DISTRIBUTED λ-CALCULUS

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Abstract. This document presents a variant of the Ordinary λ-Calculus that we call the Distributed λ-Calculus. The novelty is in addition of node annotations to terms in addition to a mobility rule. We show that, despite distributed programs, the Distributed λ-Calculus is computationally equivalent to the Ordinary λ-Calculus. That is of special importance becasuse, despite the common belief, distributed programming can happen without side-effects. A significant shortcoming of the Distributed λ-Calculus is that it does not model interaction with the real world. In other words, it only models programming a closed distributed system.

Definition 1. *Let $x, y, z, \dots, x', y', z', \dots, x_1, y_1, z_1, \dots$ range over a countably infinite set of variable V. Ordinary λ-terms (ranged over by $t_1, t_2, t_3, \dots, t', t'', \dots$) are defined as:*

$$\lambda_o \ni t ::= x \mid (\lambda x.t) \mid (t\ t).$$

Definition 2. *Define the free variables $\mathsf{fv} : \lambda_o \to 2^V$ and bound variable $\mathsf{bv} : \lambda_o \to 2^V$ as:*

$$\begin{array}{lll} \mathsf{fv}(x) &= \{x\} \\ \mathsf{fv}(\lambda x.t) &= \mathsf{fv}(t) \setminus \{x\} \\ \mathsf{fv}(t_1\ t_2) &= \mathsf{fv}(t_1) \cup \mathsf{fv}(t_2) \end{array} \qquad \begin{array}{lll} \mathsf{bv}(x) &= \varnothing \\ \mathsf{bv}(\lambda x.t) &= \mathsf{bv}(t) \cup \{x\} \\ \mathsf{bv}(t_1\ t_2) &= \mathsf{bv}(t_1) \cup \mathsf{bv}(t_2) \end{array}$$

Write x is fresh in t when $x \notin \mathsf{fv}(t) \cup \mathsf{bv}(t)$. When ambiguious, we write fv_o and bv_o for the above fv and bv. See Definition 8.

Definition 3. *Let $\mathsf{bv}(t') \cap \mathsf{fv}(t) = \varnothing$ and $x \notin \mathsf{bv}(t')$. Define substitution of an ordinary λ-term t for a variable x in t' – denoted by $t'[t/x]$ – as follows:*

$$\begin{array}{lll} x[t/x] &= t & \\ y[t/x] &= y & \text{when } y \neq x \\ (\lambda y.t'')[t/x] &= \lambda y.(t''[t/x]) & \\ (t_1\ t_2)[t/x] &= t_1[t/x]\ t_2[t/x]. & \end{array}$$

Definition 4. *Reductions $. \xrightarrow{o} .$ on the ordinary λ-terms are:*

$$\begin{array}{llll} \lambda x.t & \xrightarrow{o}_\alpha & \lambda y.t[y/x] & (\alpha\text{-conversion}) \quad \text{when } y \text{ is fresh in } t \\ (\lambda x.t_1)\ t_2 & \xrightarrow{o}_\beta & t_1[t_2/x] & (\beta\text{-reduction}) \\ (\lambda x.(t\ x)) & \xrightarrow{o}_\eta & t & (\eta\text{-reduction}) \quad \text{when } x \text{ is fresh in } t. \end{array}$$

Definition 5. *Let $a, b, c, \dots, a', b', c', \dots$ range over a set of nodes ($\mathfrak{N}$). The distributed λ-terms are defined as:*

$$\lambda_d \ni t^a = x^a \mid (\lambda x.t^a)^b \mid (t^a\ t^b)^c.$$

We drop the node annotation when not significant.

Definition 6. *The set of nodes in a distributed λ-term ($\mathsf{nodes} : \lambda_d \to \mathfrak{N}$) is defined as:*

$$\begin{array}{ll} \mathsf{nodes}(x^a) &= \{a\} \\ \mathsf{nodes}((\lambda x.t^b)^a) &= \mathsf{nodes}(t^b) \cup \{a\} \\ \mathsf{nodes}((t_1^a\ t_2^b)^c) &= \mathsf{nodes}(t_1^a) \cup \mathsf{nodes}(t_2^b) \cup \{c\}. \end{array}$$

Definition 7. *$[\![.]\!]^- : \lambda_d \to \lambda$ strips the node information from a distributed λ-term:*

$$\begin{array}{ll} [\![x^a]\!]^- &= x \\ [\![(\lambda x.t^a)^b]\!]^- &= \lambda x.[\![t^a]\!]^- \\ [\![(t^a\ t^b)^c]\!]^- &= [\![t^a]\!]^-\ [\![t^b]\!]^-. \end{array}$$

In the reverse direction, $[\![.]\!]^{+\cdot} : \lambda \times \mathfrak{N} \to \lambda_d$ *annotates an ordinary* λ*-term with a fixed node:*

$$\begin{aligned}
[\![x]\!]^{+a} &= x^a \\
[\![\lambda x.t]\!]^{+a} &= (\lambda x.[\![t]\!]^{+a})^a \\
[\![t_1\ t_2]\!]^{+a} &= ([\![t_1]\!]^{+a}\ [\![t_2]\!]^{+a})^a.
\end{aligned}$$

Definition 8. *Define* $\mathsf{fv}_d : \lambda_d \to 2^V$ *and* $\mathsf{bv}_d : \lambda_d \to 2^V$ *as* $\mathsf{fv}_d = \mathsf{fv}_o \circ [\![.]\!]^-$ *and* $\mathsf{bv}_d = \mathsf{bv}_o \circ [\![.]\!]^-$. *Unless ambiguious, write* fv *and* bv *for* fv_d *and* bv_d *– overloading the notation in Definition 2. Similarly, write* x *is fresh in* t^a *when* $x \notin \mathsf{fv}(t^a) \cup \mathsf{bv}(t^a)$.

Definition 9. *Let* $\mathsf{bv}(t') \cap \mathsf{fv}(t^a) = \varnothing$ *and* $x \notin \mathsf{bv}(t')$. *Define substitution of a distributed* λ*-term* t^a *for a variable* x *in* t' *– denoted by* $t'[t^a/x]$ *– as follows:*

$$\begin{aligned}
x^b[t^a/x] &= t^a \\
y^b[t^a/x] &= y^b && \text{when } y \neq x \\
(\lambda y.t''^b)^c[t^a/x] &= (\lambda y.t''^b[t^a/x])^c \\
(t_1^b\ t_2^c)^d[t^a/x] &= (t_1^b[t^a/x]\ t_2^c[t^a/x])^d.
\end{aligned}$$

Lemma 10. *Let* $\mathsf{bv}(t') \cap \mathsf{fv}(t) = \varnothing$ *and* $x \notin \mathsf{bv}(t')$. *Then,* $[\![t[t'/x]]\!]^{+a} = [\![t]\!]^{+a}[[\![t']\!]^{+a}/x]$.

Proof. By structural induction on t.

$t = x$:

$$\begin{aligned}
&[\![t[t'/x]]\!]^{+a} = \\
&[\![x[t'/x]]\!]^{+a} = && \text{(Definition 3)} \\
&[\![t']\!]^{+a} = && \text{(Definition 9)} \\
&[\![x]\!]^{+a}[[\![t']\!]^{+a}/x] = \\
&[\![t]\!]^{+a}[[\![t']\!]^{+a}/x]
\end{aligned}$$

$t = y$ where $x \neq y$:

$$\begin{aligned}
&[\![t[t'/x]]\!]^{+a} = \\
&[\![y[t'/x]]\!]^{+a} = && \text{(Definition 3)} \\
&[\![y]\!]^{+a} = && \text{(Definition 7)} \\
&y^a = && \text{(Definition 9)} \\
&y^a[[\![t']\!]^{+a}/x] = \\
&[\![t]\!]^{+a}[[\![t']\!]^{+a}/x]
\end{aligned}$$

$t = \lambda y.t''$:

$$\begin{aligned}
&[\![t[t'/x]]\!]^{+a} = \\
&[\![(\lambda y.t'')[t'/x]]\!]^{+a} = && \text{(Definition 3)} \\
&[\![\lambda y.(t''[t'/x])]\!]^{+a} = && \text{(Definition 7)} \\
&(\lambda y.[\![t''[t'/x]]\!]^{+a})^a = && \text{(Induction Hypothesis)} \\
&(\lambda y.[\![t'']\!]^{+a}[[\![t']\!]^{+a}/x])^a = && \text{(Definition 9)} \\
&(\lambda y.[\![t'']\!]^{+a})^a[[\![t']\!]^{+a}/x] = && \text{(Definition 7)} \\
&[\![\lambda y.t'']\!]^{+a}[[\![t']\!]^{+a}/x] = \\
&[\![t]\!]^{+a}[[\![t']\!]^{+x}/x]
\end{aligned}$$

$t = t_1\ t_2$:

$$\begin{aligned}
&[\![t[t'/x]]\!]^{+a} = \\
&[\![(t_1\ t_2)[t'/x]]\!]^{+a} = && \text{(Definition 3)} \\
&[\![t_1[t'/x]\ t_2[t'/x]]\!]^{+a} = && \text{(Definition 7)} \\
&([\![t_1[t'/x]]\!]^{+a}\ [\![t_2[t'/x]]\!]^{+a})^a = && \text{(Induction Hypothesis)} \\
&([\![t_1]\!]^{+a}[[\![t']\!]^{+a}/x]\ [\![t_2]\!]^{+a}[[\![t']\!]^{+a}/x])^a = && \text{(Definition 9)} \\
&[\![t_1\ t_2]\!]^{+a}[[\![t']\!]^{+a}/x] = \\
&[\![t]\!]^{+a}[[\![t']\!]^{+a}/x]
\end{aligned}$$

∎

Lemma 11. *Let* $\mathsf{bv}(t'^{a'}) \cap \mathsf{fv}(t^a) = \varnothing$ *and* $x \notin \mathsf{bv}(t'^{a'})$. *Then,* $[\![t^a[t'^{a'}/x]]\!]^- = [\![t^a]\!]^-[[\![t'^{a'}]\!]^-/x]$.

Proof. By structural induction on t^a.

$t^a = x^a$:

$$
\begin{array}{lll}
[\![t^a[t'^{a'}/x]]\!]^- & = & \\
[\![x^a[t'^{a'}/x]]\!]^- & = & \text{(Definition 9)} \\
[\![t'^{a'}]\!]^- & = & \text{(Definition 3)} \\
x[[\![t'^{a'}]\!]^-/x] & = & \text{(Definition 7)} \\
[\![x^a]\!]^-[[\![t'^{a'}]\!]^-/x] & = & \\
[\![t^a]\!]^-[[\![t'^{a'}]\!]^-/x] & &
\end{array}
$$

$t^a = y^a$ where $x \neq y$:

$$
\begin{array}{lll}
[\![t^a[t'^{a'}/x]]\!]^- & = & \\
[\![y^a[t'^{a'}/x]]\!]^- & = & \text{(Definition 9)} \\
[\![y^a]\!]^- & = & \text{(Definition 7)} \\
y & = & \text{(Definition 3)} \\
y[[\![t'^{a'}]\!]^-/x] & = & \text{(Definition 7)} \\
[\![y^a]\!]^-[[\![t'^{a'}]\!]^-/x] & = & \\
[\![t^a]\!]^-[[\![t'^{a'}]\!]^-/x] & &
\end{array}
$$

$t^a = (\lambda y.t''^{a''})^a$:

$$
\begin{array}{lll}
[\![t^a[t'^{a'}/x]]\!]^- & = & \\
[\![(\lambda y.t''^{a''})^a[t'^{a'}/x]]\!]^- & = & \text{(Definition 9)} \\
[\![(\lambda y.t''^{a''}[t'^{a'}/x])^a]\!]^- & = & \text{(Definition 7)} \\
\lambda y.[\![t''^{a''}[t'^{a'}/x]]\!]^- & = & \text{(Induction Hypothesis)} \\
\lambda y.[\![t''^{a''}]\!]^-[[\![t'^{a'}]\!]^-/x] & = & \text{(Definition 7)} \\
[\![(\lambda y.t''^{a''})^a]\!]^-[[\![t'^{a'}]\!]^-/x] & = & \\
[\![t^a]\!]^-[[\![t'^{a'}]\!]^-/x] & &
\end{array}
$$

$t^a = (t_1^b\ t_2^c)^a$:

$$
\begin{array}{lll}
[\![t^a[t'^{a'}/x]]\!]^- & = & \\
[\![(t_1^b\ t_2^c)^a[t'^{a'}/x]]\!]^- & = & \text{(Definition 9)} \\
[\![(t_1^b[t'^{a'}/x]\ t_2^c[t'^{a'}/x])^a]\!]^- & = & \text{(Definition 7)} \\
[\![t_1^b[t'^{a'}/x]]\!]^-\ [\![t_2^c[t'^{a'}/x]]\!]^- & = & \text{(Induction Hypothesis)} \\
([\![t_1^b]\!]^-[[\![t'^{a'}]\!]^-/x])\ ([\![t_2^c]\!]^-[[\![t'^{a'}]\!]^-/x]) & = & \text{(Definition 3)} \\
([\![t_1^b]\!]^-\ [\![t_2^c]\!]^-)[[\![t'^{a'}]\!]^-/x] & = & \text{(Definition 7)} \\
[\![(t_1^b\ t_2^c)^a]\!]^-[[\![t'^{a'}]\!]^-/x] & = & \\
[\![t^a]\!]^-[[\![t'^{a'}]\!]^-/x] & &
\end{array}
$$

□

Lemma 12. $[\![t^a]\!]^- = [\![t^{a'}]\!]^-$, *for every node a and a'.*

Proof. Three cases are possible:

$$
\begin{array}{lll}
[\![x^a]\!]^- & = x & = [\![x^{a'}]\!]^- \\
[\![(\lambda x.t^b)^a]\!]^- & = \lambda x.[\![t^b]\!]^- & = [\![(\lambda x.t^b)^{a'}]\!]^- \\
[\![(t_1^b\ t_2^c)^a]\!]^- & = [\![t_1^b]\!]^-\ [\![t_2^c]\!]^- & = [\![(t_1^b\ t_2^c)^{a'}]\!]^-
\end{array}
$$

□

Definition 13. *Reductions* $.\xrightarrow{d}.$ *on the distributed* λ*-terms are:*

$$
\begin{array}{llll}
(\lambda x.t^a)^a & \xrightarrow{d}_\alpha & (\lambda y.t^a[y^a/x])^a & (\alpha_{\mathbf{d}}\text{-conversion}) \quad \text{when } y \notin \mathsf{occur}(t^a) \\
((\lambda x.t_1^a)^a \ t_2^a)^a & \xrightarrow{d}_\beta & t_1^a[t_2^a/x] & (\beta_{\mathbf{d}}\text{-reduction}) \\
(\lambda x.(t^a \ x^a)^a)^a & \xrightarrow{d}_\eta & t^a & (\eta_{\mathbf{d}}\text{-reduction}) \quad \text{when } x \notin \mathsf{occur}(t^a) \\
t^a & \xrightarrow{d}_\mu & t^b & (\mu_{\mathbf{d}}\text{-conversion}).
\end{array}
$$

Lemma 14. *Let* $t \xrightarrow{o} t'$*. Then,* $[\![t]\!]^{+a} \xrightarrow{d} [\![t']\!]^{+a}$ *for every node* a*.*

Proof. By case distinction on $t \xrightarrow{o} t'$.

- $t = \lambda x.t'' \xrightarrow{o}_\alpha \lambda y.t''[y/x] = t'$

$$
\begin{array}{lll}
[\![t]\!]^{+a} & = & \\
[\![\lambda x.t'']\!]^{+a} & = & (\text{Definition 7}) \\
\lambda x.[\![t'']\!]^{+a} & \xrightarrow{d}_\alpha & (\text{Definition 13}) \\
\lambda y.[\![t'']\!]^{+a}[y^a/x] & = & (\text{Lemma 10}) \\
\lambda y.[\![t''[y/x]]\!]^{+a} & = & (\text{Definition 7}) \\
[\![\lambda y.t''[y/x]]\!]^{+a} & = & \\
[\![t']\!]^{+a} & &
\end{array}
$$

- $t = (\lambda x.t_1) \ t_2 \xrightarrow{o}_\beta t_1[t_2/x] = t'$

$$
\begin{array}{lll}
[\![t]\!]^{+a} & = & \\
[\![(\lambda x.t_1) \ t_2]\!]^{+a} & = & (\text{Definition 7}) \\
[\![\lambda x.t_1]\!]^{+a} \ [\![t_2]\!]^{+a} & = & (\text{Definition 7}) \\
(\lambda x.[\![t_1]\!]^{+a})^a \ [\![t_2]\!]^{+a} & \xrightarrow{d}_\beta & (\text{Definition 13}) \\
[\![t_1]\!]^{+a}[[\![t_2]\!]^{+a}/x] & = & (\text{Lemma 10}) \\
[\![t_1[t_2/x]]\!]^{+a} & = & \\
[\![t']\!]^{+a} & &
\end{array}
$$

- $t = \lambda x.(t'' \ x) \xrightarrow{o}_\eta t'' = t'$

$$
\begin{array}{lll}
[\![t]\!]^{+a} & = & \\
[\![\lambda x.(t'' \ x)]\!]^{+a} & = & (\text{Definition 7}) \\
(\lambda x.[\![t'' \ x]\!]^{+a})^a & = & (\text{Definition 7}) \\
(\lambda x.([\![t'']\!]^{+a} \ x^a)^a)^a & \xrightarrow{d}_\eta & (\text{Definition 13}) \\
[\![t'']\!]^{+a} & = & \\
[\![t']\!]^{+a} & &
\end{array}
$$

□

Definition 15. *Write* $\overset{o}{\twoheadrightarrow}$ *and* $\overset{d}{\twoheadrightarrow}$ *for the reflexive and transitive closure of* $\xrightarrow{o}$ *and* $\xrightarrow{d}$*, respectively.*

Theorem 16. *Let* $t \overset{o}{\twoheadrightarrow} t'$*. Then,* $[\![t]\!]^{+a} \overset{d}{\twoheadrightarrow} [\![t']\!]^{+a}$ *for every node* a*.*

Proof. Immediate from Lemma 14 and Definition 15. □

Lemma 17. $t^a \xrightarrow{d} t'^b$ *implies* $[\![t^a]\!]^- \overset{o}{\twoheadrightarrow} [\![t'^b]\!]^-$*, for every node* a *and* b*.*

Proof. By case distinction on $t^a \xrightarrow{d} t'^b$.

- $t^a = (\lambda x.t''^a)^a \xrightarrow{d}_\alpha (\lambda y.t''^a[y^a/x])^a = t'^b$

$$\begin{array}{lcr}
[\![t^a]\!]^- & = & \\
[\![(\lambda x.t''^a)^a]\!]^- & = & \text{(Definition 7)} \\
\lambda x.[\![t''^a]\!]^- & \xrightarrow{o}_\alpha & \text{(Definition 3)} \\
\lambda y.[\![t''^a]\!]^-[y/x] & = & \text{(Definition 7)} \\
\lambda y.[\![t''^a]\!]^-[[\![y^a]\!]^-/x] & = & \text{(Lemma 11)} \\
\lambda y.[\![t''^a[y^a/x]]\!]^- & = & \text{(Definition 7)} \\
[\![(\lambda y.t''^a[y^a/x])^a]\!]^- & = & \\
[\![t'^b]\!]^- & &
\end{array}$$

- $t^a = ((\lambda x.t_1^a)\ t_2^a)^a \xrightarrow{d}_\beta t_1^a[t_2^a/x] = t'^b$

$$\begin{array}{lcr}
[\![t^a]\!]^- & = & \\
[\![((\lambda x.t_1^a)^a\ t_2^a)^a]\!]^- & = & \text{(Definition 7)} \\
[\![(\lambda x.t_1^a)^a]\!]^-\ [\![t_2^a]\!]^- & = & \text{(Definition 7)} \\
(\lambda x.[\![t_1^a]\!]^-)\ [\![t_2^a]\!]^- & \xrightarrow{o}_\beta & \text{(Definition 4)} \\
[\![t_1^a]\!]^-[[\![t_2^a]\!]^-/x] & = & \text{(Lemma 11)} \\
[\![t_1^a[t_2^a/x]]\!]^- & = & \\
[\![t'^b]\!]^- & &
\end{array}$$

- $t^a = (\lambda x.(t^a\ x^a)^a)^a \xrightarrow{d}_\eta t^a = t'^b$

$$\begin{array}{lcr}
[\![t^a]\!]^- & = & \\
[\![(\lambda x.(t^a\ x^a)^a)^a]\!]^- & = & \text{(Definition 7)} \\
\lambda x.[\![(t^a\ x^a)^a]\!]^- & = & \text{(Definition 7)} \\
\lambda x.[\![t^a]\!]^-\ x & \xrightarrow{o}_\eta & \text{(Definition 4)} \\
[\![t^a]\!]^- & = & \\
[\![t'^b]\!]^- & &
\end{array}$$

- $t^a \xrightarrow{d}_\mu t^b = t'^b$

$$\begin{array}{lcr}
[\![t^a]\!]^- & = & \text{(Lemma 12)} \\
[\![t^b]\!]^- & = & \\
[\![t'^b]\!]^- & &
\end{array}$$

□

Theorem 18. *$t^a \overset{d}{\twoheadrightarrow} t'^b$ implies $[\![t^a]\!]^- \overset{o}{\twoheadrightarrow} [\![t'^b]\!]^-$, for every node a and b.*

Proof. Immediate from Lemma 17 and Definition 15. □