EVALUATION 2-FUNCTORS FOR KAC–MOODY 2-CATEGORIES OF TYPE A_2

MARCO MACKAAY, JAMES MACPHERSON, AND PEDRO VAZ

ABSTRACT. We construct a 2-functor from the Kac–Moody 2-category for the extended quantum affine $\mathfrak{sl}_3$ to the homotopy 2-category of bounded chain complexes with values in the Kac–Moody 2-category for quantum $\mathfrak{gl}_3$, categorifying the evaluation map between the corresponding quantum Kac–Moody algebras.

CONTENTS

1. INTRODUCTION

In the 90s, Chari and Pressley launched a systematic study of finite-dimensional representations of quantum affine algebras, starting with affine $\mathfrak{sl}_2$ in [ChPr91]. Since then, these representations have been studied intensively and continue to be an active research topic with important open questions and interesting links to other research areas, e.g. mathematical physics and cluster algebras, see e.g. [HeLe20] for more information.

In affine type A, there is a special class of irreducible finite-dimensional representations, the so-called *evaluation representations*. These are obtained by pulling back irreducible representations of finite type A through a so-called *evaluation map*, which is an algebra homomorphism $\mathrm{ev}_{a,n}\colon \mathbf{U}_{\triangle}(n) \to \mathbf{U}(n)$, where $a \in \mathbb{C}^{\times}$ is a scalar, $\mathbf{U}_{\triangle}(n)$ is the so-called *extended quantum affine* $\mathfrak{sl}_n$ and $\mathbf{U}(n) = \mathbf{U}_q(\mathfrak{gl}_n)$, see Section 2. As we will recall in more detail in that section, the level zero weight lattice of the former quantum affine Kac–Moody algebra can be identified with the $\mathfrak{gl}_n$-weight lattice. The fact that we have to pass from $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$ is important and has a categorical counterpart, as we will explain below. For more information on evaluation maps and evaluation representations in general, see e.g. [ChPr94a, ChPr94b, DuFu16]. Since we wish to categorify this construction, we must pass to the idempotented forms of the above algebras, which can be considered as categories, and the evaluation map can therefore be considered as a functor.

Quantum Kac–Moody algebras were *categorified* by Khovanov and Lauda [KhL10], and independently by Rouquier [Rou08]. Slightly altering the latter's terminology, we call these categorifications *Kac–Moody 2-categories*. In finite Dynkin types, all irreducible finite-dimensional representations can be categorified by certain quotients of the Kac–Moody 2-categories, which nowadays go under the name of *cyclotomic KLR algebras*. In other Dynkin types, e.g. affine Dynkin types, this is not true. In particular, evaluation representations in affine type A cannot be categorified by cyclotomic KLR-algebras, because the latter categorify highest weight representations and evaluation representations do not have a highest weight. However, we conjecture that the evaluation map (considered as an evaluation functor) $\mathrm{ev}_n^t := \mathrm{ev}_{q^t,n}$, for any $t \in \mathbb{Z}$ and $n \in \mathbb{N}_{>2}$, can be categorified by an evaluation 2-functor $\mathcal{E}\mathfrak{v}_n^t \colon \mathcal{U}_\triangle(n) \to \mathcal{K}^b(\mathcal{U}(n))$, which can be used to define evaluation 2-representations (i.e. categorified evaluation representations) of $\mathcal{U}_\triangle(n)$ by pulling back "irreducible" 2-representations (i.e. cyclotomic KLR algebras) of $\mathcal{U}(n)$. Here $\mathcal{K}^b(\mathcal{U}(n))$ denotes the homotopy 2-category of bounded complexes in $\mathcal{U}(n)$, so the 1-morphisms of $\mathcal{U}_\triangle(n)$ act by composing with bounded complexes in $\mathcal{U}(n)$. As a matter of fact, we not only conjecture $\mathcal{E}\mathfrak{v}_n^t$ to exist but also an extension of it to $\mathcal{K}^b(\mathcal{U}_\triangle(n))$.

In this paper, we prove the first conjecture for $\mathcal{U}_\triangle(3)$ and hope that it serves as the base case for an inductive proof for $\mathcal{U}_\triangle(n)$, when $n > 3$, in a forthcoming paper. Proving that there is no obstruction to extending $\mathcal{E}\mathfrak{v}_n^t$ to $\mathcal{K}^b(\mathcal{U}_\triangle(n))$ is not easy and certainly beyond the scope of this paper and its sequel.

There are two good reasons for publishing the case $n = 3$ separately. Firstly, our proof for $n = 3$ is shortened by the fact that it is closely related to the proofs in [ALELR24] of the well-definedness of the categorification of the internal braid group action on $\mathbf{U}_q(\mathfrak{sl}_3)$, as might be expected from the well-known relation between the evaluation map and the braid group action on the decategorified level. In principle, this relation should also categorify for $n > 3$, but only if the categorified braid group action extends to $\mathcal{K}^b(\mathcal{U}(n))$ (to include the action of longer braids), which has been conjectured but not yet proved (see [ALELR24, Conjecture 1.2]). This is why our approach for $n > 3$ will be completely different. We hope that presenting the base case $n = 3$ here will prepare the ground (and the reader) for the general case and also keep the size of the forthcoming paper withing reasonable bounds.

The second reason for publishing this case separately, is that it reveals the need to pass from $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$ once more, but now on the categorical level. Recall that the definition of a Kac–Moody 2-category depends on a choice of invertible scalars and compatible bubble parameters, see e.g. [Lauda20]. In finite type A all choices yield essentially the same 2-category, i.e. up to 2-isomorphism, but in affine type A they don't. In particular, Khovanov and Lauda's original affine type A *unsigned* Kac–Moody 2-category in [KhL10], with all scalars and bubble parameters equal to one, and the Kac–Moody 2-category defined in [MT17], with non-trivial bubble parameters depending on level zero $\widehat{\mathfrak{gl}}_n$-weights (instead of level zero $\widehat{\mathfrak{sl}}_n$-weights), are not 2-isomorphic when n is odd. This was mentioned in [KhL11a] without proof and, therefore, we prove it in Theorem A.2. Although this does not by itself imply that there is no evaluation 2-functor for trivial scalars and bubble parameters when $n = 3$, we failed to find one. More generally, it seems that one is forced to use the scalars and level zero $\widehat{\mathfrak{gl}}_n$-bubble parameters from [MT17] when n is odd. When n is even, everything is simpler because in that case both choices of scalars and bubble parameters yield essentially the same Kac–Moody 2-category, see Appendix A.

There is an analogous story for the affine Hecke algebra and its finite-dimensional representations. The categorification of the corresponding evaluation map was carried out in [MMV22] and was technically less challenging than the categorification of the evaluation map for the affine type A Kac–Moody algebra. In both cases, the target (2-)category of the evaluation (2-)functor is a homotopy category of bounded complexes and, as was argued in [MMV22], one motivation for defining and studying evaluation 2-representations is that they might provide some important clues for the development of triangulated 2-representation theory, which at the moment is very poorly understood, even at the most basic level. For example, it was shown in [MMV22] that every evaluation 2-representation of extended affine Soergel bimodules has a *finitary cover*, somehow relating finitary and triangulated 2-representations. The same might hold for the evaluation 2-representations of $\mathcal{U}_{\triangle}(n)$, but that question is outside the scope of this paper. Also in both cases, one would like to categorify tensor products of evaluation representations, which play a fundamental role in the finite-dimensional representation theory of the affine quantum algebras in question, see e.g. [ChPr94b, Chapter 12, Section 2C] for the case of $\mathcal{U}_{\triangle}(n)$. However, it is far from clear how to do that at this point. Perhaps it is possible to somehow adapt Webster's tensor algebras of Stendhal diagrams [Web17] in that case. We hope to address these and some other interesting questions about evaluation 2-representations, e.g. whether $\mathcal{E}v_t$ extends to $\mathcal{K}^b(\mathcal{U}_{\triangle}(n))$, in the future.

The structure of the paper is as follows. Section 2 reviews the evaluation map/functor ev_3^t and Section 3 presents the definitions of the affine and finite type A Kac–Moody 2-categories $\mathcal{U}_{\triangle}(3)$ and $\mathcal{U}(3)$ that we will be working with. The meat of the paper is Section 4, where we define the evaluation 2-functor $\mathcal{E}v$ and prove the paper's central theorem, Theorem 4.1, that $\mathcal{E}v$ is a well-defined 2-functor that decategorifies to ev_3^t. We finish the paper with Appendix A, where we justify our choice of the scalars and bubble parameters in the definition of $\mathcal{U}_{\triangle}(3)$ over a choice in [KhL10] by proving in Theorem A.2 that the two choices are not related by a 2-isomorphism that fixes objects and 1-morphisms.

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2. The Decategorified Setting

Our main reference for this section is [DuFu16]. Note that we are interested in the idempotented version of some of the quantum algebras in that paper, so we have to adapt Du and Fu's definitions. We use the idempotented versions because these are the ones that are categorified by Kac–Moody 2-categories.

2.1. **Finite type $\dot{\mathrm{U}}(n)$ and affine type $\dot{\mathrm{U}}_{\triangle}(n)$ of level zero.** Throughout this paper we identify both the (integral) $\mathfrak{gl}_n$-weight lattice and the level-zero (integral) $\widehat{\mathfrak{gl}}_n$-weight lattice with $\mathbb{Z}^n$, denoting either sort of (integral) weight by e.g. $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. The simple $\widehat{\mathfrak{gl}}_n$-roots $\alpha_1, \dots, \alpha_n$ are then given by

$$\alpha_i = \begin{cases} (0, \dots, 0, 1, -1, 0, \dots, 0) & 1 \leq i \leq n-1, \\ (-1, 0, \dots, 0, 1) & i = n, \end{cases}$$

where the 1 is always the ith entry. Note that $\alpha_1, \dots, \alpha_{n-1}$ are the simple $\mathfrak{gl}_n$-roots.

Under the above identification, the bilinear form on these weight lattices corresponds to the Euclidean inner product on $\mathbb{Z}^n$. Its restriction to the root lattices then reads

$$(\alpha_i, \alpha_j) = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } i \equiv j \pm 1 \bmod n, \\ 0 & \text{else}, \end{cases}$$

for $1 \leq i, j \leq n$. Note that, in the affine case, the indices $1, \dots, n$ are interpreted as representatives of the residue classes modulo n. From now on, we will always tacitly use this interpretation of the indices of affine weights and roots. We also recall the standard notation $i \cdot j := (\alpha_i, \alpha_j)$, which we will often use below.

Finally, given $\lambda \in \mathbb{Z}^n$, define $\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_n) \in \mathbb{Z}^n$, where $\overline{\lambda}_i = \lambda_i - \lambda_{i+1}$ for all $i = 1, \dots, n$. By the above convention for affine weights, we have $\overline{\lambda}_n = \lambda_n - \lambda_1$, so $\overline{\lambda}_1 + \ldots + \overline{\lambda}_n = 0$. In other words, $\overline{\lambda}$ belongs to a rank $n-1$ sublattice of $\mathbb{Z}^n$, which can be identified with the level-zero integral $\widehat{\mathfrak{sl}}_n$-weight lattice. The element $(\overline{\lambda}_1, \dots, \overline{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$ can then be identified with an (integral) $\mathfrak{sl}_n$-weight.

For the definition below, recall that the quantum integer $[m]$, for $m \in \mathbb{Z}$, is defined as

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

Definition 2.1. The *idempotented extended quantum affine* $\mathfrak{sl}_n$, denoted by $\dot{\mathrm{U}}_{\triangle}(n)$, is the associative idempotented $\mathbb{Q}(q)$-algebra generated by 1_λ, $E_i 1_\lambda$ and $F_i 1_\lambda$, for $\lambda \in \mathbb{Z}^n$ and $i = 1, \dots, n$, subject to the relations:

$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda,$$

$$E_i 1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} E_i 1_\lambda,$$

$$F_i 1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} F_i 1_\lambda,$$

$$1_\mu E_i 1_\lambda = \delta_{\mu,\lambda+\alpha_i} E_i 1_\lambda,$$

$$1_\mu F_i 1_\lambda = \delta_{\mu,\lambda-\alpha_i} F_i 1_\lambda,$$

$$E_i F_j 1_\lambda - F_j E_i 1_\lambda = \delta_{i,j} [\overline{\lambda}_i] 1_\lambda,$$

$$E_i E_j 1_\lambda = E_j E_i 1_\lambda \qquad \text{if } i \cdot j = 0,$$

$$F_i F_j 1_\lambda = F_j F_i 1_\lambda \qquad \text{if } i \cdot j = 0,$$

$$E_i^2 E_j 1_\lambda + E_j E_i^2 1_\lambda = [2] E_i E_j E_i 1_\lambda \qquad \text{if } i \cdot j = -1,$$

$$F_i^2 F_j 1_\lambda + F_j F_i^2 1_\lambda = [2] F_i F_j F_i 1_\lambda \qquad \text{if } i \cdot j = -1.$$

Note that $E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i 1_\lambda$, so we can use the notation $E_i E_j 1_\lambda := E_i 1_{\lambda+\alpha_j} \cdot E_j 1_\lambda$ without ambiguity. Similarly, we will use the notation $1_\mu E_i = 1_\mu E_i 1_{\mu-\alpha_i}$ and $1_\mu F_i = 1_\mu F_i 1_{\mu+\alpha_i}$, so that $E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i$ and $F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i$.

Definition 2.2. The *idempotented quantum* $\mathfrak{gl}_n$, denoted by $\dot{\mathbf{U}}(n)$, is the idempotented subalgebra of $\dot{\mathbf{U}}_\triangle(n)$ generated by 1_λ, $E_i 1_\lambda$ and $F_i 1_\lambda$, for $i = 1, \dots, n-1$.

Note that $\dot{\mathbf{U}}_\triangle(n)$ and $\dot{\mathbf{U}}(n)$ share the same idempotents, but, whereas $\dot{\mathbf{U}}(n) = \dot{\mathbf{U}}(\mathfrak{gl}_n)$, the idempotented algebra $\dot{\mathbf{U}}_\triangle(n)$ is only an idempotented subalgebra of $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, which is why it is called the idempotented *extended* quantum affine $\mathfrak{sl}_n$ and not the idempotented quantum affine $\mathfrak{gl}_n$, see [DuFu16, Section 2] for more details.

Remark 2.3. Recall that these idempotented algebras can be seen as linear categories whose object sets are given by the sets of weights and whose hom-spaces are given by e.g.

$$\mathrm{Hom}_{\dot{\mathbf{U}}(n)}(\lambda, \mu) = 1_\mu \dot{\mathbf{U}}(n) 1_\lambda$$

with composition corresponding to multiplication. This is why these idempotented algebras are categorified by 2-categories rather than categories.

2.2. **Evaluation maps.** Fix $t \in \mathbb{Z}$ and let $[X, Y]_{q^{\pm 1}} = XY - q^{\pm 1} YX$ be the $q^{\pm 1}$-commutator. From now on we will always assume that $n > 2$.

Definition 2.4. The *evaluation map* $\mathrm{ev}_n^t \colon \dot{\mathbf{U}}_\triangle(n) \to \dot{\mathbf{U}}(n)$ is the homomorphism of idempotented algebras defined by

$$\mathrm{ev}_n^t(1_\lambda) = 1_\lambda, \tag{1}$$

$$\mathrm{ev}_n^t(E_i 1_\lambda) = E_i 1_\lambda \quad \text{for } i \neq n, \tag{2}$$

$$\mathrm{ev}_n^t(F_i 1_\lambda) = F_i 1_\lambda \quad \text{for } i \neq n, \tag{3}$$

$$\mathrm{ev}_n^t(E_n 1_\lambda) = q^{\lambda_1+\lambda_n+t-1} [\cdots [[F_1, F_2]_q, F_3]_q \cdots]_q, F_{n-1}]_q 1_\lambda, \tag{4}$$

$$\mathrm{ev}_n^t(F_n 1_\lambda) = q^{-\lambda_1-\lambda_n-t+1} [E_{n-1}, [E_{n-2}, [\cdots [E_2, E_1]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}} 1_\lambda. \tag{5}$$

Remark 2.5. Three quick observations:

a) Note that
$$[\cdots[[F_1,F_2]_q,F_3]_q\cdots]_q,F_{n-1}]_q 1_\lambda = 1_{\lambda-\alpha_1-\cdots-\alpha_{n-1}}[\cdots[[F_1,F_2]_q,F_3]_q\cdots]_q,F_{n-1}]_q,$$
so $\mathrm{ev}_n^t(E_n 1_\lambda) = \mathrm{ev}_n^t(1_{\lambda+\alpha_n}E_n 1_\lambda)$ is well defined, because $\alpha_1+\ldots+\alpha_{n-1}+\alpha_n=0$. The same is true for $\mathrm{ev}_n^t(F_n 1_\lambda)=\mathrm{ev}_n^t(1_{\lambda-\alpha_n}F_n 1_\lambda)$.

b) Note also that our expressions for $\mathrm{ev}_n^t(E_n 1_\lambda)$ and $\mathrm{ev}_n^t(F_n 1_\lambda)$ differ from $\mathrm{ev}_{q^t}(E_n 1_\lambda)$ and $\mathrm{ev}_{q^t}(F_n 1_\lambda)$ in [DuFu16, (5.0.1)]. Recall the $\mathbb{Q}(q)$-linear involution ω on $\dot{\mathbf{U}}_\triangle(n)$, defined by
$$1_\lambda = 1_{-\lambda}, \quad \omega(E_i 1_\lambda) = F_i 1_{-\lambda}, \quad \omega(F_i 1_\lambda) = E_i 1_{-\lambda},$$
for $i=1,\ldots,n$. It naturally restricts to $\dot{\mathbf{U}}(n)$ and we use the same notation for this restriction. The relation between our evaluation map ev_n^t and Du and Fu's evaluation map ev_{q^t} can be expressed as
$$\mathrm{ev}_n^t = \omega\,\mathrm{ev}_{q^{2-t}}\,\omega. \tag{6}$$
The reason for our different choice is that we wanted to take advantage of the results in [ALELR24] to prove well-definedness of the evaluation functor for $n=3$, so we decided to adopt the conventions of that paper. It is easy to obtain the evaluation functor which categorifies Du and Fu's evaluation map from ours by categorifying the relation in (6) using Khovanov and Lauda's involution $\widetilde{\omega}$ on $\mathcal{U}_\triangle(n)$ and its restriction to $\mathcal{U}(n)$, see [KhL10, Proposition 3.28].

c) When we consider the idempotented algebras as categories as in Remark 2.3, ev_n^t becomes a linear functor. This is why it is categorified by a 2-functor rather than a functor.

The expressions for $\mathrm{ev}_n^t(E_n 1_\lambda)$ and $\mathrm{ev}_n^t(F_n 1_\lambda)$ in (4) and (5) can be written as alternating sums, which will be important later on. For $\xi=(\xi_1,\ldots,\xi_{n-2})\in\{0,1\}^{n-2}$ set
$$E_\xi 1_\lambda := E_{n-1}^{1-\xi_{n-2}}E_{n-2}^{1-\xi_{n-3}}\cdots E_2^{1-\xi_1}E_1E_2^{\xi_1}\cdots E_{n-2}^{\xi_{n-3}}E_{n-1}^{\xi_{n-2}}1_\lambda, \tag{7}$$
$$F_\xi 1_\lambda := F_{n-1}^{\xi_{n-2}}F_{n-2}^{\xi_{n-3}}\cdots F_2^{\xi_1}F_1F_2^{1-\xi_1}\cdots F_{n-2}^{1-\xi_{n-3}}F_{n-1}^{1-\xi_{n-2}}1_\lambda. \tag{8}$$
and let $|\xi|=\xi_1+\cdots+\xi_{n-2}$. The following can be obtained by direct computation.

Lemma 2.6. *We have*
$$\mathrm{ev}_n^t(E_n 1_\lambda) = q^{\lambda_1+\lambda_n+t-1}\sum_{\xi\in\{0,1\}^{n-2}}(-q)^{|\xi|}F_\xi 1_\lambda, \tag{9}$$
$$\mathrm{ev}_n^t(F_n 1_\lambda) = q^{-\lambda_1-\lambda_n-t+1}\sum_{\xi\in\{0,1\}^{n-2}}(-q)^{-|\xi|}E_\xi 1_\lambda. \tag{10}$$

For more details on the evaluation map, see [DuFu16, Section 5].

3. Kac–Moody 2-categories

We will move on to recalling in detail the 2-categories $\mathcal{U}(n)$ and $\mathcal{U}_\triangle(n)$ as defined in [MSV13, Definition 3.1] and [MT17, Definition 3.19], respectively. These decategorify to $\dot{\mathbf{U}}(n)$ and $\dot{\mathbf{U}}_\triangle(n)$.

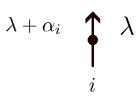
3.1. **Definition.** We define $\mathcal{U}_{\triangle}(n)$ and $\mathcal{U}(n)$ simultaneously, because only the range of the indices of the 1-morphisms and of the colors of the 2-morphisms differ. For concreteness, we will work over $\mathbb{Q}$, but any field of characteristic zero would serve equally well.

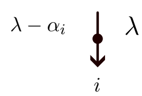
Definition 3.1. The 2-category $\mathcal{U}_{\triangle}(n)$ (resp. $\mathcal{U}(n)$) is the graded $\mathbb{Q}$-linear 2-category with:

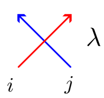
- Objects: $\lambda \in \mathbb{Z}^n$,
- 1-morphisms: formal direct sums of shifts of

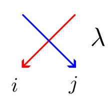
$$\mathbf{1}_\lambda, \quad \mathscr{E}_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda+\alpha_i}\mathscr{E}_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda+\alpha_i}\mathscr{E}_i, \quad \mathscr{E}_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda-\alpha_i}\mathscr{E}_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda-\alpha_i}\mathscr{E}_i,$$

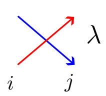
  for $\lambda \in \mathbb{Z}^n$,
- 2-morphisms: equivalence classes of $\mathbb{Q}$-linear combinations of diagrams obtained by horizontally concatenating and vertically gluing the generators below. By convention, a 2-morphism $\alpha\colon X\langle r\rangle \to Y\langle s\rangle$, for $r, s \in \mathbb{Z}$, is given by a linear combination of homogeneous diagrams of degree $s - r$, as defined in [KhL10].

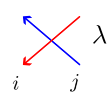
 $: \mathscr{E}_i\mathbf{1}_\lambda \to \mathscr{E}_i\mathbf{1}_\lambda\langle 2\rangle$,

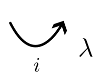
 $: \mathscr{F}_i\mathbf{1}_\lambda \to \mathscr{F}_i\mathbf{1}_\lambda\langle 2\rangle$,

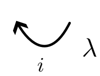
 $: \mathscr{E}_i\mathscr{E}_j\mathbf{1}_\lambda \to \mathscr{E}_j\mathscr{E}_i\mathbf{1}_\lambda\langle -i\cdot j\rangle$,

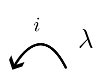
 $: \mathscr{F}_iF_j\mathbf{1}_\lambda \to \mathscr{F}_i\mathscr{F}_j\mathbf{1}_\lambda\langle -i\cdot j\rangle$,

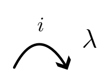
 $: \mathscr{E}_i\mathscr{F}_j\mathbf{1}_\lambda \to \mathscr{F}_j\mathscr{E}_i\mathbf{1}_\lambda$,

 $: \mathscr{F}_iE_j\mathbf{1}_\lambda \to \mathscr{E}_i\mathscr{F}_j\mathbf{1}_\lambda$,

 $: \mathbf{1}_\lambda \to \mathscr{F}_i\mathscr{E}_i\mathbf{1}_\lambda\langle 1+\overline{\lambda}_i\rangle$,

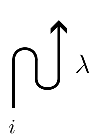
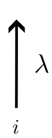
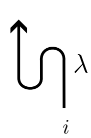
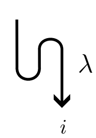
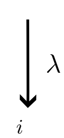
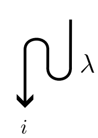
 $: \mathbf{1}_\lambda \to \mathscr{E}_i\mathscr{F}_j\mathbf{1}_\lambda\langle 1-\overline{\lambda}_i\rangle$,

 $: \mathscr{F}_i\mathscr{E}_1\mathbf{1}_\lambda \to \mathbf{1}_\lambda\langle 1+\overline{\lambda}_i\rangle$,

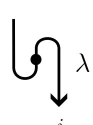
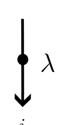
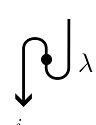
 $: \mathscr{E}_iF_i\mathbf{1}_\lambda \to \mathbf{1}_\lambda\langle 1-\overline{\lambda}_i\rangle$.

The equivalence relation is defined by the equations below.

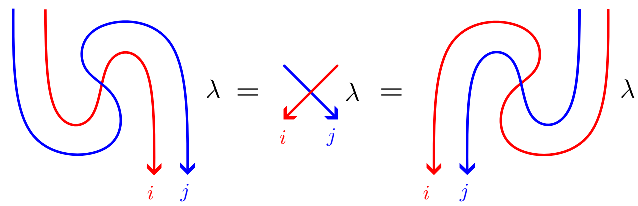
(KM1) Right and left adjunction:

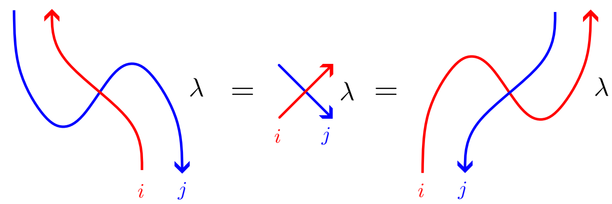
(11)  $=$  $=$   $=$  $=$ 

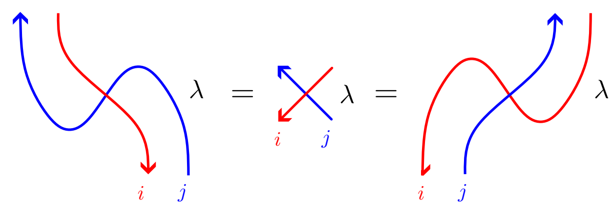
(KM2) Dot cyclicity:

(12)  $=$  $=$ 

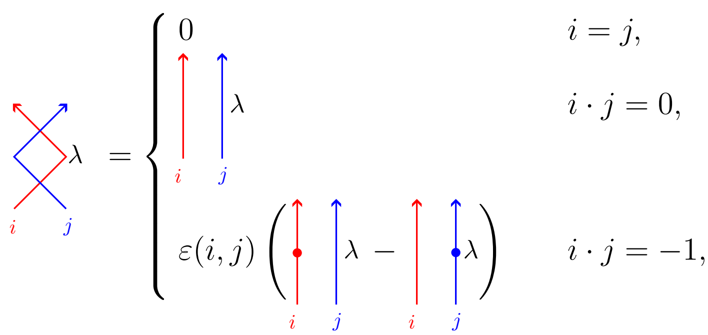
(KM3) Crossing cyclicity:

(13) 

(14) 

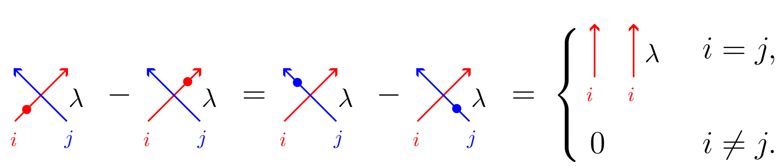
(15) 

(KM4) Quadratic KLR:

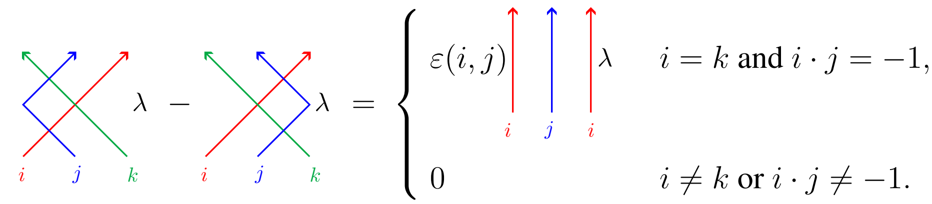
(16) 

where $\varepsilon(i,j) = \begin{cases} 1 & i = j + 1 (\operatorname{mod} n) \\ -1 & i = j - 1 (\operatorname{mod} n) \\ 0 & \text{else} \end{cases}$

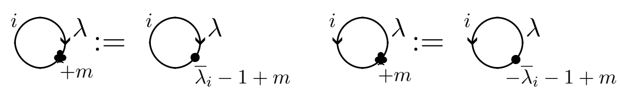
(KM5) Dot slide:

(17) 

(KM6) Cubic KLR:

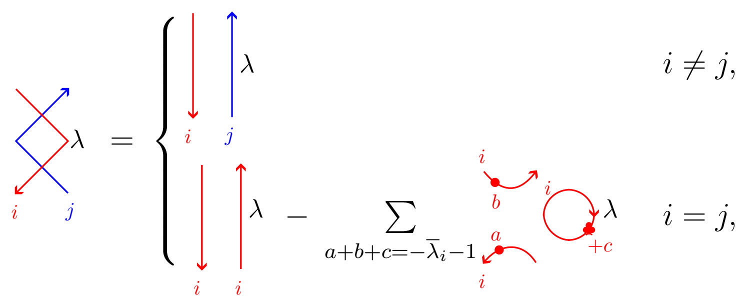
(18) 

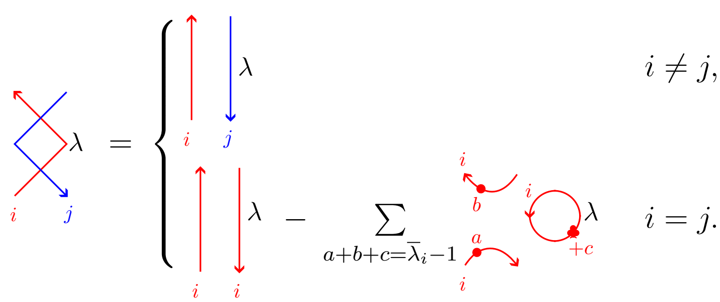
Before we list more relations, first a useful piece of notation:

(19) 

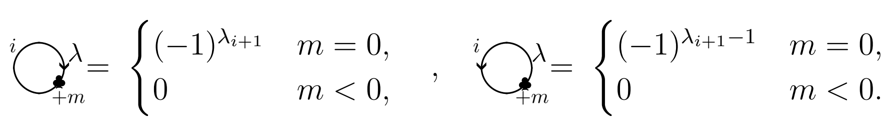
Using this notation, the other relations on diagrams are:

(KM7) Mixed EF:

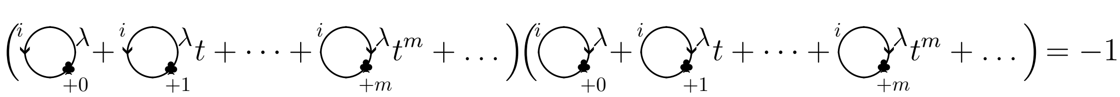
(20) 

(21) 

(KM8) Bubble relations:

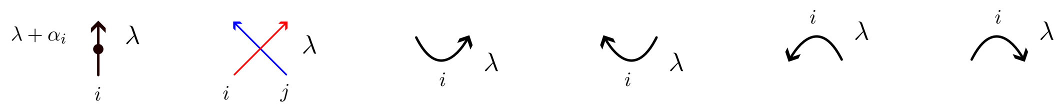
(22) 

(KM9) Infinite Grassmannian relation:

(23) 

This ends the definition of the 2-category $\mathcal{U}_{\triangle}(n)$ (resp. $\mathcal{U}(n)$).

Remark 3.2. Thanks to adjunction and cyclicity (equations (11) through (15)), the 2-morphisms of $\mathcal{U}_\triangle(n)$ are already generated by

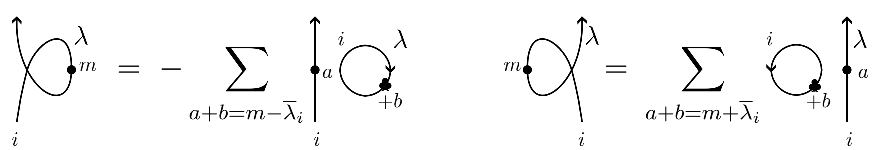


for $\lambda \in \mathbb{Z}^n$ and $i, j = 1, \ldots, n$ (and similarly for $\mathcal{U}(n)$). It therefore suffices to define the evaluation functor on this smaller set of generators.

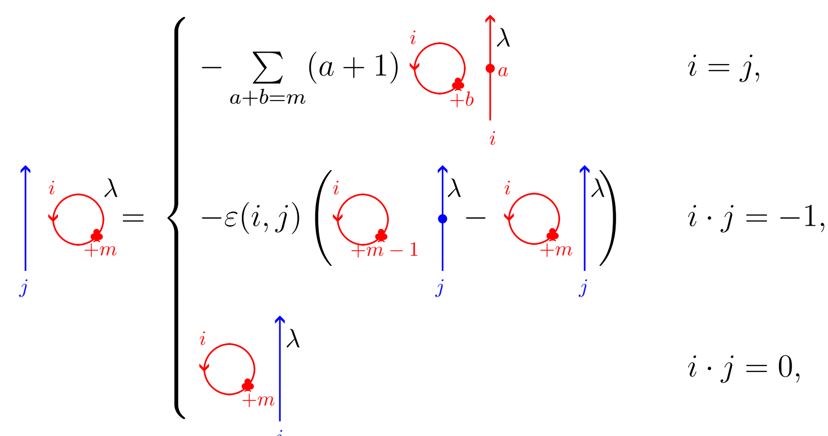
Remark 3.3. The choice of signs in Definition 3.1 is not covered by [ALELR24]. This choice of signs is referred to as a *choice of scalars* and *bubble parameters* - see [Lauda20] for an in-depth explanation. Since we will be adapting various proofs from [ALELR24] for the proof of our main result, we will therefore need to take care when translating them across the different sign conventions. We will discuss difference choices of scalars and bubble parameters further in Appendix A, since they have implications for the existence of an evaluation 2-functor.

3.2. **Some additional relations.** Some well-known consequences of the above relations are listed below. For the proofs, see e.g. [BHLW16] and references therein.

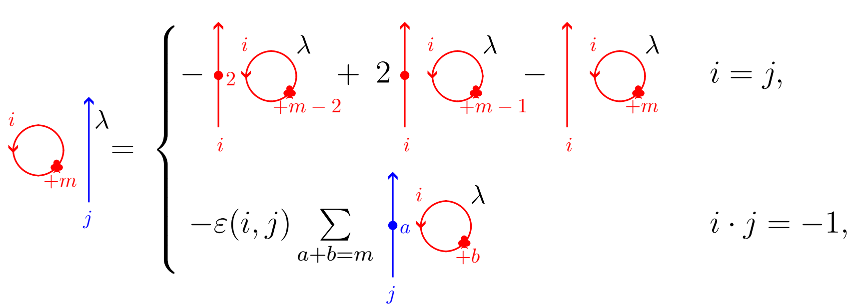
(24)

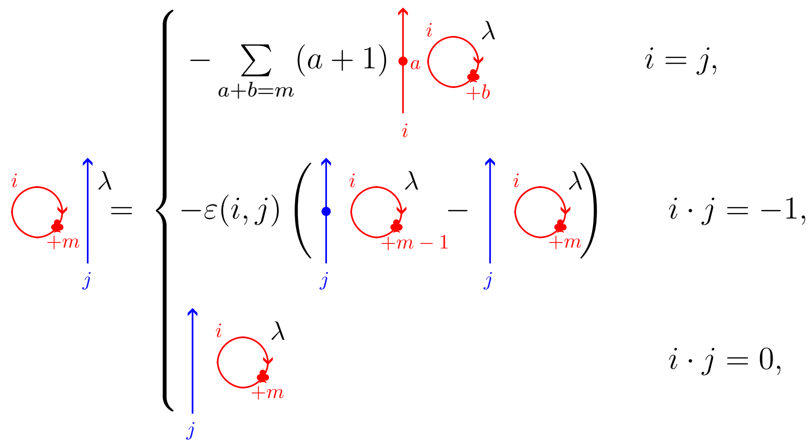


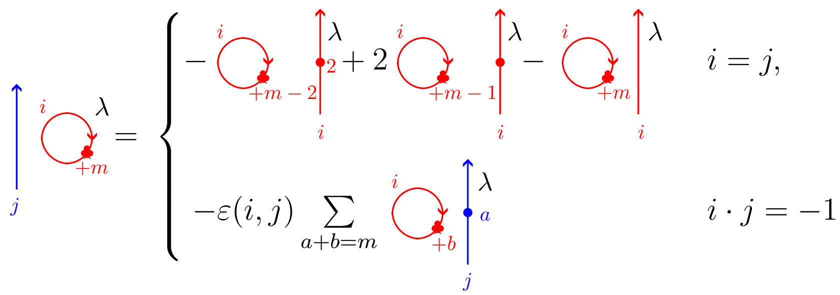
(25)

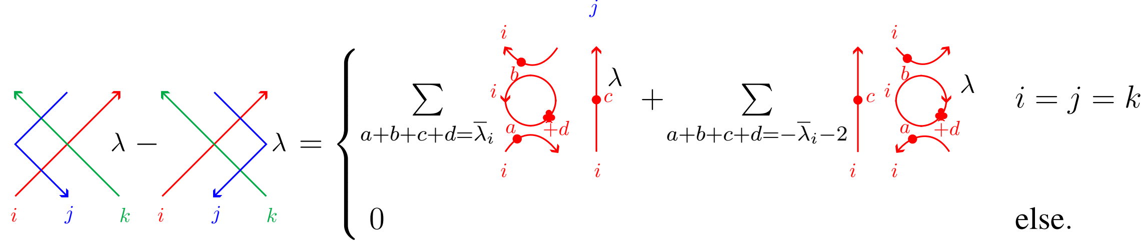


(26)



(27) 

(28) 

(29) 

4. Defining The Evaluation 2-Functor for $n = 3$

In this section, we will define an evaluation 2-functor, which is the additive $\mathbb{Q}$-linear monoidal functor

$$\mathcal{E}\mathfrak{v}_3^t \colon \mathcal{U}_\triangle(3) \to \mathcal{K}^b(\mathcal{U}(3))$$

for $t \in \mathbb{Z}$, defined in the next pages. Note that in this case, Definition 2.4 is particularly simple, because (4) and (5) only involve one q-commutator each:

$$\begin{aligned}
\mathrm{ev}_3^t(E_3 1_\lambda) &= q^{\lambda_1+\lambda_3+t-1}(F_1F_2 1_\lambda - qF_2F_1 1_\lambda),\\
\mathrm{ev}_3^t(F_3 1_\lambda) &= q^{-\lambda_1-\lambda_3-t+1}(E_2E_1 1_\lambda - q^{-1}E_1E_2 1_\lambda),
\end{aligned}$$

for $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$. For the remainder of this section, we set $S = \lambda_1 + \lambda_3 + t - 1$ and use the notation $\mathcal{E}_{i_1 i_2 \ldots i_k}\mathbf{1}_\lambda = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\ldots\mathcal{E}_{i_k}\mathbf{1}_\lambda$ and $\mathcal{F}_{i_1 i_2 \ldots i_k}\mathbf{1}_\lambda = \mathcal{F}_{i_1}\mathcal{F}_{i_2}\ldots\mathcal{F}_{i_k}\mathbf{1}_\lambda$. Since t is assumed to be fixed, we write $\mathcal{E}\mathfrak{v} := \mathcal{E}\mathfrak{v}_3^t$ to simplify notation.

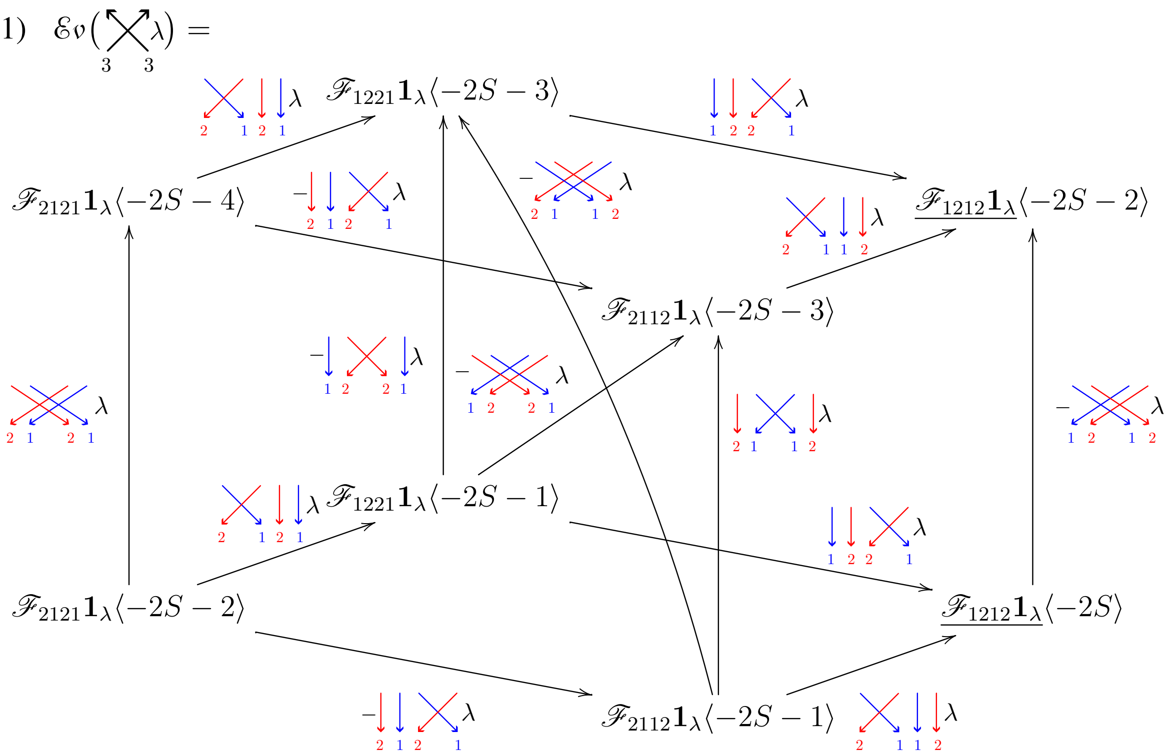
4.1. **The definition of $\mathcal{E}v$ on objects and 1-morphisms.** In the definition below, we underline the 1-morphism in homological degree zero in each complex.

- For objects, we define $\mathcal{E}v(\lambda) = \lambda$, where $\lambda \in \mathbb{Z}^3$.
- For 1-morphisms, we define the action of $\mathcal{E}v$ on generating 1-morphisms and extend it to all 1-morphisms via composition and direct sums, using the standard composition of complexes.
  - $\mathcal{E}v(\mathscr{E}_1\mathbf{1}_\lambda) = \underline{\mathscr{E}_1\mathbf{1}_\lambda}$,
  - $\mathcal{E}v(\mathscr{E}_2\mathbf{1}_\lambda) = \underline{\mathscr{E}_2\mathbf{1}_\lambda}$,
  - $\mathcal{E}v(\mathscr{F}_1\mathbf{1}_\lambda) = \underline{\mathscr{F}_1\mathbf{1}_\lambda}$,
  - $\mathcal{E}v(\mathscr{F}_2\mathbf{1}_\lambda) = \underline{\mathscr{F}_2\mathbf{1}_\lambda}$,
  - $\mathcal{E}v(\mathscr{E}_3\mathbf{1}_\lambda) = \ \mathscr{F}_{21}\mathbf{1}_\lambda\langle -S-1\rangle \longrightarrow \underline{\mathscr{F}_{12}\mathbf{1}_\lambda}\langle -S\rangle$,
  - $\mathcal{E}v(\mathscr{F}_3\mathbf{1}_\lambda) = \ \underline{\mathscr{E}_{21}\mathbf{1}_\lambda}\langle S\rangle \longrightarrow \mathscr{E}_{12}\mathbf{1}_\lambda\langle S+1\rangle$.
- We set $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$ and call it the *Schur level* of λ.

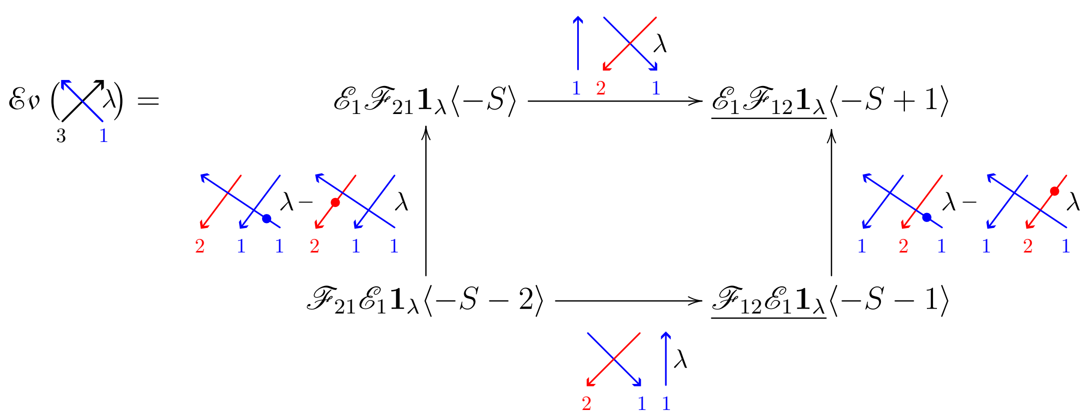
4.2. **The definition of $\mathcal{E}v$ on 2-morphisms.** For 2-morphisms consisting only of strands between $\mathscr{E}_1\mathbf{1}_\lambda$, $\mathscr{E}_2\mathbf{1}_\lambda$, $\mathscr{F}_1\mathbf{1}_\lambda$ and $\mathscr{F}_2\mathbf{1}_\lambda$, the 2-functor $\mathcal{E}v$ acts as the identity. For the remaining generating 2-morphisms, we define (see Remark 3.2), which we emphasise are commutative diagrams:

(30)
$$\mathcal{E}v\left(\uparrow_{3}\ \lambda\right) = \begin{tikzcd} \mathscr{F}_{21}\mathbf{1}_\lambda\langle -S+1\rangle \arrow[r] & \underline{\mathscr{F}_{12}\mathbf{1}_\lambda}\langle -S+2\rangle \\ \mathscr{F}_{21}\mathbf{1}_\lambda\langle -S-1\rangle \arrow[r] \arrow[u] & \underline{\mathscr{F}_{12}\mathbf{1}_\lambda}\langle -S\rangle \arrow[u] \end{tikzcd}$$

(31)


$\mathcal{E}v$
$\mathscr{F}_{1221}\mathbf{1}_\lambda\langle -2S-3\rangle$
$\mathscr{F}_{2121}\mathbf{1}_\lambda\langle -2S-4\rangle$
$\underline{\mathscr{F}_{1212}\mathbf{1}_\lambda}\langle -2S-2\rangle$
$\mathscr{F}_{2112}\mathbf{1}_\lambda\langle -2S-3\rangle$
$\mathscr{F}_{1221}\mathbf{1}_\lambda\langle -2S-1\rangle$
$\mathscr{F}_{2121}\mathbf{1}_\lambda\langle -2S-2\rangle$
$\underline{\mathscr{F}_{1212}\mathbf{1}_\lambda}\langle -2S\rangle$
$\mathscr{F}_{2112}\mathbf{1}_\lambda\langle -2S-1\rangle$

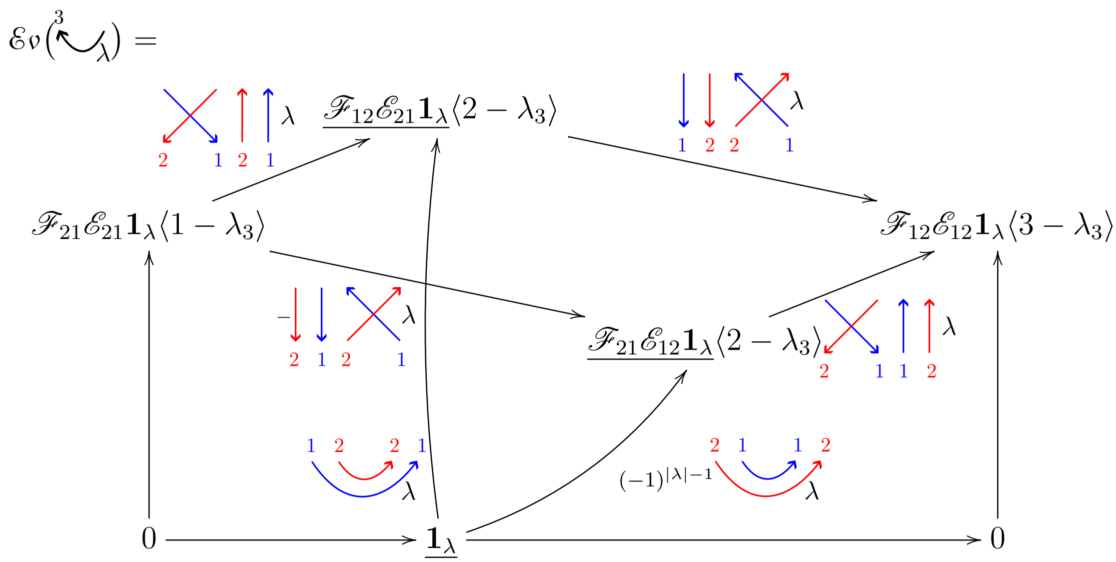
(32)

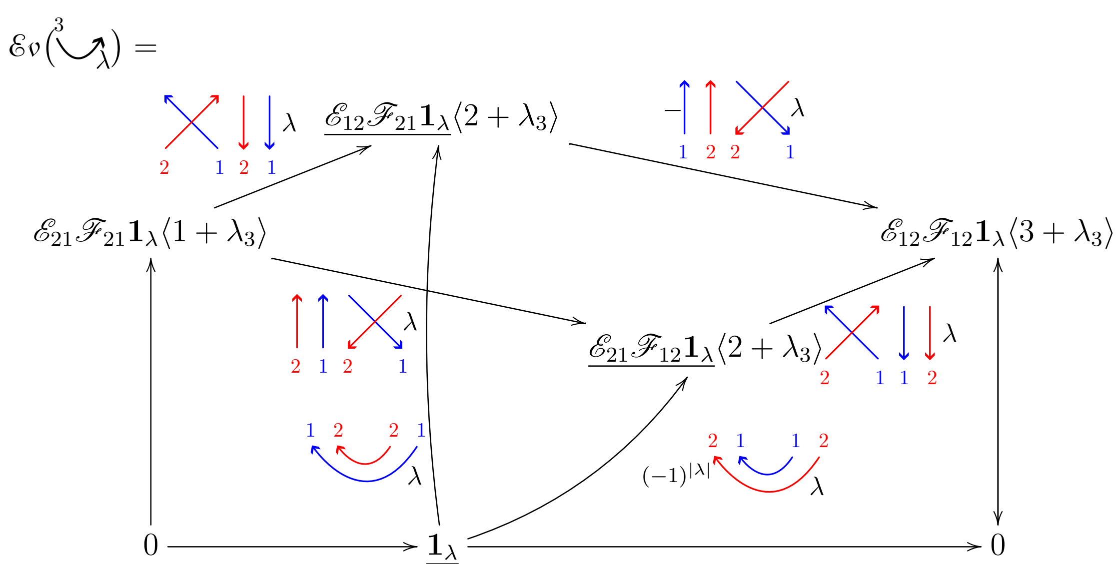

$\mathcal{E}v$
$\mathscr{E}_1\mathscr{F}_{21}\mathbf{1}_\lambda\langle -S\rangle$
$\underline{\mathscr{E}_1\mathscr{F}_{12}\mathbf{1}_\lambda}\langle -S+1\rangle$
$\mathscr{F}_{21}\mathscr{E}_1\mathbf{1}_\lambda\langle -S-2\rangle$
$\underline{\mathscr{F}_{12}\mathscr{E}_1\mathbf{1}_\lambda}\langle -S-1\rangle$

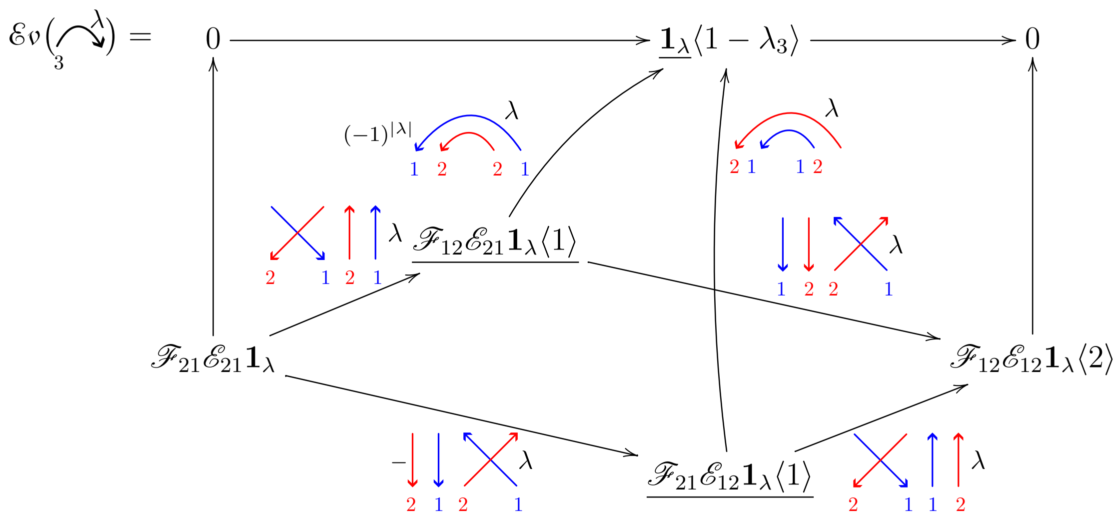
(33)
$$\mathcal{E}v\Big(\;\lambda\Big) = \begin{tikzcd} \mathscr{F}_{21}\mathscr{E}_1\mathbf{1}_\lambda\langle -S-1\rangle \arrow[r] & \underline{\mathscr{F}_{12}\mathscr{E}_1\mathbf{1}_\lambda}\langle -S\rangle \\ \mathscr{E}_1\mathscr{F}_{21}\mathbf{1}_\lambda\langle -S-1\rangle \arrow[u] \arrow[r] & \underline{\mathscr{E}_1\mathscr{F}_{12}\mathbf{1}_\lambda}\langle -S\rangle \arrow[u] \end{tikzcd}$$

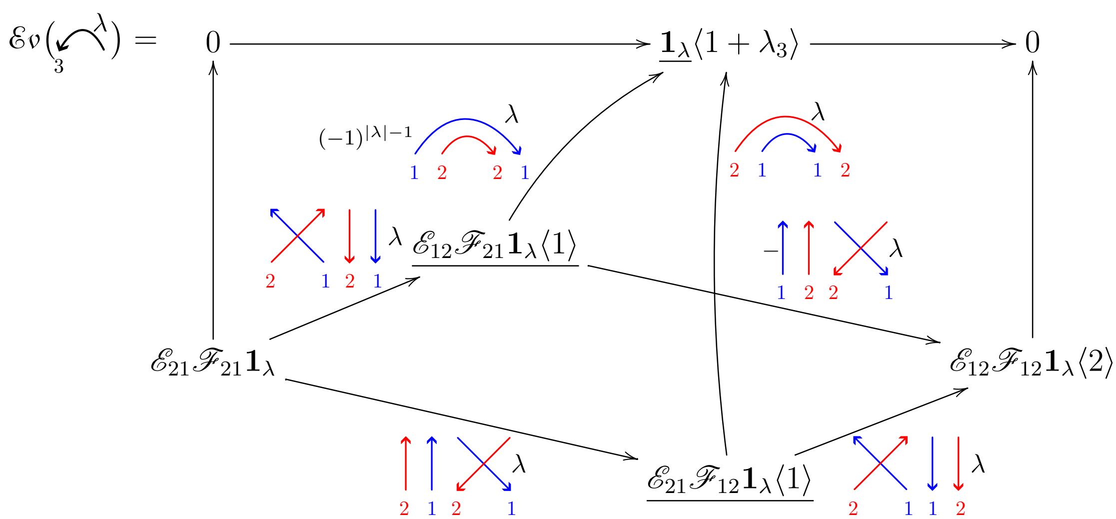
(34)
$$\mathcal{E}v\Big(\;\lambda\Big) = \begin{tikzcd} \mathscr{F}_{21}\mathscr{E}_2\mathbf{1}_\lambda\langle -S+1\rangle \arrow[r] & \underline{\mathscr{F}_{12}\mathscr{E}_2\mathbf{1}_\lambda}\langle -S+2\rangle \\ \mathscr{E}_2\mathscr{F}_{21}\mathbf{1}_\lambda\langle -S-1\rangle \arrow[u] \arrow[r] & \underline{\mathscr{E}_2\mathscr{F}_{12}\mathbf{1}_\lambda}\langle -S\rangle \arrow[u] \end{tikzcd}$$

(35)
$$\mathcal{E}v\Big(\;\lambda\Big) = \begin{tikzcd} \mathscr{E}_2\mathscr{F}_{21}\mathbf{1}_\lambda\langle -S\rangle \arrow[r] & \underline{\mathscr{E}_2\mathscr{F}_{12}\mathbf{1}_\lambda}\langle -S+1\rangle \\ \mathscr{F}_{21}\mathscr{E}_2\mathbf{1}_\lambda\langle -S\rangle \arrow[u] \arrow[r] & \underline{\mathscr{F}_{12}\mathscr{E}_1\mathbf{1}_\lambda}\langle -S+1\rangle \arrow[u] \end{tikzcd}$$

(36) 
$\mathcal{E}v\big(\;\big) =$
$\underline{\mathscr{F}_{12}\mathscr{E}_{21}\mathbf{1}_\lambda}\langle 2-\lambda_3\rangle$
$\mathscr{F}_{21}\mathscr{E}_{21}\mathbf{1}_\lambda\langle 1-\lambda_3\rangle$
$\mathscr{F}_{12}\mathscr{E}_{12}\mathbf{1}_\lambda\langle 3-\lambda_3\rangle$
$\underline{\mathscr{F}_{21}\mathscr{E}_{12}\mathbf{1}_\lambda}\langle 2-\lambda_3\rangle$
$(-1)^{|\lambda|-1}$
0
$\underline{\mathbf{1}_\lambda}$
0

(37) 
$\mathcal{E}v\big(\;\big) =$
$\underline{\mathscr{E}_{12}\mathscr{F}_{21}\mathbf{1}_\lambda}\langle 2+\lambda_3\rangle$
$\mathscr{E}_{21}\mathscr{F}_{21}\mathbf{1}_\lambda\langle 1+\lambda_3\rangle$
$\mathscr{E}_{12}\mathscr{F}_{12}\mathbf{1}_\lambda\langle 3+\lambda_3\rangle$
$\underline{\mathscr{E}_{21}\mathscr{F}_{12}\mathbf{1}_\lambda}\langle 2+\lambda_3\rangle$
$(-1)^{|\lambda|}$
0
$\underline{\mathbf{1}_\lambda}$
0

(38) 

(39) 

The second statement of the following theorem is clearly true, the proof of the first statement will be the content of Subsection 4.3.

Theorem 4.1. *$\mathcal{E}v$ is a well-defined 2-functor and there is a natural isomorphism between its decategorification and* ev_3^t.

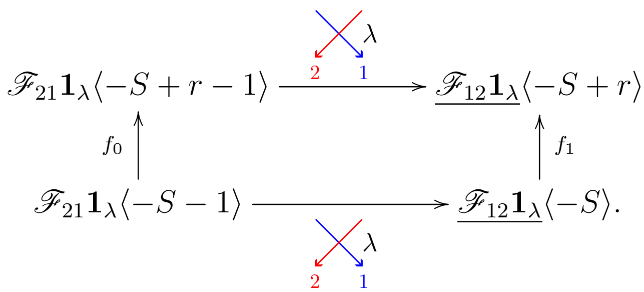
4.3. **Proof of Theorem 4.1.** Let us first show that the above choice for the image of a dotted 3-strand under $\mathcal{E}v$ is the only one possible, up to multiplication by a scalar. We will be using this on occasion in the proof of Theorem 4.1.

Lemma 4.2. *Both* $\mathrm{End}^*_{\mathcal{K}^b(\mathcal{U}(3))}(\mathcal{E}v(\mathscr{E}_3\mathbf{1}_\lambda))$ *and* $\mathrm{End}^*_{\mathcal{K}^b(\mathcal{U}(3))}(\mathcal{E}v(\mathscr{F}_3\mathbf{1}_\lambda))$ *are isomorphic to* $\mathbb{Q}[x]$, *where* $\deg x = 2$.

Proof. We only prove the result for $\mathrm{End}^*_{\mathcal{K}^b(\mathcal{U}(3))}(\mathcal{E}\mathfrak{v}(\mathscr{E}_3\mathbf{1}_\lambda))$, the proof of the other case being similar. For starters, let us work in the 2-category of bounded complexes $\mathcal{C}^b(\mathcal{U}(3))$. We claim that

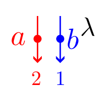
$$\mathrm{End}^*_{\mathcal{C}^b(\mathcal{U}(3))}(\mathcal{E}\mathfrak{v}(\mathscr{E}_3\mathbf{1}_\lambda)) \cong \mathbb{Q}[x_1, x_2],$$

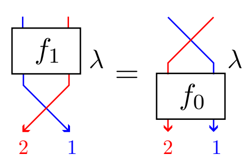
where $\deg x_1 = \deg x_2 = 2$. Note that an element of $\mathrm{End}^*_{\mathcal{C}^b(\mathcal{U}(3))}(\mathcal{E}\mathfrak{v}(\mathscr{E}_3\mathbf{1}_\lambda))$ is a commutative square of the form

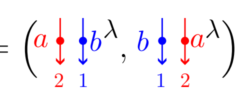


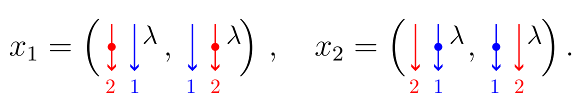
By [KhL10, Theorem 2.7], the shift r has to be even and any 2-morphism

$$\mathscr{F}_{21}\mathbf{1}_\lambda\langle -S-1\rangle \xrightarrow{f} \mathscr{F}_{21}\mathbf{1}_\lambda\langle -S+r-1\rangle$$

is of the form $f =$  , where $a + b = r/2$.
The equality

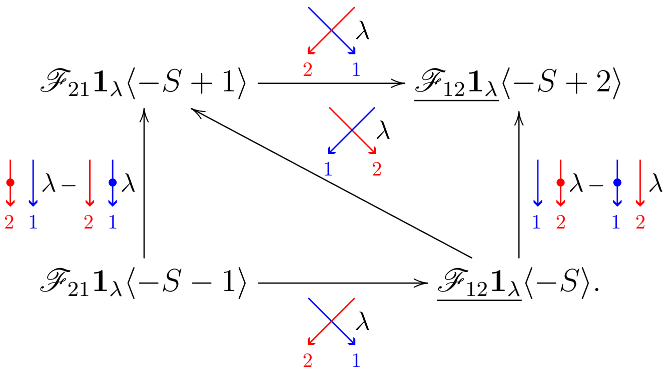


implies that f_1 is determined by the choice of f_0, i.e. $(f_0, f_1) =$  where we use the presentation used in [ALELR24] of only giving the vertical 1-morphisms as an ordered pair, for clarity of reading.
This proves the claim, with



We further claim that x_1 and x_2 are homotopic. Indeed, consider the diagram

(40)



One sees that this diagram is commutative, by the downward version of relation (16), and hence $x_1 - x_2 \simeq_h 0$, which proves the lemma. ☐

The following proofs will be extensively utilising analogies to [ALELR24], which is justified by the observation that part of the definition of $\mathcal{E}v$ can be reformulated in terms of $\mathcal{T}_1'\colon \mathcal{U}(3) \to \mathcal{K}^b(\mathcal{U}(3))$, the categorification of Lusztig's internal action of the braid generator associated to the elementary transposition $s_1 \in S_3$, e.g.

$$\mathcal{E}v(\mathscr{E}_1\mathbf{1}_\lambda) = \mathcal{T}_1'(\mathscr{F}_1\mathbf{1}_{s_1(\lambda)})\langle -\lambda_1-\lambda_2\rangle[-1], \tag{41}$$

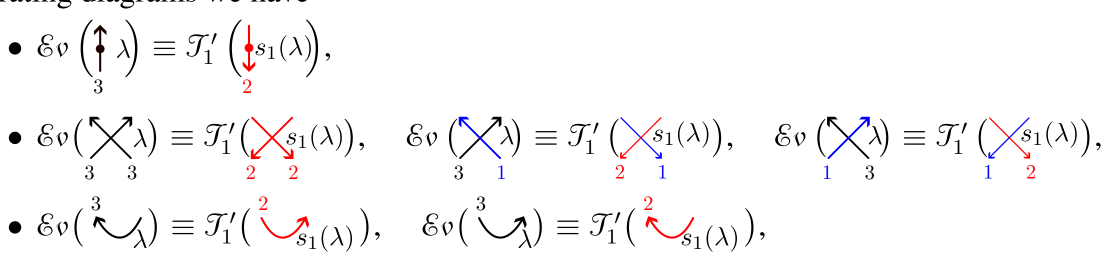
$$\mathcal{E}v(\mathscr{F}_1\mathbf{1}_\lambda) = \mathcal{T}_1'(\mathscr{E}_1\mathbf{1}_{s_1(\lambda)})\langle \lambda_1+\lambda_2+2\rangle[1], \tag{42}$$

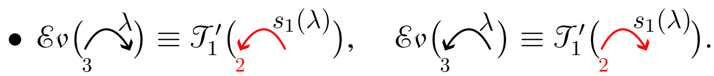
$$\mathcal{E}v(\mathscr{E}_3\mathbf{1}_\lambda) = \mathcal{T}_1'(\mathscr{F}_2\mathbf{1}_{s_1(\lambda)})\langle -S\rangle, \tag{43}$$

$$\mathcal{E}v(\mathscr{F}_3\mathbf{1}_\lambda) = \mathcal{T}_1'(\mathscr{E}_2\mathbf{1}_{s_1(\lambda)})\langle S\rangle, \tag{44}$$

see [ALELR24, Section 4A] (note our calligraphic font for $\mathcal{T}$ differs from [ALELR24]). While our scalars and bubble parameters depending on $\mathfrak{gl}_3$ differ from the ones depending on $\mathfrak{sl}_3$ considered in [ALELR24] (see Remark 3.3), a significant number of our diagrammatic relations are analogous modulo 2 to diagrammatic relations found in [ALELR24]. In each case, we can then check that our choice of scalars and bubble parameters, still gives a homotopy between maps of complexes.

Therefore, a big part of the proof that our $\mathcal{E}v$ is well-defined will follow more or less directly from the proof that $\mathcal{T}_1'$ is well-defined in [ALELR24]. For diagrams involving only strands labelled 1 and 3, this is a very direct process because the image under $\mathcal{E}v$ of such a diagram, modulo 2, is equal to the image under $\mathcal{T}_1'$ of the diagram obtained by inverting the direction of the 3 strand and relabelling it as a 2 strand, and by replacing λ with $s_1(\lambda)$. Explicitly, for the generating diagrams we have





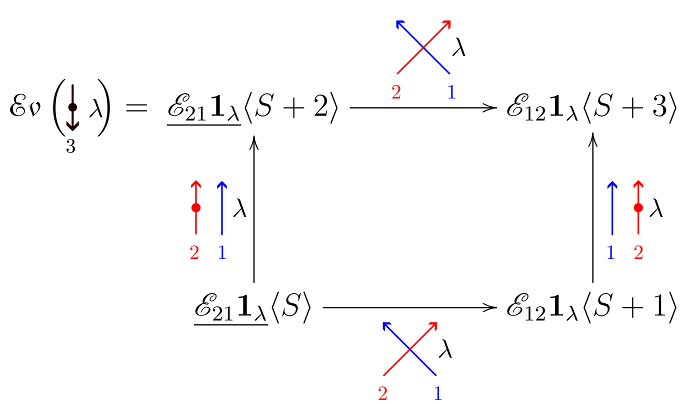
Below we will be showing that, despite this relationship only being modulo 2, all our relations do hold on the nose.

For diagrams involving only strands labelled 2 and 3, we now observe that we are still able to apply [ALELR24]. As an example, note that the definitions of

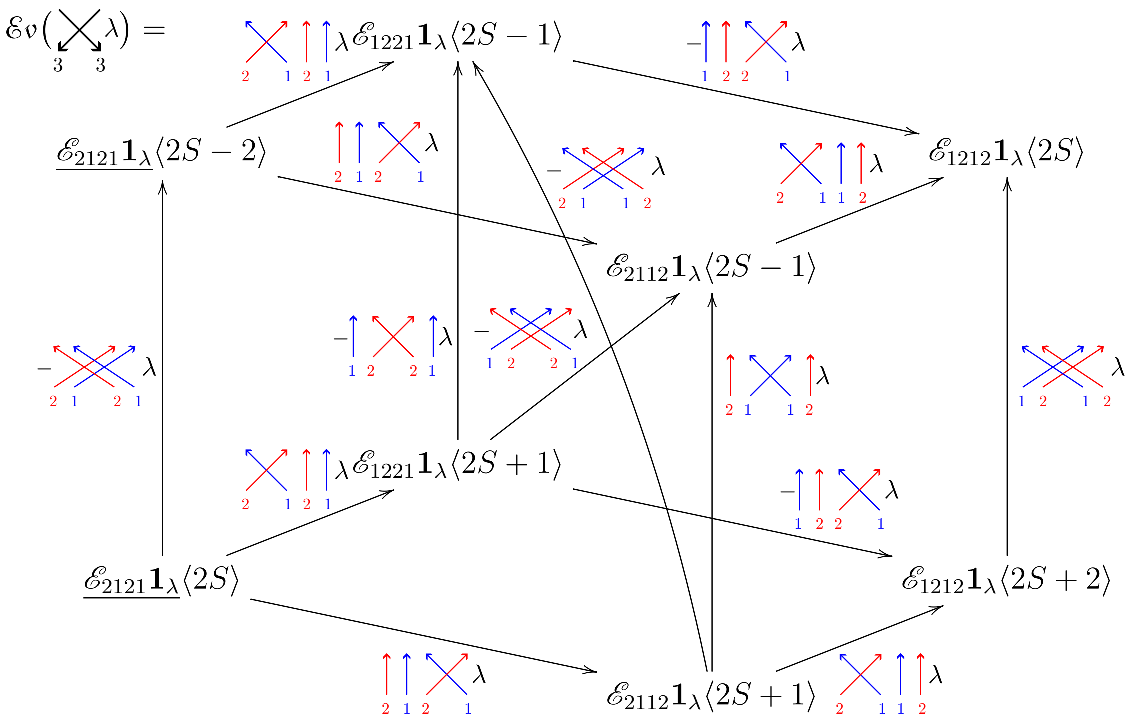
(45)


are mirror images of each other, up to swapping the strands labelled 1 and 2. Consequently, the proof of any evaluated identity using only strands labelled 2 and 3 follows directly from the proof of a suitably chosen identity using only strands labelled 1 and 3, after interchanging labels, mirroring, and a straightforward computation of signs if cups and caps are involved. Relations involving only strands labelled 1 and 2 are preserved under the evaluation 2-functor by definition, and relations involving all three strands will be proved as they appear.

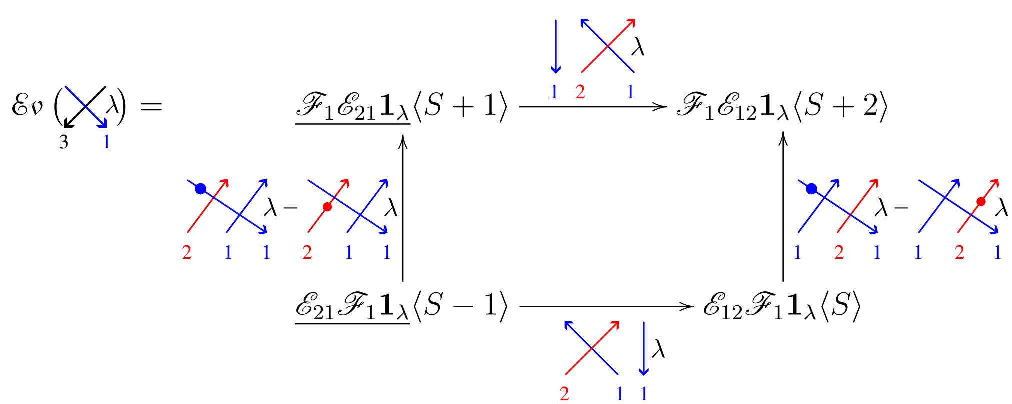
The 2-functor $\mathcal{E}v$ acts as the identity on the remaining 2-morphisms with colors 1 and 2. By [ALELR24, Propositions 5.1, 5.2 and 5.3] and the respective proofs, translated to our setting using equations (41) until (44) (which we do formally in Lemma 4.3), rotation of the above images of the generating 2-morphisms under $\mathcal{E}v$, by using leftward or rightward cups and caps, yields the following images of the remaining generating 2-morphisms of $\mathcal{U}_\triangle(3)$ with strands colored 1 and 3 in $\mathcal{K}^b(\mathcal{U}(3))$ (i.e. it yields these images up to homotopy equivalence).

(46)


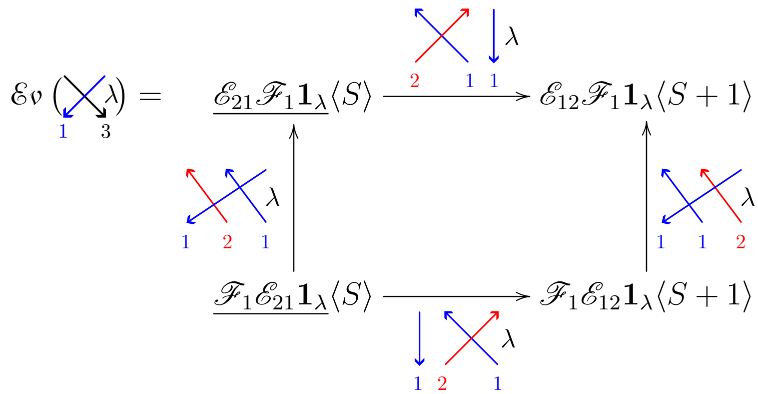
(47)


$\mathscr{E}_{1221}\mathbf{1}_\lambda\langle 2S-1\rangle$
$\mathscr{E}_{2121}\mathbf{1}_\lambda\langle 2S-2\rangle$
$\mathscr{E}_{1212}\mathbf{1}_\lambda\langle 2S\rangle$
$\mathscr{E}_{2112}\mathbf{1}_\lambda\langle 2S-1\rangle$
$\mathscr{E}_{1221}\mathbf{1}_\lambda\langle 2S+1\rangle$
$\mathscr{E}_{2121}\mathbf{1}_\lambda\langle 2S\rangle$
$\mathscr{E}_{1212}\mathbf{1}_\lambda\langle 2S+2\rangle$
$\mathscr{E}_{2112}\mathbf{1}_\lambda\langle 2S+1\rangle$

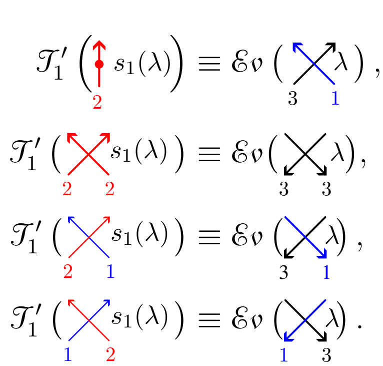
(48)


$\mathscr{F}_1\mathscr{E}_{21}\mathbf{1}_\lambda\langle S+1\rangle$
$\mathscr{F}_1\mathscr{E}_{12}\mathbf{1}_\lambda\langle S+2\rangle$
$\mathscr{E}_{21}\mathscr{F}_1\mathbf{1}_\lambda\langle S-1\rangle$
$\mathscr{E}_{12}\mathscr{F}_1\mathbf{1}_\lambda\langle S\rangle$

(49)

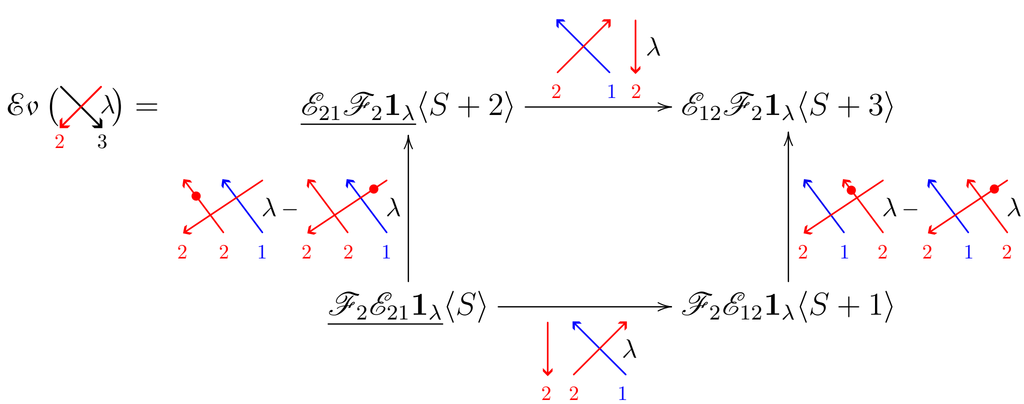


Relating these to [ALELR24], modulo 2 we have that

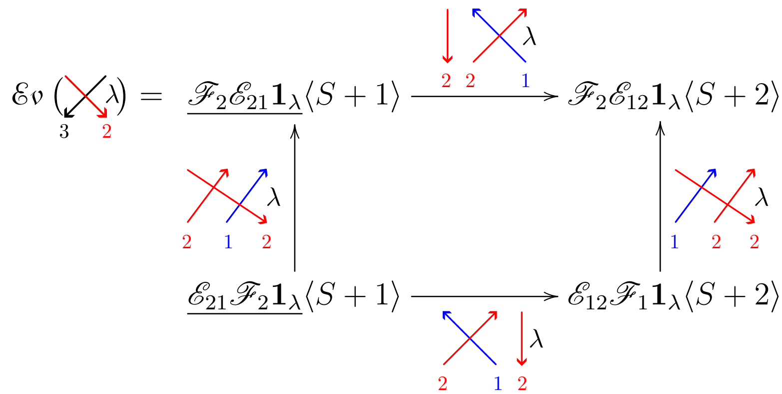


The image of the final two generating 2-morphisms with strands colored 2 and 3 can be obtained by mirroring the above ones and swapping the colors 1 and 2, as already remarked in (45):

(50)



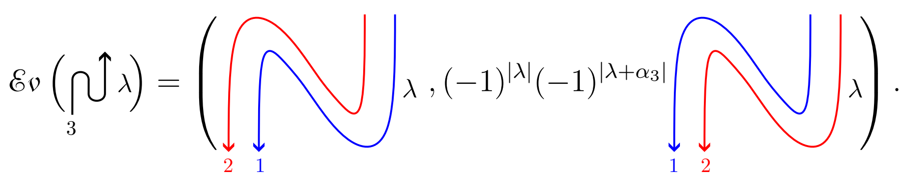
(51)



Lemma 4.3. *The evaluation 2-functor $\mathcal{E}v\colon \mathcal{U}_{\triangle}(3) \to \mathcal{K}^b(\mathcal{U}(3))$ preserves all relations on 2-morphisms in Definition 3.1.*

Proof. We will consider relations (11) through (23), and for each case identify any variations in our proof from proofs provided in [ALELR24], specifically with the choice of scalars given by $t_{i,i+1} = -1$ and $t_{ij} = 1$ otherwise.

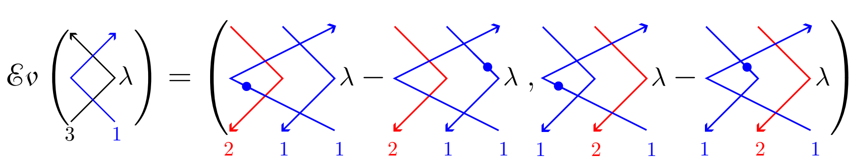
- (11), (12), (14) and (15): modulo 2, the diagrams are identical to [ALELR24, Propositions 5.1, 5.2 and 5.3]. However, we have to calculate signs ourselves, since they differ in a non-obvious fashion from [ALELR24]. For example

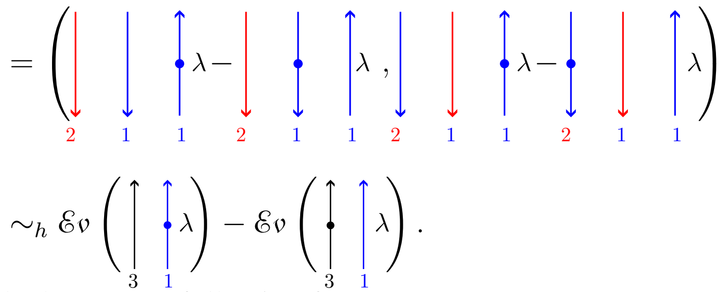


  Since the Schur level of λ is invariant under adding any multiple of any α_i, it follows that $(-1)^{|\lambda|}(-1)^{|\lambda+\alpha_3|} = 1$ as we require. The other adjunction relations, the dot cyclicity relations and the 'quarter rotation' crossing cyclicity relations are similar (using $(-1)^{|\lambda|-1}(-1)^{|\lambda+\alpha_3|-1} = 1$ when necessary).
- (13): We use [ALELR24, Proposition 5.3] modulo 2, but again we have a variation in signs and therefore have to calculate signs ourselves. For the cases where only one strand is labelled 3, the argument is analogous to the prior case. For the 3-3 crossing, we have [ALELR24, Proposition 5.3] modulo 2, and for signs we calculate e.g.

$$(-1)^{|\lambda|}(-1)^{|\lambda+\alpha_3|}(-1)^{|\lambda+2\alpha_3|}(-1)^{|\lambda+\alpha_3|} = 1,$$

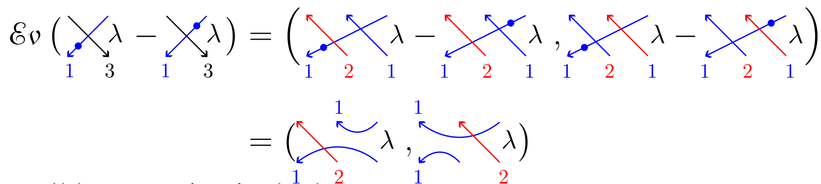
  as required.
- (16): We calculate one relation explicitly; the others are similar.



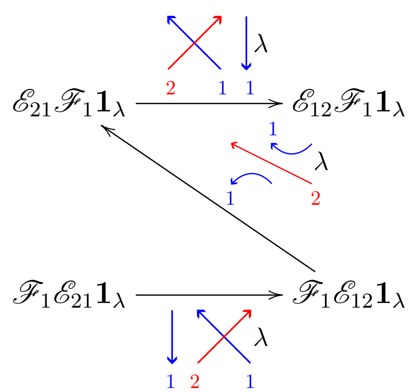


as required, with the homotopy following from Lemma 4.2.

- (17): For the mixed colour (i.e. 3-1 and by extension 3-2) nil-Hecke relations, the calculations are precisely those in [ALELR24, Proposition 5.5] up to a *Leibniz correction* flipping the sign of the homotopy. In more detail, note that the image of e.g. $\mathscr{E}_1$ under $\mathcal{T}_1'$ in [ALELR24] is $\mathscr{F}_1$ in homological degree -1, while the image of $\mathscr{E}_1$ under our $\mathcal{E}v$ is $\mathscr{E}_1$ in degree zero. Therefore, there are sign differences that arise from the Leibniz rule. For our evaluation 2-functor, we thus for example have

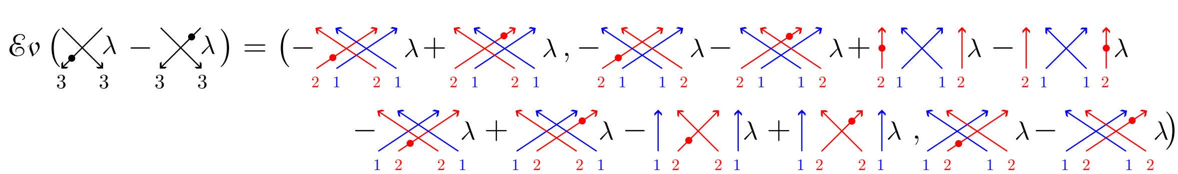


which is null homotopic via the homotopy

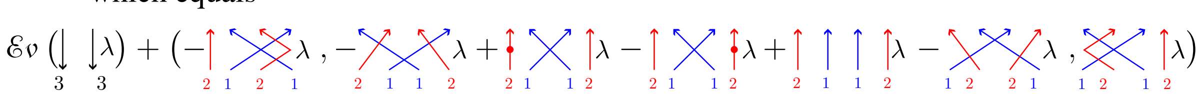


as required. This compares to the second displayed equation in the proof of [ALELR24, Proposition 5.5]. The other two-coloured nil-Hecke relations follow similarly.

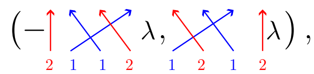
For the 3-3 nil-Hecke relations, we similarly have the same diagrams modulo 2 to [ALELR24, Proposition 5.5], but have sign differences. Specifically,



which equals



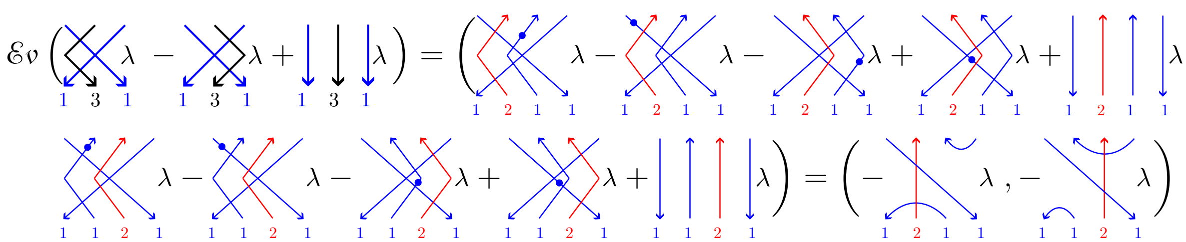
and the latter bracketed summand is null-homotopic via the homotopy



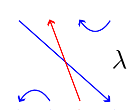
which compares to the last displayed equation in [ALELR24, Proposition 5.5]. The proof of the other 3-3 nil-Hecke relation is similar.

Remark 4.4. It is worth noting that, when considering the proof of [ALELR24, Proposition 5.5] for the 3-3 nil-Hecke relation, taking a choice of scalars of $t_{ij} = 1$ for all i and j therein gives precisely the negative of the above calculations. Since our choice of scalars and bubble parameters is not covered by that paper, it is not obvious that specialising their arguments to the aforementioned choice of scalars implies that the analogous arguments also prove the relation for our choice of parameters, but it is easy to check by hand that this is indeed true from the expressions above. This is a technique we will use further in this proof, where we utilise proofs from [ALELR24] with a specific choice of scalars therein (which is not necessarily the same choice as the one in this remark). Each time, it will again be straightforward to check that their arguments carry over to our setting despite the sign differences.

- (18): For the cubic KLR relations involving only strands labelled 3 and 1, the diagrams modulo 2 follow those in [ALELR24, Proposition 5.6]. For the signs, any relation without a 3-3 crossing which holds on the nose follows [ALELR24, Proposition 5.6] exactly for a choice of scalars given by $t_{i,i+1} = -1$ and $t_{ij} = 1$ otherwise. If the relation therein only holds up to homotopy, then there is a Leibniz correction which flips the sign of the homotopy. For example,

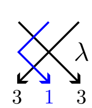


which is null homotopic via the homotopy



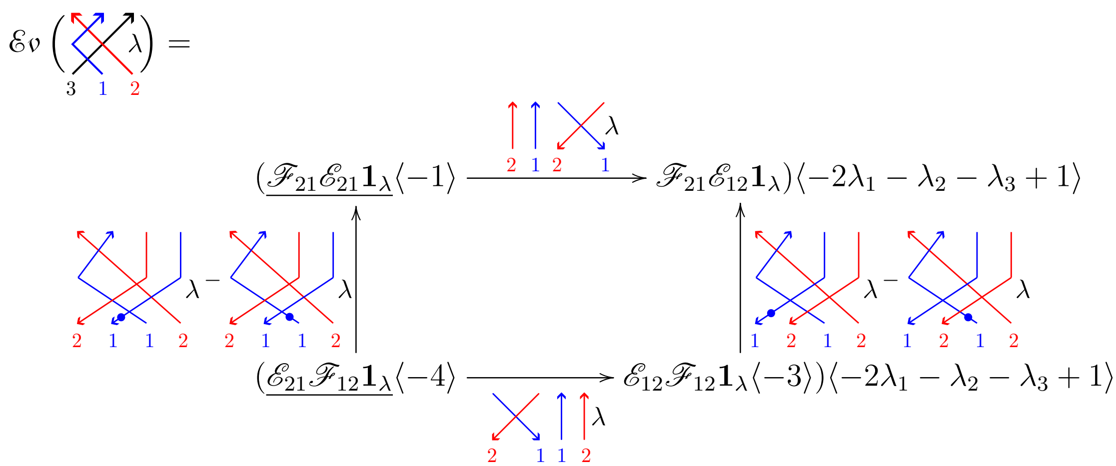
This compares to the last diagram on [ALELR24, page 50].

For relations involving 3-3 crossings, we now set $t_{ij} = 1$ for all i and j in [ALELR24], which similarly to the nil-Hecke case introduces a factor of -1 compared to [ALELR24, Proposition 5.6]. This sign difference is automatically balanced across equalities for all such relations apart from the relation involving

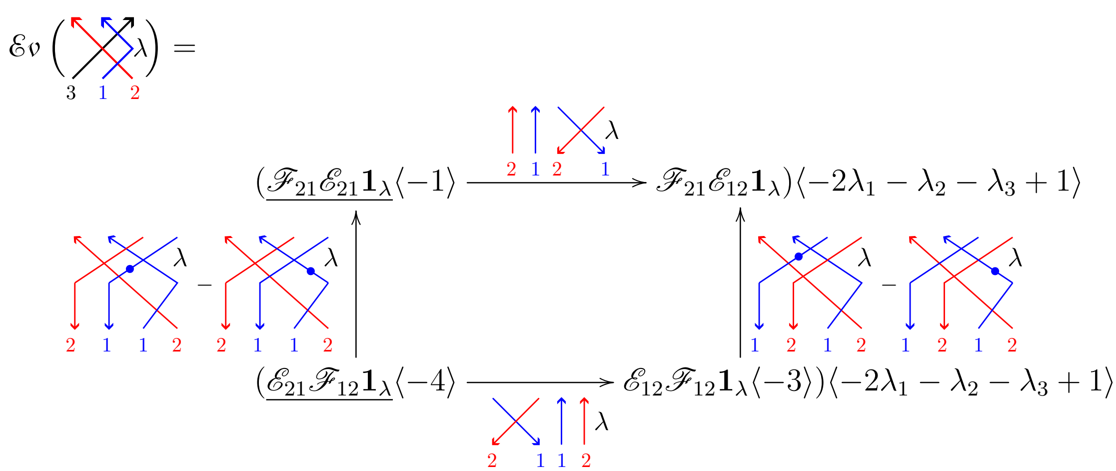


where it cancels with $\varepsilon(3,1) = -1$. Therefore all these relations hold and the relations only involving strands labelled 3 and 2 follow.

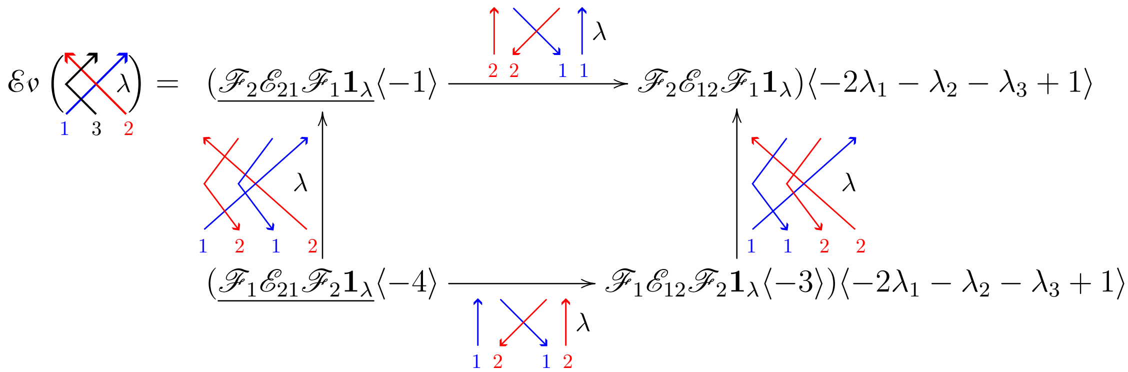
It remains to show that $\mathcal{E}v$ preserves relation (18) for three different colors, which is outside the scope of diagrams considered in [ALELR24]. We will prove three of the six possible cases explicitly, the remaining ones being similar.

(52)


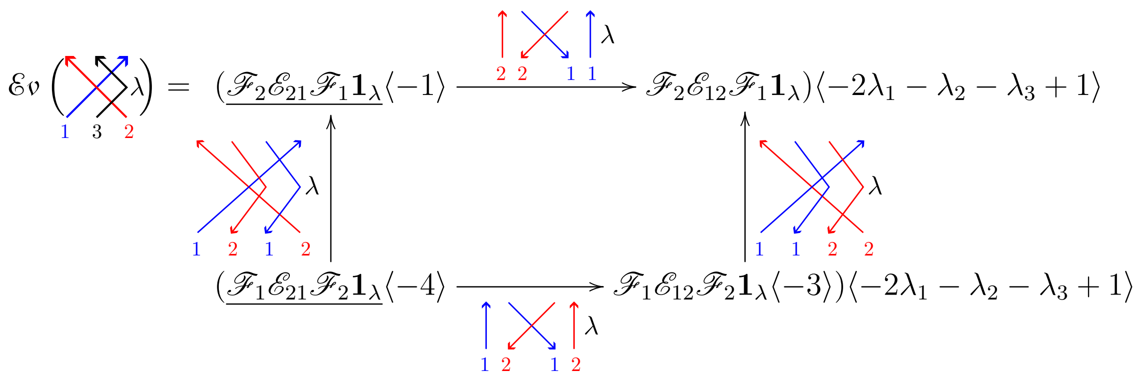
while

(53)


It is straightforward to see that these two complexes are equal by relations (17) and (29) in $\mathcal{U}(3)$.

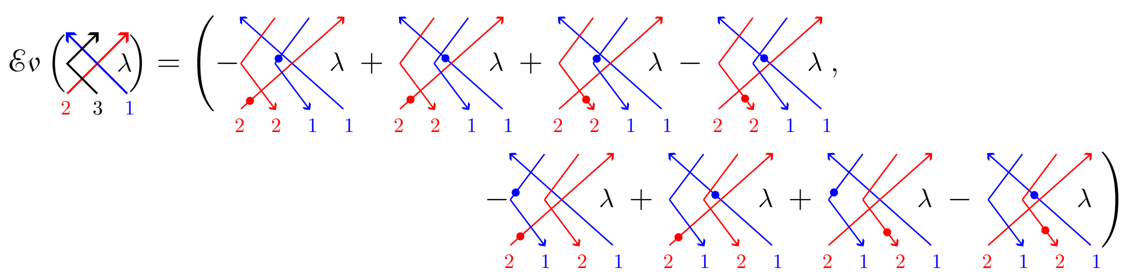
(54) 

while

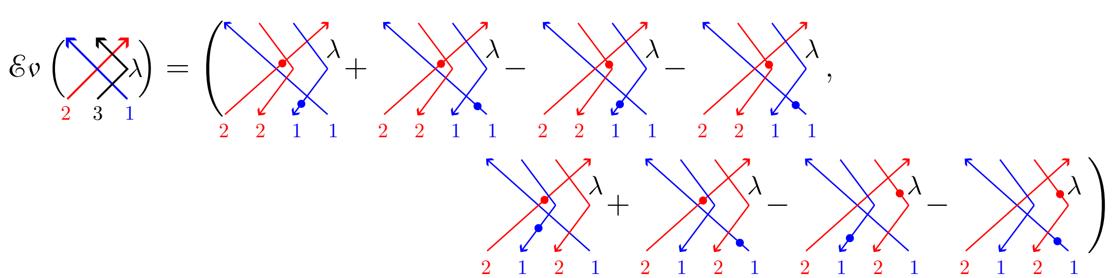
(55) 

and these two are equal by relation (29) in $\mathcal{U}(3)$.

For the third case,

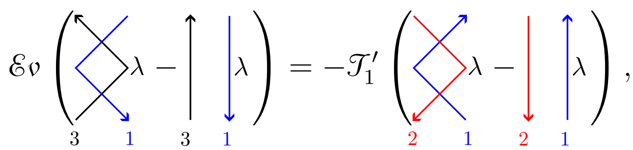
(56) 

while

(57) 

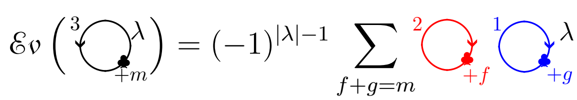
All the dots can be slid, using relation (17) in $\mathcal{U}(3)$, to either the head or tail of their arrow in a manner consistent between corresponding terms of each pair. Applying (29) in $\mathcal{U}(3)$ then shows that these diagrams are again equal.

- (20) and (21): For mixed EF relations of mixed colour, the proof that the relations hold follows directly from [ALELR24, Proposition 5.8] (with the choice $t_{12} = -1$), except that

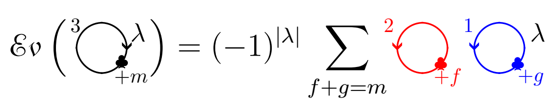


and the homotopy therefore differs by a factor of -1, which does not affect the validity of the results. For the 3-3 mixed EF relation the proof is again analogous to [ALELR24, Proposition 5.10], but the factor of $\beta_{3,\lambda} = -1$ difference between (20) and (21) in this paper and the extended $\mathfrak{sl}_2$ relations in [ALELR24, Section 3.2] similarly means that the null-homotopic relation is given by $(-\varphi_1, -\varphi_2 - \varphi_3 - \varphi_4 - \varphi_5, -\varphi_6)$ (the φ_i being as defined in the proof of [ALELR24, Proposition 5.10]), again giving that the homotopy differs only by a factor of -1.

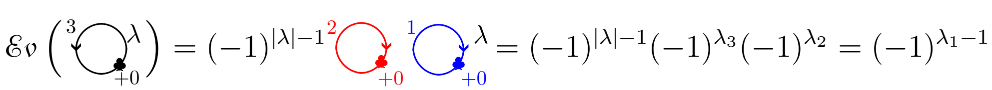
- (22): Modulo 2 these relations follow [ALELR24, Appendix A.4]. For the coefficients, it is therefore straightforward to calculate that



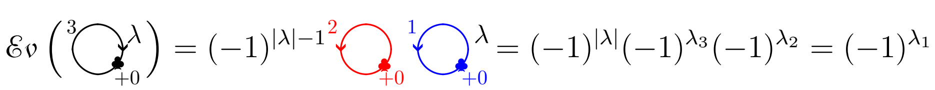
and



In particular, when $m < 0$ the right hand side of these equations is zero, and when $m = 0$,

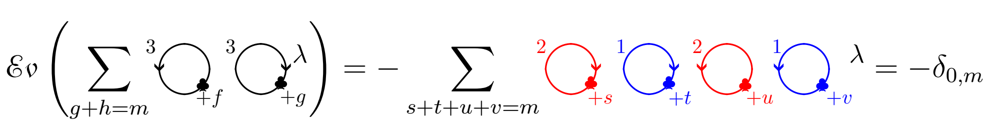


and



as required.

- (23) Modulo 2 this follows [ALELR24, Appendix A.4 Lemma 1], with the signs (at t^m) being given by



as required.

This ends the proof. $\square$

Appendix A. A remark on 2-isomorphism classes

In [KhL10, Definition 3.1], Khovanov and Lauda chose a different set of scalars and bubble parameters for the Kac–Moody 2-categories than those we use in this paper. We were unable to define an evaluation 2-functor for their choice and we conjecture that no such evaluation 2-functor exists. Although we have yet to prove this conjecture, this would be consistent with the first paragraph of [KhL11a, Page 2699], which mentions the existence of a one-parameter family of mutually non-2-isomorphic sub-2-categories of the affine type A Kac–Moody 2-categories categorifying the Borel sub-algebra in affine type A. This contrasts with the finite type A case, where [Lauda20, Theorem 3.5] proves that any two choices of scalars and bubble parameters yield 2-isomorphic Kac–Moody 2-categories. Since Khovanov and Lauda did not include a proof in the above paper, we prove here that the two different choices discussed do lead to non-2-isomorphic Kac–Moody 2-categories in affine type A.

First, a small comment on the choice of weights for the 2-categories in question. The objects of the cyclic 2-categories $\mathcal{U}(n)$ and $\mathcal{U}_{\triangle}(n)$ from Definition 3.1 are $\mathfrak{gl}_n$-weights and level-zero $\widehat{\mathfrak{gl}}_n$-weights, respectively, which coincide (as explained in 2.1). Moreover, the relations satisfied by the generating 2-morphisms of those two 2-categories really depend on those weights, and not on the induced $\mathfrak{sl}_n$-weights and level-zero $\widehat{\mathfrak{sl}}_n$-weights, respectively. Specifically, the degree-zero i-colored bubbles, in a region labeled by $\lambda \in \mathbb{Z}^n$, are equal to $(-1)^{\lambda_{i+1}}$ or $(-1)^{\lambda_{i+1}-1}$ (depending on orientation), so they cannot be expressed in terms of $\overline{\lambda}$ (unless we choose and fix a certain Schur level).

On the other hand, the cyclic 2-categories $\mathcal{U}_Q(\mathfrak{sl}_n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{sl}}_n)$, defined in [BHLW16, Definition 1.3] (generalizing [KhL10, Definition 3.1]), trivially induce cyclic 2-categories $\mathcal{U}_Q(\mathfrak{gl}_n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{gl}}_n')$ whose objects are $\mathfrak{gl}_n$-weights and level-zero $\widehat{\mathfrak{gl}}_n$-weights, respectively: Simply label the regions of the string diagrams by $\lambda \in \mathbb{Z}^n$ and let the relations be those for $\overline{\lambda}$, see Section 2.1 for the notation. We write $\mathcal{U}_Q(\widehat{\mathfrak{gl}}_n')$ to indicate that it is actually an extended version of $\mathcal{U}_Q(\widehat{\mathfrak{sl}}_n)$ rather than the full $\mathcal{U}_Q(\widehat{\mathfrak{gl}}_n)$ (whatever that would be), see remarks below Definition 2.2.

Recall that, following [Lauda20], $\mathcal{U}_Q(\mathfrak{sl}_n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{sl}}_n)$ depend on a choice of scalars $t_{ij} \in \mathbb{Q}^\times$ satisfying $t_{ii} = 1$ and $t_{ij} = t_{ji}$ when $j \neq i \pm 1 \bmod n$, and bubble parameters $\beta_i = \beta_{i,\lambda}$, $c^+_{i,\lambda}, c^-_{i,\lambda} \in \mathbb{Q}^\times$ satisfying

- $c^+_{i,\lambda} c^-_{i,\lambda} = \frac{-1}{\beta_i} = \frac{1}{t_{ii}}$.
- $c^\pm_{i,\lambda+\alpha_j} = t_{ij} c^\pm_{i,\lambda}$.

Here $i, j \in 1, \ldots, n-1$ and $\lambda \in \mathbb{Z}^{n-1}$, for $\mathfrak{sl}_n$, and $i, j \in 1, \ldots, n$ and $\lambda \in \mathbb{Z}^n$, for $\widehat{\mathfrak{sl}}_n$. For Khovanov and Lauda's original choice in [KhL10, Definition 3.1], with all scalars and bubble parameters equal to one, we will follow their notation and denote the corresponding 2-categories by $\mathcal{U}(\mathfrak{sl}_n)$ and $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$, and the trivially induced $\mathfrak{gl}_n$ versions of these by $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathcal{U}(\widehat{\mathfrak{gl}}_n')$, respectively. The 2-categories $\mathcal{U}(n)$ and $\mathcal{U}_{\triangle}(n)$ correspond to the choice $t_{ii} = -1 = t_{i,i+1} = -1$ and $t_{ij} = 1$ for all i and $j \neq i, i+1$ in the respective ranges, and $c^+_{i,\lambda} = (-1)^{\lambda_{i+1}} = -c^-_{i,\lambda}$ for all i in the respective ranges.

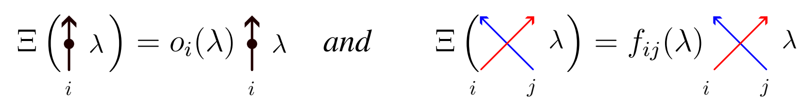
For any $n \in \mathbb{N}_{\geq 2}$, the 2-categories $\mathcal{U}(n)$ and $\mathcal{U}(\mathfrak{gl}_n)$ are 2-isomorphic, with the 2-isomorphism being obtained by composing the 2-isomorphism from [MSV13, (6)] and the 2-isomorphism Σ

from [KhL10, Section 4.2.1] (see also [KhL11b]). When $n \in \mathbb{N}_{\geq 2}$ is even, that composite 2-isomorphism extends to a 2-isomorphism between $\mathcal{U}_{\triangle}(n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{gl}}_n')$. When n is odd, it does not extend to the affine 2-categories, because Khovanov and Lauda's 2-isomorphism Σ is no longer well-defined in that case. The reason is that in the definition of Σ occur factors like $(-1)^i$, for $i = 1, \ldots, n-1$, which are not well-defined for $i \in \mathbb{Z}/n\mathbb{Z}$ when n is odd.

We are now going to show that there is no 2-isomorphism between $\mathcal{U}_{\triangle}(n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{gl}}_n')$ for odd n, for any choice of scalars and bubble parameters satisfying the above conditions.

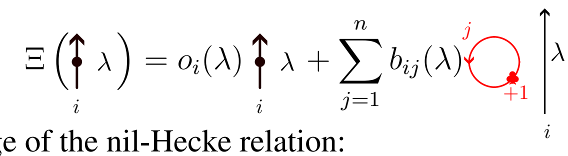
Lemma A.1. *Let Q be a choice of scalars and bubble parameters for $\widehat{\mathfrak{sl}}_n$ and let $\Xi : \mathcal{U}_Q(\widehat{\mathfrak{gl}}_n') \to \mathcal{U}_{\triangle}(n)$ be a 2-isomorphism which is the identity on objects and 1-morphisms. Then*

(58)

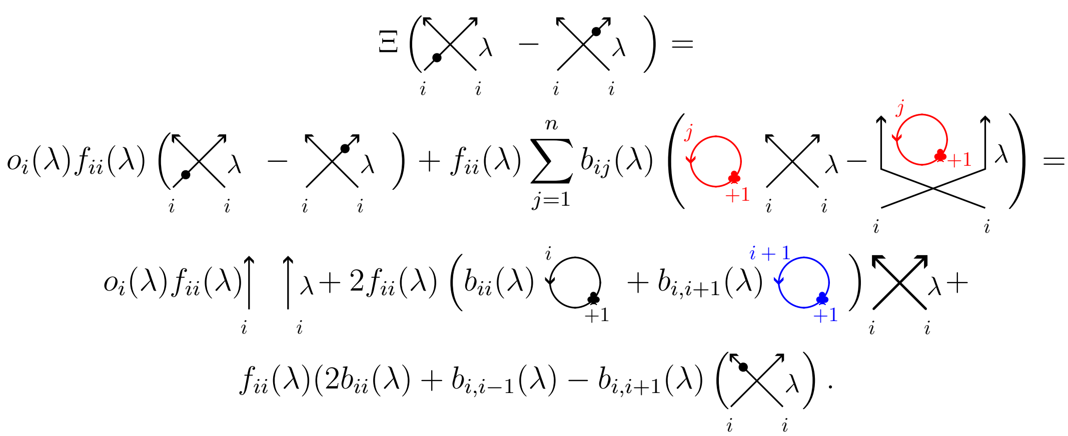


for some $o_i(\lambda), f_{ij}(\lambda) \in \mathbb{Q}^{\times}$ and for all $i, j \in \{1, \ldots, n\}$ and all $\lambda \in \mathbb{Z}^n$. Moreover, these scalars satisfy $o_i(\lambda) f_{ii}(\lambda) = 1$ for all $i \in \widehat{I}$ and all $\lambda \in \mathbb{Z}^n$.

Proof. For degree reasons, the second equality in (58) is immediate, but the first one requires an argument. A priori, we have

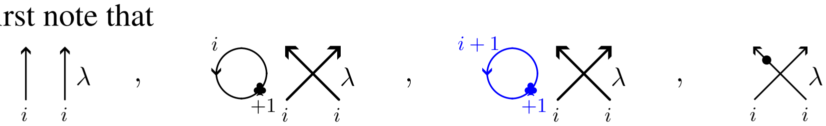


Now consider the image of the nil-Hecke relation:



The fact that Ξ has to preserve the nil-Hecke relation implies that $o_i(\lambda) f_{ii}(\lambda) = 1$ and $b_{ii}(\lambda) = b_{i,i-1}(\lambda) = b_{i,i+1}(\lambda) = 0$ for all $i \in \widehat{I}$ and all $\lambda \in \mathbb{Z}^n$.

To see this, first note that



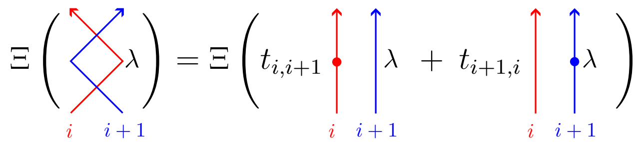
are linearly independent in $\mathrm{Hom}_{\mathcal{U}_{\triangle}(n)}(\mathscr{E}_{ii}\mathbf{1}_{\lambda}, \mathscr{E}_{ii}\mathbf{1}_{\lambda})$. Just as in the proof of [KhL10, Lemma 6.16], this follows from looking at their images under the 2-representation $\mathscr{F}_{\mathrm{Bim}}$ from [MSV13, Section 4.2] (in particular, see (45) in that paper), and its extension for the affine case in [MT17, Definition 5.6].

The condition $o_i(\lambda)f_{ii}(\lambda) = 1$ is therefore immediate. Further, for each $i \in \widehat{I}$, linear independence of the two degree-two bubbles above, colored $i-1$ and i implies that $b_{i,i+1} = b_{ii} = 0$. Using this and

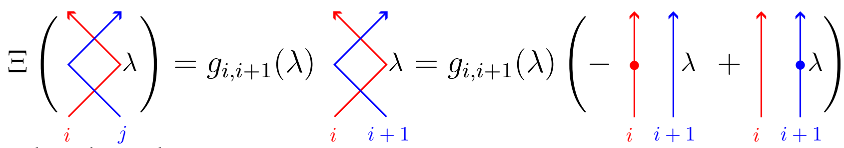
$$2b_{ii}(\lambda) + b_{i,i-1}(\lambda) - b_{i,i+1}(\lambda) = 0,$$

we see that $b_{i,i-1} = 0$ as well.

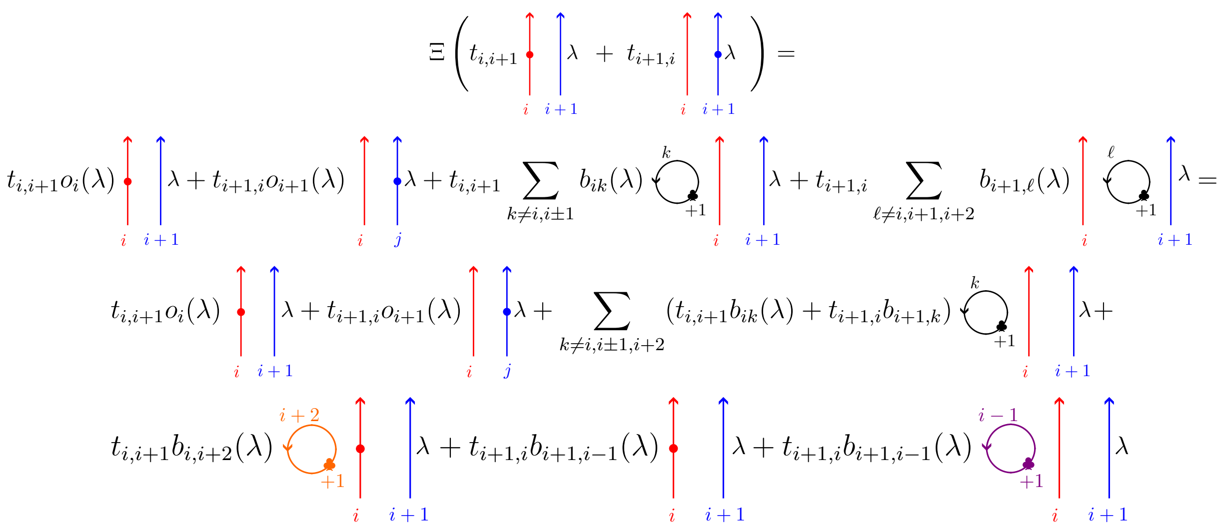
Next we are going to show that $b_{ij}(\lambda) = 0$ for all $i, j \in \widehat{I}$, using the fact that Ξ has to satisfy



for $j = i+1$. To shorten notation, put $g_{ij}(\lambda) := f_{ij}(\lambda)f_{ji}(\lambda)$ for all $i, j \in \widehat{I}$. On the one hand, we have



and on the other hand, we have



By linear independence of the different terms of each expression and comparing corresponding terms in both expressions, as above, we get

$$b_{i,i+2}(\lambda) = 0, \quad b_{i+1,i-1}(\lambda) = 0, \quad t_{i,i+1}b_{ik}(\lambda) + t_{i+1,i}b_{i+1,k}(\lambda) = 0$$

for all $i, k \in \widehat{I}$ such that $k \neq i, i \pm 1, i+2$. Together with the previous results, these equations imply that $b_{ij}(\lambda) = 0$ for all $i, j \in \widehat{I}$ and all $\lambda \in \mathbb{Z}^n$. $\square$

Note that the above proof also shows that the following equations must hold:

$$g_{i,i+1}(\lambda) = -t_{i,i+1}o_i(\lambda) = t_{i+1,i}o_{i+1}(\lambda) \tag{59}$$

for all $i \in \widehat{I}$.

Theorem A.2. *When n is odd, there does not exist a 2-isomorphism $\Xi : \mathcal{U}_Q(\widehat{\mathfrak{gl}}_n') \to \mathcal{U}_\triangle(n)$ which is the identity on objects and 1-morphisms, for any choice of scalars Q with compatible bubble parameters.*

Proof. Assume for contradiction that such a 2-isomorphism Ξ exists for some choice of scalars with compatible bubble parameters, with Ξ having parameters $\{o_i, f_{ij} | i, j = 1, \dots, n\}$ as above. Recall that $o_i(\lambda), g_{ij}(\lambda), t_{ij} \in \mathbb{Q}^\times$ and that $g_{ij} = g_{ji}$ for all $i, j \in \widehat{I}$. Thus, suppressing λ for readability reasons, we get

$$o_1 = -g_{12}t_{12}^{-1} = -o_2 t_{21}t_{12}^{-1} = (-1)^2 g_{23}t_{23}^{-1}t_{21}t_{12}^{-1} = \cdots = (-1)^n g_{n,1}t_{n,1}^{-1} \dots t_{21}t_{12}^{-1}$$

$$= (-1)^n o_1 \prod_{i \in \widehat{I}} t_{i+1,i}t_{i,i+1}^{-1}.$$

This implies that $\prod_{i\in\widehat{I}} t_{i+1,i}t_{i,i+1}^{-1} = (-1)^n$ has to hold. But by the definition of Q and the fact that $\sum_{k=1}^n \alpha_k = 0$ in the (level zero) $\widehat{\mathfrak{sl}}_n$-root lattice, we have the following:

- $t_{ii} = 1$ for all $i = 1, \dots, n$, so in particular $\prod_{i=1}^n t_{ii} = 1$,
- $t_{ij} = t_{ji}$ whenever $|i - j| > 1 \bmod n$, so in particular $\displaystyle\prod_{\substack{i,j=1,\dots,n \\ |i-j|>1}} t_{ij} = x^2$ for some $x \in \mathbb{Q}^\times$,
- $1 = \frac{c_{i,\overline{\lambda}}}{c_{i,\overline{\lambda}}} = \frac{c_{i,\overline{\lambda}+\sum_{k=1}^n \alpha_k}}{c_{i,\overline{\lambda}}} = \prod_{j=1}^n \frac{c_{i,\overline{\lambda}+\sum_{k=1}^j \alpha_k}}{c_{i,\overline{\lambda}+\sum_{k=1}^{j-1} \alpha_k}} = \prod_{j=1}^n t_{ij}$ for any $\lambda \in \mathbb{Z}^n$ and any $i = 1, \dots, n$.

Therefore, for any $\lambda \in \mathbb{Z}^n$

$$\prod_{i=1}^n \frac{c_{i,\overline{\lambda}}}{c_{i,\overline{\lambda}}} = \prod_{i,j=1,\dots,n} t_{ij} = 1.$$

But

$$\prod_{i,j=1,\dots,n} t_{ij} = (\prod_{i=1}^n t_{ii})(\prod_{\substack{i,j=1,\dots,n \\ |i-j|>1}} t_{ij})(\prod_{\substack{i=1,\dots,n \\ |i-j|=1}} t_{ij}) = x^2 \prod_{\substack{i=1,\dots,n \\ |i-j|=1}} t_{ij},$$

for some $x \in \mathbb{Q}^\times$, by the above remarks. Multiplying this by $\prod_{i\in\widehat{I}} t_{i+1,i}t_{i,i+1}^{-1}$ yields

$$\prod_{i=1}^n t_{i+1,i}^2 = (-1)^n x^{-2},$$

which implies n has to be even, completing our proof. $\square$

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