	5.97
15	2.66	1.50	3.65	2.43	2.72	3.66	2.42	2.61	3.17
25	1.06	-0.74	-0.28	1.12	1.13	1.10	1.36	1.53	1.48
50	0.52	-0.99	-0.66	0.45	0.57	0.63	0.72	0.98	0.98
75	0.31	-0.92	-0.75	0.17	0.53	0.43	0.46	0.80	0.76
100	0.26	-0.86	-0.59	0.09	0.54	0.35	0.38	0.79	0.67
T = 2500		α_0=0.03			α_0=0.05			α_0=0.10	
5	6.32	5.85	6.08	6.47	6.19	6.28	5.92	5.74	5.81
15	2.07	1.43	1.59	2.14	2.42	2.31	2.22	2.28	2.58
25	1.35	0.72	0.87	1.30	1.35	1.31	1.32	1.39	1.36
50	0.70	-0.11	0.10	0.58	0.54	0.56	0.66	0.71	0.72
75	0.48	-0.23	-0.11	0.37	0.45	0.50	0.44	0.57	0.49
100	0.42	-0.09	-0.15	0.28	0.50	0.26	0.35	0.56	0.41
T = 1000		β_0=0.95			β_0=0.90			β_0=0.85	
5	-0.49	-0.52	-0.49	-0.74	-0.91	-0.81	-0.54	-0.65	-0.59
15	-0.52	-0.50	-0.52	-0.64	-0.84	-0.72	-0.46	-0.63	-0.56
25	-0.52	-0.41	-0.43	-0.67	-0.81	-0.76	-0.47	-0.63	-0.58
50	-0.53	-0.41	-0.43	-0.66	-0.78	-0.75	-0.46	-0.61	-0.59
75	-0.54	-0.42	-0.43	-0.65	-0.78	-0.73	-0.45	-0.60	-0.57
100	-0.54	-0.42	-0.43	-0.65	-0.76	-0.71	-0.46	-0.59	-0.55
T = 2500		β_0=0.95			β_0=0.90			β_0=0.85	
5	-0.20	-0.18	-0.19	-0.29	-0.29	-0.28	-0.22	-0.21	-0.21
15	-0.18	-0.13	-0.14	-0.26	-0.33	-0.30	-0.19	-0.25	-0.25
25	-0.20	-0.15	-0.16	-0.26	-0.31	-0.29	-0.19	-0.24	-0.22
50	-0.20	-0.14	-0.15	-0.25	-0.29	-0.28	-0.18	-0.23	-0.22
75	-0.20	-0.15	-0.14	-0.25	-0.31	-0.28	-0.18	-0.24	-0.20
100	-0.20	-0.15	-0.15	-0.25	-0.31	-0.27	-0.18	-0.25	-0.22

Table 2: Relative biases of estimators of α and β, scalar Re-cDCC model

n	L	CL2	CL3	L	CL2	CL3	L	CL2	CL3
T = 1000		α_0=0.03			α_0=0.05			α_0=0.10	
5	2.97	2.46	2.50	3.52	3.59	3.57	4.51	4.59	4.61
15	1.32	0.83	1.00	1.16	1.91	1.74	1.68	2.17	2.09
25	0.58	-1.19	-0.71	0.52	0.64	0.61	0.99	1.25	1.22
50	0.23	-1.33	-0.93	0.06	0.15	0.20	0.46	0.76	0.74
75	0.09	-0.97	-0.70	-0.10	0.53	0.41	0.28	0.81	0.74
100	0.08	-1.35	-1.01	-0.13	0.18	0.07	0.22	0.60	0.53
T=2500		$\alpha_0 = 0.03$			$\alpha_0 = 0.05$			$\alpha_0 = 0.10$	
5	4.16	3.69	3.91	3.79	3.57	3.68	4.54	4.39	4.50
15	1.53	1.36	1.42	1.27	1.62	1.53	1.62	1.84	1.80
25	0.93	0.31	0.48	0.76	0.86	0.83	1.01	1.12	1.11
50	0.47	-0.53	-0.25	0.29	0.07	0.16	0.48	0.46	0.50
75	0.33	-0.42	-0.25	0.15	0.17	0.18	0.31	0.40	0.41
100	0.29	-0.34	-0.20	0.10	0.15	0.16	0.24	0.35	0.35
T=1000		$\beta_0 = 0.95$			$\beta_0 = 0.90$			$\beta_0 = 0.85$	
5	-0.67	-0.71	-0.66	-1.03	-1.20	-1.10	-1.09	-1.22	-1.16
15	-0.59	-0.56	-0.56	-0.76	-0.94	-0.88	-0.66	-0.83	-0.78
25	-0.56	-0.44	-0.46	-0.72	-0.87	-0.81	-0.58	-0.75	-0.70
50	-0.55	-0.42	-0.45	-0.68	-0.79	-0.76	-0.51	-0.65	-0.62
75	-0.55	-0.43	-0.45	-0.66	-0.82	-0.77	-0.48	-0.65	-0.62
100	-0.55	-0.42	-0.44	-0.66	-0.79	-0.73	-0.48	-0.63	-0.59
T=2500		$\beta_0 = 0.95$			$\beta_0 = 0.90$			$\beta_0 = 0.85$	
5	-0.35	-0.33	-0.33	-0.52	-0.53	-0.52	-0.74	-0.75	-0.75
15	-0.25	-0.24	-0.24	-0.34	-0.41	-0.38	-0.36	-0.43	-0.41
25	-0.23	-0.18	-0.19	-0.31	-0.36	-0.34	-0.30	-0.35	-0.34
50	-0.21	-0.13	-0.15	-0.27	-0.28	-0.28	-0.23	-0.26	-0.26
75	-0.21	-0.15	-0.16	-0.26	-0.30	-0.29	-0.21	-0.26	-0.25
100	-0.21	-0.15	-0.16	-0.26	-0.29	-0.28	-0.20	-0.25	-0.24

Table 3: Ratios of RMSE of estimators of α and β, scalar Re-DCC model

n	**T = 1000**	α_0=0.03	β_0=0.95	α_0=0.05	β_0=0.90	α_0=0.10	β_0=0.85
5	CL2/L	1.17	1.22	1.23	1.33	1.15	1.29
5	CL3/L	1.08	1.09	1.08	1.12	1.05	1.11
5	CL3/CL2	0.93	0.89	0.87	0.84	0.91	0.86
15	CL2/L	2.02	1.34	2.04	1.73	1.65	1.80
15	CL3/L	1.54	1.13	1.59	1.28	1.44	1.37
15	CL3/CL2	0.76	0.85	0.78	0.74	0.87	0.76
25	CL2/L	3.15	1.20	3.05	1.75	2.30	1.89
25	CL3/L	2.32	1.04	2.24	1.42	1.77	1.53
25	CL3/CL2	0.73	0.87	0.74	0.81	0.77	0.81
50	CL2/L	6.34	1.14	6.08	1.69	3.97	1.90
50	CL3/L	4.35	0.99	4.31	1.42	2.94	1.59
50	CL3/CL2	0.69	0.87	0.71	0.84	0.74	0.84
75	CL2/L	8.80	1.13	9.18	1.75	5.70	1.97
75	CL3/L	6.15	0.96	6.26	1.40	3.97	1.56
75	CL3/CL2	0.70	0.85	0.68	0.80	0.70	0.80
100	CL2/L	10.77	1.08	11.36	1.67	6.71	1.88
100	CL3/L	7.88	0.95	7.95	1.36	4.77	1.53
100	CL3/CL2	0.73	0.89	0.70	0.81	0.71	0.82
n	**T = 2500**	α_0=0.03	β_0=0.95	α_0=0.05	β_0=0.90	α_0=0.10	β_0=0.85
5	CL2/L	1.19	1.27	1.12	1.27	1.05	1.26
5	CL3/L	1.04	1.08	1.03	1.10	1.01	1.09
5	CL3/CL2	0.88	0.85	0.92	0.86	0.96	0.87
15	CL2/L	1.93	1.63	1.85	1.90	1.42	1.97
15	CL3/L	1.50	1.31	1.48	1.51	1.28	1.51
15	CL3/CL2	0.78	0.80	0.80	0.79	0.90	0.77
25	CL2/L	2.66	1.57	2.58	2.13	1.88	2.28
25	CL3/L	1.99	1.27	1.96	1.66	1.50	1.77
25	CL3/CL2	0.75	0.81	0.76	0.78	0.80	0.78
50	CL2/L	4.55	1.48	4.82	2.30	3.08	2.59
50	CL3/L	3.22	1.19	3.37	1.74	2.25	1.96
50	CL3/CL2	0.71	0.80	0.70	0.76	0.73	0.76
75	CL2/L	6.87	1.53	7.18	2.36	4.58	2.74
75	CL3/L	4.62	1.17	5.06	1.78	3.16	2.02
75	CL3/CL2	0.67	0.77	0.70	0.75	0.69	0.74
100	CL2/L	7.78	1.45	8.86	2.27	5.32	2.64
100	CL3/L	5.46	1.12	6.05	1.68	3.66	1.93
100	CL3/CL2	0.70	0.77	0.68	0.74	0.69	0.73

A value larger (smaller) than 1 indicates that the estimator in the numerator is less (more) efficient than the estimator in the denominator.

Table 4: Ratios of RMSE of estimators of α and β, scalar Re-cDCC model

n	**T = 1000**	α_0=0.03	β_0=0.95	α_0=0.05	β_0=0.90	α_0=0.10	β_0=0.85
5	CL2/L	1.16	1.18	1.29	1.29	1.20	1.22
5	CL3/L	1.08	1.07	1.11	1.11	1.08	1.09
5	CL3/CL2	0.93	0.91	0.86	0.86	0.90	0.89
15	CL2/L	2.33	1.28	2.35	1.60	1.92	1.56
15	CL3/L	1.76	1.13	1.81	1.34	1.56	1.34
15	CL3/CL2	0.76	0.88	0.77	0.84	0.81	0.86
25	CL2/L	3.35	1.16	3.33	1.70	2.62	1.72
25	CL3/L	2.46	1.02	2.44	1.39	1.99	1.44
25	CL3/CL2	0.74	0.88	0.73	0.82	0.76	0.84
50	CL2/L	6.23	1.10	6.08	1.65	4.59	1.79
50	CL3/L	4.42	0.98	4.34	1.38	3.34	1.50
50	CL3/CL2	0.71	0.89	0.71	0.84	0.73	0.84
75	CL2/L	9.27	1.12	9.36	1.75	7.26	1.92
75	CL3/L	6.42	0.99	6.48	1.43	5.12	1.58
75	CL3/CL2	0.69	0.88	0.69	0.82	0.71	0.82
100	CL2/L	11.26	1.07	10.75	1.67	8.15	1.85
100	CL3/L	7.79	0.95	7.42	1.37	5.69	1.52
100	CL3/CL2	0.69	0.89	0.69	0.82	0.70	0.82
n	**T = 2500**	α_0=0.03	β_0=0.95	α_0=0.05	β_0=0.90	α_0=0.10	β_0=0.85
5	CL2/L	1.20	1.17	1.20	1.23	1.09	1.15
5	CL3/L	1.06	1.06	1.07	1.09	1.03	1.06
5	CL3/CL2	0.89	0.90	0.89	0.88	0.95	0.92
15	CL2/L	2.11	1.46	2.16	1.76	1.64	1.60
15	CL3/L	1.63	1.23	1.68	1.43	1.37	1.35
15	CL3/CL2	0.77	0.85	0.78	0.81	0.84	0.84
25	CL2/L	3.05	1.45	3.12	1.96	2.18	1.82
25	CL3/L	2.27	1.20	2.33	1.56	1.72	1.49
25	CL3/CL2	0.74	0.83	0.75	0.80	0.79	0.82
50	CL2/L	5.50	1.38	5.80	2.15	3.73	2.19
50	CL3/L	3.89	1.12	4.13	1.67	2.73	1.72
50	CL3/CL2	0.71	0.81	0.71	0.77	0.73	0.79
75	CL2/L	7.77	1.42	8.49	2.19	5.35	2.28
75	CL3/L	5.45	1.14	5.94	1.68	3.83	1.80
75	CL3/CL2	0.70	0.80	0.70	0.77	0.72	0.79
100	CL2/L	9.39	1.42	11.13	2.17	6.69	2.34
100	CL3/L	6.59	1.14	7.77	1.68	4.74	1.82
100	CL3/CL2	0.70	0.80	0.70	0.77	0.71	0.78

A value larger (smaller) than 1 indicates that the estimator in the numerator is less (more) efficient than the estimator in the denominator.

in the data about these parameters than in the Re-(c)DCC models. Moreover, whereas in the Re-(c)DCC cases the efficiency ratios are much different between α_0 and β_0, in the Re-BEKK they are very similar to each other, and comparable in magnitude to the results for α_0 in the tables above.

6.4 Computing times, models and estimators

The computing time for estimating the scalar Re-DCC model is smaller than for the scalar Re-cDCC model. For QML estimation, the ratio of the former to the latter is approximately 0.5 for all n (with 2500 observations). For CL2, the ratios are 0.15 for $n \leq 10$, 0.4 for $n = 50$ and 0.5 for $n = 100$. For CL3, they are higher than for CL2 but below one. This pleads in favor of using the scalar Re-DCC model in empirical work, given the comparable bias and efficiency properties of the two models. The computing time for estimating the scalar Re-DCC model by CL2 is 8 percent of the time for QML for $n = 5$ and increases until 30 percent for $n = 100$. For CL3 the corresponding percentages are 10 and 80.

7 Empirical illustration

We consider stock returns of fifty assets traded in the NYSE and NASDAQ, their tickers being shown in Table 5. The sample period spans January 5, 1999 to May 22, 2007, which amounts to 2084 trading days. The dataset has been cleaned from weekends, holidays and early closing days. Days with many consecutive missing values or constant prices have also been removed. Rare missing values have been linearly interpolated. The realized conditional covariances are based on intraday returns computed from 6-minute intervals last mid-quotes. Since the daily trading period of the NYSE and NASDAQ is 6.5 hours, this amounts to 65 intraday observations per day. Relying on the arguments of Andersen, Bollerslev, Frederiksen, and Nielsen (2010), we estimate the scalar Re-BEKK and the two scalar Realized DCC models using daily open-to-close realized covariances.

Results are reported for the three scalar specifications proposed, namely the Re-BEKK, defined by (1) and (4), the Re-DCC defined by (1), (5), (6) and (7) and the Re-cDCC

Table 5: Tickers

AAPL	BMY	CSCO	EXC	HD	JNJ	MMM	SLB	ORCL	WFC
ABT	BP	CVX	F	HNZ	JPM	MOT	T	PEP	WMT
AXP	C	DELL	FDX	HON	KO	MRK	TWX	PFE	WYE
BA	CAT	DIS	GE	IBM	LLY	MS	UN	PG	XOM
BAC	CL	EK	GM	INTC	MCD	MSFT	VZ	QCOM	XRX

defined by (1), (5), (8), (9) and (10).

Scalar Re-BEKK

Table 6 reports (L and CL) estimates and standard errors of the parameters a and b of the covariance matrix equation (4) for portfolios of 3, 5, 15, 30 and 50 (i.e. all) assets. For the portfolios of less than fifty, the choice of assets is arbitrary. The estimation imposes covariance targeting, i.e., $\bar{S}_T = \sum_t C_t/T$. All estimates are significant at standard confidence levels. Strikingly, while the CL estimator seems to be insensitive to the cross-sectional dimension, the L estimator of the parameter a decreases towards zero and that of b increases towards one as the cross-sectional dimension increases.

Table 6: Scalar Re-BEKK Models: L and CL estimates

	L		CL2	
n	a	b	a	b
3	0.2384	0.7516	0.2537	0.7364
	(0.0236)	(0.0255)	(0.0257)	(0.0276)
5	0.2286	0.7614	0.2835	0.7051
	(0.0185)	(0.0201)	(0.0249)	(0.0271)
15	0.1593	0.8307	0.2761	0.7125
	(0.0106)	(0.0120)	(0.0181)	(0.0196)
30	0.1196	0.8704	0.2717	0.7172
	(0.0086)	(0.0103)	(0.0144)	(0.0155)
50	0.0958	0.8942	0.2691	0.7206
	(0.0147)	(0.0183)	(0.0135)	(0.0145)

Robust standard errors in parentheses. The L estimator is based on the Wishart likelihood and the CL2 estimator on the corresponding CL2 function.

Scalar Re-DCC and Re-cDCC

For these models, the elements on the diagonal of S_t in (5) are specified as

$$S_{ii,t} = \omega_i + \gamma_i C_{ii,t-1} + \delta_i S_{ii,t-1}, \tag{26}$$

for $i = 1$ to n. Table 7 (top panel) summarizes the estimates of the parameters of these equations for the fifty stocks. The individual estimates depend neither on the model used (Re-DCC or Re-cDCC) for the second step nor on the dimension n. The results show that the set of assets considered here are characterized by a strong degree of heterogeneity of their individual dynamics. The bottom panel reports (L and CL) estimates and standard errors of the estimates of α and β of the correlation equations for the same set of portfolios (3, 5, 15, 30 and 50 assets) that are used for the scalar Re-BEKK. Both models are estimated using a two-step approach and correlation targeting (Re-DCC) or profiling (Re-cDCC).

All estimates are significant at standard confidence levels. As noted for the Re-BEKK, the L estimator of the innovation parameter, α, approaches zero as the cross-sectional dimension increases, while the smoothing parameter β converges to one. These moves are more pronounced in the Re-DCC case than in Re-cDCC. Like in the Re-BEKK, the CL estimates seem to be insensitive to the cross-sectional dimension.

As a consequence, for large portfolios, the L estimates produce fitted conditional correlations that are close to be constant. A random sample of correlation paths generated by the L and the CL estimators for the Re-DCC model are shown in Figure 2. Figure 1 shows conditional variance and correlation paths generated by the Re-DCC model for the portfolio of three assets (AAPL, ABT, AXP).

L vs CL: A large scale empirical experiment

To better illustrate the contrasting behavior of the L and CL estimators, we computed them for a large number of random portfolios of various dimensions (n =2, 3, 5, 10 and

Table 7: Scalar Re-DCC Models: L and CL estimates

Summary of parameter estimates for the conditional variances[1]

γ_i			δ_i			ω_i		
Min.	Med.	Max.	Min.	Med.	Max.	Min.	Med.	Max.
0.0986	0.1505	0.3302	0.4376	0.6898	0.7965	0.001	0.1089	0.6233

Parameter estimates of the conditional correlation for various portfolio dimensions[2]

	Re-DCC				Re-cDCC			
	L		CL2		L		CL2	
n	α	β	α	β	α	β	α	β
3	0.0564	0.8988	0.0591	0.8831	0.0531	0.9368	0.0538	0.9361
	(0.0101)	(0.0203)	(0.0108)	(0.0211)	(0.0092)	(0.0119)	(0.0089)	(0.0114)
5	0.0412	0.9308	0.0625	0.8952	0.0511	0.9388	0.0536	0.9363
	(0.0077)	(0.0153)	(0.0098)	(0.0174)	(0.0081)	(0.0104)	(0.0079)	(0.0100)
15	0.0201	0.9664	0.0574	0.9052	0.0447	0.9452	0.0539	0.9360
	(0.0022)	(0.0045)	(0.0060)	(0.0110)	(0.0056)	(0.0076)	(0.0065)	(0.0082)
30	0.0139	0.9743	0.0591	0.9024	0.0385	0.9514	0.0541	0.9358
	(0.0010)	(0.0026)	(0.0053)	(0.0098)	(0.0041)	(0.0060)	(0.0058)	(0.0074)
50	0.0105	0.9794	0.0575	0.9058	0.0333	0.9567	0.0547	0.9352
	(0.0007)	(0.0022)	(0.0049)	(0.0090)	(0.0032)	(0.0049)	(0.0060)	(0.0076)

[1]Summary statistics based on the parameter estimates of conditional variances for the fifty stocks. See (26) for definitions of the parameters. [2]Robust standard errors in parentheses. The L estimator is based on (15). The CL2 estimator is based on (23).

20 respectively)[5] selected from the pool of fifty assets considered in this section. The aim is to check whether the discrepancies observed between the L and the CL estimates appear systematically as the cross-sectional dimension increases and independently of the composition of the portfolios, and to assess to what extent these estimators are affected by parameter heterogeneity under (possible) model misspecification, i.e., when imposing common dynamics for the conditional correlations.[6] To this end, in Figure 3 we report box-plot representations of the parameter estimates for each portfolio size. Results are compared with the estimates obtained using the fifty available assets, which are indicated by a straight dashed line.

Consistently with the results reported in Table 7, Figure 3 shows that, as the cross-sectional dimension increases, the CL estimator clearly tends to average correlation dynamics. Indeed, average dynamics, measured by means (or medians) of the correlation dynamic parameters estimated for different portfolio compositions of fixed size, are consistent across cross-sectional dimensions and in line with the estimates obtained for our portfolio of fifty assets. On the contrary, when n is sufficiently large, the L estimator seems to be unable to capture correlation dynamics irrespectively of the model specification. The discrepancy between the L and CL estimators becomes striking. The L estimator appears unable to absorb the heterogeneity in the correlation dynamics.

Engle, Shephard, and Sheppard (2008) report that for GARCH-DCC models, L estimates of the dynamic parameters are biased when the cross-sectional dimension n becomes large relatively to the sample size T, while CL estimates remain stable.[7] They explain the difference by an incidental parameter problem: as n approaches T, the targeting estimator of the constant matrix term of the correlation process, which involves $O(n^2)$ parameters, gets more and more ill-conditioned, and the use of this targeting estimator impacts the L

[5]For $n = 2$ and 3, all portfolios are used (1,225 and 19,600). For Re-DCC (Re-cDCC), the numbers used are: 150,212 (186,983) for $n = 5$; 60,505 (95,206) for $n = 10$; 30,653 (29,817) for $n = 20$.

[6]Notice that when $n = 2$, parameter heterogeneity is captured to the highest extent since we model individually each correlation. Also, when considering all bivariate combinations the L and the CL estimators coincide.

[7]In their context, the L and CL functions are based on the Gaussian distribution.

estimates considerably. This explanation does not apply in our context, since n is equal to 50 and T to 2084. The condition number (ratio of largest to smallest eigenvalue) of our targeting estimator $\bar{P}_T$ in the scalar Re-DCC model is about 25, a value which is not indicative that the matrix is close to be singular.[8] Similarly, for the scalar Re-BEKK model, the apparent bias in the L estimator cannot be explained by ill-conditioning of the targeting estimator $\bar{S}_T$ of the unconditional covariance matrix, since that estimator is regular by construction. Moreover, it is consistent (and even unbiased), which suggests that the inconsistency of the targeting estimator of the constant term of the scalar Re-DCC model is probably not the source of the bias for the L estimates of that model.

As a further check, we computed the score contribution of each observation and did not find signs of influential observations.

8 Conclusions

We have proposed a new dynamic model for realized covariance matrices. The model can be specified and estimated in two steps, the first one for the variances, and the second for the dynamic correlation matrix. The first step can also be split into individual steps. This enables to apply the model to matrices of large dimension, where large in this context means of the order of fifty. This is a significant progress relative to existing models. The possibility to split the estimation in steps comes from the use of a scalar DCC model, and from the use of the Wishart distribution. The latter assumption also allows us to use a composite likelihood approach which might be especially relevant for very large dimensions since the usual ML estimator of the dynamic parameters of the correlation process seems to be biased. The Wishart assumption should not be viewed as a big drawback given that the estimation has a quasi-likelihood interpretation. Several extensions are on our

[8]This is not surprising since $\bar{P}_T$ is an average of $T = 2084$ correlation matrices P_t, each of which is regular. Even if some of the P_t matrices were singular, this would be washed out by averaging. We also checked the condition numbers of the individual P_t matrices and found them to be of the order of 150 when constructed from 65 intra-day observations.

research agenda, among which non-scalar DCC models, tests of the scalar restriction, and forecasting.

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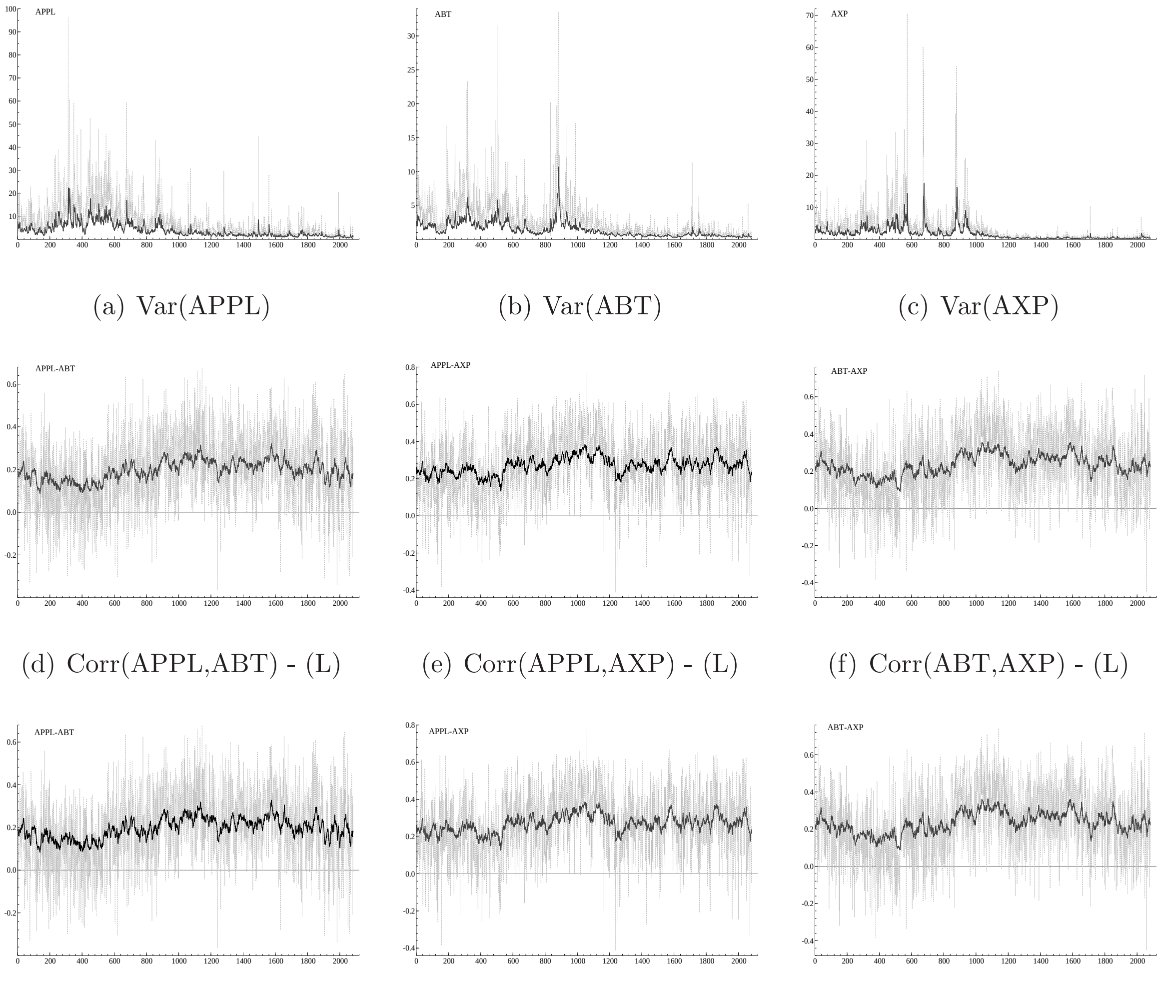
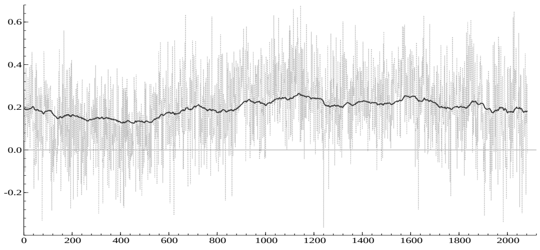


(a) Var(APPL) (b) Var(ABT) (c) Var(AXP)

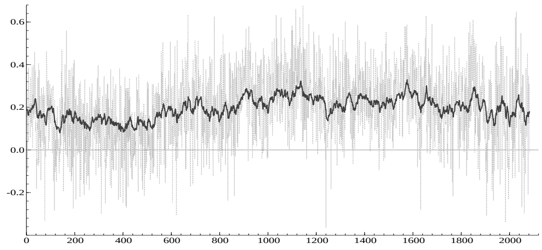
(d) Corr(APPL,ABT) - (L) (e) Corr(APPL,AXP) - (L) (f) Corr(ABT,AXP) - (L)

(g) Corr(APPL,ABT) - (CL) (h) Corr(APPL,AXP) - (CL) (i) Corr(ABT-AXP) - (CL)

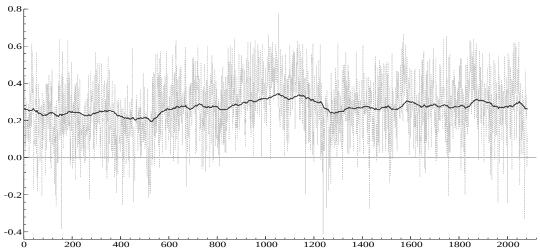
Figure 1: Fitted variances and correlations (solid) vs. realizations (dashed) for trivariate Re-DCC model



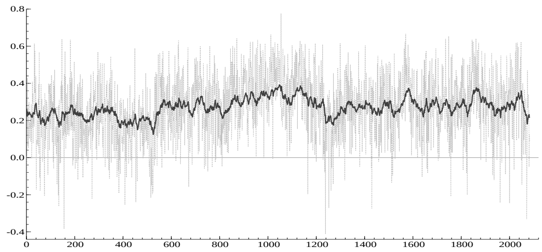
(a) Corr 1 - L



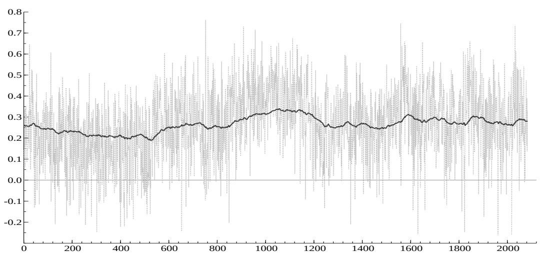
(b) Corr 1 - CL



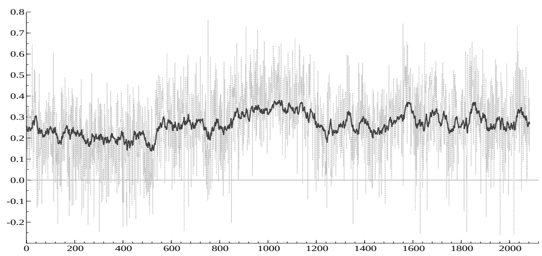
(c) Corr 2 - L



(d) Corr 2 - CL

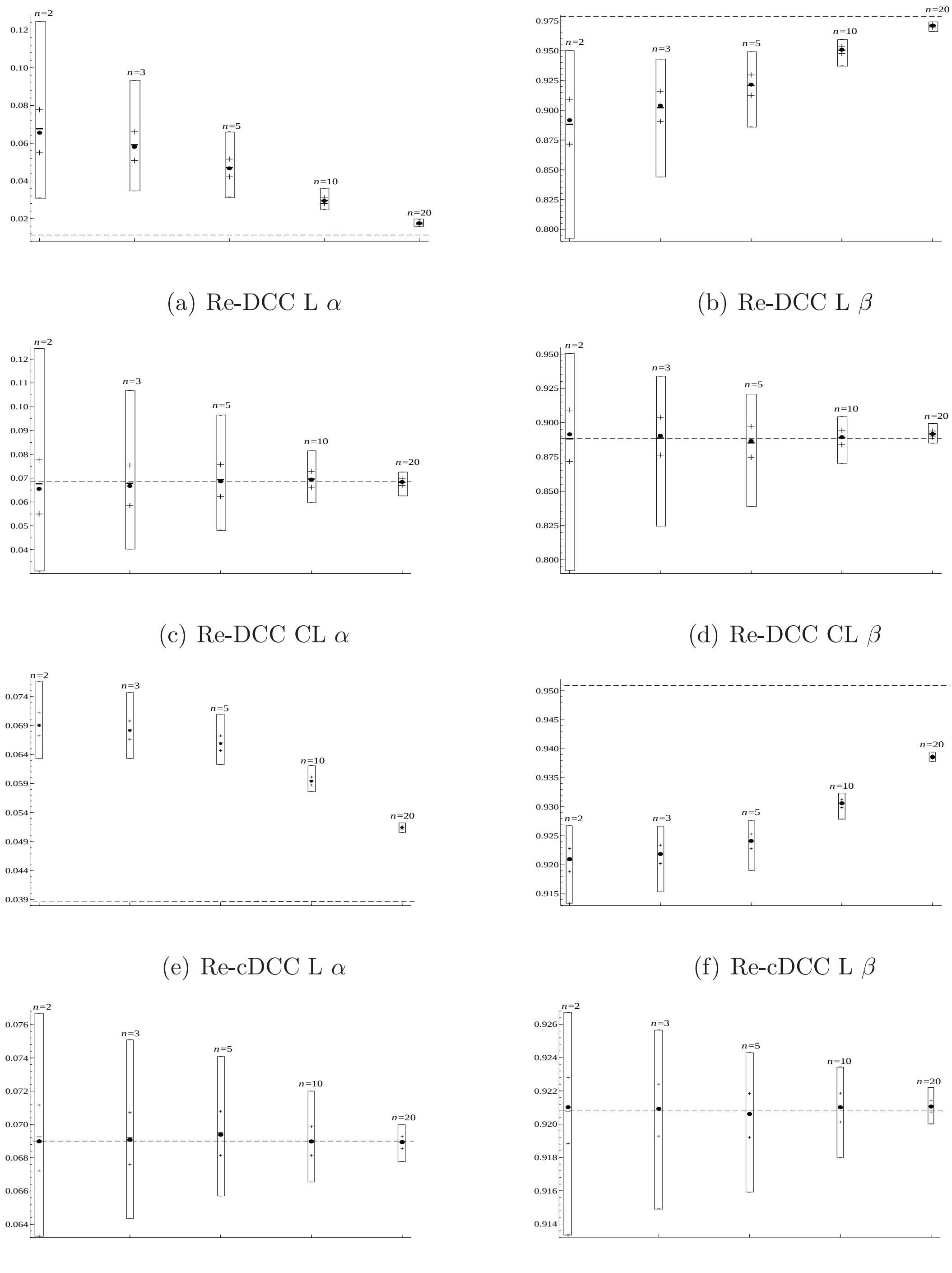


(e) Corr 3 - L



(f) Corr 3 - CL

Figure 2: Selection of fitted correlations (solid) vs. realizations (dashed) for Re-DCC model of fifty stocks. Parameters estimated using L (left) and CL (right)



(a) Re-DCC L α (b) Re-DCC L β

(c) Re-DCC CL α (d) Re-DCC CL β

(e) Re-cDCC L α (f) Re-cDCC L β

(g) Re-cDCC CL α (h) Re-cDCC CL β

Figure 3: Box-plots of the correlation parameters. Box bounds represent the 1% and 99% quantiles respectively, + the 25% and 75% quantiles, the dot and the dash the median and mean, respectively

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