AN ASYMPTOTIC EXPANSION OF THE EMPIRICAL ANGULAR MEASURE FOR BIVARIATE EXTREMAL DEPENDENCE

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An asymptotic expansion of the empirical angular measure for bivariate extremal dependence

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Abstract

The angular measure on the unit sphere characterizes the first-order dependence structure of the components of a random vector in extreme regions and is defined in terms of standardized margins. Its statistical recovery is an important step in learning problems involving observations far away from the center. In the common situation that the components of the vector have different distributions, the rank transformation offers a convenient and robust way of standardizing data in order to build an empirical version of the angular measure based on the most extreme observations. We provide a functional asymptotic expansion for the empirical angular measure in the bivariate case based on the theory of weak convergence in the space of bounded functions. From the expansion, not only can the known asymptotic distribution of the empirical angular measure be recovered, it also enables to find expansions and weak limits for other statistics based on the associated empirical process or its quantile version.

1 Introduction and main result

A standard assumption on the joint upper tail of the distribution of a random vector $X = (X_1, X_2)$ in $\mathbb{R}^2$ is that its distribution belongs to the *maximal domain of attraction* of a multivariate extreme value distribution. This hypothesis comprises two parts:

1. the marginal distributions of X belong to the maximal domains of attraction of some univariate extreme value distributions;
2. after marginal transformation of X_1 and X_2 through the probability integral transform or a variation thereof, the joint distribution of the transformed vector belongs to the maximal domain of attraction of a multivariate extreme value distribution with pre-specified margins.

Under the side assumption that the marginal cumulative distribution functions F_1 and F_2 of X are continuous, which we make in this paper, point 2 above only involves the copula of X in view of Sklar's theorem. Point 2 can be imposed on its own, that is, independently of the assumptions on the margins in point 1, and this is what we will do in this paper.

Regular variation. Our working hypothesis is that the distribution of the standardized vector

$$(Z_1, Z_2) := \left(\frac{1}{1 - F_1(X_1)}, \frac{1}{1 - F_2(X_2)} \right)$$

is *multivariate regularly varying*, i.e., there exists a Radon measure ν on $E := [0, +\infty]^2 \setminus \{(0,0)\}$ such that

$$s\,\mathbb{P}\left[(Z_1, Z_2) \in s\,\cdot\,\right] \xrightarrow{M_0} \nu(\cdot), \qquad \text{when } s \to \infty, \tag{1}$$

where $\xrightarrow{M_0}$ denotes M_0-*convergence* in E [8], i.e., for every $f : E \to \mathbb{R}$ which is continuous, bounded and vanishes in a neighborhood of the origin, we have $s\,\mathbb{E}\left[f(s^{-1}(Z_1, Z_2))\right] \to \int_E f \, \mathrm{d}\nu$ as $s \to \infty$. The *exponent measure* ν appears in the exponent of the cumulative distribution function of the multivariate extreme value distribution to which (Z_1, Z_2) is attracted [4, Definition 6.1.7].

It can be shown [8, Theorem 2.4] that the M_0-convergence (1) is equivalent to the following convergence of probabilities: if B is a Borel set of E such that $0 \notin \overline{B}$, where $\overline{B}$ denotes the closure of B, and $\nu(\partial B) = 0$, then

$$\lim_{s \to \infty} s\,\mathbb{P}\left[(Z_1, Z_2) \in sB\right] = \nu(B). \tag{2}$$

Intuitively, this relation permits to describe probabilities of events far away from the origin under the law of the standardized random vector (Z_1, Z_2) based on the knowledge of ν. Consequently, statistical inference on the exponent measure ν is fundamental to model the joint upper tail of (X_1, X_2).

Angular measure. It can be shown that ν is concentrated on $[0, \infty)^2 \setminus \{(0,0)\}$ and satisfies the following homogeneity property: for B a Borel set in E,

$$\nu(cB) = c^{-1}\nu(B), \qquad \text{for every } c > 0, \tag{3}$$

where $cB := \{cx \in E : x \in B\}$. Property (3) motivates a transformation to pseudo-polar coordinates. Given any L^p-norm, $p \in [1, \infty]$, on $\mathbb{R}^2$, consider $T : (z_1, z_2) \in [0, \infty)^2 \setminus \{(0,0)\} \mapsto T(z_1, z_2) = (r(z_1, z_2), \theta(z_1, z_2)) \in (0, \infty) \times [0, \pi/2]$ with

$$r(z_1, z_2) := \|(z_1, z_2)\|_p \quad \text{and} \quad \theta(z_1, z_2) := \arctan(z_1/z_2).$$

By (3), the push-forward measure of ν under T is a product measure on $(0, \infty) \times [0, \pi/2]$:

$$T\#\nu := \nu \circ T^{-1} = \nu_{-1} \otimes \Phi_p, \tag{4}$$

where ν_{-1} is the Borel measure on $(0,\infty)$ determined by $\nu_{-1}([r,\infty)) = r^{-1}$ for any $r > 0$ and where the *angular measure* Φ_p on $[0,\pi/2]$ is defined by

$$\Phi_p(A) := \nu\left(\{(z_1,z_2) \in [0,\infty)^2 \setminus \{(0,0)\} : \|r(z_1,z_2)\|_p \geq 1, \theta(z_1,z_2) \in A\}\right), \qquad (5)$$

for every Borel set $A \subseteq [0,\pi/2]$. The representation (4) implies that the angular measure Φ_p fully characterizes ν. Consequently, we focus on inference for the angular measure Φ_p. We refer to the monographs [1, 9] for background on the subject.

Empirical angular measure. Given is an independent random sample (X_{i1}, X_{i2}), for $i \in \{1,\ldots,n\}$, whose common distribution has continuous margins and angular measure Φ_p. Our estimate for Φ_p relies on the probabilistic interpretation of its definition (5) obtained by (2), i.e., for any Borel set $A \subseteq [0,\pi/2]$,

$$\Phi_p(A) = \lim_{s\to\infty} s\,\mathbb{P}\left[\|(Z_1,Z_2)\|_p \geq s, \arctan(Z_1/Z_2) \in A\right].$$

Let $k = k(n) \in \{1,\ldots,n\}$ be such that $k \to \infty$ and $k = \mathrm{o}(n)$ as $n \to \infty$. Choose $s = n/k$ and replace the induced law of (Z_1,Z_2) on E by its empirical counterpart on the empirically standardized sample

$$(\hat{Z}_{i1}, \hat{Z}_{i2}) := \left(\frac{1}{1-\hat{F}_1(X_{i1})}, \frac{1}{1-\hat{F}_2(X_{i2})}\right) = \left(\frac{n}{n+1-R_{i1}}, \frac{n}{n+1-R_{i2}}\right),$$

where $\hat{F}_j(x_j) := n^{-1}\sum_{i=1}^n \mathbb{1}\{X_{ij} < x_j\}$ and where R_{ij} denotes the rank of X_{ij} in the marginal sample $X_{1j},\ldots,X_{nj}$. The *empirical angular (spectral) measure* of $\Phi_p(\theta) := \Phi_p([0,\theta])$, $\theta \in [0,\pi/2]$ is defined by

$$\begin{aligned}\widehat{\Phi}_p(\theta) &:= \frac{1}{k}\sum_{i=1}^n \mathbb{1}\left\{\|(\hat{Z}_{i1},\hat{Z}_{i2})\|_p \geq \tfrac{n}{k}, \arctan(\hat{Z}_{i1}/\hat{Z}_{i2}) \leq \theta\right\} \\ &= \frac{1}{k}\sum_{i=1}^n \mathbb{1}\left\{(n+1-R_{i1})^{-p} + (n+1-R_{i2})^{-p} \geq k^{-p}, \frac{n+1-R_{i2}}{n+1-R_{i1}} \leq \tan\theta\right\},\end{aligned}$$

see [6, 7, 2]. The estimator $\widehat{\Phi}_p$ depends on the data only through their ranks, posing challenges for asymptotic distribution theory.

Asymptotic distribution theory. The asymptotic distribution of the empirical process

$$\left\{\sqrt{k}\left(\widehat{\Phi}_p(\theta) - \Phi_p(\theta)\right) : \theta \in [0,\pi/2]\right\}$$

was derived in [6] for the case $p = \infty$ and [7] for a general $p \in [1,\infty]$. We recall the assumptions and the precise statement below since they form the backbone for our main result.

The assumptions are described in terms of a measure Λ, equivalent to the exponent measure ν, and defined as follows: for Borel sets $B \subseteq E^{-1} := [0,\infty]^2 \setminus \{(\infty,\infty)\}$,

$$\Lambda(B) := \nu\left(\{(z_1,z_2) \in E : (1/z_1, 1/z_2) \in B\}\right).$$

It corresponds to a uniform standardization of the margins in the multivariate regular variation assumption (1), i.e., Λ, arises as the M_0-limit of

$$s\,\mathbb{P}\left[(1-F_1(X_1), 1-F_2(X_2)) \in s^{-1}\cdot\right] \xrightarrow{M_0} \Lambda(\cdot), \qquad \text{when } s \to \infty.$$

Assumption 1 (Smoothness). The measure Λ is absolutely continuous with respect to the Lebesgue measure with a density λ which is continuous on $[0,\infty)^2\backslash\{(0,0)\}$. Furthermore, $\Lambda\left([0,\infty)\times\{\infty\}\right) = \Lambda\left(\{\infty\}\times[0,\infty)\right) = 0$.

Assumption 1 implies that Φ_p is concentrated on $(0,\pi/2)$, thereby excluding asymptotic independence. Indeed, a simple calculation shows that $\Phi_p(\{0\}) = \Lambda(\{\infty\}\times[0,\infty)) = 0$ and $\Phi_p(\{\pi/2\}) = \Lambda([0,\infty)\times\{\infty\}) = 0$. It also implies that the map $\theta \in (0,\pi/2) \mapsto \Phi_p(\theta)$ has a continuous derivative φ_p on $(0,\pi/2)$. In particular, λ and φ_p are related through the following expression: for every $x, y > 0$,

$$\lambda(x,y) = \frac{y}{x^2+y^2}\left(1+(y/x)^p\right)^{-1/p}\varphi_p(\arctan(y/x)). \tag{6}$$

The proof of this identity, which is of independent interest, is given in the Appendix. In addition, we see that

$$\lambda(ax, ay) = a^{-1}\lambda(x,y), \qquad a > 0, (x,y) \in [0,\infty)^2 \setminus \{(0,0)\}. \tag{7}$$

Assumption 2 (Vanishing bias). If c denotes the density of $(1-F_1(X_1), 1-F_2(X_2))$, the quantity

$$\mathcal{D}_T(t) := \iint_{\mathcal{L}_T} |tc(tu_1, tu_2) - \lambda(u_1,u_2)|\,\mathrm{d}u_1\,\mathrm{d}u_2, \quad 1 \le T < \infty,\ t > 0,$$

where $\mathcal{L}_T := \{(u_1,u_2) \in [0,T]^2 : u_1 \wedge u_2 \le 1\}$, satisfies $\mathcal{D}_{1/t}(t) \to 0$ as $t \to 0$ and $\sqrt{k}\mathcal{D}_{n/k}(k/n) \to 0$ as $n \to \infty$.

The asymptotic normality result in [7] also involves quantities related to the geometry of sets we are considering. In particular, the sets

$$C_{p,\theta} := \begin{cases} ([0,\infty]\times\{0\}) \cup (\{\infty\}\times[0,1]) & \text{if } \theta = 0, \\ \{(x,y) : 0 \le x \le \infty, 0 \le y \le \min\{x\tan\theta, y_p(x)\}\} & \text{if } 0 < \theta < \pi/2, \\ \{(x,y) : 0 \le x \le \infty, 0 \le y \le y_p(x)\} & \text{if } \theta = \pi/2, \end{cases} \tag{8}$$

where

$$y_p(x) := \begin{cases} \infty & \text{if } x \in [0,1), \\ \left(1+\frac{1}{x^p-1}\right)^{1/p} & \text{if } x \in [1,\infty] \text{ and } p \in [1,\infty), \\ 1 & \text{if } x \in [1,\infty] \text{ and } p = \infty, \end{cases}$$

are such that $\Phi_p(\theta) = \Lambda(C_{p,\theta})$ for $\theta \in [0,\pi/2]$. Note that $x\tan\theta < y_p(x)$ if and only if $x < x_p(\theta)$, where for $\theta \in [0,\pi/2]$, we have defined $x_p(\theta) := \|(1,\cot\theta)\|_p$. We illustrate these quantities in Figure 1 in case $p = 1$ and $0 < \theta < \pi/2$.

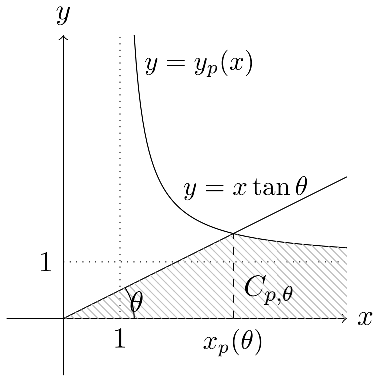

Figure 1: Illustration of the set $C_{p,\theta}$ in (8) and related quantities for $p = 1$ and $0 < \theta < \pi/2$.

Let W_Λ be a centered Wiener process indexed by Borel sets on $E^{-1} = [0,\infty]^2 \setminus \{(\infty,\infty)\}$ with "time" Λ and covariance function $\mathbb{E}[W_\Lambda(C)W_\Lambda(C')] = \Lambda(C \cap C')$. In particular, note that if $=_d$ means equality in distribution, we have

$$(W_\Lambda(C_{p,\theta}))_{\theta\in[0,\pi/2]} =_d (W(\Phi_p(\theta)))_{\theta\in[0,\pi/2]},$$

where W is a standard Wiener process on $[0,\infty)$.

We are now ready to formulate Theorem 3.1 in [7]. Here and below, the symbol $\rightsquigarrow$ denotes weak convergence in a metric space as in [11]. Let $\ell^\infty(\mathcal{F})$ denote the Banach space of bounded functions from a set $\mathcal{F}$ to $\mathbb{R}$, equipped with the supremum norm.

Theorem 1 (Asymptotic distribution of the empirical angular measure)**.** *Let $p \in [1,\infty]$. Under Assumptions 1 and 2, we have in $\ell^\infty([0,\pi/2])$ the weak convergence*

$$\left\{\sqrt{k}\left(\widehat{\Phi}_p(\theta) - \Phi_p(\theta)\right)\right\}_{\theta\in[0,\pi/2]} \rightsquigarrow \{W_\Lambda(C_{p,\theta}) + Z_p(\theta) =: \alpha_p(\theta)\}_{\theta\in[0,\pi/2]},$$

where the process Z_p is defined by

$$\begin{aligned} Z_p(\theta) &:= \int_0^{x_p(\theta)} \lambda(x, x\tan\theta)\,[W_1(x)\tan\theta - W_2(x\tan\theta)]\ \mathrm{d}x \\ &+ \begin{cases} \displaystyle\int_{x_p(\theta)}^{\infty} \lambda(x, y_p(x))\,\left[W_1(x)y_p'(x) - W_2(y_p(x))\right]\ \mathrm{d}x & \text{if } p < \infty, \\ \displaystyle -W_1(1)\int_1^{1\vee\tan\theta} \lambda(1,y)\ \mathrm{d}y - W_2(1)\int_{1\vee\cot\theta}^{\infty} \lambda(x,1)\ \mathrm{d}x & \text{if } p = \infty, \end{cases} \end{aligned}$$

with y_p' the derivative of y_p and with $W_1(x) := W_\Lambda((0,x]\times(0,\infty])$ and $W_2(y) := W_\Lambda((0,\infty]\times(0,y])$ the marginal processes.

Often, it is not enough to only have the limiting distribution at our disposal, and it is more convenient to have an asymptotic expansion of the process. For example, some statistic may involve the quantile process rather than the process itself, and an expansion of such a statistic can be obtained through the Functional Delta Method [10, Theorem 20.8], provided we have an expansion of the original process. The main aim of this paper is to provide such an expansion and the result is presented in Theorem 2 below. In contrast to the proof of Theorem 1 in [6, 7], the proof of our main result, Theorem 2, does not require a Skorokhod construction and is shown on the original probability space directly.

Theorem 2 (Asymptotic expansion of the empirical angular measure)**.** *Let P be the law of the uniform standardization of X given by $U := (1 - F_1(X_1), 1 - F_2(X_2))$ and let P_n denote the empirical measure of an independent sample $U_1, \dots, U_n$ of the law of U. For $j \in \{1, 2\}$ and $u \in [0, 1]$, let the associated marginal empirical cumulative distribution functions be denoted by $\Gamma_{jn}(u) := n^{-1} \sum_{i=1}^{n} \mathbb{1}\{U_{ij} \le u\}$. We set $\Gamma_{jn}(u) := u$ for $u > 1$. We let Q_{jn} denote the quantile function associated to Γ_{jn} with $Q_{jn}(y) = 0$ for $0 \le y \le (2n)^{-1}$, by convention. We define the tail marginal empirical processes $w_{jn}(x) := \sqrt{k}\{\frac{n}{k}\Gamma_{jn}(\frac{k}{n}x) - x\}$ and $v_{jn}(x) := \sqrt{k}\{\frac{n}{k}Q_{jn}(\frac{k}{n}x) - x\}$, for $j \in \{1, 2\}$ and $x \ge 0$.*

Under Assumptions 1 and 2, we have for any $p \in [1, \infty]$

$$\sup_{\theta \in [0, \pi/2]} \left| \sqrt{k} \left(\widehat{\Phi}_p(\theta) - \Phi_p(\theta) \right) - E_{n,p}(\theta) \right| = o_{\mathbb{P}}(1),$$

as $n \to \infty$, where

$$\begin{aligned} E_{n,p}(\theta) &:= \sqrt{k} \left(\tfrac{n}{k} P_n \left(\tfrac{k}{n} C_{p,\theta} \right) - \tfrac{n}{k} P \left(\tfrac{k}{n} C_{p,\theta} \right) \right) \\ &+ \int_0^{x_p(\theta)} \lambda(x, x \tan\theta) \left\{ w_{1n}(x) \tan\theta - w_{2n}(x \tan\theta) \right\} \mathrm{d}x \\ &+ \begin{cases} \displaystyle\int_{x_p(\theta)}^{\infty} \lambda(x, y_p(x)) \left\{ y_p'(x) w_{1n}(x) - w_{2n}(y_p(x)) \right\} \mathrm{d}x & \text{if } p < \infty, \\ \displaystyle -w_{1n}(1) \int_1^{1 \vee \tan\theta} \lambda(1, y)\, \mathrm{d}y - w_{2n}(1) \int_{1 \vee \cot\theta}^{\infty} \lambda(x, 1)\, \mathrm{d}x & \text{if } p = \infty. \end{cases} \end{aligned}$$

The proof of Theorem 2 is long and is presented in detail in Section 2. In Section 3, we study the empirical quantile function of the normalized angular measure.

2 Proof of Theorem 2

The key idea behind the proof is the following decomposition that can be found in [6] or [7]: for each $\theta \in [0, \pi/2]$,

$$\begin{aligned} \sqrt{k} \left(\widehat{\Phi}_p(\theta) - \Phi_p(\theta) \right) &= \sqrt{k} \left(\tfrac{n}{k} P_n \left(\tfrac{k}{n} \hat{C}_{p,\theta} \right) - \tfrac{n}{k} P \left(\tfrac{k}{n} \hat{C}_{p,\theta} \right) \right) \\ &\quad + \sqrt{k} \left(\tfrac{n}{k} P \left(\tfrac{k}{n} \hat{C}_{p,\theta} \right) - \Lambda \left(\hat{C}_{p,\theta} \right) \right) \\ &\quad + \sqrt{k} \left(\Lambda \left(\hat{C}_{p,\theta} \right) - \Lambda \left(C_{p,\theta} \right) \right) \\ &=: V_{n,p}(\theta) + r_{n,p}(\theta) + Z_{n,p}(\theta), \end{aligned} \tag{9}$$

where the set $C_{p,\theta}$ is estimated by

$$\hat{C}_{p,\theta} := \Big\{(x,y) : 0 \leq x \leq \infty, 0 \leq y \leq \tfrac{n}{k} Q_{2n}\left(\Gamma_{1n}\left(\tfrac{k}{n}x\right)\tan\theta\right),\\ y \leq \tfrac{n}{k} Q_{2n}\left(\tfrac{k}{n} y_p\left(\tfrac{n}{k}\Gamma_{1n}\left(\tfrac{k}{n}x\right)\right)\right)\Big\}. \tag{10}$$

The three terms in the decomposition (9) will be called the *stochastic term*, *bias term*, and *random set term*, respectively. In Section 2.3, we treat each term separately. First we show that, by symmetry, we can reduce the analysis to the case $\theta \in [0, \pi/4]$ (Section 2.1) and we establish a weak convergence of an underlying tail empirical process (Section 2.2).

2.1 Reduction to $[\mathbf{0}, \pi/\mathbf{4}]$

Similarly to the papers [6, 7] on which our proof is based, we argue that we can focus on the region $\theta \in [0, \pi/4]$ by exploiting symmetry.

Let $S : (x, y) \in \mathbb{R}^2 \mapsto (y, x) \in \mathbb{R}^2$ be the symmetry map. By extension, we also define the set-valued map $S(A) := \{S(x, y) : (x, y) \in A\}$ for $A \subseteq \mathbb{R}^2$. Clearly, the measures $S\#P_n$ and $S\#P$ underlying the statement of the theorem satisfy Assumptions 1 and 2 with respect to the measure $\Lambda_S := S\#\Lambda$, which has density $\lambda_S := \lambda \circ S$. The smoothness assumption is trivial. For the bias assumption, letting $c_S = c \circ S$ denote the density of $S\#P$, it suffices to note that $S(\mathcal{L}_T) = \mathcal{L}_T$ for any $T \geq 1$ and thus

$$\iint_{\mathcal{L}_T} |tc_S(tu_1, tu_2) - \lambda_S(u_1, u_2)| \, \mathrm{d}u_1 \, \mathrm{d}u_2 = \mathcal{D}_T(t),$$

for any $t > 0$. We may thus apply our result for $S\#P_n$ and $S\#P$ for $\theta \in [0, \pi/4]$ once it has been proved for P_n and P.

Now consider $\theta \in [\pi/4, \pi/2]$. Then, up to sets of Hausdorff dimension 1, sets which are asymptotically negligible for our purposes, we have

$$C_{p,\theta} = C_{p,\pi/4} \cup \left(S(C_{p,\pi/4}) \setminus S(C_{p,\pi/2-\theta})\right). \tag{11}$$

Consequently, letting $\widehat{\Phi}_p^S$ denote the empirical angular measure associated with $S\#P_n$ and Φ_p^S the angular measure associated with $S\#P$, we have for such $\theta \in [\pi/4, \pi/2]$ that asymptotically, by continuity of the limiting process,

$$\sqrt{k}\left(\widehat{\Phi}_p(\theta) - \Phi_p(\theta)\right) = \sqrt{k}\left(\widehat{\Phi}_p(\pi/4) - \Phi_p(\pi/4)\right) + \sqrt{k}\left(\widehat{\Phi}_p^S(\pi/4) - \Phi_p^S(\pi/4)\right)\\ - \sqrt{k}\left(\widehat{\Phi}_p^S(\pi/2-\theta) - \Phi_p^S(\pi/2-\theta)\right) + \mathrm{o}_{\mathbb{P}}(1),$$

where the $\mathrm{o}_{\mathbb{P}}(1)$ term is uniform in $\theta \in [\pi/4, \pi/2]$. Assuming that the theorem has been shown to hold for $\theta \in [0, \pi/4]$, if $E_{n,p}^S(\theta)$ denotes the asymptotic expansion obtained on the basis of the measures $S\#P_n$ and $S\#P$, we have for $\theta \in [\pi/4, \pi/2]$,

$$\sqrt{k}\left(\widehat{\Phi}_p(\theta) - \Phi_p(\theta)\right) = E_{n,p}(\pi/4) + E_{n,p}^S(\pi/4) - E_{n,p}^S(\pi/2-\theta) + \mathrm{o}_{\mathbb{P}}(1),$$

where the $o_{\mathbb{P}}(1)$ term is uniform in $\theta \in [\pi/4, \pi/2]$. It is now sufficient to show that

$$E_{n,p}(\pi/4) + E^S_{n,p}(\pi/4) - E^S_{n,p}(\pi/2 - \theta) = E_{n,p}(\theta), \qquad \theta \in [\pi/4, \pi/2]. \tag{12}$$

To do so, we expand $E_{n,p}$ and $E^S_{n,p}$ into three terms as in Theorem 2. The expansions of $E_{n,p}(\pi/4)$ and $E_{n,p}(\theta)$ are provided in the theorem. We compute the expansions of $E^S_{n,p}(\pi/4)$ and $E^S_{n,p}(\pi/2 - \theta)$.

For $E^S_{n,p}(\pi/4)$, the first term in the expansion is trivially given by

$$\sqrt{k}\left(\tfrac{n}{k}P_n\left(\tfrac{k}{n}S(C_{p,\pi/4})\right) - \tfrac{n}{k}P\left(\tfrac{k}{n}S(C_{p,\pi/4})\right)\right).$$

For the second term in the expansion, the integral over $x < x_p(\pi/4)$ becomes

$$\int_0^{2^{1/p}} \lambda_S(x,x)\{w_{2n}(x) - w_{1n}(x)\}\,\mathrm{d}x = -\int_0^{2^{1/p}} \lambda(x,x)\{w_{1n}(x) - w_{2n}(x)\}\,\mathrm{d}x,$$

since $\tan(\pi/4) = \cot(\pi/4) = 1$ and the measure $S\#P_n$ has first marginal tail empirical process w_{2n} and second marginal tail empirical process w_{1n}. We see that it equals the opposite of the integral over $x < x_p(\pi/4)$ in the expansion of $E_{n,p}(\pi/4)$ so that they will cancel out. The third term in the expansion is an integral over $x \geq x_p(\pi/4)$, for which we only consider the more complicated case $p < \infty$. It equals

$$\begin{aligned}\int_{2^{1/p}}^{\infty} &\lambda_S(x, y_p(x))\left\{w_{2n}(x)y_p'(x) - w_{1n}(y_p(x))\right\}\,\mathrm{d}x \\ &= \int_1^{2^{1/p}} \lambda(z, y_p(z))\left\{w_{1n}(z)y_p'(z) - w_{2n}(y_p(z))\right\}\,\mathrm{d}z,\end{aligned}$$

where the second line follows from the change of variable $x = y_p(z)$ which is an involution and satisfies $y_p'(y_p(z))y_p'(z) = 1$.

For $E^S_{n,p}(\pi/2 - \theta)$, the reasoning is similar. The first term in the expansion is

$$\sqrt{k}\left(\tfrac{n}{k}P_n\left(\tfrac{k}{n}S\big(C_{p,\pi/2-\theta}\big)\right) - \tfrac{n}{k}P\left(\tfrac{k}{n}S\big(C_{p,\pi/2-\theta}\big)\right)\right).$$

Since $\tan(\pi/2 - \theta) = \cot(\theta)$, the second term in the expansion is

$$\begin{aligned}\int_0^{x_p(\pi/2-\theta)} &\lambda_S(x, x\tan(\pi/2-\theta))\{w_{2n}(x)\tan(\pi/2-\theta) - w_{1n}(x\tan(\pi/2-\theta))\}\,\mathrm{d}x \\ &= -\int_0^{x_p(\theta)} \lambda(y, y\tan\theta)\left\{w_{1n}(y)\tan\theta - w_{2n}(y\tan\theta)\right\}\,\mathrm{d}y.\end{aligned}$$

The third term is an integral over $x \geq x_p(\pi/2 - \theta)$. Again assuming $p < \infty$, using the same change of variable as before and the fact that $y_p(x_p(\pi/2 - \theta)) = x_p(\theta)$, we get

$$\int_1^{x_p(\theta)} \lambda(z, y_p(z))\left\{w_{1n}(z)y_p'(z) - w_{2n}(y_p(z))\right\}\,\mathrm{d}z.$$

Combining everything, we find (12). Consequently, once the asymptotic expansion stated in the theorem has been shown to hold on the region $\theta \in [0, \pi/4]$, it holds on $\theta \in [0, \pi/2]$.

2.2 Weak convergence of a tail empirical process

Unlike [6, 7], we prove our asymptotic expansion without requiring a Skorokhod construction but work with the original sample space and random variables directly. This is based on a more general result, holding in arbitrary dimension and which may be of independent interest.

Proposition 1 (Weak convergence of the tail empirical process)**.** *Let $\mathcal{A}$ be a VC class of Borel sets on $[0,\infty)^d$ such that there exists $M > 0$ with*

$$A \subseteq L_M := \{x \in [0,\infty)^d : \min(x_1, \dots, x_d) \le M\}, \qquad A \in \mathcal{A}.$$

Let P be a Borel probability measure on $[0,1]^d$ with uniform margins such that there exists a Radon measure Λ on $[0,\infty]^d \setminus \{(\infty, \dots, \infty)\}$ satisfying

$$\lim_{t\to 0} \left| t^{-1} P(tB) - \Lambda(B) \right| = 0,$$

for any Borel set bounded away from infinity with $\Lambda(\partial B) = 0$, and

$$\lim_{t\to 0} \sup_{A,A'\in\mathcal{A}} \left| t^{-1} P(t(A \triangle A')) - \Lambda(A \triangle A') \right| = 0, \tag{13}$$

Let P_n be the empirical measure of an i.i.d. random sample of size $n \in \mathbb{N}$ from P. Let $k = k(n) \in \mathbb{N}$ be such that $k \to \infty$ and $k = \mathrm{o}(n)$ as $n \to \infty$. In $\ell^\infty(\mathcal{A})$, we have the weak convergence

$$\left(W_{n,k}(A) := \sqrt{k} \left\{ \tfrac{n}{k} P_n \left(\tfrac{k}{n} A \right) - \tfrac{n}{k} P \left(\tfrac{k}{n} A \right) \right\} \right)_{A\in\mathcal{A}} \rightsquigarrow \left(W_\Lambda(A) \right)_{A\in\mathcal{A}}, \tag{14}$$

where W_Λ is tight, centred Gaussian process on $\mathcal{A}$ with covariance function $\mathbb{E}[W_\Lambda(A)\, W_\Lambda(A')] = \Lambda(A \cap A')$. Moreover, the process $W_{n,k}$ is asymptotically equicontinuous in probability with respect to the semi-metric $\rho(A, A') := \Lambda(A \triangle A')$, i.e., for all $\epsilon > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \mathbb{P}^* \left(\sup_{A,A'\in\mathcal{A}, \rho(A,A')<\delta} \left| W_{n,k}(A) - W_{n,k}(A') \right| > \epsilon \right) = 0. \tag{15}$$

Proof. The proof is an application of Theorem 19.28 in [10]. For any $n \in \mathbb{N}$ and $A \in \mathcal{A}$, let $\mathcal{F}_n$ be the class of functions $f_{n,A} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$f_{n,A}(x) := \sqrt{\tfrac{n}{k}} \mathbb{1} \left\{ x \in \tfrac{k}{n} A \right\}.$$

The VC dimension of $\mathcal{F}_n$ equals the one of $\mathcal{A}$, which is finite, by assumption. Using similar notation as in [10, Section 19.5], we have

$$\sqrt{k} \left\{ \tfrac{n}{k} P_n \left(\tfrac{k}{n} A \right) - \tfrac{n}{k} P \left(\tfrac{k}{n} A \right) \right\} = \sqrt{n} \left\{ P_n \left(f_{n,A} \right) - P \left(f_{n,A} \right) \right\} =: G_n \left(f_{n,A} \right),$$

for any $A \in \mathcal{A}$. Let us verify that the technical conditions underlying Theorem 19.28 in [10] are satisfied here.

We show that the metric space $(\mathcal{A}, \rho)$ is totally bounded. Indeed, noting that $\Lambda(L_M) \leq d \times M$, since each element of $\mathcal{A}$ is included in L_M, we may construct the standard empirical process with respect to the probability measure

$$P_M(\cdot) = \frac{\Lambda(\cdot \cap L_M)}{\Lambda(L_M)}, \qquad \text{on } L_M,$$

indexed by functions $f_A := \mathbb{1}_A$ for $A \in \mathcal{A}$ and use the fact that $\mathcal{A}$ is a VC-class with $|f_A| \leq F_M := \mathbb{1}_{L_M}$ pointwise for each $A \in \mathcal{A}$ to apply Theorem 19.14 in [10] to conclude that the considered process converges in $\ell^\infty(\mathcal{A})$ (where we identify $\mathcal{A}$ with its class of indicator functions). In view of Theorem 18.14 and Lemma 18.15 in [10], it must imply that the metric space $(\mathcal{A}, \rho_M)$ with $\rho_M := \rho/\Lambda(L_M)$ is totally bounded, which also guarantees that $(\mathcal{A}, \rho)$ is totally bounded.

Moreover, we have

$$\lim_{n\to\infty} \sup_{A,A' \in \mathcal{A}: \rho(A,A') < \delta_n} P\left((f_{n,A} - f_{n,A'})^2\right) = 0 \tag{16}$$

whenever $\delta_n \downarrow 0$ as $n \to \infty$. Indeed,

$$\begin{aligned}
&\sup_{A,A' \in \mathcal{A}: \rho(A,A') < \delta_n} P\left((f_{n,A} - f_{n,A'})^2\right) \\
&= \sup_{A,A' \in \mathcal{A}: \rho(A,A') < \delta_n} \tfrac{n}{k} P\left((\mathbb{1}_{(k/n)A} - \mathbb{1}_{(k/n)A'})^2\right) \\
&= \sup_{A,A' \in \mathcal{A}: \rho(A,A') < \delta_n} \tfrac{n}{k} P\left(\tfrac{k}{n}(A \triangle A')\right).
\end{aligned}$$

Fix $\epsilon > 0$. It follows from our assumptions that there exists $N_1(\epsilon) \in \mathbb{N}$ such that for $n \geq N_1(\epsilon)$,

$$\sup_{A,A' \in \mathcal{A}: \rho(A,A') < \delta_n} \left| \tfrac{n}{k} P\left(\tfrac{k}{n}(A \triangle A')\right) - \Lambda(A \triangle A') \right| \leq \epsilon/2,$$

while, since $\delta_n \to 0$ as $n \to \infty$, we may find $N_2(\epsilon) \in \mathbb{N}$ such that for $n \geq N_2(\epsilon)$,

$$\sup_{A,A' \in \mathcal{A}: \rho(A,A') < \delta_n} \Lambda(A \triangle A') \leq \epsilon/2.$$

It follows that for $n \geq \max\{N_1(\epsilon), N_2(\epsilon)\}$, we have

$$\sup_{A,A' \in \mathcal{A}: \rho(A,A') < \delta_n} P\left((f_{n,A} - f_{n,A'})^2\right) \leq \epsilon,$$

which yields (16).

The second condition to verify is that the classes $\mathcal{F}_n$ possess envelope functions F_n satisfying the Lindeberg condition

$$P\left(F_n^2\right) = \mathrm{O}(1) \qquad \text{and} \qquad P\left(F_n^2\, \mathbb{1}\{F_n > \epsilon\sqrt{n}\}\right) = \mathrm{o}(1), \quad \epsilon > 0,$$

as $n \to \infty$. It suffices to note that for every $n \in \mathbb{N}$ and $A \in \mathcal{A}$,

$$f_{n,A}(x) = \sqrt{\tfrac{n}{k}}\mathbb{1}\left\{x \in \tfrac{k}{n}A\right\} \leq \sqrt{\tfrac{n}{k}}\mathbb{1}\left\{x \in \tfrac{k}{n}L_M\right\} =: F_n(x).$$

The first part of the Lindeberg condition clearly holds since

$$\lim_{n\to\infty} P\left(F_n^2\right) = \lim_{n\to\infty} \tfrac{n}{k} P\left(\tfrac{k}{n} L_M\right) = \Lambda\left(L_M\right) \leq d \cdot M.$$

The second part of the Lindeberg condition is immediate since for any $\epsilon > 0$, the indicator $\mathbb{1}\{F_n > \epsilon\sqrt{n}\}$ is zero as soon as $1/\sqrt{k} \leq \epsilon$, which is the case for n large enough, since, by assumption, $k = k(n) \to \infty$.

The last condition to verify is the one associated to the uniform entropy integral. Here, since the VC dimension of any $\mathcal{F}_n$ equals the one of $\mathcal{A}$, which is finite by assumption, we may simply apply [11, Example 2.11.24].

Consequently, Theorem 19.28 in [11] applies and the covariance of the limiting Gaussian process is obtained by computing the limit

$$\lim_{n\to\infty} P\left(f_{n,A} f_{n,A'}\right) - P\left(f_{n,A}\right) P\left(f_{n,A'}\right) = \Lambda(A \cap A'),$$

for $A, A' \in \mathcal{A}$, which follows from our assumption.

Finally, we show (15). By Example 1.5.10 in [11] the processes $W_{n,k}$ are asymptotically equicontinuous in probability with respect to the standard deviation semi-metric $\rho_2(A, A') := (\mathbb{E}[|W_\Lambda(A) - W_\Lambda(A')|^2])^{1/2}$. Since

$$\mathbb{E}[|W_\Lambda(A) - W_\Lambda(A')|^2] = \Lambda(A) + \Lambda(A') - 2\Lambda(A \cap A') = \Lambda(A \triangle A') = \rho(A, A'),$$

the said equicontinuity property holds with respect to ρ as well. □

For fixed $\Delta \in \{1, 1/2, 1/3, \ldots\}$ and $M > 1$, let $\mathcal{A} = \mathcal{A}(\Delta, M)$ be the VC class defined on page 2974 of [7]. Every set in $\mathcal{A}$ is contained in L_M if $M > 1$ is picked large enough (which is not restrictive since we will consider it large later). Furthermore, the conditions underlying Proposition 1 are easily shown to hold in view of the assumptions. In particular, the uniform regular variation condition (13) is satisfied thanks to the convergence in total variation in Assumption 2. Consequently, we can conclude that (14) holds in the setting of Theorem 2.

In view of the nature of the sets A in the class $\mathcal{A}$, we have in particular the marginal weak convergences

$$\{w_{jn}(x) : x \in [0, M]\} \rightsquigarrow \{W_j(x) : x \in [0, M]\}, \tag{17}$$

jointly in $j = 1, 2$ on any $M > 1$, while, by the Functional Delta Method, we also have

$$\sup_{x\in[0,M]} |v_{jn}(x) + w_{jn}(x)| = o_{\mathbb{P}}(1). \tag{18}$$

We will also need some weighted versions of those marginal convergence results that can be derived from [5] and references therein: for $0 \leq \delta < 1/2 < \eta$ we have, jointly in $j = 1, 2$ (with the convention $0/0 = 0$),

$$\left\{\frac{w_{jn}(x)}{x^\delta \vee x^\eta} : x \in [0, \infty)\right\} \rightsquigarrow \left\{\frac{W_j(x)}{x^\delta \vee x^\eta} : x \in [0, \infty)\right\}, \qquad \text{in } \ell^\infty([0, \infty)). \tag{19}$$

The proof of (19) consists of checking finite dimensional convergence and asymptotic tightness. For the latter, partition $[0,\infty)$ into $[0,\epsilon]$, (ϵ,M) and $[M,\infty)$; on the two exterior intervals the processes can be made arbitrarily small in probability as $\epsilon \downarrow 0$ and $M \to \infty$, while on the middle interval use (17). Moreover, for $0 \le \delta < 1/2$, the empirical tail quantile processes v_{jn} satisfy [3, Section 4.3]

$$\sup_{x\in[0,M]} \frac{|v_{jn}(x)+w_{jn}(x)|}{x^{\delta}} = \mathrm{o}_{\mathbb{P}}(1). \tag{20}$$

2.3 Asymptotic expansions of the three terms in (9)

The stochastic term $V_{n,p}(\theta)$. Recall $W_{n,k}$ in Proposition 1. Note that $V_{n,p}(\theta) = W_{n,k}(\hat{C}_{p,\theta})$. Our aim is to show that, as $n\to\infty$,

$$\sup_{\theta\in[0,\pi/4]} \left| W_{n,k}(\hat{C}_{p,\theta}) - W_{n,k}(C_{p,\theta}) \right| = \mathrm{o}_{\mathbb{P}}(1). \tag{21}$$

We will consider separately $V_{n,p,1}$ and $V_{n,p,2}$, where

$$V_{n,p,i}(\theta) := W_{n,k}(\hat{C}_{p,\theta,i})$$

for $i=1,2$, with

$$\hat{C}_{p,\theta,1} := \hat{C}_{p,\theta} \cap \{(x,y) : x < x_p(\theta)\}, \quad \hat{C}_{p,\theta,2} := \hat{C}_{p,\theta} \cap \{(x,y) : x \ge x_p(\theta)\}.$$

In particular, it would follow from the triangle inequality that the expansion (21) is satisfied, provided that

$$\sup_{\theta\in[0,\pi/4]} |V_{n,p,i}(\theta) - W_{n,k}(C_{p,\theta,i})| = \mathrm{o}_{\mathbb{P}}(1), \qquad i\in\{1,2\}, \tag{22}$$

where

$$C_{p,\theta,1} := C_{p,\theta} \cap \{(x,y) : x < x_p(\theta)\}, \quad C_{p,\theta,2} := C_{p,\theta} \cap \{(x,y) : x \ge x_p(\theta)\}.$$

Let us first consider $i=1$. Recall equation (10) and observe that, since $\frac{n}{k}\Gamma_{jn}(\frac{k}{n}x) = x + w_{jn}(x)/\sqrt{k}$ and $\frac{n}{k}Q_{jn}(\frac{k}{n}x) = x + v_{jn}(x)/\sqrt{k}$,

$$\begin{aligned} \tfrac{n}{k}Q_{2n}\left(\Gamma_{1n}\left(\tfrac{k}{n}x\right)\tan\theta\right) &= x\tan\theta + \frac{z_{n,\theta}(x)}{\sqrt{k}}, \\ \tfrac{n}{k}Q_{2n}\left(\tfrac{k}{n}y_p\left(\tfrac{n}{k}\Gamma_{1n}\left(\tfrac{k}{n}x\right)\right)\right) &= y_p(x) + \frac{s_{n,p}(x)}{\sqrt{k}}, \end{aligned}$$

where

$$z_{n,\theta}(x) := w_{1n}(x)\tan\theta + v_{2n}\left(x\tan\theta + \frac{w_{1n}(x)\tan\theta}{\sqrt{k}}\right), \tag{23}$$

$$s_{n,p}(x) := \sqrt{k}\left\{y_p\left(x+\frac{w_{1n}(x)}{\sqrt{k}}\right) - y_p(x)\right\} + v_{2n}\left(y_p\left(x+\frac{w_{1n}(x)}{\sqrt{k}}\right)\right). \tag{24}$$

Now set, for any $m \in \{0, 1, 2, \ldots, 1/\Delta - 1\}$, with the convention that $0/0 = 0$,

$$V^{+}_{m,\Delta,\theta} := \sup_{x \in I_\Delta(m,\theta)} \frac{z_{n,\theta}(x) \wedge \Big(s_{n,p}(x) + \sqrt{k}(y_p(x) - x\tan\theta)\Big)}{(x\tan\theta)^{1/16}}$$

$$V^{-}_{m,\Delta,\theta} := \inf_{x \in I_\Delta(m,\theta)} \frac{z_{n,\theta}(x) \wedge \Big(s_{n,p}(x) + \sqrt{k}(y_p(x) - x\tan\theta)\Big)}{(x\tan\theta)^{1/16}},$$

where the intervals $I_\Delta(m,\theta)$ are defined by

$$I_\Delta(m,\theta) := \begin{cases} [m\Delta x_p(\theta), (m+1)\Delta x_p(\theta)] & \text{if } \theta \in (0, \pi/4], \\ [0,\infty) & \text{if } \theta = 0 \text{ and } m = 0, \\ \emptyset & \text{if } \theta = 0 \text{ and } m > 0. \end{cases}$$

For fixed θ, these intervals cover $[0, x_p(\theta)]$, on which $y_p(x) \geq x\tan\theta$. It follows that on an event whose probability tends to one, $V^{\pm}_{m,\Delta,\theta}$ is equal to the same expression as in its definition but with the numerator simplified to $z_{n,\theta}(x)$. In the following, we work with these simplified definitions. Moreover, we have the weak convergence

$$\left\{ \frac{|z_{n,\theta}(x)|}{(x\tan\theta)^{1/16}} : \theta \in [0,\pi/4], x \in [0, x_p(\theta)] \right\} \rightsquigarrow \left\{ \frac{W_1(x)\tan\theta - W_2(x\tan\theta)}{(x\tan\theta)^{1/16}} : \theta \in [0,\pi/4], x \in [0, x_p(\theta)] \right\}, \quad (25)$$

in the Banach space of bounded functions on the indicated domain. To see (26), recall the definition of $z_{n,\theta}$ in (23) and note that for any $\theta \in [0, \pi/4]$,

$$\frac{z_{n,\theta}(x)}{(x\tan\theta)^{1/16}} = \frac{w_{1n}(x)}{x^{1/16}}(\tan\theta)^{15/16} + \frac{v_{2n}\left(x\tan\theta + \frac{w_{1n}(x)\tan\theta}{\sqrt{k}}\right)}{\left(x\tan\theta + \frac{w_{1n}(x)\tan\theta}{\sqrt{k}}\right)^{1/4}} \left((x\tan\theta)^{3/4} + \frac{(\tan\theta)^{3/4}}{\sqrt{k}} \frac{w_{1n}(x)}{x^{1/4}} \right)^{1/4}.$$

Using equations (19) and (20) together with the fact that $x\tan\theta \leq 2^{1/p}$ on the said domain yields (25). In particular, we have

$$\sup_{\theta \in [0,\pi/4]} \sup_{x \in [0, x_p(\theta)]} \frac{|z_{n,\theta}(x)|}{(x\tan\theta)^{1/16}} = \mathrm{O}_{\mathbb{P}}(1), \qquad n \to \infty. \quad (26)$$

Defining

$$M^{\pm}_{\Delta,\theta} := \bigcup_{m=0}^{\Delta-1} \left\{ (x,y) : x \in I_\Delta(m,\theta), 0 \leq y \leq x\tan\theta + \frac{V^{\pm}_{m,\Delta,\theta}}{\sqrt{k}}(x\tan\theta)^{1/16} \right\},$$

it is easy to see that for any $\theta \in [0, \pi/4]$ we have $M^-_{\Delta,\theta} \subseteq \hat{C}_{\theta,p,1} \subseteq M^+_{\Delta,\theta}$ and thus

$$V^-_{n,\Delta,1}(\theta) - R_{n,\Delta,1}(\theta) \leq V_{n,p,1}(\theta) \leq V^+_{n,\Delta,1}(\theta) + R_{n,\Delta,1}(\theta), \tag{27}$$

where

$$V^\pm_{n,\Delta,1}(\theta) := \sqrt{k}\left\{\tfrac{n}{k} P_n\left(\tfrac{k}{n} M^\pm_{\Delta,\theta}\right) - \tfrac{n}{k} P\left(\tfrac{k}{n} M^\pm_{\Delta,\theta}\right)\right\}, \tag{28}$$

$$R_{n,\Delta,1}(\theta) := \sqrt{k}\tfrac{n}{k} P\left(\tfrac{k}{n}(M^+_{\Delta,\theta} \setminus M^-_{\Delta,\theta})\right). \tag{29}$$

To this end, observe that, due to Assumption 2 and the behavior of the marginal tail empirical and quantile processes,

$$\sup_{\theta\in[0,\pi/4]} \left| R_{n,\Delta,1}(\theta) - \sqrt{k}\Lambda(M^+_{\Delta,\theta} \setminus M^-_{\Delta,\theta}) \right| = o_{\mathbb{P}}(1).$$

Similar computations as in [6, Section 3.1.1] show that

$$\sqrt{k}\Lambda(M^+_{\Delta,\theta} \setminus M^-_{\Delta,\theta}) \leq 16 \times 2^{1/p} \sup_{y\geq 0} \lambda(1,y) \max_{m=0,\dots,\Delta-1} (V^+_{m,\Delta,\theta} - V^-_{m,\Delta,\theta}),$$

where $\sup_{y\geq 0} \lambda(1,y) < \infty$ since λ is continuous on $[0,\infty)^2 \setminus \{(0,0)\}$ thanks to Assumption 1 and $\lim_{y\to\infty} \lambda(1,y) = 0$ (the latter limit following from the homogeneity of λ in (7)). Since $V^+_{m,\Delta,0} = V^-_{m,\Delta,0} = 0$ by convention for any value of m and Δ, we deduce from (25) that, for every $\epsilon > 0$,

$$\lim_{\Delta\to 0} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{\theta\in[0,\pi/4]} \max_{m=0,\dots,\Delta-1} (V^+_{m,\Delta,\theta} - V^-_{m,\Delta,\theta}) > \epsilon \right) = 0.$$

Collecting the latter three equations, we conclude that, for every $\epsilon > 0$,

$$\lim_{\Delta\to 0} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{\theta\in[0,\pi/4]} R_{n,\Delta,1}(\theta) > \epsilon \right) = 0. \tag{30}$$

Furthermore, since (26) implies that on an event with probability tending to one we have $M^\pm_{\Delta,\theta} \in \mathcal{A}$ for all $\theta \in [0, \pi/4]$, it follows that for all $\epsilon, \delta > 0$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\left(\max_{\sigma\in\{-,+\}} \sup_{\theta\in[0,\pi/4]} \left| W_{n,k}(M^\sigma_{\Delta,\theta}) - W_{n,k}(C_{p,\theta,1}) \right| > \epsilon \right) \\
&\qquad \leq \limsup_{n\to\infty} \mathbb{P}\left(\max_{\sigma\in\{-,+\}} \sup_{\theta\in[0,\pi/4]} \rho(M^\sigma_{\Delta,\theta}, C_{p,\theta,1}) \geq \delta \right) \\
&\qquad\qquad + \limsup_{n\to\infty} \mathbb{P}\left(\sup_{A,A'\in\mathcal{A},\rho(A,A')<\delta} \left| W_{n,k}(A) - W_{n,k}(A') \right| > \epsilon \right).
\end{aligned} \tag{31}$$

Standard computations lead to

$$\sup_{\theta\in[0,\pi/4]} \rho(M^{\pm}_{\Delta,\theta}, C_{p,\theta,1}) \\ \leq \frac{16\times 2^{1/p}}{\sqrt{k}} \sup_{y\geq 0}\lambda(1,y) \sup_{\theta\in[0,\pi/4]} \max_{m=0,\dots,\Delta-1} |V^{\pm}_{m,\Delta,\theta}| = \mathrm{o}_{\mathbb{P}}(1),$$

so that the first limsup on the right hand side of (31) is zero. The second limsup can be made arbitrary close to zero as $\delta \downarrow 0$ by (15). Combining everything leads to (22) for $i = 1$: for $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{\theta\in[0,\pi/4]} |V_{n,p,1}(\theta) - W_{n,k}(C_{p,\theta,1})| > \epsilon\right) \\ \leq \sum_{\sigma\in\{-,+\}} \mathbb{P}\left(\sup_{\theta\in[0,\pi/4]} \left|V^{\sigma}_{n,\Delta,1}(\theta) - W_{n,k}(C_{p,\theta,1})\right| > \epsilon/2\right) \\ + 2\,\mathbb{P}\left(\sup_{\theta\in[0,\pi/4]} R_{n,\Delta,1}(\theta) > \epsilon/2\right);$$

the first term on the right-hand side vanishes as $n\to\infty$, while the limsup over $n\to\infty$ of the second term on the right-hand side can be made arbitrarily small as $\Delta\to 0$.

Let us now consider $i = 2$ in (22). Similarly to $i = 1$, it will be useful to note that for any $\theta\in[0,\pi/4]$,

$$\hat{C}_{p,\theta,2} = \left\{(x,y) : x \geq x_p(\theta), 0 \leq y \leq \left(x\tan\theta + \frac{z_{n,\theta(x)}}{\sqrt{k}}\right) \wedge \left(y_p(x) + \frac{s_{n,p}(x)}{\sqrt{k}}\right)\right\},$$

with $z_{n,\theta}$ and $s_{n,p}$ the processes defined in (23) and (24) respectively. For each $m \in \{0, 1, 2, \dots, 1/\Delta - 1\}$, we let

$$S^{+}_{m,\Delta,\theta} := \sup_{x\in J_\Delta(m)} \frac{s_{n,p}(x) \wedge \left(z_{n,\theta}(x) + \sqrt{k}(x\tan\theta - y_p(x))\right)}{y_p(x)}$$

$$S^{-}_{m,\Delta,\theta} := \inf_{x\in J_\Delta(m)} \frac{s_{n,p}(x) \wedge \left(z_{n,\theta}(x) + \sqrt{k}(x\tan\theta - y_p(x))\right)}{y_p(x)},$$

where the intervals $J_\Delta(m)$ are defined by

$$J_\Delta(m) := [y_p(1 + (2^{1/p} - 1)(m+1)\Delta), y_p(1 + (2^{1/p} - 1)m\Delta)].$$

These intervals cover $[2^{1/p}, \infty]$, on which $x\tan\theta \geq x_p(\theta)$ for any $\theta\in[0,\pi/4]$. It follows that on an event whose probability tends to one, $S^{\pm}_{m,\Delta,\theta}$ is equal to the same expression as in its definition but with the numerator simplified to $s_{n,p}(x)$. In the following, we work with these simplified definitions. Note that, by (19) and (20), the mean value theorem and the extended continuous mapping theorem [11, Theorem 1.11.1]

$$\left\{s_{n,p}(x) : x \in [2^{1/p}, \infty)\right\} \rightsquigarrow \left\{W_1(x)y_p'(x) - W_2(y_p(x)) : x \in [2^{1/p}, \infty)\right\}, \tag{32}$$

in $\ell^\infty([2^{1/p}, \infty))$. In particular, since $y_p(x) \geq 1$ when $x \geq 2^{1/p}$, we have

$$\sup_{x\in[2^{1/p},\infty]} \frac{|s_{n,p}(x)|}{y_p(x)} = \mathrm{O}_{\mathbb{P}}(1). \tag{33}$$

Defining

$$N^{\pm}_{\Delta,\theta} := \bigcup_{m=0}^{\Delta-1} \left\{(x, y) : x \geq x_p(\theta), x \in J_\Delta(m), 0 \leq y \leq y_p(x)\left(1 + \frac{S^{\pm}_{m,\Delta,\theta}}{\sqrt{k}}\right)\right\},$$

we see that for any $\theta \in [0, \pi/4]$ we have $N^-_{\Delta,\theta} \subseteq \hat{C}_{\theta,p,2} \subseteq N^+_{\Delta,\theta}$ and thus

$$V^-_{n,\Delta,2}(\theta) - R_{n,\Delta,2}(\theta) \leq V_{n,p,2}(\theta) \leq V^+_{n,\Delta,2}(\theta) + R_{n,\Delta,2}(\theta), \tag{34}$$

where

$$V^{\pm}_{n,\Delta,2}(\theta) := \sqrt{k}\left\{\tfrac{n}{k}P_n\left(\tfrac{k}{n}N^{\pm}_{\Delta,\theta}\right) - \tfrac{n}{k}P\left(\tfrac{k}{n}N^{\pm}_{\Delta,\theta}\right)\right\}, \tag{35}$$

$$R_{n,\Delta,2}(\theta) := \sqrt{k}\tfrac{n}{k}P\left(\tfrac{k}{n}(N^+_{\Delta,\theta} \setminus N^-_{\Delta,\theta})\right). \tag{36}$$

Due to Assumption 2 and the behavior of the marginal tail empirical and quantile processes,

$$\sup_{\theta\in[0,\pi/4]} \left|R_{n,\Delta,2}(\theta) - \sqrt{k}\Lambda(N^+_{\Delta,\theta} \setminus N^-_{\Delta,\theta})\right| = \mathrm{o}_{\mathbb{P}}(1).$$

Computations in [7, Paragraph A.1] show that with probability tending to one

$$\sup_{\theta\in[0,\pi/4]} \sqrt{k}\Lambda(N^+_{\Delta,\theta} \setminus N^-_{\Delta,\theta}) \leq 3 \sup_{\theta\in[0,\pi/4]} \max_{m=0,\ldots,\Delta-1} (S^+_{m,\Delta,\theta} - S^-_{m,\Delta,\theta}).$$

In view of (32), we have, for any $\epsilon > 0$,

$$\lim_{\Delta\to 0} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{\theta\in[0,\pi/4]} \max_{m=0,\ldots,\Delta-1} (S^+_{m,\Delta,\theta} - S^-_{m,\Delta,\theta}) > \epsilon\right) = 0,$$

which implies (30) with $R_{n,\Delta,1}$ replaced by $R_{n,\Delta,2}$. By a similar argument as for $i = 1$, we have

$$\sup_{\theta\in[0,\pi/4]} \rho(N^{\pm}_{\Delta,\theta}, C_{p,\theta,2}) \leq \frac{3}{\sqrt{k}} \sup_{\theta\in[0,\pi/4]} \max_{m=0,\ldots,\Delta-1} |S^{\pm}_{m,\Delta,\theta}| = \mathrm{o}_{\mathbb{P}}(1),$$

and thus (22) holds for $i = 2$.

The bias term $r_{n,p}(\theta)$. Note that all sets $\hat{C}_{p,\theta}$ for $\theta \in [0, \pi/4]$ are contained in the set $\hat{C}_{p,\pi/4}$ and, with probability tending to one, the latter is contained in the set $\{(x, y) : 0 \leq x \wedge y \leq M\}$ for $M = M(p) = 2^{1/p}$. It follows that, with probability tending to one,

$$\sup_{\theta \in [0,\pi/4]} |r_{n,p}(\theta)| \leq \sqrt{k} \int_{0 \leq x \wedge y \leq M} \left| \tfrac{k}{n} c\left(\tfrac{k}{n}x, \tfrac{k}{n}y\right) - \lambda(x, y) \right| \mathrm{d}(x, y).$$

By Assumption 2 and a change of variables, the right-hand side tends to zero. It follows that

$$\sup_{\theta \in [0,\pi/4]} |r_{n,p}(\theta)| = \mathrm{o}_{\mathbb{P}}(1).$$

The random set term $Z_{n,p}(\theta)$. Similarly as for the stochastic term, we work on

$$Z_{n,p,1}(\theta) := \sqrt{k} \left(\Lambda\left(\hat{C}_{p,\theta,1}\right) - \Lambda\left(C_{p,\theta,1}\right) \right) \quad \text{and}$$
$$Z_{n,p,2}(\theta) := \sqrt{k} \left(\Lambda\left(\hat{C}_{p,\theta,2}\right) - \Lambda\left(C_{p,\theta,2}\right) \right)$$

separately. In particular, our aim is to show that, as $n \to \infty$,

$$\sup_{\theta \in [0,\pi/4]} \left| Z_{n,p,1}(\theta) - \int_0^{x_p(\theta)} \lambda(x, x\tan\theta) \left\{ w_{1n}(x)\tan\theta - w_{2n}(x\tan\theta) \right\} \mathrm{d}x \right| = \mathrm{o}_{\mathbb{P}}(1) \quad (37)$$

and, for $p < \infty$ (since this is the most involved case),

$$\sup_{\theta \in [0,\pi/4]} \left| Z_{n,p,2}(\theta) - \int_{x_p(\theta)}^{\infty} \lambda(x, y_p(x)) \left\{ y_p'(x) w_{1n}(x) - w_{2n}(y_p(x)) \right\} \mathrm{d}x \right| = \mathrm{o}_{\mathbb{P}}(1) \quad (38)$$

We start with (37). Since, as pointed out earlier, $y_p(x) \geq x\tan\theta$ on $[0, x_p(\theta)]$, we have, on an event with probability tending to one,

$$Z_{n,p,1}(\theta) = \sqrt{k} \int_0^{x_p(\theta)} \int_{x\tan\theta}^{x\tan\theta + \frac{z_{n,\theta}(x)}{\sqrt{k}}} \lambda(x, y) \, \mathrm{d}y \, \mathrm{d}x,$$

where $z_{n,\theta}$ was defined in (23). Changing variables $y(z) = x\tan\theta + z/\sqrt{k}$ and making use of the mean value theorem leads to

$$Z_{n,p,1}(\theta) = \int_0^{x_p(\theta)} \lambda\left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}}\right) z_{n,\theta}(x) \, \mathrm{d}x,$$

where $y(x, \theta)$ is between 0 and $z_{n,\theta}(x)$. Observing that

$$\begin{aligned}
&\lambda\left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}}\right) z_{n,\theta}(x) - \lambda(x, x\tan\theta) \left\{ w_{1n}(x)\tan\theta - w_{2n}(x\tan\theta) \right\} \\
&\quad = \left\{ \lambda\left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}}\right) - \lambda(x, x\tan\theta) \right\} z_{n,\theta}(x) \\
&\qquad + \left[z_{n,\theta}(x) - \left\{ w_{1n}(x)\tan\theta - w_{2n}(x\tan\theta) \right\} \right] \lambda(x, x\tan\theta),
\end{aligned}$$

we see that sufficient conditions for relation (37) to hold are provided by

$$\sup_{\theta\in[0,\pi/4]} \int_0^{x_p(\theta)} \left| \lambda \left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}} \right) - \lambda(x, x\tan\theta) \right| |z_{n,\theta}(x)| \, \mathrm{d}x = \mathrm{o}_{\mathbb{P}}(1) \quad (39)$$

$$\sup_{\theta\in[0,\pi/4]} \int_0^{x_p(\theta)} \lambda(x, x\tan\theta) \, |z_{n,\theta}(x) - \{w_{1n}(x)\tan\theta - w_{2n}(x\tan\theta)\}| \, \mathrm{d}x = \mathrm{o}_{\mathbb{P}}(1). \quad (40)$$

Let us first consider (39). Since we will need uniform continuity of λ, we will need to work on a compact domain. Hence, for some $M > 1$, we consider separately the case $\theta \in [0, \arctan(1/M)]$, on which we have no control on the value of $x_p(\theta)$ which can be arbitrary large, and the case $\theta \in [\arctan(1/M), \pi/4]$ on which it is easy to see that $x_p(\theta) \le 1 + M$. It will be useful to note that as $n \to \infty$, by (26),

$$\|z_n\| := \sup_{\theta\in[0,\pi/4]} \sup_{x\in[0,x_p(\theta)]} |z_{n,\theta}(x)| = \mathrm{O}_{\mathbb{P}}(1). \quad (41)$$

Consider the supremum of (39) on the region $\theta \in [\arctan(1/M), \pi/4]$ first, for which the domain of integration is included in $x \in [0, 1 + M]$. We will split the domain of integration further on $x \in [0, \delta]$ and $x \in [\delta, 1 + M]$ for some $\delta > 0$. The latter case will be considered first. Observe that, by (41),

$$\sup_{\theta\in[\arctan(1/M),\pi/4]} \sup_{x\in[\delta,1+M]} \left\| \left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}} \right) - (x, x\tan\theta) \right\|_1 \le \frac{\|z_{n,\theta}\|}{\sqrt{k}} = \mathrm{o}_{\mathbb{P}}(1),$$

which, by uniform continuity of λ on the compact set $[\delta, 1+M] \times [\delta/(2M), 2+M]$ implies that

$$\sup_{\theta\in[\arctan(1/M),\pi/4]} \sup_{x\in[\delta,1+M]} \left| \lambda \left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}} \right) - \lambda(x, x\tan\theta) \right| = \mathrm{o}_{\mathbb{P}}(1).$$

Consequently,

$$\sup_{\theta\in[\arctan(1/M),\pi/4]} \int_\delta^{x_p(\theta)} \left| \lambda \left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}} \right) - \lambda(x, x\tan\theta) \right| |z_{n,\theta}(x)| \, \mathrm{d}x = \mathrm{o}_{\mathbb{P}}(1).$$

The next step is to consider the same region for θ but $x \in [0, \delta]$. The idea is then to bound the integral as follows: using the homogeneity of λ in (7), we have

$$\sup_{\theta\in[\arctan(1/M),\pi/4]} \int_0^{\delta} \left| \lambda \left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}} \right) - \lambda(x, x\tan\theta) \right| |z_{n,\theta}(x)| \, \mathrm{d}x \le 32 \sup_{y\ge0} \lambda(1,y) \sup_{\theta\in[\arctan(1/M),\pi/4]} \sup_{x\in[0,\delta]} \frac{|z_{n,\theta}(x)|}{(x\tan\theta)^{1/16}} (\tan\theta)^{1/16} \delta^{1/16}.$$

Using (26) and picking δ small enough permits to conclude and to show that (39) is verified on the region $\theta \in [\arctan(1/M), \pi/4]$. For the region $\theta \in [0, \arctan(1/M)]$, we split again the domain of integration in two parts : $x \in [0,1]$ and $x \in [1, x_p(\theta)]$. We start with the region near zero. Similar computations as what we have done previously lead to

$$\sup_{\theta\in[0,\arctan(1/M)]} \int_0^1 \left|\lambda\left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}}\right) - \lambda(x, x\tan\theta)\right| |z_{n,\theta}(x)| \,\mathrm{d}x$$
$$\leq 32 \sup_{y\geq 0} \lambda(1,y) \sup_{\theta\in[0,\arctan(1/M)]} \sup_{x\in[0,1]} \frac{|z_{n,\theta}(x)|}{(x\tan\theta)^{1/16}} \times (\tan\theta)^{1/16}.$$

Picking $M > 1$ large enough permits to conclude for this region. For the region $x \in [1, x_p(\theta)]$, using the homogeneity of λ in (7) and factoring out $(x\tan\theta)^{1/16}$, we can show that

$$\sup_{\theta\in[0,\arctan(1/M)]} \int_1^{x_p(\theta)} \left|\lambda\left(x, x\tan\theta + \frac{y(x,\theta)}{\sqrt{k}}\right) - \lambda(x, x\tan\theta)\right| |z_{n,\theta}(x)| \,\mathrm{d}x$$
$$\leq 64 \sup_{\theta\in[0,\arctan(1/M)]} \sup_{x\in[1,x_p(\theta)]} \frac{|z_{n,\theta}(x)|}{(x\tan\theta)^{1/16}}$$
$$\times \sup_{\theta\in[0,\arctan(1/M)]} \sup_{|z|\leq\|z_n\|} \lambda(1, \tan\theta + z/\sqrt{k}).$$

Again using (41) and the fact that $\lim_{y\to 0}\lambda(1,y) = 0$ permits to conclude for this region too by picking $M > 1$ large enough. Combining everything, we have proven (39).

We now turn to (40). By arguments similar to those leading up to (25), we have

$$\sup_{\theta\in[0,\pi/4]} \sup_{x\in[0,x_p(\theta)]} \frac{|z_{n,\theta}(x) - \{w_{1n}(x)\tan\theta - w_{2n}(x\tan\theta)\}|}{(x\tan\theta)^{1/16}}$$
$$= \sup_{\theta\in[0,\pi/4]} \sup_{x\in[0,x_p(\theta)]} \frac{\left|v_{2n}\left(x\tan\theta + \frac{w_{1n}(x)\tan\theta}{\sqrt{k}}\right) + w_{2n}(x\tan\theta)\right|}{(x\tan\theta)^{1/16}} = o_{\mathbb{P}}(1).$$

It follows from the homogeneity (7) and a computation similar to the previous ones that (40) holds.

Finally, we show (38). Since $y_p(x) \leq x\tan\theta$ on $[x_p(\theta), \infty]$, we have, on an event with probability tending to one,

$$Z_{n,p,2}(\theta) = \sqrt{k}\int_{x_p(\theta)}^{\infty} \int_{y_p(x)}^{y_p(x)+s_{n,p}(x)/\sqrt{k}} \lambda(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

where $s_{n,p}$ was defined in (24). Changing variables $y(s) = y_p(x) + s/\sqrt{k}$ and making use of the mean value theorem leads to

$$Z_{n,p,2}(\theta) = \int_{x_p(\theta)}^{\infty} \lambda\left(x, y_p(x) + \frac{s(x,p)}{\sqrt{k}}\right) s_{n,p}(x)\,\mathrm{d}x,$$

where $s(x,p)$ is between 0 and $s_{n,p}(x)$. Observing that

$$\lambda\left(x, y_p(x)+\frac{s(x,p)}{\sqrt{k}}\right) s_{n,p}(x) - \lambda(x,y_p(x))\left\{y_p'(x)w_{1n}(x) - w_{2n}\left(y_p(x)\right)\right\}$$
$$= \left\{\lambda\left(x, y_p(x)+\frac{s(x,p)}{\sqrt{k}}\right) - \lambda(x,y_p(x))\right\} s_{n,p}(x)$$
$$+ \left[s_{n,p}(x) - \left\{y_p'(x)w_{1n}(x) - w_{2n}\left(y_p(x)\right)\right\}\right]\lambda(x,y_p(x)),$$

and that for any $\theta \in [0,\pi/4]$ and $1 \le p < \infty$ we have $x_p(\theta) \ge 2^{1/p}$, we see that sufficient conditions for (38) to hold are provided by

$$\int_{2^{1/p}}^{\infty} \left|\lambda\left(x, y_p(x)+\frac{s(x,p)}{\sqrt{k}}\right) - \lambda(x,y_p(x))\right| |s_{n,p}(x)|\,\mathrm{d}x = \mathrm{o}_{\mathbb{P}}(1), \tag{42}$$

$$\int_{2^{1/p}}^{\infty} \lambda(x,y_p(x))\left|s_{n,p}(x) - \left\{y_p'(x)w_{1n}(x) - w_{2n}\left(y_p(x)\right)\right\}\right|\,\mathrm{d}x = \mathrm{o}_{\mathbb{P}}(1). \tag{43}$$

Let us first consider (42). Since we will need uniform continuity of λ again, we will need to work on a compact domain. Hence, for some $M > 1$, we consider separately the cases $x \in [2^{1/p}, M]$ and $x \in [M,\infty)$. It will be useful to note that, by (32), as $n \to \infty$,

$$\|s_{n,p}\| := \sup_{x\in[2^{1/p},\infty)} |s_{n,p}(x)| = \mathrm{O}_{\mathbb{P}}(1). \tag{44}$$

and, by the mean value theorem, for each $x \in [2^{1/p},\infty)$,

$$s_{n,p}(x) = y_p'(c_n(x))w_{1n}(x) + v_{2n}\left(y_p\left(x+\frac{w_{1n}(x)}{\sqrt{k}}\right)\right), \tag{45}$$

where $c_n(x)$ is between x and $x + w_{1n}(x)/\sqrt{k}$ and $y_p'(x) = -1/(x^p-1)^{1+1/p}$. Since

$$\sup_{x\in[2^{1/p},\infty)} \left\|\left(x, y_p(x)+\frac{s(x,p)}{\sqrt{k}}\right) - (x,y_p(x))\right\|_1 \le \frac{\|s_{n,p}\|}{\sqrt{k}} = \mathrm{o}_{\mathbb{P}}(1),$$

it follows from the uniform continuity of λ on compact sets included in $[0,\infty)^2 \setminus \{(0,0)\}$ that for any $M > 1$,

$$\sup_{x\in[2^{1/p},M]} \left|\lambda\left(x, y_p(x)+\frac{s(x,p)}{\sqrt{k}}\right) - \lambda(x,y_p(x))\right| = \mathrm{o}_{\mathbb{P}}(1)$$

and (42) follows easily for the region $x \in [2^{1/p}, M]$. For the region $x \in [M,\infty]$, just observe that by (44),

$$\int_M^{\infty} \left|\lambda\left(x, y_p(x)+\frac{s(x,p)}{\sqrt{k}}\right) - \lambda(x,y_p(x))\right| |s_{n,p}(x)|\,\mathrm{d}x$$
$$\le \|s_{n,p}\|\left\{\int_M^{\infty} \lambda\left(x, y_p(x)+\frac{s(x,p)}{\sqrt{k}}\right)\mathrm{d}x + \int_M^{\infty}\lambda(x,y_p(x))\,\mathrm{d}x\right\}. \tag{46}$$

We first consider the second integral which is easier to deal with. By homogeneity (7) and a change of variables,

$$\begin{aligned}\int_M^\infty \lambda(x, y_p(x))\,\mathrm{d}x &= \int_M^\infty \lambda((x^p-1)^{1/p}, 1)(y_p(x))^{-1}\,\mathrm{d}x \\ &\le \int_{(M^p-1)^{1/p}}^\infty \lambda(u,1)\left(\frac{u^p}{u^p+1}\right)^{1-1/p}\,\mathrm{d}u \\ &\le \int_{(M^p-1)^{1/p}}^\infty \lambda(u,1)\,\mathrm{d}u.\end{aligned}$$

Since $\int_0^\infty \lambda(u,1)\,\mathrm{d}u = 1$, we may apply the dominated convergence theorem to ensure that the above integral can be made as small as desired provided we pick $M > 1$ large enough. Now consider the second integral on the right-hand side of (46). Since $0 < s(x,p) < s_{n,p}(x)$ or $0 > s(x,p) > s_{n,p}(x)$, we have, by (32), for all $M > 2^{1/p}$,

$$\int_M^\infty \lambda\left(x, y_p(x) + \frac{s(x,p)}{\sqrt{k}}\right)\,\mathrm{d}x = \int_M^\infty \lambda(x, y_p(x))\,\mathrm{d}x + \mathrm{o}_{\mathbb{P}}(1),$$

and since the integral on the right-hand side vanishes as $M \to \infty$, we have shown (42).

Let us now consider (43). Using (17), (18), the continuity of y_p and its derivative y_p', we observe that,

$$\begin{aligned}&\sup_{x\in[2^{1/p},\infty)} \left|s_{n,p}(x) - \left\{y_p'(x)w_{1n}(x) - w_{2n}(y_p(x))\right\}\right| \\ &\le \sup_{x\in[2^{1/p},\infty)} \left|\left(y_p'(c_n(x)) - y_p'(x)\right) w_{1n}(x)\right| \\ &\quad + \sup_{x\in[2^{1/p},\infty)} \left|v_{2n}\left(y_p\left(x + \frac{w_{1n}(x)}{\sqrt{k}}\right)\right) + w_{2n}(y_p(x))\right| \\ &= \mathrm{o}_{\mathbb{P}}(1).\end{aligned}$$

Hence by integrability of the function $x \in [2^{1/p}, \infty) \mapsto \lambda(x, y_p(x))$,

$$\int_{2^{1/p}}^\infty \lambda(x, y_p(x)) \left|s_{n,p}(x) - \left\{y_p'(x)w_{1n}(x) - w_{2n}\left(y_p(x)\right)\right\}\right|\,\mathrm{d}x = \mathrm{o}_{\mathbb{P}}(1),$$

so that (43) holds.

Conclusion of the proof of Theorem 2. Adding the asymptotic expansions of the three terms in (9) permits to conclude the proof for $\theta \in [0, \pi/4]$. By the symmetry argument in Section 2.1, we can extend the conclusion to all $\theta \in [0, \pi/2]$. The proof of Theorem 2 is complete.

3 The quantile process of the angular probability measure

Many statistics of interest, for example, in goodness-of-fit testing based on the Wasserstein distance, do not involve the original process directly, but rather its quantile version.

To find the limit distribution of such a test statistic, one needs an expansion of the quantile process. Since the total mass of Φ_p is unknown, it is convenient to consider the *angular probability measure* and its empirical counterpart,

$$Q_p(\cdot) := \frac{\Phi_p(\cdot)}{\Phi_p(\pi/2)} \qquad \text{and} \qquad \widehat{Q}_p(\cdot) := \frac{\widehat{\Phi}_p(\cdot)}{\widehat{\Phi}_p(\pi/2)},$$

respectively. The asymptotic expansion of $\widehat{Q}_p$ follows from the one of $\widehat{\Phi}_p$ in Theorem 2 and Slutsky's lemma. The proof of the following corollary is standard and omitted for brevity.

Corollary 1 (Asymptotic expansion of the empirical angular probability measure)**.** *Under the same assumptions as Theorem 2, we have, as $n \to \infty$,*

$$\sup_{\theta\in[0,\pi/2]} \left| \sqrt{k}\left(\widehat{Q}_p(\theta)) - Q_p(\theta)\right) - \frac{E_{n,p}(\theta)\Phi_p(\frac{\pi}{2}) - \Phi_p(\theta)E_{n,p}(\frac{\pi}{2})}{\Phi_p(\frac{\pi}{2})^2} \right| = o_{\mathbb{P}}(1).$$

The expansion of the quantile process $\{\sqrt{k}(\widehat{Q}_p^{-1}(u) - Q_p^{-1}(u)) : u \in (0,1)\}$ is a corollary of the Functional Delta Method.

Corollary 2 (Asymptotic expansion of the quantile process)**.** *Under the same assumptions as Theorem 2 and the side assumption that $\varphi_p > 0$ on $(0,\pi/2)$, we have, as $n \to \infty$,*

$$\sup_{u\in(0,1)} \left| \sqrt{k}\left(\widehat{Q}_p^{-1}(u) - Q_p^{-1}(u)\right) - \frac{\Phi_p(Q_p^{-1}(u))E_{n,p}(\frac{\pi}{2}) - E_{n,p}(Q_p^{-1}(u))\Phi_p(\frac{\pi}{2})}{\varphi_p(Q_p^{-1}(u))\Phi_p(\frac{\pi}{2})} \right| = o_{\mathbb{P}}(1).$$

Proof. Let

$$\psi : F \in \mathbb{D}_\psi \subset \ell^\infty([0,\pi/2]) \mapsto \phi(F) := F^{-1} \in \ell^\infty\big((0,1)\big)$$

be the generalized inverse map, where $\mathbb{D}_\psi$ is the set of cumulative distribution functions supported on $[0,\pi/2]$, vanishing at 0, which are continuously differentiable on their domain with strictly positive derivative. By [10, Lemma 21.4], ψ is Hadamard differentiable tangentially to $C([0,\pi/2])$, the set of continuous functions on $[0,\pi/2]$. Furthermore, its derivative at $F \in \mathbb{D}_\psi$ in the direction $\gamma \in C([0,\pi/2])$ is

$$\psi'_F(\gamma) = -\frac{\gamma}{f} \circ F^{-1},$$

where f is the derivative of F. Consequently, the map ψ is Hadamard differentiable at $Q_p \in \mathbb{D}_\psi$, and its derivative, ψ'_{Q_p}, is defined and continuous on the whole space $\ell^\infty([0,\pi/2])$. Therefore, it follows from [10, Theorem 20.8] that

$$\sup_{u\in(0,1)} \left| \sqrt{k}\left(\widehat{Q}_p^{-1}(u) - Q_p^{-1}(u)\right) - \psi'_{Q_p}\left(\sqrt{k}\big(\widehat{Q}_p(u) - Q_p(u)\big)\right) \right| = o_{\mathbb{P}}(1),$$

as $n \to \infty$. Evaluating the derivative yields the desired result. ☐

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References

[1] Beirlant, J., Goegebeur, Y., Segers, J., and Teugels, J. *Statistics of Extremes: Theory and Applications.* Wiley Series in Probability and Statistics, 2005.

[2] Clémençon, S., Jalalzaï, H., Lhaut, S., Sabourin, A., and Segers, J. Concentration bounds for the empirical angular measure with statistical learning applications. *Bernoulli* (2022). forthcoming.

[3] Csörgo, M., Csörgo, S., Horvath, L., and Mason, D. Weighted empirical and quantile processes. *The Annals of Probability 14*, 1 (1986), 31–85.

[4] de Haan, L., and Ferreira, A. *Extreme Value Theory: An Introduction.* Springer Series in Operations Research and Financial Engineering. Springer-Verlag New York, 2006.

[5] Einmahl, J. H. J. Poisson and Gaussian approximation of weighted local empirical processes. *Stochastic Processes and their Applications 70* (1997), 31–58.

[6] Einmahl, J. H. J., de Haan, L., and Piterbarg, V. I. Nonparametric estimation of the spectral measure of an extreme value distribution. *The Annals of Statistics 29*, 5 (2001), 1401–1423.

[7] Einmahl, J. H. J., and Segers, J. Maximum empirical likelihood estimation of the spectral measure of an extreme-value distribution. *The Annals of Statistics 37*, 5B (2009), 2953–2989.

[8] Hult, H., and Lindskog, F. Regular variation for measures on metric spaces. *Publications de l'Institut Mathematique*, 94 (2006), 121–140.

[9] Resnick, S. I. *Extreme Values, Regular Variation, and Point Processes.* Springer Series in Operations Research and Financial Engineering. Springer-Verlag, New York, 1987.

[10] van der Vaart, A. *Asymptotic Statistics.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 1998.

[11] van der Vaart, A., and Wellner, J. *Weak Convergence and Empirical Processes.* Springer Series in Statistics. Springer-Verlag New York, 1996.

Appendix: Link between Λ and Φ_p densities

For any $x, y > 0$, we have

$$\lambda(x,y) = \frac{\partial^2}{\partial x \partial y}\Lambda\left((0,x] \times (0,y]\right) = \frac{\partial^2}{\partial x \partial y}\nu\left([x^{-1},\infty) \times [y^{-1},\infty)\right),$$

where ν factorizes in polar coordinates under T in (4), i.e., $T\#\nu = \nu_{-1} \otimes \Phi_p$, so that

$$\begin{aligned}
&\nu\left([x^{-1},\infty) \times [y^{-1},\infty)\right)\\
&= (\nu_{-1} \otimes \Phi_p)\left(\left\{T(u,v) : u \geq x^{-1}, v \geq y^{-1}\right\}\right)\\
&= (\nu_{-1} \otimes \Phi_p)\left(\left\{(r,\theta) : \frac{r\sin\theta}{\|(\sin\theta,\cos\theta)\|_p} \geq x^{-1}, \frac{r\cos\theta}{\|(\sin\theta,\cos\theta)\|_p} \geq y^{-1}\right\}\right)\\
&= \int_0^{\pi/2}\left(\frac{x\sin\theta}{\|(\sin\theta,\cos\theta)\|_p} \wedge \frac{y\cos\theta}{\|(\sin\theta,\cos\theta)\|_p}\right)\varphi_p(\theta)\,\mathrm{d}\theta\\
&= x\int_0^{\arctan(y/x)}\frac{\sin\theta}{\|(\sin\theta,\cos\theta)\|_p}\varphi_p(\theta)\,\mathrm{d}\theta\\
&\qquad + y\int_{\arctan(y/x)}^{\pi/2}\frac{\cos\theta}{\|(\sin\theta,\cos\theta)\|_p}\varphi_p(\theta)\,\mathrm{d}\theta.
\end{aligned}$$

Differentiating this expression with respect to x and y using the Leibniz differentiation rule yields the desired formula (6) for λ, showing that its continuity on $(0,\infty)^2$ is equivalent to the one of φ_p on $(0,\pi/2)$.