Nonlinear Schrödinger equation and Schrödinger-Poisson system in the semiclassical limit

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Contents

Introduction

This thesis is devoted, firstly, to semilinear elliptic equations of the form

$$\text{(NLS)} \qquad -\varepsilon^2 \Delta u + V(x)u = |u|^{p-1}\, u \qquad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a small parameter, Δ denotes the Laplace operator, $V : \mathbb{R}^N \to \mathbb{R}^+$ is called the potential and $p > 1$. In this introduction, unless otherwise stated, p will always be assumed subcritical, i.e. $p < (N+2)/(N-2)$ if $N \geqslant 3$. The equation (NLS) is usually referred to as the (stationary) nonlinear Schrödinger equation. The second focus of the thesis will be a variant of (NLS), the nonlinear Schrödinger-Poisson system:

$$\text{(NLSP)} \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi u = |u|^{p-1}\, u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2. \end{cases}$$

Both problems have been the subject of extensive research in the last decades. This popularity comes on the one hand from the numerous and important applications of (NLS) and (NLSP), mostly in physics, and on the other hand from the fact that they give rise to interesting mathematical problems.

Physical motivations

Various physical phenomena can be modelled by the equation

$$\text{(NLSE)} \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + W(x)\psi - |\psi|^{p-1}\, \psi, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

which is often called the nonlinear Schrödinger equation because of its formal resemblance to the usual (linear) Schrödinger equation. However, it does not necessarily describe the time evolution of a quantum state. The equation (NLSE) appears for example in the theory of Bose-Einstein condensates, where it is also called the Gross-Pitaevskii equation (with $p = 3$). In this case, the nonlinear term models the interaction between the particles and $\hbar$ represents the Planck constant. In other contexts, equations of the form (NLSE) arise in the description of nonlinear waves, for example light waves propagating in a medium whose index of refraction depends on the wave amplitude, or water waves [141].

Heuristically, the nonlinear term in (NLSE) has a focusing effect that compensates the dispersive effect of the linear terms. Therefore, it can be expected that (NLSE) has solitary waves as solutions, namely solutions whose energy travels as localized packets and which preserve their form under perturbations. These solitary waves behave like particles. When the nonlinear term in (NLSE) has a positive sign, the equation is called defocusing. See [142] for more information about the evolution equation (NLSE).

The simplest solutions of (NLSE) are the standing waves, which are solutions of the form

$$\psi(t,x) = e^{-iEt/\hbar}u(x),$$

where E is the energy of the wave. We compute that ψ is a standing wave solution of (NLSE) if and only if u is a solution of (NLS) with $\varepsilon^2 = \hbar^2/2m$ and $V(x) = W(x) - E$.

The correspondence principle states that quantum mechanics reduces to classical mechanics in the semiclassical limit, i.e. as $\hbar \to 0$. Thus we can expect that the solutions of (NLS) concentrate around a classical equilibrium as $\varepsilon \to 0$. We can guess that this classical equilibrium is a critical point of the potential V. More details can be found in [23].

Solutions concentrating at points

First results

The study of (NLS) was initiated by Floer and Weinstein [83] in 1986 in the particular case $N = 1$ and $p = 3$. Assuming that V is globally bounded and that $\inf_{\mathbb{R}} V > 0$, they proved the existence, for $\varepsilon > 0$ small enough, of a positive solution u_ε of (NLS) concentrating at any given nondegenerate critical point of V, say $x = 0$, in the sense that most of the support of u_ε is contained in a neighborhood of 0 that shrinks to a point as $\varepsilon \to 0$.

The result of Floer and Weinstein was later extended to higher dimensions and to more general potentials by Oh [121], who also proved in [122] that for any finite collection of nondegenerate critical points of V, (NLS) possesses, for $\varepsilon > 0$ small enough, a positive solution concentrating simultaneously around these critical points. These solutions are called multi-bump or multi-peak solutions.

The arguments in all these works are based on the Lyapunov-Schmidt reduction. Since then, this method has been considerably developed by

Ambrosetti, Malchiodi and others (see [6, 8]). Although it has proved very fruitful, we have not adopted this approach in this thesis. Its main drawback is the fact that it requires additional assumptions, for example nondegeneracy of the critical point of V.

The variational approach

The equation (NLS) and the system (NLSP) have a variational structure, which means that their solutions are the critical points of an associated functional. For example, the functional associated to (NLS) is given by

$$I_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N}\left(\varepsilon^2\left|\nabla u(x)\right|^2 + V(x)\left|u(x)\right|^2\right)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,dx.$$

Well-known variational methods like the mountain-pass theorem can be used to find critical points. However, the problem of existence is not straightforward when the domain is unbounded because the functional I_ε does not in general satisfy the required compactness condition (the Palais-Smale condition).

A ground state or a least-energy solution of (NLS) is by definition a solution u such that $I_\varepsilon(u)$ is minimal among all the positive critical values of I_ε. A bound state of (NLS) is a solution u such that the quadratic part of $I_\varepsilon(u)$ is finite.

The variational approach was initiated by Rabinowitz [126], who showed that if $V \in C^1(\mathbb{R}^N)$ satisfies the condition

$$(V_1) \qquad \liminf_{x\to\infty} V(x) > \inf_{\mathbb{R}^N} V > 0,$$

then for $\varepsilon > 0$ small enough, (NLS) has a positive ground state.

Concentration of the ground states

In the spirit of the definition of Ambrosetti and Malchiodi [8], we say that a solution u_ε of (NLS) concentrates at x_0 as $\varepsilon \to 0$ if there is a sequence $(x_\varepsilon)_{\varepsilon>0}$ of maximum points of u_ε such that

$$\forall\delta > 0,\ \exists\varepsilon_0 > 0,\ \exists R > 0\ :\ |x - x_\varepsilon| \geqslant \varepsilon R,\ 0 < \varepsilon < \varepsilon_0 \Rightarrow u_\varepsilon(x) \leqslant \delta.$$

It is often possible to give more information about the behaviour of the solution as $\varepsilon \to 0$. Setting $v_\varepsilon(x) = u_\varepsilon(x_0 + \varepsilon x)$, equation (NLS) becomes

$$-\Delta v_\varepsilon + V(x_0 + \varepsilon x)v_\varepsilon = |v_\varepsilon|^{p-1}\,v_\varepsilon \qquad \text{in } \mathbb{R}^N.$$

Thus we may expect that for ε small, the solution u_ε of (NLS) will locally look like a rescaled solution of the limit equation

$$-\Delta v + V(x_0)v = v^p, \quad v > 0 \quad \text{in } \mathbb{R}^N.$$

In particular, $u_\varepsilon(x_0)$ is bounded away from zero as $\varepsilon \to 0$.

Under assumption (V_1), Wang [143] proved that the ground states of (NLS) concentrate at a global minimum point of V. More precisely, any sequence of positive ground states contains a subsequence concentrating at a global minimum point of V as $\varepsilon \to 0$. In particular, if the global minimum point of V is unique, then all positive ground states concentrate at that point as $\varepsilon \to 0$.

Moreover, Wang proved that if $(u_{\varepsilon_n})_n$ is a sequence of ground states and if there is a sequence of local maximum points of u_{ε_n} moving toward a certain point x_0 as $\varepsilon \to 0$, then x_0 is a global minimum point of V and $(u_{\varepsilon_n})_n$ concentrates at x_0 as $\varepsilon \to 0$. Thus if V has no global minimum, the positive ground states, if any, do not concentrate. Wang also proved that if $N \geqslant 2$ and V is radial, then the positive radial ground states concentrate at the origin. Finally, under a growth condition on the gradient of V, Wang obtained a necessary condition for concentration: a point at which a sequence of positive bound states concentrate must be a critical point of V.

If V achieves its global minimum at several points, Grossi and Pistoia [87] discovered that the ground states concentrate at a point where the potential V is the flatter. Roughly speaking, the order of flatness of V at the point x_0 is defined as the order of the first nonzero term in the Taylor development of V at x_0. Under some assumptions on V, Grossi and Pistoia show that if the global minimum points of V have different orders of flatness, then the ground states concentrate, as $\varepsilon \to 0$, at a point where the order of flatness is maximal. Moreover, if all the global minimum points of V are nondegenerate (i.e. their order of flatness is 2), then the ground states concentrate, as $\varepsilon \to 0$, at a global minimum point of V where ΔV is minimal.

Equation with a constant potential

It is natural to wonder where the ground states concentrate when the potential V is constant. If the domain is $\mathbb{R}^N$, the problem is invariant by translations. It is more interesting to consider the problem on a bounded domain $\Omega \subset \mathbb{R}^N$:

$$(\mathcal{N}) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^p, \quad u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here ν denotes the unit outer normal to $\partial\Omega$. This kind of equation arises in models of pattern formation in biology or chemistry [115] and has also been extensively studied. Ni and Takagi [116, 117] proved that the least-energy solutions of $(\mathcal{N})$ concentrate as $\varepsilon \to 0$ at a point of $\partial\Omega$ at which the mean curvature of the boundary is maximal.

The Dirichlet problem

$$(\mathcal{D}) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^p, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

was studied by Ni and Wei [118]. As ε goes to 0, the least-energy solutions of $(\mathcal{D})$ concentrate at a point of Ω at which the distance to the boundary achieves its global maximum. del Pino and Felmer considerably simplified the proofs of the concentration of the least-energy solutions of $(\mathcal{N})$ and $(\mathcal{D})$ in [62].

Let us also mention the work of Berestycki and Wei [25] about the Robin problem

$$(\mathcal{R}) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^p, \quad u > 0 & \text{in } \Omega, \\ \varepsilon \frac{\partial u}{\partial \nu} + \lambda u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let u_ε be a least-energy solution of $(\mathcal{R})$. There exists a constant $\lambda^* > 1$ depending only on N and p such that if $\lambda \leqslant \lambda^*$, the peak of u_ε is located near a point of $\partial\Omega$ where the mean curvature is maximal while if $\lambda > \lambda^*$, the peak of u_ε is located near the innermost part of Ω.

Byeon and Park [37] studied the same equation on a connected compact smooth Riemannian manifold $\mathcal{M}$ of dimension $N \geqslant 3$ (with or without a boundary):

$$(\mathcal{M}) \qquad \begin{cases} -\varepsilon^2 \Delta_{\mathcal{M}} u + u = u^p, \quad u > 0 & \text{on } \mathcal{M}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{M}, \end{cases}$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$. They showed that for sufficiently small $\varepsilon > 0$, the problem $(\mathcal{M})$ possesses a least-energy solution which concentrates either at a maximum point of the scalar curvature of $\mathcal{M}$ if $\partial\mathcal{M} = \emptyset$, or at a maximum of the mean curvature of $\partial\mathcal{M}$ if $\partial\mathcal{M} \neq \emptyset$.

A feature of all the problems considered above is the existence of a finite-dimensional concentration function (potential V, mean curvature of the boundary, distance to the boundary,...) which controls the existence and multiplicity of solutions. We will see other examples below.

Equation with a competing function

Wang and Zeng [144] considered the problem

$$(\text{NLS'}) \qquad -\varepsilon^2 \Delta u + V(x) u = K(x) u^p \qquad \text{in } \mathbb{R}^N,$$

where K is called the competing function. The role of the potential V in Rabinowitz result is now played by the auxiliary function

$$\mathcal{A}(x) := [V(x)]^{\frac{p+1}{p-1}-\frac{N}{2}} [K(x)]^{-\frac{2}{p-1}}.$$

More precisely, under the assumption that V is bounded away from zero, K is bounded and

$$(\mathcal{A}_1) \qquad \inf_{\mathbb{R}^N} \mathcal{A} < \frac{\liminf_{|x|\to\infty} [V(x)]^{\frac{p+1}{p-1}-\frac{N}{2}}}{\limsup_{|x|\to\infty} [K(x)]^{\frac{2}{p-1}}},$$

Wang and Zeng proved, by variational methods, that for $\varepsilon > 0$ small enough, (NLS') has a positive ground state concentrating at a global minimum point of $\mathcal{A}$. Thus the concentration occurs in the middle ground between the minima of V and the maxima of K.

The assumptions (V_1) or $(\mathcal{A}_1)$ are needed in order to ensure the Palais-Smale condition but are slightly unsatisfactory because they are global while the results are local in nature. It is natural to try to remove them.

The penalization method

A solution of (NLS) concentrating at x_0 decays to 0 outside a small neighborhood of x_0. Since $p > 1$, the nonlinear term u^p will be much smaller than the linear term $V(x)u$ outside a compact set containing x_0, at least if $\inf_{\mathbb{R}^N} V > 0$. Therefore it is natural to try to construct a solution of (NLS) by first solving a modified problem in which the nonlinearity is replaced outside a compact set by a function of the form

$$f(s) := \begin{cases} s^p & \text{if } s \leqslant s_0, \\ ks & \text{if } s > s_0, \end{cases}$$

where s_0 and k are parameters. Then one proves, using the maximum principle, that the solution of the modified problem is indeed small outside the compact set, so that it satisfies (NLS). An advantage of the modified problem is that it is compact, whereas the original problem is not in general.

This penalization method was devised by del Pino and Felmer [61]. Assuming that V is bounded away from zero and that there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$(V_2) \qquad \inf_{\Lambda} V < \inf_{\partial\Lambda} V,$$

they proved that for $\varepsilon > 0$ small enough, (NLS) possesses a solution concentrating at a global minimum point of V in Λ. A precise statement and proof of the del Pino-Felmer theorem will be given in Chapter I. It can be seen as a local analogue of Wang's theorem stated above.

Let us also mention the work of Spradlin [138] who assumes that $V \in C^2(\mathbb{R}^N)$ has a (perhaps unbounded) component of local minima, along which ΔV achieves a strict local minimum. Using the penalization method, he proves that (NLS) has a solution concentrating at a local minimum of V where ΔV achieves a local minimum.

In both works, the critical points of V are allowed to be degenerate. The only global assumption on the potential V is $\inf_{\mathbb{R}^N} V > 0$. After a brief discussion of sign-changing potentials, we will consider the so-called "critical frequency case", namely $\inf_{\mathbb{R}^N} V = 0$.

Sign-changing potential

When V is allowed to be negative in some region, new types of solutions can arise. In dimension $N = 1$, Felmer and Torres [81] showed that, if V is negative in an interval and positive outside, then, under some further assumptions, for every $c > 0$, there exist sequences $\varepsilon_n \to 0$ and $(u_{\varepsilon_n})_n$ solutions of (NLS) with $\varepsilon = \varepsilon_n$, such that $\|u_{\varepsilon_n}\|_{L^2} = c$ and the number of zeroes of u_{ε_n} goes to infinity as $n \to \infty$. Thus u_{ε_n} is a highly oscillatory sign-changing solution. This result was generalized to higher dimensions in the radial case by Castro and Felmer [43]. The existence and multiplicity of solutions was studied in [73–75]. We see that the situation is drastically different from the case of positive potentials.

Concentration at zeroes of the potential

Byeon and Wang [38, 39] assumed that V vanishes at some points but remains bounded away from zero at infinity. They showed that there exists a solution of (NLS') which is trapped in a neighborhood of isolated minimum points of V and whose amplitude goes to 0 as $\varepsilon \to 0$. Moreover, the limiting profile of this solution depends on the local behaviour of the potential V near its minimum point. This is in sharp contrast to the non-critical frequency case. Ambrosetti and Wang [15] generalized this result to potentials that both vanish and decay to zero at infinity.

In [41], Byeon and Wang obtained solutions of (NLS') concentrating at a point which can be a zero or a singularity of V or K. The location of this point depends on the zero set of an auxiliary potential involving both V and K.

The case of decaying potentials

When V is positive but is allowed to decay to 0 at infinity, the arguments of del Pino and Felmer cannot be used in a direct way. The functional setting is not the same, the Palais-Smale condition is more delicate to establish and sharp decay estimates of the solutions are needed in order to go back to the original problem.

The first attempt to solve this problem is credited to Ambrosetti, Felli and Malchiodi [7], who used the variational method to study (NLS'). They assume that $N \geqslant 3$ and that there exist constants $A, a, \alpha, B, \beta > 0$ with $\alpha < 2$ such that

$$\frac{a}{1+|x|^{\alpha}} \leqslant V(x) \leqslant A \quad \text{and} \quad 0 < K(x) \leqslant \frac{B}{1+|x|^{\beta}}.$$

Defining

$$\sigma = \begin{cases} \frac{N+2}{N-2} - \frac{4\beta}{\alpha(N-2)} & \text{if } \alpha > \beta, \\ 1 & \text{if } \alpha \leqslant \beta, \end{cases}$$

they prove the existence for every $\varepsilon > 0$ of a positive ground state in $H^1(\mathbb{R}^N)$ provided $\sigma < p < (N+2)/(N-2)$. This ground state concentrates as $\varepsilon \to 0$ at a global minimum of the auxiliary function $\mathcal{A}$. The hypothesis on p implies that $\mathcal{A}(x) \to \infty$ as $|x| \to \infty$.

Another result was obtained by Ambrosetti, Malchiodi and Ruiz [12] by the perturbative method. Under the more restrictive assumptions that K is bounded and

$$\liminf_{|x|\to\infty} V(x)\,|x|^2 > 0,$$

they construct, for sufficiently small ε, bound state solutions concentrating at any isolated stable stationary point of the auxiliary function $\mathcal{A}$.

The penalization method of del Pino and Felmer was adapted by Bonheure and Van Schaftingen [30] to treat positive but decaying potentials and unbounded competing functions. They assume one of the three sets of growth conditions

($\mathcal{G}^1_\infty$) there exist $\alpha \in [0,2)$ and $\lambda > 0$ such that

$$\liminf_{|x|\to\infty} V(x)\,|x|^{\alpha} > 0 \qquad \text{and} \qquad \limsup_{|x|\to\infty} \exp(-\lambda\,|x|^{\frac{2-\alpha}{2}})\frac{K(x)}{V(x)} < \infty;$$

($\mathcal{G}^2_\infty$) there exists $\lambda > 0$ such that

$$\liminf_{|x|\to\infty} V(x)\,|x|^{2} > 0 \qquad \text{and} \qquad \limsup_{|x|\to\infty} |x|^{-\lambda}\,\frac{K(x)}{V(x)} < \infty;$$

($\mathcal{G}^3_\infty$) $N > 2$ and

$$\limsup_{|x|\to\infty} |x|^{-(p-1)(N-2)} \frac{K(x)}{V(x)} < \infty.$$

If in addition there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$\inf_{\Lambda} \mathcal{A} < \inf_{\partial\Lambda} \mathcal{A}, \tag{$\mathcal{A}_2$}$$

then for $\varepsilon > 0$ small enough, (NLS') possesses a bound state concentrating at a global minimum point of $\mathcal{A}$ in Λ. In this result, the potential V can possibly decay very fastly at infinity, provided the competing function K compensates this bad behaviour by either decaying itself or at least by not growing too fast. Unlike the previous result, here $\mathcal{A}$ is allowed to decay to zero at infinity. The penalized nonlinearity is defined for $x \notin \Lambda$ by

$$g(x,u) := \min\left\{\mu V(x)u, u^p\right\},$$

where $\mu \in (0,1)$. In a subsequent paper [31], Bonheure and Van Schaftingen studied the existence of ground-states and gave conditions so that the solution is in $L^2(\mathbb{R}^N)$.

The decay rate of the potential V at infinity is important because it influences the decay rate of solutions at infinity. When $\inf_{\mathbb{R}^N} V > 0$ or when V decays to 0 in such a way that

$$\liminf_{|x|\to\infty} V(x)\,|x|^2 = \infty,$$

the positive solutions of (NLS) decay exponentially at infinity. On the other hand, when V decays faster, the solutions only have a polynomial decay at infinity.

The case of compactly supported potentials

The case of compactly supported potentials is even more difficult than the case of positive decaying potentials. Clearly, the penalization scheme of Bonheure and Van Schaftingen no longer works. A new idea was found by Moroz and Van Schaftingen [113] and consists in adding to V a Hardy-type potential defined outside a bounded open set Λ by

$$H(x) := \frac{\kappa}{|x|^2\left(\log\frac{|x|}{\rho}\right)^{1+\beta}},$$

where $\kappa > 0$, $\rho > 0$ and $\beta > 0$ must be chosen adequately. The potential H enjoys two important properties. Firstly, by the Hardy inequality, the quadratic form associated to $-\Delta - H$ is positive. As a consequence, the operator $-\Delta - H$ satisfies the comparison principle. Secondly, the minimal positive solution to the operator $-\Delta - H$ in $\mathbb{R}^N \setminus \Lambda$ decays at infinity as $|x|^{-(N-2)}$, namely as the fundamental solution of the laplacian.

Moroz and Van Schaftingen prove that if $N \geqslant 3$, $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, $V \in C(\mathbb{R}^N)$ is nonnegative and there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$0 < \inf_{\Lambda} V < \inf_{\partial\Lambda} V,$$

then equation (NLS) has at least one positive solution concentrating at a global minimum point of V in Λ. This result can be extended to dimension 2 and to the equation (NLS'). The condition on p is sharp if V decays faster than $1/|x|^2$.

Let us mention that this result encompasses the case of fastly decaying potentials, i.e. potentials for which

$$\liminf_{|x|\to\infty} V(x)\,|x|^2 = 0,$$

with $K \equiv 1$, which had not been treated previously.

Similar results have been obtained by similar methods by Yin and Zhang [146] and by Fei and Yin [82].

Multi-peak solutions

Still using the penalization method, del Pino and Felmer [65] and Gui [88] proved the existence of multi-peak solutions concentrating at any prescribed finite set of local minima of the potential V. In [64], del Pino and Felmer found multi-peak solutions concentrating at any prescribed finite set of topologically nontrivial critical points of V. Multi-peak solutions were also found by Ba, Deng and Peng [18] in the case of compactly supported or unbounded potentials. Let us also mention that Byeon and Oshita [36] and Sato [135] obtained multi-peak solutions concentrating at points where V vanishes, with the limiting profiles of the peaks depending on the local behaviours of V.

Multi-peak solutions also exist for the Neumann problem ($\mathcal{N}$), as shown in [89] or [90].

Another type of multi-bump solutions were constructed by Kang and Wei [99]. They proved that, if x_0 is an isolated local maximum point of V and $k \in \mathbb{N}$, then for ε small enough, there exists a solution u_ε of (NLS) having exactly k maximum points $x_\varepsilon^1, x_\varepsilon^2, \ldots, x_\varepsilon^k$ such that $x_\varepsilon^i \to x_0$, $i = 1, \ldots, k$ as $\varepsilon \to 0$. The authors also show that such a solution does not exist near a nondegenerate minimum point of V. In contrast to the previous works, here the bumps are interacting. Cao and

Peng [42] generalized this result to equation (NLS') and to potentials vanishing at infinity.

Multiplicity of positive solutions

The preceding results suggest that if V has multiple critical points, then the number n_ε of positive solutions of (NLS) increases as $\varepsilon \to 0$. Felmer, Martínez and Tanaka [78] showed that, if $N = 1$ and V has k local maxima, then there exists a constant $c_1 = c_1(V)$ such that

$$\liminf_{\varepsilon \to 0} \varepsilon^k n_\varepsilon \geqslant c_1.$$

A similar result was found by Lin, Ni and Wei [104] for problem $(\mathcal{N})$ in higher dimension.

In [79], Felmer, Martínez and Tanaka considered the one-dimensional equation

$$-\varepsilon^2 u'' + V(x)u = |u|^{p-1} u \quad \text{ in } I,$$

where I is an interval, with Neumann boundary conditions. They explain that, as ε approaches 0, the behaviour of the solutions becomes highly oscillatory and can be described by means of an envelope, which determines the asymptotic amplitude of the solutions. They prove that, for every prescribed envelope, there is a family of solutions exhibiting such an asymptotic behaviour.

Sign-changing solutions

The results stated up to now concerned positive solutions. Alves and Soares [2] found nodal solutions by a variant of the penalization method of del Pino-Felmer. They proved that, if $\inf_{\mathbb{R}^N} V > 0$ and V satisfies (V_2), then for $\varepsilon > 0$ small enough, there is a solution of (NLS) with exactly one positive maximum point and one negative minimum point, which both concentrate around a global minimum of V in Λ. Using a dynamical systems approach, Bartsch, Clapp and Weth [20] established a lower bound on the number of solutions with precisely two nodal domains.

Sign-changing solutions with multiple spikes have also been studied. In dimension $N = 1$, del Pino, Felmer and Tanaka proved that, given any finite set of points $x_1 < x_2 < \cdots < x_m$ constituted by isolated local minima or maxima of V, and corresponding arbitrary integers n_i, $i = 1, \dots, m$, there is a bound state of (NLS) possessing clusters of n_i spikes concentrating around each x_i as $\varepsilon \to 0$. The clusters consist of spikes with alternating sign if the point is a minimum, and of constant sign if it is a maximum. On the other hand, assuming symmetry conditions on the potential, D'Aprile and Pistoia [56, 57] constructed sign-changing solutions with multiple peaks that concentrate near critical points of V.

Assuming that the zero set of V has a non-empty, connected and smooth interior Ω, Felmer and Mayorga-Zambrano [80] proved that (NLS) and the limit equation

$$\begin{cases} -\Delta u = |u|^{p-1} u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

both have an infinite number of solutions. Moreover, the solutions of (NLS) concentrate around Ω and, up to a rescaling and up to a subsequence, they converge to corresponding solutions of the limit equation.

Noussair and Wei [119] established the existence of a least-energy nodal solution of $(\mathcal{D})$ having exactly one positive peak and one negative peak, and that the peak points converge, as $\varepsilon \to 0$, to two distinct points of Ω, whose locations depend on the geometry of Ω. They obtained the same kind of result for problem $(\mathcal{N})$ in [120].

More general nonlinearities

Most of the results obtained by variational methods can easily be generalized to a nonlinearity $f : \mathbb{R}^+ \to \mathbb{R}^+$ which is continuous and satisfies

(f_1) $f(s) = o(s)$ as $s \to 0^+$;

(f_2) there exists $1 < p < (N+2)/(N-2)$ if $N \geqslant 3$ or $1 < p < \infty$ if $N = 1, 2$, such that

$$\lim_{s\to\infty} \frac{f(s)}{s^p} = 0;$$

(f_3) there exists $2 < \theta \leqslant p+1$ such that

$$0 < \theta F(s) \leqslant f(s)s, \qquad \text{for } s > 0,$$

where $F(s) := \int_0^s f(\sigma)\, d\sigma$;

(f_4) the function $s \mapsto f(s)/s$ is nondecreasing.

On the contrary, the results obtained by the Liapunov-Schmidt reduction rely on the uniqueness and non-degeneracy of the ground-states of the limit equation. These properties require restrictive assumptions on the nonlinearity f.

Avila and Jeanjean [16] and Byeon and Jeanjean [32, 33] improved the variational method in order to find an analogue of the del Pino-Felmer theorem with the almost optimal assumptions on f. They assumed that $N \geqslant 3$, V is bounded away from zero and satisfies (V_2), and that f verifies (f_1), (f_2) and

(f_5) there exists $T > 0$ such that $\frac{1}{2}mT^2 < F(T)$, where $m := \inf_\Lambda V$.
Under these assumptions, they found, for $\varepsilon > 0$ sufficiently small, positive solutions of

$$-\varepsilon^2\Delta u + V(x)u = f(u) \qquad \text{in } \mathbb{R}^N,$$

concentrating at a global minimum point of V in Λ. In a work with Tanaka [35], they extended this result to the one and two-dimensional cases. Byeon and Jeanjean also found multi-peak solutions with a general nonlinearity [34].

Let us also mention a paper of Grossi [86], in which nonlinearities of the form $f(x,u)$ are studied.

Concentration at other critical points

When searching for solutions concentrating at a local minimum of the potential V, the main tool is the mountain pass theorem. In order to find solutions concentrating at other critical points, it is necessary to use more sophisticated variational methods, like the general minimax principle [145]. For potentials bounded away from zero, del Pino and Felmer adapted their penalization method to construct solutions concentrating at any topologically nontrivial critical point of the potential V [63, 64]. To put it simply, this means a critical point having a minimax characterization. Similar methods were used for example by Cingolani and Lazzo [48] or by Ishiwata [94] to find multiple solutions.

Let us add that del Pino, Felmer and Wei obtained similar results for problem ($\mathcal{N}$) in [66] and problem ($\mathcal{D}$) in [67]. They showed that for every topologically nontrivial critical point of the mean curvature of the boundary (respectively of the distance to the boundary), there is a single-peaked solution of ($\mathcal{N}$) (respectively of ($\mathcal{D}$)) concentrating at that point as $\varepsilon \to 0$.

Our result

In this thesis, we have adapted the penalization method of Moroz and Van Schaftingen to find solutions of (NLS) concentrating at a local maximum of the potential V, in the case of a fastly decaying or a compactly supported potential. This result will be presented in Chapter IV. We assume that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$\sup_\Lambda V > \inf_\Lambda V = \sup_{\partial\Lambda} V$$

and

$$\sup_\Lambda V^{\frac{p+1}{p-1}-\frac{N}{2}} < 2 \inf_{\overline{\Lambda}} V^{\frac{p+1}{p-1}-\frac{N}{2}} .$$

Note that the first assumption implies that

$$\sup_{\partial\Lambda} V = \inf_{\partial\Lambda} V \ ,$$

that is, $\partial\Lambda$ is a level line of V. We will prove the following result.

Theorem. *Let $N \geqslant 3$, $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and $V \in C(\mathbb{R}^N, \mathbb{R}^+)$. Assume that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ satisfying the preceding assumptions. Then for $\varepsilon > 0$ small enough, equation* (NLS) *possesses a positive solution $u_\varepsilon \in C^1(\mathbb{R}^N)$ such that u_ε attains its maximum at $x_\varepsilon \in \Lambda$,*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0,$$

and

$$\lim_{\varepsilon\to 0} V(x_\varepsilon) = \sup_{x\in\Lambda} V(x).$$

The proof is purely variational and relies on the general minimax principle [145].

Solutions concentrating on manifolds

One can ask if there exist solutions of (NLS) concentrating on a higher dimensional set. The simplest case is the one of radial solutions concentrating on a sphere, under the assumption that V is radial. This kind of solutions have been found by different methods and under various assumptions [10, 11, 19, 23]. Solutions concentrating simultaneously on several spheres have been constructed by Bartsch and Peng [22]. We point out that, as long as we are concerned with solutions concentrating on spheres, any power $p > 1$ is allowed. This is not really a consequence of the fact that the solutions are radial, but rather that the limiting problem is one-dimensional. Solutions concentrating on circles [52] and on $(N-2)$-dimensional spheres [111] have also been investigated.

All the papers cited above assume that V is bounded from below by a positive constant. Byeon and Wang [40] dealt with the critical frequency case. They obtained solutions concentrating on spheres near zeros of the potential. These solutions exhibit quite a different behaviour, as explained above. On the other hand, Ambrosetti and Ruiz [13] assume that $p > 1$, that $V \in C^1(\mathbb{R}^N)$ is a positive bounded radially symmetric potential, that ∇V is bounded and that

$$\liminf_{|x|\to\infty} V(x)\,|x|^2 > 0.$$

They prove that, if there exists r^* such that the auxiliary potential $\mathcal{M} : (0, \infty) \to \mathbb{R}$ defined for $r > 0$ by

$$\mathcal{M}(r) := r^{N-1} \left[V(r)\right]^{\frac{p+1}{p-1}-\frac{1}{2}}$$

has an isolated local maximum or minimum at $r = r^*$, then, for $\varepsilon > 0$ small enough, equation (NLS) has a positive radially symmetric solution $u_\varepsilon \in H^1(\mathbb{R}^N)$ that concentrates at the sphere $|x| = r^*$.

The auxiliary potential $\mathcal{M}$ quantifies the fact that the energy of a solution concentrating on a sphere depends not only on the potential V as in the case of concentration at points, but also on the volume of the sphere. In a region where V is decreasing, there could be a balance between these two effects, yielding a favourable place for concentration on a sphere.

Another kind of solutions to (NLS) with p supercritical and V decaying superquadratically have been found in [60].

Related results have been obtained for problem $(\mathcal{D})$ by Dancer and Yan [59], who found solutions concentrating on spheres on the boundary of the domain.

Without symmetry assumptions, the first result of concentration on a manifold was obtained by del Pino, Kowalczyk and Wei [70] in dimension 2. Assuming that V is bounded away from zero and that Γ is a nondegenerate closed stationary curve for the weighted length functional

$$\int_\Gamma V^{\frac{p+1}{p-1}-\frac{1}{2}},$$

they show that, for $\varepsilon > 0$ small enough and away from certain numbers, there exists a solution of (NLS) concentrating along the whole of Γ. This result was generalized by Mahmoudi, Malchiodi and Montenegro [106] to higher dimensions (and to compact manifolds).

There are similar results for problem $(\mathcal{N})$ [107, 108]. All these works use the perturbative method.

Our result

The Ambrosetti-Ruiz theorem relies on the Liapunov-Schmidt reduction method. One of the problem considered in this thesis was to try to apply the penalization method in order to improve this theorem. This is the subject of Chapter II. Our results include the following simple particular case.

Theorem. *Let $N \geqslant 3$, $p > \frac{N}{N-2}$ and $V \in C(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^+)$ be a radial potential. If there exists $r^* > 0$ such that the auxiliary potential $\mathcal{M}$:*

$(0,\infty) \to \mathbb{R}$ *defined for* $r > 0$ *by*

$$\mathcal{M}(r) := r^{N-1} \left[V(r)\right]^{\frac{p+1}{p-1}-\frac{1}{2}}$$

has an isolated local minimum at $r = r^*$ *such that* $\mathcal{M}(r^*) > 0$*, then for* ε *small enough, the equation* (NLS) *has a positive radially symmetric solution* u_ε *that concentrates on the sphere of radius* r^**.*

The novelty of our results consists in treating potentials V with a fast decay or even a compact support, and in allowing V or ∇V to be unbounded. In particular, V could be singular at the origin. Moreover, we can deal with a nonlinearity which is neither necessarily homogeneous nor autonomous. We assume that the nonlinearity satisfies assumptions (f_1)-(f_4). In this respect, we improve the Ambrosetti-Ruiz theorem [13] mentioned above. Furthermore, we will find solutions concentrating on a k-dimensional sphere, $1 \leqslant k \leqslant N-1$.

Other equations

Spike solutions have been found for other equations as well. Quasilinear elliptic equations were studied by Squassina [139] and elliptic equations in divergence form were investigated by Pomponio and Secchi [125], both extending the penalization method of del Pino-Felmer. Results for the p-laplacian are also available [69, 71]. On the other hand, del Pino and Flores [68] studied the nonlinear boundary value problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = 0 & \text{in } \Omega, \\ \varepsilon \frac{\partial u}{\partial \nu} = |u|^{p-1} u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. They found that the least-energy solutions concentrate around a point of $\partial\Omega$ of maximum mean curvature.

The semiclassical limit for the Dirac equation was studied by Ding [72], see also [76]. New problems arise from the fact that the associated energy functional is strongly indefinite.

Higher-order equations and systems

Pimenta and Soares [124] considered the equation

$$-\varepsilon^4 \Delta^2 u + V(x) u = |u|^{p-1} u \qquad \text{in } \mathbb{R}^N,$$

where Δ^2 is the biharmonic operator and $N \geqslant 5$. Assuming that V is positive and bounded away from zero, they show that the preceding problem has a solution concentrating at a global minimum point of V. The proof relies on the penalization method of del Pino and Felmer, with adaptations in order to deal with the lack of a general maximum principle for the biharmonic operator.

Several authors studied the Hamiltonian system

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = |v|^{p-1} v & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + W(x)v = |u|^{q-1} u & \text{in } \mathbb{R}^N, \end{cases}$$

where $N \geqslant 3$ and $p, q > 1$ are below the critical hyperbola, that is

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}.$$

In the case $V = W$, Ramos and Soares [128] proved that, for sufficiently small $\varepsilon > 0$, this system has a ground-state solution $(u_\varepsilon, v_\varepsilon)$ with the property that u_ε and v_ε attain their maximum at the same point x_ε, which converges, up to a subsequence, to a minimum point of V. Ramos and Tavares [129] found solutions with multiple peaks concentrating at local minima of V. The same problem with non-autonomous nonlinearities has been investigated by Ramos [127].

When the potentials V and W are different, Alves and Soares [4] proved the existence, for $\varepsilon > 0$ small enough, of a ground-state $(u_\varepsilon, v_\varepsilon)$. They showed that u_ε and v_ε possess just one maximum point and that these two points converge, along a subsequence, to the same point which is a global minimum of a certain concentration function.

Gradient systems have also been studied by variational methods [1,3].

Nonlinear Schrödinger equation with a magnetic potential

Let us now introduce some other elliptic equations exhibiting the same kind of concentration behaviour. We first consider the nonlinear Schrödinger equation with an electromagnetic field

$$\text{(MNLS)} \qquad -\left(\varepsilon \nabla + iA(x)\right)^2 u + V(x)u = |u|^{p-1} u, \qquad \text{in } \mathbb{R}^N,$$

where $A : \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic vector potential and u is now complex-valued. This equation was first studied by Esteban and Lions [77] in the case $\varepsilon = 1$. In the semiclassical limit, a result analogous to Wang's theorem was obtained by Kurata [101]. Under assumption (V_1), he proves that (MNLS) has a ground-state which concentrates, as $\varepsilon \to 0$, near a global minimum point of V. Thus the magnetic potential A doesn't play any role in the location of the spikes of the solutions of (MNLS). It only contributes to the phase of the solutions. Cingolani and Secchi [49, 50] considered (MNLS) with a competing function K in front of the nonlinearity. They showed that for every nondegenerate critical point x_0 of the auxiliary function $\mathcal{A}$, this equation has, for ε small enough, at least one solution concentrating at x_0. The multiplicity of semiclassical states was investigated in [44].

When the nonlinearity in (MNLS) is replaced by a more general term $f(|u|^2)u$, the smoothness properties of the auxiliary function $\mathcal{A}$ are lesser-known. In this context, Secchi and Squassina [136] studied the location of spikes and gave necessary conditions for concentration. They showed that in general, the location of the concentration points might also depend on the magnetic potential A.

Concerning multipeak solutions, let us mention the works of Bartsch, Dancer and Peng [21] and of Cingolani, Jeanjean and Secchi [47]. The latter paper gives nearly optimal conditions on the nonlinearity.

Finally, we mention the recent work of Cingolani and Clapp [45, 46] in the case of symmetric potentials. The number of solutions depends both on the symmetries and on the potential V.

The Schrödinger-Poisson system

The Schrödinger-Poisson system

$$\text{(NLSP)} \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi u = |u|^{p-1} u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2. \end{cases}$$

arises for example as a model of the interaction of a charged particle with the electrostatic field [24]. It also appears as an approximation to the Hartree-Fock system and is used as a model of quantum transport in semiconductor devices [100]. It is interesting to find out whether the concentration phenomena observed for the nonlinear Schrödinger equation still occur for the solutions of (NLSP). Moreover, one could ask if there are solutions of (NLSP) exhibiting another type of behaviour, not known for the solutions of (NLS).

Let us first review the results about the system (NLSP) with a constant potential V. Assuming $3 \leqslant p < 5$, D'Aprile and Mugnai [54] established the existence of a radially symmetric solution for every $\varepsilon > 0$ and Coclite [51] proved the existence of an infinity of such solutions. D'Aprile and Mugnai [55] also proved that there does not exist nontrivial solutions when $0 \leqslant p < 1$ or $p \geqslant 5$. Kikuchi [100] proved that when $2 < p < 5$, the system (NLSP) has at least one radial solution for every $\varepsilon > 0$ and no nontrivial solution in $H^1(\mathbb{R}^3)$ when $1 < p < 2$ and ε is large. Ruiz [131, 132] completed the study of the existence and non-existence results of positive solutions of (NLSP) as a function of p and of the fact that ε is small or not. He summed up his results in the following table:

	ε small		ε large	
$1 < p < 2$	Two solutions	$\inf J > -\infty$	No solution	$\inf J = 0$
$p = 2$	One solution	$\inf J = -\infty$	No solution	$\inf J = 0$
$2 < p < 5$	One solution	$\inf J = -\infty$	One solution	$\inf J = -\infty$

Here "One solution" means that there exists at least one positive radial solution, and J denotes the associated energy functional.

From the variational point of view, one of the difficulties is the lack of a bound on the Palais-Smale sequences when $p < 3$. A first method to overcome this problem is the "monotonicity trick" introduced by Struwe [140] (see also [95, 97]). The second method, which works only for ε small, is a truncation method due to Jeanjean and Le Coz [96] and used by Kikuchi [100] to show the existence of a solution of (NLSP) when $1 < p < 5$.

Radial semiclassical states concentrating around a sphere were found under various assumptions [53, 58, 130].

In the case of a nonconstant potential V bounded away from zero, solutions concentrating at a nondegenerate local minimum or maximum of V were found by Ianni and Vaira [92] for $p \in (1, 5)$. They also have results for degenerate critical points. Ianni and Vaira also studied radial solutions concentrating on spheres for a similar problem [91, 93]. All these works rely on the perturbative method. The existence of solutions to problem (NLSP) for all $\varepsilon > 0$ was established by variational methods and under various assumptions by Zhao and Zhao [147] and by Azzolini and Pomponio [17].

The results of Ambrosetti, Felli and Malchiodi [7] about decaying potentials for the nonlinear Schrödinger equation were extended by Mercuri [110] to the Schrödinger-Poisson system. Other existence results were obtained by Bonheure and Mercuri [28] using embedding theorems for weighted Sobolev spaces. Assuming that V is a radial potential decaying to zero at infinity, Ambrosetti and Ruiz [14] showed that the system (NLSP) has infinitely many nontrivial radial solutions. Sign-changing potentials have also been investigated [98].

Ruiz and Vaira [134] proved the existence of multi-bump solutions of (NLSP), whose bumps concentrate around a local minimum of the potential V. This kind of solutions do not exist for the nonlinear Schrödinger equation.

The Schrödinger-Poisson system has also been studied in a bounded domain with Dirichlet boundary conditions [133, 137].

Our result

In this thesis, we have applied the penalization method to the Schrödinger-Poisson system. We have found solutions concentrating at points and solutions concentrating on circles. Choose any 1-dimensional linear subspace $d \subset \mathbb{R}^3$. We denote by π the orthogonal complement of d. If $x \in \mathbb{R}^3$, we will write $x = (x', x'')$ with $x' \in d$ and $x'' \in \pi$. As a particular case of our main result, we have the following theorem:

Theorem. *Let $p > 3$ and $V \in C(\mathbb{R}^3 \setminus \{0\}, \mathbb{R}^+)$ be a radial potential. Write $V(x) = \tilde{V}(x', |x''|)$. If there exists $r^* > 0$ such that the function*

$$\mathcal{M}(r) := r\left[\tilde{V}(0,r)\right]^{\frac{2}{p-1}}$$

has an isolated local minimum at $r = r^*$ such that $\mathcal{M}(r^*) > 0$, then for ε small enough, the system

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi u = u^p, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, \end{cases}$$

has a positive cylindrically symmetric solution u_ε that concentrates on the circle of radius r^* centered at the origin and contained in the plane π.

Again we have no global assumption on the potential V. In particular, V could be compactly supported. This is an improvement on previous works, see e.g. [5].

Outline of the thesis

The first chapter gives a detailed proof of the del Pino-Felmer theorem. It does not contain anything new, but serves as an introduction to the subject and as a basis for the subsequent chapters. The ideas are those of del Pino and Felmer [61] but the proof is rewritten in the spirit of the thesis.

Chapter II is devoted to solutions concentrating on k-dimensional spheres, $1 \leqslant k \leqslant N-1$. We improve previous results in the sense that we require weaker assumptions on the potential V. Moreover, we study carefully the case of concentration on lower-dimensional spheres, which up to now had only been discussed in remarks in [8, 9, 40]. This is a joint work with Denis Bonheure and Jean Van Schaftingen and a paper [27] is to appear in Journal of Differential Equations.

The subject of Chapter III is the Schrödinger-Poisson system. We show that the penalization by a Hardy-type potential is flexible enough to be adapted to a different problem and we obtain new results about solutions concentrating on a circle. A paper [26] written with Denis Bonheure and Carlo Mercuri containing these results has been accepted for publication in Communications in Contemporary Mathematics.

In Chapter IV, we prove the existence of solutions concentrating at a maximum point of the potential V, with few assumptions on V. In particular, V can decay fastly or be compactly supported. We employ the same penalization scheme as before. The difficulty lies in the application of the general minimax principle. In particular, we give a careful proof of an essential strict inequality between energy levels. A paper with Jean Van Schaftingen is in preparation.

The different chapters reproduce the structure of the aforementioned publications. Therefore there can be an overlap in some definitions and statements.

CHAPTER I

A model problem

The aim of this introductory chapter is to study a model problem for which the penalization method can be applied without too much technicalities. We consider the semilinear elliptic equation

$$\text{(NLS)} \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x) u = u^p & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N),\ u > 0, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive potential which is bounded away from zero, i.e. $\inf_{\mathbb{R}^N} V > 0$, and $p > 1$ is a subcritical exponent, i.e. $p < (N+2)/(N-2)$ if $N \geqslant 3$. We are interested in solutions of (NLS) concentrating at a local minimum of the potential V. This chapter is devoted to the proof of the following theorem, which is by now classical.

Theorem (del Pino-Felmer [61]**).** *Assume that V is a positive locally Hölder continuous potential which is bounded away from zero and that $\Lambda \subset \mathbb{R}^N$ is a bounded open set satisfying*

$$V_0 := \inf_{x \in \Lambda} V(x) < \inf_{x \in \partial\Lambda} V(x). \tag{1.1}$$

Then, there exist $\varepsilon_0 > 0$ and a family of solutions $\{u_\varepsilon \in H^1(\mathbb{R}^N) : 0 < \varepsilon < \varepsilon_0\}$ of (NLS) *with the property that each u_ε possesses a single maximum point x_ε such that $V(x_\varepsilon) \to V_0$ as $\varepsilon \to 0$. Moreover,*

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \|u_\varepsilon\|_{L^\infty(\Lambda \setminus B(x_\varepsilon, \varepsilon R))} = 0,$$

and the limiting profile is given by

$$u_\varepsilon(x) = v\left(\frac{x - x_\varepsilon}{\varepsilon}\right) + w_\varepsilon(x),$$

where v is the unique positive radial solution of

$$-\Delta v + V_0 v = |v|^{p-1} v \quad \text{in } \mathbb{R}^N$$

and $w_\varepsilon \to 0$ in $C^2_{loc}(\mathbb{R}^N)$ and in $L^\infty(\mathbb{R}^N)$ as $\varepsilon \to 0$.

The proof that we give in this chapter follows the ideas of del Pino and Felmer [61] but is written in the spirit of the subsequent works [30] and [113].

Notice that we can be more precise about the decay of the solution. One can prove that there exist $C, \lambda > 0$ such that for every $x \in \mathbb{R}^N$,

$$u_\varepsilon(x) \leqslant Ce^{-\lambda|x-x_\varepsilon|/\varepsilon}.$$

1. The penalized problem

Let $\mu \in (0,1)$. We denote by χ_Λ the characteristic function of the set Λ. The penalized nonlinearity is the function $g : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$g(x,s) := \chi_\Lambda(x)s^p + (1-\chi_\Lambda(x))\min\{\mu V(x)s, s^p\}. \tag{1.2}$$

We set

$$G(x,s) := \int_0^s g(x,\sigma)d\sigma.$$

Although not continuous, g is a Carathéodory function with the following properties :

(g_1) $g(x,s) = o(s)$ as $s \to 0^+$, uniformly in compact subsets of $\mathbb{R}^N$;

(g_2) there exists $\lambda > 1$ such that $\frac{1}{\lambda+1} > \frac{1}{2} - \frac{1}{N}$ and

$$\lim_{s\to\infty} \frac{g(x,s)}{s^\lambda} = 0;$$

(g_3) (i) $0 < (p+1)G(x,s) \leqslant g(x,s)s$, for all $x \in \Lambda$ and all $s > 0$,
(ii) $0 < 2G(x,s) \leqslant g(x,s)s \leqslant \mu V(x)s^2$, for all $x \notin \Lambda$ and all $s > 0$;

(g_4) the function

$$s \mapsto \frac{g(x,s)}{s}$$

is nondecreasing for all $x \in \mathbb{R}^N$.

We are going to solve the penalized problem

$$(\mathcal{P}_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u = g(x,u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N),\ u > 0. \end{cases}$$

The weak solutions of $(\mathcal{P}_\varepsilon)$ are the critical points of the functional J_ε defined by

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(\varepsilon^2 |\nabla u(x)|^2 + V(x)|u(x)|^2\right) dx - \int_{\mathbb{R}^N} G(x,u(x))\,dx.$$

In view of (g_3), the functional J_ε is well-defined on the Hilbert space

$$H_V^1 := \left\{ u \in H^1(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} V(x)\,|u(x)|^2 \ dx < \infty \right\}$$

endowed with the norm

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} \left(\varepsilon^2 \,|\nabla u(x)|^2 + V(x)\,|u(x)|^2 \right) \ dx \right)^{1/2}.$$

Notice that, since V is bounded away from zero, H_V^1 is continuously embedded in $H^1(\mathbb{R}^N)$.

Standard arguments show that $J_\varepsilon \in C^1(H_V^1, \mathbb{R})$ and its Fréchet derivative is given by

$$J_\varepsilon'(u)(v) = \int_{\mathbb{R}^N} \left(\varepsilon^2 \nabla u \cdot \nabla v + V u v \right) - \int_{\mathbb{R}^N} g(x,u) v.$$

The existence of a critical point of J_ε will be deduced from the well-known mountain pass theorem of Ambrosetti and Rabinowitz (see for example [145]). The first step is to show that J_ε has a mountain pass geometry.

Lemma 1.1. *Let g be defined by* (1.2)*. Then,*

(i) *The functional J_ε achieves a local minimum at* 0 *in H_V^1;*
(ii) *The infimum of J_ε over H_V^1 is $-\infty$.*

Proof. Thanks to (g_3), we have for all $u \in H_V^1$,

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(\varepsilon^2 \,|\nabla u|^2 + V\,|u|^2 \right) - \int_\Lambda G(x,u) - \int_{\mathbb{R}^N \setminus \Lambda} G(x,u) \\ &\geqslant \frac{1-\mu}{2} \,\|u\|_\varepsilon^2 - \frac{1}{p+1} \int_\Lambda |u|^{p+1}. \end{aligned}$$

The Sobolev inequality implies that there exist constants $c_1, c_2 > 0$ such that

$$J_\varepsilon(u) \geqslant c_1 \,\|u\|_\varepsilon^2 - c_2 \,\|u\|_\varepsilon^{p+1}.$$

Since $p > 1$, there exist $c, r > 0$ such that for every $u \in H_V^1$ with $\|u\|_\varepsilon \leqslant r$, $J_\varepsilon(u) \geqslant c\,\|u\|_\varepsilon^2$. Thus assertion (i) is proved.

For the second assertion, let $u \in H_V^1$, $u \geqslant 0$, $u \not\equiv 0$, be a function supported in Λ. We compute that

$$J_\varepsilon(tu) = t^2 \left[\frac{1}{2} \int_\Lambda \left(\varepsilon^2 \,|\nabla u|^2 + V u^2 \right) - \frac{t^{p-1}}{p+1} \int_\Lambda |u|^{p+1} \right].$$

Hence, $J_\varepsilon(tu) \to -\infty$ as $t \to \infty$. □

Now we define the minimax level

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} J_\varepsilon(\gamma(t)), \tag{1.3}$$

where Γ_ε is the family of paths

$$\Gamma_\varepsilon := \left\{\gamma \in C([0,1], H^1_V) \ : \ \gamma(0) = 0,\ J_\varepsilon(\gamma(1)) < 0\right\}.$$

The mountain pass theorem ensures the existence of a Palais-Smale sequence for J_ε at level c_ε, namely a sequence $(u_n)_n \subset H^1_V$ such that

$$J_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad J'_\varepsilon(u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

The functional J_ε is said to satisfy the Palais-Smale condition (at level c_ε) if every Palais-Smale sequence for J_ε (at level c_ε) contains a convergent subsequence. We will now show that J_ε satisfies the Palais-Smale condition (at every level), in contrast to the functional associated to the original problem (NLS). We point out that the nonlinearity u^p in (NLS) fulfills the properties given for g, except for (g_3) (ii). Firstly we prove a useful lemma.

Lemma 1.2. *Let $g : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying (g_3) and let $u \in H^1_V$. Then,*

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)(1-\mu)\|u\|^2_\varepsilon \leqslant J_\varepsilon(u) - \frac{1}{p+1} J'_\varepsilon(u)(u).$$

Proof. We compute that

$$\begin{aligned} J_\varepsilon(u) - \frac{1}{p+1} J'_\varepsilon(u)(u) = &\left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|^2_\varepsilon \\ &+ \frac{1}{p+1}\int_{\mathbb{R}^N} \left(g(x,u)u - (p+1)G(x,u)\right). \end{aligned}$$

In view of (g_3),

$$\begin{aligned} \frac{1}{p+1}\int_{\mathbb{R}^N} \left(g(x,u)u - (p+1)G(x,u)\right) &\geqslant -\frac{p-1}{p+1}\int_{\mathbb{R}^N\setminus\Lambda} G(x,u) \\ &\geqslant -\left(\frac{1}{2} - \frac{1}{p+1}\right)\mu \int_{\mathbb{R}^N\setminus\Lambda} V(x)u^2 \\ &\geqslant -\left(\frac{1}{2} - \frac{1}{p+1}\right)\mu\,\|u\|^2_\varepsilon. \end{aligned}$$

The lemma follows. □

Lemma 1.3. *Let g be defined by* (1.2) *and let $V : \mathbb{R}^N \to \mathbb{R}^+_0$ be a continuous potential bounded away from zero. Then, the functional $J_\varepsilon : H^1_V \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Let $(u_n)_n \subset H^1_V$ be a Palais-Smale sequence.

Claim 1: the sequence $(u_n)_n$ is bounded in H^1_V. Since $\|J'_\varepsilon(u_n)\|$ is bounded, we have

$$-J'_\varepsilon(u_n)(u_n) \leqslant \left|J'_\varepsilon(u_n)(u_n)\right| \leqslant \left\|J'_\varepsilon(u_n)\right\| \|u_n\|_\varepsilon \leqslant C_1 \|u_n\|_\varepsilon .$$

As $J_\varepsilon(u_n)$ is also bounded, we can write

$$J_\varepsilon(u_n) - \frac{1}{p+1} J'_\varepsilon(u_n)(u_n) \leqslant C_2 + C_3 \|u_n\|_\varepsilon .$$

The preceding lemma implies that

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)(1-\mu) \|u_n\|^2_\varepsilon \leqslant C_2 + C_3 \|u_n\|_\varepsilon .$$

This proves the claim because the coefficient of $\|u_n\|^2_\varepsilon$ is positive.

Claim 2: for all $\delta > 0$, there exists $R > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B(0,R)} \left(\varepsilon^2 |\nabla u_n|^2 + V u_n^2\right) \leqslant \delta. \tag{1.4}$$

Let $\delta > 0$ be given. We choose a cut-off function $\eta_R \in C^\infty(\mathbb{R}^N)$ such that $\eta_R = 0$ in $B(0, R/2)$, $\eta_R = 1$ in $\mathbb{R}^N \setminus B(0,R)$, $0 \leqslant \eta_R \leqslant 1$ and $|\nabla \eta_R| \leqslant C/R$. Since $(u_n)_n$ is bounded in H^1_V,

$$\left|J'_\varepsilon(u_n)(\eta_R u_n)\right| \leqslant \left\|J'_\varepsilon(u_n)\right\| \|\eta_R u_n\|_\varepsilon = o(1)$$

as $n \to \infty$. Therefore,

$$J'_\varepsilon(u_n)(\eta_R u_n) = o(1) \quad \text{as } n \to \infty.$$

We compute that

$$\begin{aligned}\int_{\mathbb{R}^N} \left(\varepsilon^2 |\nabla u_n|^2 + V u_n^2\right) \eta_R = \int_{\mathbb{R}^N} & g(x,u_n) u_n \eta_R \\ & - \int_{\mathbb{R}^N} \varepsilon^2 u_n \nabla u_n \cdot \nabla \eta_R + o(1).\end{aligned}$$

If we choose R large enough so that $\Lambda \subset B(0, R/2)$, we get

$$\int_{\mathbb{R}^N} g(x,u_n) u_n \eta_R \leqslant \mu \int_{\mathbb{R}^N} V u_n^2 \eta_R.$$

On the other hand, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \varepsilon^2 u_n \nabla u_n \cdot \nabla \eta_R\right| &\leqslant \frac{C}{R} \int_{\mathbb{R}^N} \varepsilon^2 |u_n| \, |\nabla u_n| \\ &\leqslant \frac{C'}{R} \|u_n\|^2_\varepsilon .\end{aligned}$$

We conclude that

$$(1-\mu)\int_{\mathbb{R}^N\setminus B(0,R)}\left(\varepsilon^2\left|\nabla u_n\right|^2+Vu_n^2\right)\leqslant\frac{C'}{R}\left\|u_n\right\|_\varepsilon^2+o(1).$$

The claim follows.

Conclusion. We deduce from Claim 1 that, up to a subsequence, $(u_n)_n$ converges weakly in H_V^1 to a function $u\in H_V^1$. Let us show that the convergence is actually strong. We fix $\delta>0$ and we choose R large enough so that (1.4) holds and

$$\int_{\mathbb{R}^N\setminus B(0,R)}Vu^2\leqslant\delta. \tag{1.5}$$

We can write

$$\begin{aligned}\|u_n-u\|_\varepsilon^2=J_\varepsilon'(u_n)(u_n-u)-J_\varepsilon'(u)(u_n-u)\\ +\int_{\mathbb{R}^N}\left(g(x,u_n)-g(x,u)\right)(u_n-u)\,.\end{aligned}$$

The first term in the right-hand side of this equality goes to 0 because $J_\varepsilon'(u_n)\to 0$ and u_n-u is bounded. The second term also converges to 0 because $J_\varepsilon'(u)$ is a linear functional on H_V^1. For the third term, we deduce from (g_3), (1.4) and (1.5) that

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B(0,R)}\left|g(x,u_n)-g(x,u)\right|\left|u_n-u\right|\\ &\qquad\leqslant 2\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B(0,R)}\left(\left|g(x,u_n)u_n\right|+\left|g(x,u)u\right|\right)\\ &\qquad\leqslant 2\mu\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B(0,R)}V\left(u_n^2+u^2\right)\\ &\qquad\leqslant 4\mu\delta.\end{aligned}$$

On the other hand, (g_2) implies that $|g(x,u)|\leqslant c(1+|u|^\lambda)$ and by Rellich's compactness theorem, $u_n\to u$ in $L^{\lambda+1}(B(0,R))$), so that by continuity of the superposition operator, $g(x,u_n)\to g(x,u)$ in $L^q(B(0,R))$, where $q:=(\lambda+1)/\lambda$. It follows from the Hölder inequality that

$$\begin{aligned}&\int_{B(0,R)}\left|g(x,u_n)-g(x,u)\right|\left|u_n-u\right|\\ &\qquad\leqslant\left\|g(\cdot,u_n)-g(\cdot,u)\right\|_{L^q(B(0,R))}\left\|u_n-u\right\|_{L^{\lambda+1}(B(0,R))}\to 0.\end{aligned}$$

We conclude that

$$\limsup_{n\to\infty} \|u_n - u\|_\varepsilon^2 \leqslant 4\mu\delta.$$

Since δ is arbitrary, this proves that $u_n \to u$ in H^1_V. □

We can now state an existence result for the penalized problem $(\mathcal{P}_\varepsilon)$.

Theorem 1.4. *Let g be defined by (1.2) and let $V : \mathbb{R}^N \to \mathbb{R}^+_0$ be a continuous potential bounded away from zero. Then, for every $\varepsilon > 0$, the functional $J_\varepsilon : H^1_V \to \mathbb{R}$ has a critical point $u_\varepsilon \in H^1(\mathbb{R}^N)$ with $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ and u_ε is a weak positive solution of $(\mathcal{P}_\varepsilon)$.*

The function u_ε found in Theorem 1.4 is called a least-energy solution of $(\mathcal{P}_\varepsilon)$. The regularity of u_ε can be investigated by standard arguments.

Proposition 1.5. *Under the assumptions of Theorem 1.4, any solution $u_\varepsilon \in H^1(\mathbb{R}^N)$ of $(\mathcal{P}_\varepsilon)$ satisfies $u_\varepsilon \in W^{2,q}_{loc}(\mathbb{R}^N)$ for every $q < \infty$. In particular, $u_\varepsilon \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for every $0 < \alpha < 1$.*

This result cannot be improved, even if V is more regular, because even for a smooth u, $g(.,u)$ does not need to be continuous.

We end this section with two standard characterizations of c_ε, see [145] for details. We define the Nehari manifold associated to the functional J_ε by

$$\mathcal{N}_\varepsilon := \left\{u \in H^1_V \ : \ u \geqslant 0, u \not\equiv 0 \text{ and } J'_\varepsilon(u)(u) = 0\right\}.$$

Proposition 1.6. *Under the assumptions of Proposition 1.4, we have*

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{\substack{u\in H^1_V \\ u\geqslant 0, u\not\equiv 0}} \max_{t>0} J_\varepsilon(tu). \tag{1.6}$$

2. The limit equation

Let $a > 0$. The problem

$$\begin{cases} -\Delta u + au = u^p & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ u > 0, \end{cases} \tag{1.7}$$

is called the limit problem associated with (NLS). The weak solutions of (1.7) are critical points of the functional $\mathcal{I}_a : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{I}_a(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + au^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}. \tag{1.8}$$

Any nontrivial critical point $u \in H^1(\mathbb{R}^N)$ of $\mathcal{I}_a$ belongs to the Nehari manifold

$$\mathcal{M}_a := \left\{u \in H^1(\mathbb{R}^N) \ : \ u \not\equiv 0 \text{ and } \mathcal{I}'_a(u)(u) = 0\right\}.$$

A function $u \in H^1(\mathbb{R}^N)$ is a least-energy solution of (1.7) if

$$\mathcal{I}_a(u) = \inf_{v \in \mathcal{M}_a} \mathcal{I}_a(v).$$

The ground-energy function is defined by

$$\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}^+ : a \mapsto \mathcal{E}(a) := \inf_{u \in \mathcal{M}_a} \mathcal{I}_a(u),$$

and the concentration function $\mathcal{C} : \mathbb{R}^N \to (0, +\infty]$ by

$$\mathcal{C}(\xi) = \mathcal{E}(V(\xi)).$$

The following lemma states some properties of the ground-energy function.

Lemma 1.7. *Assume that* $1 < p < (N+2)/(N-2)$ *if* $N \geqslant 3$, $1 < p < \infty$ *if* $N = 1, 2$. *Then, for every* $a > 0$, $\mathcal{E}(a)$ *is a critical value of* $\mathcal{I}_a$ *and we have*

$$\mathcal{E}(a) = \inf_{\substack{u \in H^1(\mathbb{R}^N) \\ u \neq 0}} \max_{t \geqslant 0} \mathcal{I}_a(tu). \tag{1.9}$$

If $u \in \mathcal{N}_a$ *and* $\mathcal{E}(a) = \mathcal{I}_a(u)$, *then* $u \in C^1(\mathbb{R}^N)$ *and up to a translation, u is a radial function with a unique and nondegenerate maximum at the origin. Moreover,*

$$\mathcal{E}(a) = \mathcal{E}(1)\, a^{\frac{p+1}{p-1} - \frac{N}{2}}. \tag{1.10}$$

In particular, $\mathcal{E}$ *is a continuous and strictly increasing function.*

Proof. The fact that $\mathcal{E}(a)$ is a critical value of $\mathcal{I}_a$ and the minimax characterization date back at least to Rabinowitz [126], see also [145]. The radial symmetry of u follows from the Gidas-Ni-Nirenberg theorem.

Let $a > 0$. For $u \in H^1(\mathbb{R}^N)$, we set $u_a(x) := a^{1/(p-1)} u(a^{1/2} x)$. It is straightforward to compute that

$$\mathcal{I}_a(u_a) = a^{\frac{p+1}{p-1} - \frac{N}{2}} \mathcal{I}_1(u).$$

Since $u \mapsto u_a$ is a bijection of $H^1(\mathbb{R}^N)$, this implies (1.10). □

Let us point out that, since $\mathcal{C}(\xi)$ is proportional to $V(\xi)^{\frac{p+1}{p-1} - \frac{N}{2}}$, assumption (1.1) is equivalent to

$$\inf_{\Lambda} \mathcal{C} < \inf_{\partial \Lambda} \mathcal{C}. \tag{1.11}$$

3. Asymptotics of the solutions

The goal of this section is to study the asymptotic behaviour of the solutions u_ε of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to 0$. In a first step, we are going to give an upper estimate of the critical value c_ε given by (1.3).

Lemma 1.8. *Suppose that the assumptions of Theorem 1.4 are satisfied. Then,*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} c_\varepsilon \leqslant \inf_\Lambda \mathcal{C}.$$

Proof. Let $x_0 \in \Lambda$ be such that $V(x_0) = V_0 = \inf_\Lambda V$ and let u_0 be the solution of (1.7) with $a = V_0$ such that $u(0) = \max_{\mathbb{R}^N} u$. We choose a cut-off function $\eta \in C^\infty_{\mathrm{c}}(\Lambda)$ such that $0 \leqslant \eta \leqslant 1$ and $\eta = 1$ in a neighborhood of x_0. We define the test function

$$u(x) := \eta(x) u_0\left(\frac{x - x_0}{\varepsilon}\right). \tag{1.12}$$

Since $u \in H^1_V$, $u \not\equiv 0$ and $u \geqslant 0$, we deduce from (1.6) that

$$c_\varepsilon \leqslant \max_{t>0} J_\varepsilon(tu).$$

We are going to show that, as $\varepsilon \to 0$,

$$\varepsilon^{-N} J_\varepsilon(tu) = I_0(tu_0) + o(1), \tag{1.13}$$

where $I_0 := \mathcal{I}_{V_0}$. The lemma will follow readily from these two inequalities and (1.9) because

$$\varepsilon^{-N} c_\varepsilon \leqslant \max_{t>0} \varepsilon^{-N} J_\varepsilon(tu) \leqslant \max_{t>0} I_0(tu_0) + o(1) = c_0 + o(1).$$

The proof of (1.13) is a standard computation but we make it once for the sake of completeness. We do the rescaling $y = (x - x_0)/\varepsilon$ in (1.12) and we compute that

$$\begin{aligned}
\int_{\mathbb{R}^N} \varepsilon^2 \left|\nabla u\right|^2 \, dx &= \varepsilon^N \int_{\mathbb{R}^N} \left|\nabla u(x_0 + \varepsilon y)\right|^2 \, dy \\
&= \varepsilon^N \int_{\mathbb{R}^N} \left|\nabla u_0(y)\right|^2 \left|\eta(x_0 + \varepsilon y)\right|^2 \, dy \\
&\quad + \varepsilon^{N+1} \int_{\mathbb{R}^N} 2\nabla\eta(x_0 + \varepsilon y) \cdot \nabla u_0(y) \eta(x_0 + \varepsilon y) u_0(y) \, dy \\
&\quad + \varepsilon^{N+2} \int_{\mathbb{R}^N} \left|\nabla\eta(x_0 + \varepsilon y)\right|^2 \left|u_0(y)\right|^2 \, dy \\
&= \varepsilon^N \int_{\mathbb{R}^N} \left|\nabla u_0(y)\right|^2 \left|\eta(x_0 + \varepsilon y)\right|^2 \, dy + o(\varepsilon^N).
\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^N} V(x)\,|u(x)|^2\,dx = \varepsilon^N \int_{\mathbb{R}^N} V_0\,|u_0(y)|^2\,|\eta(x_0+\varepsilon y)|^2\,dy$$
$$+\,\varepsilon^N \int_{\mathbb{R}^N} (V(x_0+\varepsilon y)-V(x_0))\,|u_0(y)|^2\,|\eta(x_0+\varepsilon y)|^2\,dy.$$

Let $\delta > 0$. Since $u_0 \in L^2(\mathbb{R}^N)$, there exists a compact K containing x_0 such that

$$\int_{\mathbb{R}^N\setminus K} |V(x_0+\varepsilon y)-V(x_0)|\,|u_0(y)|^2\,|\eta(x_0+\varepsilon y)|^2\,dy \leqslant \frac{\delta}{2}.$$

By uniform continuity of V on K, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$,

$$\int_K |V(x_0+\varepsilon y)-V(x_0)|\,|u_0(y)|^2\,|\eta(x_0+\varepsilon y)|^2\,dy \leqslant \frac{\delta}{2}.$$

We conclude that for every $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{\mathbb{R}^N} V(x)\,|u(x)|^2\,dx = \varepsilon^N \int_{\mathbb{R}^N} V_0\,|u_0(y)|^2\,|\eta(x_0+\varepsilon y)|^2\,dy + o(\varepsilon^N).$$

Combining the preceding estimates, we obtain

$$\int_{\mathbb{R}^N} \left(\varepsilon^2\,|\nabla u|^2 + V u^2\right)\,dx = \varepsilon^N \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + V_0 u_0^2\right)\,dy$$
$$+\,\varepsilon^N \int_{\mathbb{R}^N} \left(|\nabla u_0(y)|^2 + V_0\,|u_0(y)|^2\right)\left(1-|\eta(x_0+\varepsilon y)|^2\right)\,dy + o(\varepsilon^N).$$

Let $\delta > 0$. Since $u_0 \in H^1(\mathbb{R}^N)$, there exists a compact K containing x_0 such that

$$\int_{\mathbb{R}^N\setminus K} \left(|\nabla u_0(y)|^2 + V_0\,|u_0(y)|^2\right)\left(1-|\eta(x_0+\varepsilon y)|^2\right)\,dy \leqslant \frac{\delta}{2}.$$

Since $1-|\eta(x_0+\varepsilon y)|^2$ converges uniformly to 0 on K as $\varepsilon \to 0$, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$,

$$\int_K \left(|\nabla u_0(y)|^2 + V_0\,|u_0(y)|^2\right)\left(1-|\eta(x_0+\varepsilon y)|^2\right)\,dy \leqslant \frac{\delta}{2}.$$

We conclude that for every $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{\mathbb{R}^N} \left(\varepsilon^2\,|\nabla u|^2 + V u^2\right)\,dx = \varepsilon^N \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + V_0 u_0^2\right)\,dy + o(\varepsilon^N).$$

Arguing in the same way, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned}\int_{\mathbb{R}^N} G(x, u(x))\, dx &= \varepsilon^N \int_{\mathbb{R}^N} |u_0(y)|^{p+1}\, |\eta(x_0 + \varepsilon y)|^{p+1}\ dy \\ &= \varepsilon^N \int_{\mathbb{R}^N} u_0^{p+1}\, dy + o(\varepsilon^N).\end{aligned}$$

The estimate (1.13) follows. □

Corollary 1.9. *Suppose that the assumptions of Theorem 1.4 are satisfied and let $u_\varepsilon \in H_V^1$ be the solution of $(\mathcal{P}_\varepsilon)$ found in that theorem. Then, there exists $C > 0$ such that, for every ε small enough,*

$$\|u_\varepsilon\|_\varepsilon \leqslant C\varepsilon^{N/2}.$$

Proof. The previous lemma implies that, as $\varepsilon \to 0$,

$$c_\varepsilon = J_\varepsilon(u_\varepsilon) - \frac{1}{p+1} J'_\varepsilon(u_\varepsilon)(u_\varepsilon) \leqslant \varepsilon^N \left(c_0 + o(1)\right).$$

We deduce from Lemma 1.2 that

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)(1-\mu)\, \|u_\varepsilon\|_\varepsilon^2 \leqslant \varepsilon^N \left(c_0 + o(1)\right).$$

This proves the required estimate. □

Lemma 1.10. *Suppose that the assumptions of Theorem 1.4 are satisfied. Let $u_\varepsilon \in H_V^1$ be the positive solution of $(\mathcal{P}_\varepsilon)$ found in that theorem, $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \mathbb{R}^N$ be sequences such that $\varepsilon_n \to 0$ and $x_n \to \bar{x}$ as $n \to \infty$. Set*

$$v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x).$$

Then, there exists $v \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ such that, along a subsequence that we still denote by $(v_n)_n$,

$$v_n \xrightarrow{C^1_{\mathrm{loc}}(\mathbb{R}^N)} v.$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v|^2 &= \lim_{R\to\infty} \lim_{n\to\infty} \varepsilon_n^{-N} \int_{B(x_n, \varepsilon_n R)} \varepsilon_n^2\, |\nabla u_{\varepsilon_n}|^2, \\ \int_{\mathbb{R}^N} V(\bar{x}) v^2 &= \lim_{R\to\infty} \lim_{n\to\infty} \varepsilon_n^{-N} \int_{B(x_n, \varepsilon_n R)} V(x) u_{\varepsilon_n}^2.\end{aligned}$$

Proof. We deduce from the previous corollary and from the continuous embedding of H^1_V in $H^1(\mathbb{R}^N)$ that the sequence $(v_n)_n$ is bounded in $H^1(\mathbb{R}^N)$. Using the Sobolev inequality and the L^p-estimates, we can do a bootstrap argument and prove that for every $R > 0$ and every $q > 1$, the sequence $(v_n)_n$ is bounded in $W^{2,q}(B(0,R))$. Now, we may extract from $(v_n)_n$ a subsequence $(v_{n_k})_k$, that converges weakly in $H^1(\mathbb{R}^N)$ to some function $v \in H^1(\mathbb{R}^N)$. The Sobolev embedding theorem implies that v_{n_k} converges to v in $C^1(B(0,R))$ for every $R > 0$. By a diagonal argument, we conclude that, up to a subsequence, v_n converges to v in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$. The last estimates follow from the continuity of V and the fact that $v \in H^1(\mathbb{R}^N)$. □

In the next three lemmas we will estimate from below the energy of the solution u_ε as a function of the number and the location of its local maxima. We prove separately the estimates inside and outside small balls, then we combine these two estimates.

Lemma 1.11. *Suppose that the assumptions of Theorem 1.4 are satisfied. Let $u_\varepsilon \in H^1_V$ be the positive solution of $(\mathcal{P}_\varepsilon)$ found in that theorem, $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \mathbb{R}^N$ be sequences such that $\varepsilon_n \to 0$, $x_n \to \bar{x}$ as $n \to \infty$ and*

$$\liminf_{n\to\infty} u_{\varepsilon_n}(x_n) > 0. \tag{1.14}$$

Then, up to a subsequence, we have

(1.15)

$$\liminf_{R\to\infty} \liminf_{n\to\infty} \varepsilon_n^{-N} \left(\int_{B_n(R)} \frac{1}{2} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V u_{\varepsilon_n}^2 \right) - G(x, u_{\varepsilon_n}) \right) \geqslant \mathcal{C}(\bar{x}),$$

where $B_n(R) := B(x_n, \varepsilon_n R)$.

Proof. Let $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x)$. Up to a subsequence, we can assume that there exists $v \in H^1(\mathbb{R}^N)$ such that $v_n \to v$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$. Since Λ is smooth, still going to a subsequence if necessary, the sequence of characteristic functions $\chi_n(x) = \chi_\Lambda(x_n + \varepsilon_n x)$ converges almost everywhere to a measurable function χ satisfying $0 \leqslant \chi(x) \leqslant 1$. Let

$$\tilde{g}(x,s) := \chi(x) s^p + (1 - \chi(x)) \min\left\{\mu V(\bar{x}) s, s^p\right\}.$$

We see that v solves the limiting equation

$$-\Delta v + V(\bar{x}) v = \tilde{g}(x, v) \qquad \text{in } \mathbb{R}^N.$$

By (1.14), $v(0) = \lim_{n\to\infty} v_n(0) > 0$, so that $v \not\equiv 0$. Let $\mathcal{F} : H^1(\mathbb{R}^N) \to \mathbb{R}$ be the functional defined by

$$\mathcal{F}(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(\varepsilon^2 \left|\nabla u(x)\right|^2 + V(\bar{x}) \left|u(x)\right|^2\right) \, dx - \int_{\mathbb{R}^N} \tilde{G}(x, u(x)) \, dx,$$

where

$$\tilde{G}(x, s) := \int_0^s \tilde{g}(x, \sigma) d\sigma.$$

We have $\tilde{g}(x, s) \leqslant s^p$ for every $x \in \mathbb{R}^N$ and thus, for all $u \in H^1(\mathbb{R}^N)$,

$$\mathcal{F}(u) \geqslant \frac{1}{2}\int_{\mathbb{R}^N} \left(\varepsilon^2 \left|\nabla u\right|^2 + V(\bar{x}) \left|u\right|^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1} = \mathcal{I}_{V(\bar{x})}(u).$$

Since v belongs to the Nehari manifold associated to $\mathcal{F}$, it follows that

$$\mathcal{F}(v) \geqslant \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ u\neq 0}} \max_{t\geqslant 0} \mathcal{F}(tu) \geqslant \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ u\neq 0}} \max_{t\geqslant 0} \mathcal{I}_{V(\bar{x})}(tu) = \mathcal{C}(\bar{x}).$$

We now claim that

$$\begin{aligned} \liminf_{R\to\infty} \liminf_{n\to\infty} \frac{1}{2}\int_{B(0,R)} &\left(\left|\nabla v_n\right|^2 + V(x_n + \varepsilon_n x) \left|v_n\right|^2\right) \\ &- \int_{B(0,R)} G(x_n + \varepsilon_n x, v_n) \geqslant \mathcal{F}(v). \end{aligned}$$

Let

$$h_n(x) := \frac{1}{2}\left(\left|\nabla v_n\right|^2 + V(x_n + \varepsilon_n x) \left|v_n\right|^2\right) - G(x_n + \varepsilon_n x, v_n).$$

For every $R > 0$, the uniform convergence of v_n in $C^1(B(0, R))$ implies that

$$\begin{aligned} \lim_{n\to\infty} \int_{B(0,R)} h_n &= \int_{B(0,R)} \lim_{n\to\infty} h_n \\ &= \frac{1}{2}\int_{B(0,R)} \left(\left|\nabla v\right|^2 + V(\bar{x}) \left|v\right|^2\right) - \int_{B(0,R)} \tilde{G}(x, v). \end{aligned}$$

Since $v \in H^1(\mathbb{R}^N)$, for every $\delta > 0$, we can find R_0 large enough so that for all $R \geqslant R_0$,

$$\lim_{n\to\infty} \int_{B(0,R)} h_n \geqslant \mathcal{F}(v) - \delta.$$

The claim follows. $\square$

Lemma 1.12. *Suppose that the assumptions of Theorem 1.4 are satisfied. Let $u_\varepsilon \in H^1_V$ be the positive solution of $(\mathcal{P}_\varepsilon)$ found in that theorem, $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x^i_n)_n \subset \mathbb{R}^N$, $1 \leqslant i \leqslant K$ be sequences such that $\varepsilon_n \to 0$, $x^i_n \to \bar{x}^i \in \bar{\Lambda}$ as $n \to \infty$. Then, up to a subsequence, we have*

(1.16)

$$\liminf_{R\to\infty} \liminf_{n\to\infty} \varepsilon_n^{-N} \left(\int_{\mathbb{R}^N \setminus \mathcal{B}_n(R)} \frac{1}{2} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V u_{\varepsilon_n}^2 \right) - G(x, u_{\varepsilon_n}) \right) \geqslant 0,$$

where $\mathcal{B}_n(R) := \bigcup_{i=1}^K B(x^i_n, \varepsilon_n R)$.

Proof. Let $\eta_{R,\varepsilon_n} \in C^\infty(\mathbb{R}^N)$ be a cut-off function such that $\eta_{R,\varepsilon_n} \equiv 0$ in $\mathcal{B}_n(R/2)$, $\eta_{R,\varepsilon_n} \equiv 1$ in $\mathcal{B}_n(R)$, $0 \leqslant \eta_{R,\varepsilon_n} \leqslant 1$ and $|\nabla \eta_{R,\varepsilon_n}| \leqslant 2/(\varepsilon_n R)$. Using $\eta_{R,\varepsilon_n} u_{\varepsilon_n}$ as a test function in $(\mathcal{P}_\varepsilon)$, we obtain

$$\int_{\mathbb{R}^N} \varepsilon_n^2 \nabla u_{\varepsilon_n} \cdot \nabla \left(\eta_{R,\varepsilon_n} u_{\varepsilon_n} \right) + V \left|u_{\varepsilon_n}\right|^2 \eta_{R,\varepsilon_n} - g(x, u_{\varepsilon_n}) u_{\varepsilon_n} \eta_{R,\varepsilon_n} = 0.$$

We deduce from (g_3) and from the last equality that

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^N \setminus \mathcal{B}_n(R)} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V \left|u_{\varepsilon_n}\right|^2 \right) - \int_{\mathbb{R}^N \setminus \mathcal{B}_n(R)} G(x, u_{\varepsilon_n}) \\
&\quad \geqslant \frac{1}{2} \int_{\mathbb{R}^N \setminus \mathcal{B}_n(R)} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V \left|u_{\varepsilon_n}\right|^2 - g(x, u_{\varepsilon_n}) u_{\varepsilon_n} \right) \\
&\quad = -\frac{1}{2} \int_{\mathcal{A}_n(R)} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V \left|u_{\varepsilon_n}\right|^2 - g(x, u_{\varepsilon_n}) u_{\varepsilon_n} \right) \eta_{R,\varepsilon_n} \\
&\qquad - \frac{1}{2} \int_{\mathcal{A}_n(R)} \varepsilon_n^2 u_{\varepsilon_n} \nabla u_{\varepsilon_n} \cdot \nabla \eta_{R,\varepsilon_n},
\end{aligned}$$

where $\mathcal{A}_n(R) := \mathcal{B}_n(R) \setminus \mathcal{B}_n(R/2)$. For the last term, using the Cauchy-Schwarz inequality and Corollary 1.9, we compute that

$$\begin{aligned}
\left| \varepsilon_n^{-N} \int_{\mathcal{A}_n(R)} \varepsilon_n^2 u_{\varepsilon_n} \nabla u_{\varepsilon_n} \cdot \nabla \eta_{R,\varepsilon_n} \right| &\leqslant \frac{2}{R}\, \varepsilon_n^{-N} \int_{\mathcal{A}_n(R)} \varepsilon_n u_{\varepsilon_n} \left|\nabla u_{\varepsilon_n}\right| \\
&\leqslant \frac{2}{R}\, \varepsilon_n^{-N} \left(\int_{\mathcal{A}_n(R)} \left|u_{\varepsilon_n}\right|^2 \right)^{1/2} \left(\int_{\mathcal{A}_n(R)} \varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 \right)^{1/2} \\
&\leqslant \frac{2}{R} \sup_{\mathcal{A}_n(R)} V^{-1/2}\, \varepsilon_n^{-N} \left(\int_{\mathbb{R}^N} V \left|u_{\varepsilon_n}\right|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} \varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 \right)^{1/2} \\
&\leqslant \frac{C}{R}.
\end{aligned}$$

On the other hand, using the inequality $g(x,u) \leqslant u^p$ for all $x \in \mathbb{R}^N$ and the Sobolev inequality, we infer that

$$\left| \liminf_{n\to\infty} \varepsilon_n^{-N} \int_{\mathcal{A}_n(R)} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V \left|u_{\varepsilon_n}\right|^2 - g(x, u_{\varepsilon_n}) u_{\varepsilon_n} \right) \eta_{R,\varepsilon_n} \right|$$
$$\leqslant \liminf_{n\to\infty} \varepsilon_n^{-N} \sum_{i=1}^{K} C_i \left(I_{i,n,R}^2 + I_{i,n,R}^{p+1} \right),$$

where the constants C_i only depend on $\bar{x}_i$, and

$$I_{i,n,R} := \left(\int_{B(x_n^i, \varepsilon_n R) \setminus B(x_n^i, \varepsilon_n R/2)} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V \left|u_{\varepsilon_n}\right|^2 \right) \eta_{R,\varepsilon_n} \right)^{1/2}.$$

The estimates of Lemma 1.10 imply that

$$\lim_{R\to\infty} \lim_{n\to\infty} I_{i,n,R} = 0.$$

Hence, the conclusion follows from the previous estimates. $\square$

Lemma 1.13. *Suppose that the assumptions of Theorem 1.4 are satisfied. Let $u_\varepsilon \in H_V^1$ be the positive solution of $(\mathcal{P}_\varepsilon)$ found in that theorem, $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n^i)_n \subset \mathbb{R}^N$, $1 \leqslant i \leqslant K$ be sequences such that $\varepsilon_n \to 0$, $x_n^i \to \bar{x}^i \in \bar{\Lambda}$ as $n \to \infty$. If for every $1 \leqslant i \leqslant j \leqslant K$, we have*

$$\limsup_{n\to\infty} \frac{|x_n^i - x_n^j|}{\varepsilon_n} = \infty$$

and if for every $1 \leqslant i \leqslant K$,

$$\liminf_{n\to\infty} u_{\varepsilon_n}(x_n^i) > 0,$$

then

$$\liminf_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geqslant \sum_{i=1}^{K} \mathcal{C}(\bar{x}^i). \tag{1.17}$$

Proof. Going to a subsequence if necessary, we can assume that for every $1 \leqslant i \leqslant j \leqslant K$, we have

$$\lim_{n\to\infty} \frac{|x_n^i - x_n^j|}{\varepsilon_n} = \infty.$$

Let $\delta > 0$. By the two previous lemmas, we can choose R large enough so that

$$\liminf_{n\to\infty} \varepsilon_n^{-N} \left(\int_{\mathbb{R}^N \setminus \mathcal{B}_n(R)} \frac{1}{2} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V u_{\varepsilon_n}^2 \right) - G(x, u_{\varepsilon_n}) \right) \geqslant -\delta,$$

where $\mathcal{B}_n(R) := \bigcup_{i=1}^K B(x_n^i, \varepsilon_n R)$ and for every $1 \leqslant i \leqslant K$,

$$\liminf_{n\to\infty} \varepsilon_n^{-N} \left(\int_{B_n^i(R)} \frac{1}{2} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V u_{\varepsilon_n}^2 \right) - G(x, u_{\varepsilon_n}) \right) \geqslant \mathcal{C}(\bar{x}^i) - \delta,$$

where $B_n^i(R) := B(x_n^i, \varepsilon_n R)$. Since for n sufficiently large, the balls $B_n^i(R)$ are mutually disjoint, we may write that

$$\begin{aligned} J_{\varepsilon_n}(u_{\varepsilon_n}) = \sum_{i=1}^K & \left(\int_{B_n^i(R)} \frac{1}{2} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V u_{\varepsilon_n}^2 \right) - G(x, u_{\varepsilon_n}) \right) \\ & + \int_{\mathbb{R}^N \setminus \mathcal{B}_n(R)} \frac{1}{2} \left(\varepsilon_n^2 \left|\nabla u_{\varepsilon_n}\right|^2 + V u_{\varepsilon_n}^2 \right) - G(x, u_{\varepsilon_n}). \end{aligned}$$

We conclude that

$$\liminf_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geqslant \sum_{i=1}^K \mathcal{C}(\bar{x}^i) - (K+1)\delta.$$

Since this can be done for every subsequence, the conclusion holds for the whole sequence. $\square$

Let us show that there indeed exists a sequence $(x_n)_n$ satisfying the assumptions of the previous lemma.

Lemma 1.14. *If $u_\varepsilon \in H_V^1$ is a weak positive solution of $(\mathcal{P}_\varepsilon)$, then*

$$\left\|u_\varepsilon\right\|_{L^\infty(\Lambda)} > (\mu V_0)^{1/(p-1)}.$$

Proof. Assume by contradiction that $u_\varepsilon \leqslant (\mu V_0)^{1/(p-1)}$ in Λ. Then, for every $x \in \mathbb{R}^N$, we have

$$g(x, u_\varepsilon(x)) \leqslant \mu V(x) u_\varepsilon(x).$$

This implies that u_ε satisfies the inequality

$$-\varepsilon^2 \Delta u_\varepsilon + (1-\mu) V u_\varepsilon \leqslant 0 \qquad \text{in } \mathbb{R}^N.$$

Multiplying by $(u_\varepsilon)_+$ and integrating, we get

$$\int_{\mathbb{R}^N} \left(\varepsilon^2 \left|\nabla (u_\varepsilon)_+\right|^2 + (1-\mu) V (u_\varepsilon)_+^2 \right) \leqslant 0,$$

contradicting the positivity of u_ε. $\square$

We can now prove a first concentration result.

Lemma 1.15. *Suppose that the assumptions of Theorem 1.4 are satisfied. Let $u_\varepsilon \in H^1_V$ be the positive solution of $(\mathcal{P}_\varepsilon)$ found in that theorem and let $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ be such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Then,

(i) $\lim_{\varepsilon\to 0} V(x_\varepsilon) = \inf_\Lambda V$,

(ii) $\liminf_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \partial\Lambda) > 0$,

(iii) for every $\delta > 0$, there exists $\varepsilon_0 > 0$ and $R > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$,

$$\|u_\varepsilon\|_{L^\infty(\Lambda\setminus B(x_\varepsilon, \varepsilon R))} \leqslant \delta.$$

Proof. Assume by contradiction that the first assertion does not hold. By compactness, there exists a sequence $(\varepsilon_n)_{n\geqslant 1}$ such that $\varepsilon_n \to 0$, $x_{\varepsilon_n} \to \bar{x} \in \bar{\Lambda}$ and $\mathcal{C}(\bar{x}) > \inf_\Lambda \mathcal{C}$. We infer from Lemma 1.13 that

$$\liminf_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geqslant \mathcal{C}(\bar{x}).$$

The last two inequalities contradict Lemma 1.8.

For the second assertion, assume by contradiction that there exists a sequence $(\varepsilon_n)_{n\geqslant 1}$ such that $\varepsilon_n \to 0$ and $\lim_{n\to\infty} \operatorname{dist}(x_{\varepsilon_n}, \partial\Lambda) = 0$. Then, by Lemma 1.13 and (1.11),

$$\liminf_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geqslant \liminf_{n\to\infty} \mathcal{C}(x_{\varepsilon_n}) \geqslant \inf_{\partial\Lambda} \mathcal{C} > \inf_{\Lambda} \mathcal{C}.$$

But this contradicts Lemma 1.8.

For the third assertion, assume by contradiction that there exist sequences $(\varepsilon_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ such that $y_n \in \Lambda$,

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad u_{\varepsilon_n}(y_n) \geqslant \delta \qquad \text{and} \qquad \lim_{n\to\infty} \frac{|x_{\varepsilon_n} - y_n|}{\varepsilon_n} = \infty.$$

Then, by Lemma 1.13,

$$\liminf_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geqslant \liminf_{n\to\infty} \big(\mathcal{C}(x_{\varepsilon_n}) + \mathcal{C}(y_n)\big) \geqslant 2 \inf_\Lambda \mathcal{C}.$$

Since $\inf_\Lambda \mathcal{C} > 0$, this is again a contradiction with Lemma 1.8. □

4. Solution of the original problem

We can at last prove an existence theorem for the original problem (NLS).

Theorem 1.16. *Assume that V is a continuous potential bounded away from zero and that $\Lambda \subset \mathbb{R}^N$ is a bounded open set satisfying*

$$V_0 := \inf_{x\in\Lambda} V(x) < \inf_{x\in\partial\Lambda} V(x).$$

Then, there exists $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$*, problem* (NLS) *has at least one positive solution* $u_\varepsilon \in H^1_V \cap C^{2,\alpha}(\mathbb{R}^N)$*.*

Proof. Let $u_\varepsilon \in H^1_V$ be the positive solution of $(\mathcal{P}_\varepsilon)$ found in Theorem 1.4 and let $x_\varepsilon \in \bar{\Lambda}$ be such that $u_\varepsilon(x_\varepsilon) = \max_{\bar{\Lambda}} u_\varepsilon$. By Lemma 1.14,

$$\liminf_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Let

$$\delta := \Big(\mu \inf_{\mathbb{R}^N} V\Big)^{1/(p-1)}.$$

In view of Lemma 1.15, there exists $\varepsilon_0 > 0$ and $R > 0$ such that

$$\varepsilon_0 R \leqslant \frac{1}{2} \liminf_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \partial\Lambda)$$

and for every $\varepsilon \in (0, \varepsilon_0)$,

$$\|u_\varepsilon\|_{L^\infty(\Lambda \setminus B(x_\varepsilon, \varepsilon R))} \leqslant \delta.$$

We deduce that $u_\varepsilon \leqslant \delta$ on $\partial\Lambda$. Since u_ε satisfies the inequality

$$-\varepsilon^2 \Delta u_\varepsilon + (1-\mu) V u_\varepsilon \leqslant 0 \qquad \text{in } \mathbb{R}^N \setminus \Lambda,$$

the maximum principle implies that $u_\varepsilon \leqslant \delta$ in $\mathbb{R}^N \setminus \Lambda$. We conclude that $u_\varepsilon^p \leqslant \mu V u_\varepsilon$ in $\mathbb{R}^N \setminus \Lambda$ and thus $g(x, u_\varepsilon) = u_\varepsilon^p$ in $\mathbb{R}^N \setminus \Lambda$. □

In order to complete the proof of the del Pino-Felmer Theorem, we first improve Lemma 1.10 about the convergence of sequences of rescaled solutions.

Lemma 1.17. *Suppose that the assumptions of Theorem 1.16 are satisfied and assume furthermore that* V *is locally Hölder continuous in* Λ*. Let* $u_\varepsilon \in C^{2,\alpha}(\mathbb{R}^N)$ *be the positive solution of* (NLS) *found in that theorem,* $(\varepsilon_n)_n \subset \mathbb{R}^+$ *and* $(x_n)_n \subset \mathbb{R}^N$ *be sequences such that* $\varepsilon_n \to 0$ *and* $x_n \to \bar{x}$ *as* $n \to \infty$*. Set*

$$v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x).$$

Then, there exists $v \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ *such that, along a subsequence that we still denote by* $(v_n)_n$*,*

$$v_n \overset{C^2_{\mathrm{loc}}(\mathbb{R}^N)}{\longrightarrow} v.$$

Proof. This follows from Lemma 1.10 and from the Schauder regularity estimates. □

Lemma 1.18. *Suppose that the assumptions of Theorem 1.16 are satisfied and assume furthermore that V is locally Hölder continuous in Λ. Let $u_\varepsilon \in C^{2,\alpha}(\mathbb{R}^N)$ be the positive solution of* (NLS) *found in that theorem. There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, u_ε has exactly one local (hence global) maximum $x_\varepsilon \in \Lambda$.*

Proof. First we notice that if x_ε is a local maximum of u_ε, then $x_\varepsilon \in \Lambda$. Indeed, since $\Delta u_\varepsilon(x_\varepsilon) \leqslant 0$, we must have $V(x_\varepsilon)u_\varepsilon(x_\varepsilon) \leqslant u_\varepsilon^p(x_\varepsilon)$. But the fact that u_ε also solves the penalized problem $(\mathcal{P}_\varepsilon)$ implies that for every $x \in \mathbb{R}^N \setminus \Lambda$, $u_\varepsilon^p(x) \leqslant \mu V(x)u_\varepsilon(x)$.

By the same argument, if x_ε is a local maximum of u_ε, then $u_\varepsilon(x_\varepsilon) \geqslant V_0^{1/(p-1)}$.

Now we assume by contradiction that there exists a sequence $\varepsilon_n \to 0$ such that u_{ε_n} has at least two distinct local maxima $x_n^1, x_n^2 \in \Lambda$. We may assume that $x_n^i \to \bar{x}^i \in \bar{\Lambda}$, $i = 1, 2$. Set $v_n(x) := u_{\varepsilon_n}(x_n^1 + \varepsilon_n x)$. By the previous lemma, there exists $v \in H^1(\mathbb{R}^N)$ such that $v_n \to v$ in $C^2_{\text{loc}}(\mathbb{R}^N)$ and v is a solution of the limit problem (1.7) with $a = V(\bar{x}^1)$. We set

$$z_n := \frac{x_n^2 - x_n^1}{\varepsilon_n}.$$

By Lemma 1.15, $\limsup_{n\to\infty} |z_n| < \infty$. Up to a subsequence, $z_n \to z \in \mathbb{R}^N$. Since $v_n(z_n) = u_{\varepsilon_n}(x_n^2)$, z_n is a local maximum of v_n. Hence $\nabla v_n(z_n) = 0$ and

$$\nabla v(z) = \lim_{n\to\infty} \nabla v_n(z_n) = 0.$$

By Lemma 1.7, $z = 0$. We conclude that $x_n^2 - x_n^1 \to 0$, but this contradicts the nondegeneracy of the maximum of v. □

CHAPTER II

Concentration on k-dimensional spheres

We are concerned with solutions concentrating on spheres for the nonlinear Schrödinger equation

$$\text{(NLS)} \qquad -\varepsilon^2 \Delta u + V(x)u = u^p, \quad \text{in } \mathbb{R}^N, \quad u > 0.$$

Our results include the following simple particular case.

Theorem. *Let $N \geqslant 3$, $p > \frac{N}{N-2}$ and $V \in C(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^+)$ be a radial potential. If there exists $r^* > 0$ such that the auxiliary potential $\mathcal{M} : (0, \infty) \to \mathbb{R}$ defined for $r > 0$ by*

$$\mathcal{M}(r) := r^{N-1} \left[V(r)\right]^{\frac{p+1}{p-1} - \frac{1}{2}}$$

has an isolated local minimum at $r = r^*$ such that $\mathcal{M}(r^*) > 0$, then for ε small enough, the equation (NLS) *has a positive radially symmetric solution u_ε that concentrates on the sphere of radius r^*.*

The previous theorem is a particular case of Theorem 2.2 below, which deals with a nonlinearity which is neither necessarily homogeneous nor autonomous, see equation (2.1) below. As explained in the introduction, the main interest of our results is that they require weaker assumptions on the potential compared to the previous works. Furthermore, we will find solutions concentrating on a k-dimensional sphere, $1 \leqslant k \leqslant N - 1$. We also obtain results for $N = 2$ with a little more care, see Section 5. If $N \geqslant 5$, one has that $u_\varepsilon \in L^2(\mathbb{R}^N)$ (see Corollary 2.20).

Let us give an example of a potential V satisfying the assumptions of our theorem and not treated in other works. Let $N = 3$ and $p = 5$. We define $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^+$ by

$$V(x) := \begin{cases} \frac{1}{|x|^2} + 1 & \text{if } 0 < |x| < \frac{1}{2}, \\ \frac{1}{|x|^2} + \left(1 - \frac{1}{|x|}\right)^2 & \text{if } \frac{1}{2} \leqslant |x| \leqslant 2, \\ \frac{8}{|x|^4} & \text{if } 2 < |x|. \end{cases}$$

We compute that $\mathcal{M}(r) = 1 + (r-1)^2$ if $\frac{1}{2} \leqslant r \leqslant 2$, so that $\mathcal{M}$ has an isolated local minimum at $r = 1$.

Let us point out that if V is compactly supported and $p \leqslant \frac{N}{N-2}$, then equation (NLS) has no positive solution in the neighborhood of infinity, in accordance with a result of Gidas and Spruck [85], see also the discussion in [113].

Assuming that the potential V is cylindrically symmetric, we can reduce (NLS) to a problem in $\mathbb{R}^{N-k}$. The single-peaked solutions of this problem can then be extended to $\mathbb{R}^N$ by symmetry. In this way, we obtain a solution of (NLS) concentrating around a k-dimensional sphere. Observe that since the reduced problem is in $\mathbb{R}^{N-k}$, the critical exponent to be considered is the one in dimension $N-k$, i.e. $p_k = \frac{N-k+2}{N-k-2}$ if $N-k \geqslant 3$, $p_k = \infty$ if $N-k = 1, 2$. This allows for example to treat critical or supercritical problems by looking for cylindrically symmetric (non necessarily radial) solutions.

1. Assumptions and main result

We shall study the equation with a more general nonlinearity

$$-\varepsilon^2 \Delta u + V(x) u = K(x) f(u), \qquad x \in \mathbb{R}^N. \tag{2.1}$$

Let k be a fixed integer such that $1 \leqslant k \leqslant N-1$. This number k is the dimension of the sphere on which we want to construct concentrating solutions. Let us choose any $(N-k-1)$-dimensional linear subspace $\mathcal{H} \subset \mathbb{R}^N$. We denote by $\mathcal{H}^\perp$ the orthogonal complement of $\mathcal{H}$.

1.1. The potentials

We consider a nonnegative potential $V \in C(\mathbb{R}^N \setminus \{0\})$ and a nonnegative competing function $K \in C(\mathbb{R}^N \setminus \{0\})$, $K \not\equiv 0$. We assume that for every $R \in \mathbf{O}(N)$ such that $R(\mathcal{H}) = \mathcal{H}$, we have $V \circ R = V$ and $K \circ R = K$. This will be the case if for example V and K are radial functions. Notice that if we write $x = (x', x'')$ with $x' \in \mathcal{H}$ and $x'' \in \mathcal{H}^\perp$, this assumption is equivalent to the fact that V and K only depend on $|x'|$ and $|x''|$.

1.2. The nonlinearity

We make classical assumptions on f that lead to a good minimax characterization of the infimum on the Nehari manifold. Namely, we assume that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and that

- (f_1) there exists $q > 1$ such that $f(s) = O(s^q)$ as $s \to 0^+$,
- (f_2) there exists $p > 1$ such that $\frac{1}{p+1} > \frac{1}{2} - \frac{1}{N-k}$ and $f(s) = O(s^p)$ as $s \to \infty$,

(f_3) there exists $2 < \theta \leqslant p+1$ such that

$$0 < \theta F(s) \leqslant f(s)s \qquad \text{for } s > 0,$$

where $F(s) := \int_0^s f(\sigma)d\sigma$,

(f_4) the function

$$s \mapsto \frac{f(s)}{s}$$

is nondecreasing.

Notice that (f_2) is nothing but the subcriticality condition in dimension $N-k$.

1.3. The growth conditions

Following [30, 113] we impose one of the three sets of growth conditions at infinity :

($\mathcal{G}^1_\infty$) there exists $\sigma < (N-2)q - N$ such that

$$\limsup_{|x|\to\infty} \frac{K(x)}{|x|^\sigma} < \infty;$$

($\mathcal{G}^2_\infty$) there exists $\sigma \in \mathbb{R}$ such that

$$\liminf_{|x|\to\infty} V(x)\,|x|^2 > 0 \qquad \text{and} \qquad \limsup_{|x|\to\infty} \frac{K(x)}{|x|^\sigma} < \infty;$$

($\mathcal{G}^3_\infty$) there exist $\alpha < 2$ and $\sigma \in \mathbb{R}$ such that

$$\liminf_{|x|\to\infty} V(x)\,|x|^\alpha > 0 \qquad \text{and} \qquad \limsup_{|x|\to\infty} \frac{K(x)}{\exp(\sigma\,|x|^{\frac{2-\alpha}{2}})} < \infty.$$

Note that in comparison with [30], in ($\mathcal{G}^2_\infty$) and ($\mathcal{G}^3_\infty$), V might vanish somewhere. We also impose one of the three sets of growth conditions at the origin, which mirror those at infinity :

($\mathcal{G}^1_0$) there exists $\tau > -2$, such that

$$\limsup_{|x|\to 0} \frac{K(x)}{|x|^\tau} < \infty,$$

($\mathcal{G}^2_0$) there exists $\tau \in \mathbb{R}$ such that

$$\liminf_{|x|\to 0} V(x)\,|x|^2 > 0 \qquad \text{and} \qquad \limsup_{|x|\to 0} \frac{K(x)}{|x|^\tau} < \infty;$$

($\mathcal{G}_0^3$) there exist $\gamma > 2$ and $\tau \in \mathbb{R}$ such that

$$\liminf_{|x|\to 0} V(x)\,|x|^\gamma > 0 \qquad \text{and} \qquad \limsup_{|x|\to 0} \frac{K(x)}{\exp(\tau\,|x|^{-\frac{\gamma-2}{2}})} < \infty.$$

By Kelvin transform, there is a duality between the conditions at the origin and the conditions at infinity, at least in the case where $f(t) = t^p$. If one defines $\hat{u}$ to be the Kelvin transform of u, i.e.,

$$\hat{u}(x) = \frac{1}{|x|^{N-2}} u\Big(\frac{x}{|x|^2}\Big)$$

and the transformed potentials

$$\hat{V}(x) = \frac{1}{|x|^4} V\Big(\frac{x}{|x|^2}\Big)$$

and

$$\hat{K}(x) = \frac{1}{|x|^{N+2-p(N-2)}} K\Big(\frac{x}{|x|^2}\Big),$$

the function u_ε solves (2.1) if and only if $\hat{u}_\varepsilon$ solves the same problem with $\hat{V}$ and $\hat{K}$ in place of V and K. One sees that V, K satisfy ($\mathcal{G}_0^i$) if and only if $\hat{V}$ and $\hat{K}$ satisfy ($\mathcal{G}_\infty^i$).

The problem at the origin is in a sense in duality with the one at infinity. Whereas a slow decay of V at infinity does allow a lot of freedom for K, a strong singularity at the origin allows for very singular K's too. The critical threshold growth is $1/|x|^2$ both at the origin and at infinity. This can be made clearer if we observe that the optimal barrier functions at the origin are the optimal ones at infinity mapped by Kelvin transform.

1.4. The auxiliary potential

Before we can state our last assumption, we need a few preliminaries. Let $a, b > 0$. The equation

$$-\Delta u + au = bf(u) \qquad \text{in } \mathbb{R}^{N-k} \tag{2.2}$$

is called the *limit equation* associated with (2.1). The weak solutions of (2.2) are critical points of the functional $\mathcal{I}_{a,b} : H^1(\mathbb{R}^{N-k}) \to \mathbb{R}$ defined by

$$\mathcal{I}_{a,b}(u) := \frac{1}{2}\int_{\mathbb{R}^{N-k}} \Big(|\nabla u|^2 + au^2\Big)\,dx - b\int_{\mathbb{R}^{N-k}} F(u)\,dx. \tag{2.3}$$

Any nontrivial critical point $u \in H^1(\mathbb{R}^{N-k})$ of $\mathcal{I}_{a,b}$, belongs to the Nehari manifold

$$\mathcal{N}_{a,b} := \left\{ u \in H^1(\mathbb{R}^{N-k}) \mid u \not\equiv 0 \text{ and } \langle \mathcal{I}'_{a,b}(u), u \rangle = 0 \right\}.$$

A solution $u \in H^1(\mathbb{R}^{N-k})$ is a *least-energy solution* of (2.2) if

$$\mathcal{I}_{a,b}(u) = \inf_{v \in \mathcal{N}_{a,b}} \mathcal{I}_{a,b}(v).$$

The *ground-energy function* is defined by

$$\mathcal{E} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ : (a,b) \mapsto \mathcal{E}(a,b) := \inf_{u \in \mathcal{N}_{a,b}} \mathcal{I}_{a,b}(u),$$

and the *auxiliary potential* $\mathcal{M} : \mathbb{R}^N \to (0, +\infty]$ by

$$x = (x', x'') \mapsto \mathcal{M}(x', x'') := \begin{cases} |x''|^k \, \mathcal{E}\left(V(x), K(x)\right) & \text{if } K(x) > 0, \\ +\infty & \text{if } K(x) = 0. \end{cases}$$

The following lemma states some properties of the ground-energy function, see [30, Lemma 3].

Lemma 2.1. *Assume $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that fulfills assumptions (f_1)-(f_4). Then, for every $(a,b) \in \mathbb{R}^+_0 \times \mathbb{R}^+_0$, $\mathcal{E}(a,b)$ is a critical value of $\mathcal{I}_{a,b}$ and we have*

$$\mathcal{E}(a,b) = \inf_{\substack{u \in H^1(\mathbb{R}^N) \\ u \neq 0}} \max_{t \geqslant 0} \mathcal{I}_{a,b}(tu).$$

If $u \in \mathcal{N}_{a,b}$ and $\mathcal{E}(a,b) = \mathcal{I}_{a,b}(u)$, then $u \in C^1(\mathbb{R}^N)$ and up to a translation, u is a radial function such that $\nabla u(x) \cdot x < 0$ for every $x \in \mathbb{R}^N \setminus \{0\}$. Moreover, the following properties hold:

(i) *$\mathcal{E}$ is continuous in $\mathbb{R}^+_0 \times \mathbb{R}^+_0$;*
(ii) *for every $b^* \in \mathbb{R}^+_0$, $a \to \mathcal{E}(a, b^*)$ is strictly increasing;*
(iii) *for every $a^* \in \mathbb{R}^+_0$, $b \to \mathcal{E}(a^*, b)$ is strictly decreasing;*
(iv) *for every $\lambda > 0$, $\mathcal{E}(\lambda a, \lambda b) = \lambda^{1-N/2} \mathcal{E}(a,b)$;*
(v) *if $f(u) = u^p$ with $\frac{1}{2} - \frac{1}{N-k} < \frac{1}{p+1} < \frac{1}{2}$, then*

$$\mathcal{E}(a,b) = \mathcal{E}(1,1) a^{\frac{p+1}{p-1} - \frac{N}{2}} b^{-\frac{2}{p-1}}.$$

If $f(u) = u^p$ with $\frac{1}{2} - \frac{1}{N-k} < \frac{1}{p+1} < \frac{1}{2}$, the last property of the preceding lemma implies the following explicit form of the auxiliary potential:

$$\mathcal{M}(x', x'') = \mathcal{E}(1,1) \left| x'' \right|^k \left[V(x) \right]^{\frac{p+1}{p-1} - \frac{N-k}{2}} \left[K(x) \right]^{\frac{-2}{p-1}}.$$

Due to the symmetry that we shall impose on the solution (see (2.11)), the concentration can only occur in the space $\mathcal{H}^{\perp}$. We assume that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$\bar{\Lambda} \cap \mathcal{H} = \emptyset, \ \Lambda \cap \mathcal{H}^{\perp} \neq \emptyset, \tag{2.4}$$

for every $R \in \mathbf{O}(N)$ such that $R(\mathcal{H}) = \mathcal{H}$,

$$R(\Lambda) = \Lambda \tag{2.5}$$

and

$$0 < \inf_{\Lambda \cap \mathcal{H}^{\perp}} \mathcal{M} < \inf_{\partial\Lambda \cap \mathcal{H}^{\perp}} \mathcal{M}. \tag{2.6}$$

In the case where $k = N-2$, we shall need the condition

$$\inf_{\Lambda \cap \mathcal{H}^{\perp}} \mathcal{M} < 2 \inf_{\Lambda} \mathcal{M}. \tag{2.7}$$

By continuity of $\mathcal{M}$ in Λ, this condition is not restrictive. Indeed, we can always replace Λ by

$$\tilde{\Lambda} := \left\{ x \in \Lambda \ : \ \mathcal{M}(0, x'') < 2 \inf_{\Lambda} \mathcal{M} \right\}.$$

Similarly, we can also assume that $V > 0$ on $\overline{\Lambda}$ and that $\mathcal{M}$ is continuous on $\overline{\Lambda}$.

Our main result is the following theorem.

Theorem 2.2. *Let $N \geqslant 2$, $V, K \in C(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^+)$ satisfy one set $(\mathcal{G}_0^i)$ of growth conditions at the origin and one set $(\mathcal{G}_\infty^j)$ of growth conditions at infinity, and f satisfy assumptions (f_1)-(f_4). Assume there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that* (2.4), (2.5), (2.6) *and, if $k = N-2$,* (2.7) *hold. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, problem* (NLS) *has at least one positive solution u_ε. Moreover, for every $0 < \varepsilon < \varepsilon_0$, there exists $x_\varepsilon \in \Lambda \cap \mathcal{H}^{\perp}$ such that u_ε attains its maximum at x_ε,*

$$\liminf_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) > 0,$$
$$\lim_{\varepsilon \to 0} \mathcal{M}(x_\varepsilon) = \inf_{\Lambda \cap \mathcal{H}^{\perp}} \mathcal{M},$$

and there exist $C > 0$ and $\lambda > 0$ such that

$$u_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d(x, S_\varepsilon^k)}{1 + d(x, S_\varepsilon^k)} \right) \left(1 + |x|^2\right)^{\frac{-(N-2)}{2}}, \qquad \forall x \in \mathbb{R}^N,$$

where S_ε^k is the k-sphere centered at the origin and of radius $|x''_\varepsilon|$.

In the special case where $x_0 \in \Lambda \cap \mathcal{H}^\perp$ is the unique minimizer of $\mathcal{M}$ on $\Lambda \cap \mathcal{H}^\perp$, then $x_\varepsilon \to x_0$, and the solution concentrates around a k–dimensional sphere of radius $|x_0|$ centered at the origin.

One should note that the theorem is valid in dimension 2, but the solutions that are obtained do not decay at infinity in general.

The theorem stated in the introduction follows from Theorem 2.2 by taking $K \equiv 1$, $f(u) = u^p$ and $k = N-1$. Indeed, we notice that the growth condition $(\mathcal{G}_0^1)$ is always satisfied whereas the condition $(\mathcal{G}_\infty^1)$ holds if and only if $(N-2)p - N > 0$, i.e. $p > \frac{N}{N-2}$.

The sequel of the paper is devoted to the proof of Theorem 2.2. In Section 2, we introduce a penalized problem and prove that it has a least-energy solution. In Section 3, we study the asymptotics of this solution and in Section 4, we obtain decay estimates of the solution and show that it also solves the original problem. In all these sections, we assume that $N \geqslant 3$. The modifications for the case $N = 2$ will be addressed in Section 5.

2. The penalization scheme

We assume that $N \geqslant 3$. The homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of the set of compactly supported smooth functions $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm

$$\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{\frac{1}{2}}.$$

Thanks to Sobolev inequality, we have $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$. Let us also recall Hardy's inequality in $\mathbb{R}^N$. One has

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}\, dx \leqslant \int_{\mathbb{R}^N} |\nabla u|^2,$$

for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Following [113], we define the *penalization potential* $H : \mathbb{R}^N \to \mathbb{R}$ by

$$H(x) := \frac{\kappa}{|x|^2 \left(\left(\log |x|\right)^2 + 1\right)^{\frac{1+\beta}{2}}}$$

where $\beta > 0$ and $0 < \kappa < \frac{1}{2}(\frac{N-2}{2})^2$. Notice that for all $x \in \mathbb{R}^N$, we have

$$H(x) \leqslant \frac{\kappa}{|x|^2}.$$

By Hardy's inequality, we deduce that the quadratic form associated to $-\Delta - H$ is positive, i.e.

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 - Hu^2\right) \geqslant \left(\left(\frac{N-2}{2}\right)^2 - \kappa\right) \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}\, dx \geqslant 0, \tag{2.8}$$

for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

This inequality implies the following comparison principle.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^N \setminus \{0\}$ be a smooth domain. Let $v, w \in H^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ be such that $\nabla(w-v)_- \in L^2(\Omega)$, $(w-v)_-/\,|x| \in L^2(\Omega)$ and*

$$-\Delta w - Hw \geqslant -\Delta v - Hv, \qquad \forall x \in \Omega. \tag{2.9}$$

If $\partial\Omega \neq \emptyset$, assume also that $w \geqslant v$ on $\partial\Omega$. Then $w \geqslant v$ in Ω.

Proof. It suffices to multiply the inequality (2.9) by $(w-v)_-$, integrate by parts and use (2.8). $\square$

Fix $\mu \in (0, \frac{1}{2})$. We define the penalized nonlinearity $g_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\begin{aligned} g_\varepsilon(x,s) &:= \chi_\Lambda(x) K(x) f(s) \\ &\qquad + (1 - \chi_\Lambda(x)) \min\left\{\left(\varepsilon^2 H(x) + \mu V(x)\right) s, K(x) f(s)\right\}. \end{aligned}$$

Let $G_\varepsilon(x,s) := \int_0^s g_\varepsilon(x,\sigma) d\sigma$. One can check that g_ε is a Carathéodory function with the following properties :

(g_1) $g_\varepsilon(x,s) = o(s)$, $s \to 0^+$, uniformly in compact subsets of $\mathbb{R}^N$.

(g_2) there exists $p > 1$ such that $\frac{1}{p+1} > \frac{1}{2} - \frac{1}{N-k}$ and

$$\lim_{s \to \infty} \frac{g_\varepsilon(x,s)}{s^p} = 0,$$

(g_3) there exists $2 < \theta \leqslant p+1$ such that

$$\begin{aligned} &0 < \theta G_\varepsilon(x,s) \leqslant g_\varepsilon(x,s)s && \forall x \in \Lambda,\ \forall s > 0, \\ &0 < 2G_\varepsilon(x,s) \leqslant g_\varepsilon(x,s)s \leqslant \left(\varepsilon^2 H(x) + \mu V(x)\right) s^2 && \forall x \notin \Lambda,\ \forall s > 0, \end{aligned}$$

(g_4) the function

$$s \mapsto \frac{g_\varepsilon(x,s)}{s}$$

is nondecreasing for all $x \in \mathbb{R}^N$.

We look for a positive solution of the penalized equation

$$-\varepsilon^2 \Delta u + V(x) u = g_\varepsilon(x,u) \qquad \text{in } \mathbb{R}^N. \tag{$\mathcal{P}_\varepsilon$}$$

Let us define the Hilbert space

$$H^1_V(\mathbb{R}^N) := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} V u^2 < \infty \right\}$$

endowed with the norm

$$\|u\|^2_\varepsilon := \int_{\mathbb{R}^N} \left(\varepsilon^2 \left|\nabla u\right|^2 + V u^2 \right). \tag{2.10}$$

We will search for a solution of $(\mathcal{P}_\varepsilon)$ in the closed subspace

(2.11)

$$H^1_{V,\mathcal{H}}(\mathbb{R}^N) := \left\{ u \in H^1_V(\mathbb{R}^N) \ : \ \forall R \in \mathbf{O}(N) \text{ s.t. } R(\mathcal{H}) = \mathcal{H}, u \circ R = u \right\}.$$

In other words, if $x = (x', x'')$ with $x' \in \mathcal{H}$ and $x'' \in \mathcal{H}^\perp$, the functions in $H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ depend only on $|x'|$ and $|x''|$.

Define $J_\varepsilon : H^1_{V,\mathcal{H}}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^N} \left(\varepsilon^2 \left|\nabla u(x)\right|^2 + V(x) \left|u(x)\right|^2 \right) dx - \int_{\mathbb{R}^N} G_\varepsilon(x, u(x)) \, dx.$$

The functional J_ε is well defined and of class $C^1(H^1_{V,\mathcal{H}}(\mathbb{R}^N), \mathbb{R})$. By the principle of symmetric criticality [123], critical points are weak solutions of $(\mathcal{P}_\varepsilon)$. Furthermore,

Lemma 2.4. *The functional J_ε has the mountain pass geometry, i.e.*

(i) *J_ε achieves a local minimum at 0 in $H^1_{V,\mathcal{H}}(\mathbb{R}^N)$;*

(ii) *The infimum of J_ε over $H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ is $-\infty$.*

Proof. We do exactly the same proof as in Lemma 1.1 in Chapter I, using the estimate

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus \Lambda} G_\varepsilon(x, u) &\leqslant \frac{1}{2} \int_{\mathbb{R}^N} \left(\varepsilon^2 H + \mu V \right) u^2 \\
&\leqslant \frac{\kappa}{2} \left(\frac{2}{N-2} \right)^2 \int_{\mathbb{R}^N} \varepsilon^2 \left|\nabla u\right|^2 + \frac{\mu}{2} \int_{\mathbb{R}^N} V u^2 \\
&\leqslant \frac{\nu}{2} \|u\|^2_\varepsilon,
\end{aligned}$$

where $0 < \nu < 1$. □

It remains to show that J_ε satisfies the Palais-Smale condition. The proof below is inspired from [30]. Recall that a sequence $(u_n)_n \subset H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ is a Palais-Smale sequence for J_ε if

$$J_\varepsilon(u_n) \leqslant C \qquad \text{and} \qquad J'_\varepsilon(u_n) \to 0 \quad \text{as } n \to \infty.$$

Proposition 2.5. *For ε sufficiently small, every Palais-Smale sequence for J_ε contains a convergent subsequence.*

Proof. Let $(u_n)_n \subset H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ be a Palais-Smale sequence for J_ε. It is standard to check, using (g_3), that for ε sufficiently small, the sequence $(u_n)_n$ is bounded in $H^1_{V,\mathcal{H}}(\mathbb{R}^N)$. We infer that, up to a subsequence, $u_n \rightharpoonup u$ in $H^1_{V,\mathcal{H}}(\mathbb{R}^N)$.

For $\lambda \in \mathbb{R}^+$, set $A_\lambda := \overline{B(0,e^\lambda)} \setminus B(0,e^{-\lambda})$. Note that

$$H(x) \leqslant \frac{\kappa}{|x|^2 \left|\log|x|\right|^{1+\beta}}.$$

By Hardy's inequality, we have for $\lambda \geqslant 0$,

$$\int_{\mathbb{R}^N\setminus A_\lambda} H u_n^2 \leqslant \frac{\kappa}{\lambda^{1+\beta}} \int_{\mathbb{R}^N} \frac{|u_n(x)|^2}{|x|^2}\,dx \leqslant \frac{\kappa}{\lambda^{1+\beta}} \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 .$$

Since $(u_n)_n$ is bounded in $H^1_{V,\mathcal{H}}(\mathbb{R}^N)$, for every $\delta > 0$, there exists $\bar\lambda \geqslant 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus A_{\bar\lambda}} H u_n^2 < \delta. \tag{2.12}$$

Now we claim that for all $\delta > 0$, there exists $\tilde\lambda > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus A_{\tilde\lambda}} V u_n^2 < \delta. \tag{2.13}$$

The arguments are similar to those in [30, Lemma 6]. Since $\bar\Lambda \subset \mathbb{R}^N \setminus \{0\}$ is compact, there exists $\lambda_0 \geqslant 0$ such that

$$\bar\Lambda \subset A_{\lambda_0}.$$

Let $\zeta \in C^\infty(\mathbb{R})$ be such that $0 \leqslant \zeta \leqslant 1$ and

$$\zeta(s) = \begin{cases} 0 & \text{if } |s| \leqslant \frac12, \\ 1 & \text{if } |s| \geqslant 1. \end{cases}$$

Define a cut-off function $\eta_\lambda \in C^\infty(\mathbb{R}^N, \mathbb{R})$ by

$$\eta_\lambda(x) := \zeta\left(\frac{\log|x|}{\lambda}\right).$$

Since $\langle J'_\varepsilon(u_n), \eta_\lambda u_n\rangle = o(1)$ as $n \to \infty$, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} \left(\varepsilon^2 |\nabla u_n|^2 + V u_n^2\right)\eta_\lambda = \int_{\mathbb{R}^N} g_\varepsilon(x, u_n(x)) u_n(x) \eta_\lambda(x)\,dx \\ - \varepsilon^2 \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \eta_\lambda + o(1),\end{aligned} \tag{2.14}$$

as $n \to \infty$. If $\lambda \geqslant 2\lambda_0$, $\eta_\lambda = 0$ on Λ and it follows from (g_3) that

$$\int_{\mathbb{R}^N} g_\varepsilon(x, u_n(x))u_n(x)\eta_\lambda(x)\,dx \leqslant \int_{\mathbb{R}^N} \left(\varepsilon^2 H + \mu V\right) u_n^2 \eta_\lambda. \tag{2.15}$$

On the other hand,

$$\int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \eta_\lambda = \int_{\mathbb{R}^N} \frac{u_n}{|x|} \nabla u_n \cdot (|x|\, \nabla \eta_\lambda).$$

The Hardy inequality ensures that

$$\varepsilon^2 \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^2} \leqslant C \left\|u_n\right\|_\varepsilon^2.$$

Furthermore, we compute

$$\left\| |x|\, \nabla \eta_\lambda \right\|_\infty \leqslant \sup_{0<r<\infty} \left| \frac{1}{\lambda} \zeta' \left(\frac{\log r}{\lambda} \right) \right| \leqslant \frac{C}{\lambda}.$$

The Cauchy-Schwarz inequality and the previous estimates imply

$$\left| \varepsilon^2 \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \eta_\lambda \right| \leqslant \frac{C}{\lambda} \left\|u_n\right\|_\varepsilon^2. \tag{2.16}$$

Combining (2.14), (2.15) and (2.16), we get, for every $\lambda \geqslant 2\lambda_0$,

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus A_\lambda} &\left(\varepsilon^2 \left|\nabla u_n\right|^2 + (1-\mu) V u_n^2 \right) \eta_\lambda \\
&\leqslant \int_{\mathbb{R}^N} \left(\varepsilon^2 \left|\nabla u_n\right|^2 + (1-\mu) V u_n^2 \right) \eta_\lambda \\
&\qquad\leqslant \int_{\mathbb{R}^N} \varepsilon^2 H u_n^2 \eta_\lambda + \frac{C}{\lambda} \left\|u_n\right\|_\varepsilon^2 + o(1).
\end{aligned}$$

By (2.12), for λ large enough,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} H u_n^2 \eta_\lambda \leqslant \limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus A_{\bar\lambda}} H u_n^2 < \frac{\delta}{2};$$

the claim follows.

Conclusion. We can write

$$\begin{aligned}
\left\|u_n - u\right\|_\varepsilon^2 &= J_\varepsilon'(u_n)(u_n - u) - J_\varepsilon'(u)(u_n - u) \\
&\quad + \int_{\mathbb{R}^N} \left(g_\varepsilon(x, u_n(x)) - g_\varepsilon(x, u(x))\right)\left(u_n(x) - u(x)\right)\,dx.
\end{aligned} \tag{2.17}$$

We notice that the first two terms in the right-hand side tend to 0 as $n \to \infty$. Fix $\delta > 0$ and let $\lambda > 0$ be such that (2.12) and (2.13) hold. We evaluate the integral in the third term of (2.17) separately on Λ, $A_\lambda \setminus \Lambda$ and $\mathbb{R}^N \setminus A_\lambda$, where $\lambda = \max\{\tilde\lambda, \bar\lambda\}$.

By (g_2), one has $|g_\varepsilon(x, u_n(x))| \leqslant C\,|u_n(x)|^p$. By Rellich Theorem, the embedding $H^1_{V,\mathcal{H}}(\Lambda) \hookrightarrow L^q(\Lambda)$ is compact for all $q > 1$ such that $\frac{1}{q} > \frac{1}{2} - \frac{1}{N-k}$. We can thus assume that $u_n \to u$ in $L^{p+1}(\Lambda)$. We deduce that $g_\varepsilon(x, u_n) \to g_\varepsilon(x, u)$ in $L^q(\Lambda)$ as $n \to \infty$, where $q := \frac{p+1}{p}$. We conclude from Hölder inequality that

$$\int_\Lambda (g_\varepsilon(x, u_n(x)) - g_\varepsilon(x, u(x)))\,(u_n(x) - u(x))\;dx \to 0, \;\text{ as }\; n \to \infty.$$

By (g_3), one has $|g_\varepsilon(x, u_n(x))| \leqslant \left(\varepsilon^2 H(x) + \mu V(x)\right)|u_n(x)|$ for $x \in A_\lambda \setminus \Lambda$. By Rellich Theorem, we can assume that $u_n \to u$ in $L^2(A_\lambda \setminus \Lambda)$. We deduce that $g_\varepsilon(x, u_n) \to g_\varepsilon(x, u)$ in $L^2(A_\lambda \setminus \Lambda)$ as $n \to \infty$. We conclude as above that

$$\int_{A_\lambda \setminus \Lambda} (g_\varepsilon(x, u_n(x)) - g_\varepsilon(x, u(x)))\,(u_n(x) - u(x))\;dx \to 0, \;\text{ as }\; n \to \infty.$$

Finally, using (g_3), (2.12) and (2.13), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} &\int_{\mathbb{R}^N \setminus A_\lambda} |g_\varepsilon(x, u_n(x)) - g_\varepsilon(x, u(x))|\,|u_n(x) - u(x)|\;dx \\
&\leqslant 2\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus A_\lambda} (|g_\varepsilon(x, u_n(x))u_n(x)| + |g_\varepsilon(x, u(x))u(x)|)\;dx \\
&\leqslant 2\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus A_\lambda} \left(\varepsilon^2 H + \mu V\right)\left(u_n^2 + u^2\right) \\
&\leqslant 4(1+\mu)\delta,
\end{aligned}$$

since $\lambda \geqslant \bar{\lambda}$ and $\lambda \geqslant \tilde{\lambda}$.

Since $\delta > 0$ is arbitrary, we conclude

$$\lim_{n\to\infty} \|u_n - u\|_\varepsilon = 0,$$

which ends the proof. □

We can now state an existence theorem for the penalized problem $(\mathcal{P}_\varepsilon)$. The proof follows from standard arguments.

Theorem 2.6. *Let $g : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ be a Carathéodory function satisfying $(g_1) - (g_4)$ and $V \in C(\mathbb{R}^N \setminus \{0\})$ be a nonnegative function. Then, for all $\varepsilon > 0$, the functional J_ε possesses a nontrivial critical point $u_\varepsilon \in H^1_{V,\mathcal{H}}(\mathbb{R}^N)$, which is characterized by*

$$c_\varepsilon := J_\varepsilon(u_\varepsilon) = \inf_{u \in H^1_{V,\mathcal{H}}(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} J_\varepsilon(tu). \tag{2.18}$$

The function u_ε found in Theorem 2.6 is called a *least energy solution* of $(\mathcal{P}_\varepsilon)$. By standard regularity theory, if $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ is a solution of $(\mathcal{P}_\varepsilon)$, then $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $q \in (1,\infty)$. In particular, $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $\alpha \in (0,1)$. Since g_ε is not continuous, we cannot achieve a better regularity. Notice also that, by the strong maximum principle, any nontrivial nonnegative solution $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ of $(\mathcal{P}_\varepsilon)$ is positive in $\mathbb{R}^N$.

3. Asymptotics of solutions

In this section we study the asymptotic behaviour as $\varepsilon \to 0$ of the solution found in Theorem 2.6. We follow closely the arguments in [30, §6]. We first prove an energy estimate which is the counterpart of Lemma 1.8 in Chapter I. Let $\mathbb{R}^{N-k}_+ := \mathbb{R}^{N-k-1} \times \mathbb{R}^+$.

Proposition 2.7 (Upper estimate of the critical value). *Suppose that the assumptions of Theorem 2.6 are satisfied. For ε small enough, the critical value c_ε defined in* (2.18) *satisfies*

$$c_\varepsilon \leqslant \varepsilon^{N-k}\bigl(\omega_k \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M} + o(1)\bigr) \ \textit{as}\ \varepsilon \to 0,$$

where ω_k is the volume of the unit sphere in $\mathbb{R}^{k+1}$. Moreover, the solution u_ε of $(\mathcal{P}_\varepsilon)$ found in Theorem 2.6 satisfies, for some $C > 0$,

$$\|u_\varepsilon\|_\varepsilon^2 \leqslant C\varepsilon^{N-k}.$$

Proof. Let $x_0 = (0, x_0'') \in \Lambda \cap \mathcal{H}^\perp$ be such that $\mathcal{M}(x_0) = \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M}$. Denote by I_0 the functional defined by (2.3) with $a = V(x_0)$ and $b = K(x_0)$ and by w a ground state of (2.2). Take $\eta \in \mathcal{D}\left(\mathbb{R}^{N-k}_+\right)$ to be a cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in a neighbourhood of $(0, |x_0''|)$ and $\|\nabla\eta\|_\infty \leqslant C$. Consider the test function

$$u(x) := \eta(x', |x''|)w\left(\frac{x'}{\varepsilon}, \frac{|x''| - |x_0''|}{\varepsilon}\right).$$

Setting

$$u(x', x'') =: v\left(\frac{x'}{\varepsilon}, \frac{|x''| - |x_0''|}{\varepsilon}\right),$$

we compute by a change of variable

$$\begin{aligned}
&J_\varepsilon(tu)\\
&= \omega_k \frac{t^2}{2} \int_{\mathbb{R}^{N-k-1}} \int_{-\frac{|x_0''|}{\varepsilon}}^{\infty} \Big(|\nabla v|^2 + V v^2\Big) (\varepsilon\rho + |x_0''|)^k \varepsilon \, d\rho \, \varepsilon^{N-k-1} dy'\\
&- \omega_k \int_{\mathbb{R}^{N-k-1}} \int_{-\frac{|x_0''|}{\varepsilon}}^{\infty} G(\varepsilon y', \varepsilon\rho + |x_0''|, tv)(\varepsilon\rho + |x_0''|)^k \varepsilon \, d\rho \, \varepsilon^{N-k-1} dy',
\end{aligned}$$

where $v = v(y', \rho)$ and $V = V(\varepsilon y', \varepsilon\rho + |x_0''|)$. For ε small enough, we obtain by a computation similar to that in Lemma 1.8 in Chapter I

$$\varepsilon^{-(N-k)} J_\varepsilon(tu) \leqslant \omega_k |x_0''|^k I_0(tw) + o(1). \tag{2.19}$$

We deduce from (2.18) that

$$\begin{aligned}
\varepsilon^{-(N-k)} c_\varepsilon &\leqslant \max_{t>0} \varepsilon^{-(N-k)} J_\varepsilon(tu)\\
&\leqslant \omega_k |x_0''|^k \max_{t>0} I_0(tw) + o(1)\\
&= \omega_k \mathcal{M}(x_0) + o(1),
\end{aligned}$$

which is the desired conclusion. □

Proposition 2.8 (No uniform convergence to 0 in Λ). *Suppose that the assumptions of Theorem 2.6 are satisfied and let $(u_\varepsilon)_\varepsilon \subset H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ be positive solutions of $(\mathcal{P}_\varepsilon)$ obtained in Theorem 2.6. Then there exists $\delta > 0$ such that*

$$\|u_\varepsilon\|_{L^\infty(\Lambda)} \geqslant \delta.$$

Proof. Suppose by contradiction that $\|u_\varepsilon\|_{L^\infty(\Lambda)} \to 0$ as $\varepsilon \to 0$. Then, (f_1) implies that, for all ε sufficiently small, $K f(u_\varepsilon) \leqslant \mu V u_\varepsilon$ in Λ. By (g_3), we deduce that

$$-\varepsilon^2 (\Delta u_\varepsilon + H u_\varepsilon) + (1-\mu) V u_\varepsilon \leqslant 0 \quad \text{in } \mathbb{R}^N.$$

Proposition 2.3 then implies that $u_\varepsilon \equiv 0$ for all ε sufficiently small, which is impossible. □

By the symmetry imposed on the solution u_ε, one can write

$$u_\varepsilon(x', x'') = \tilde{u}_\varepsilon(x', |x''|)$$

with $\tilde{u}_\varepsilon : \mathbb{R}^{N-k}_+ \to \mathbb{R}$. Since the H^1_V-norm of u_ε is of the order $\varepsilon^{(N-k)/2}$, it is natural to rescale $\tilde{u}_\varepsilon(x', |x''|)$ as

$$\tilde{u}_\varepsilon(x'_\varepsilon + \varepsilon y', |x''_\varepsilon| + \varepsilon |y''|)$$

around a well-chosen family of points $x_\varepsilon = (x'_\varepsilon, x''_\varepsilon) \in \mathbb{R}^N$.

The next lemma shows that the sequences of rescaled solutions converge, up to a subsequence, in $C^1_{\mathrm{loc}}(\mathbb{R}^{N-k})$ to a function $v \in H^1(\mathbb{R}^{N-k})$. In contrast to Lemma 1.10 in Chapter I, we only know that $u_\varepsilon \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Lemma 2.9. *Suppose that the assumptions of Theorem 2.6 are satisfied. Let $u_\varepsilon \in H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ be positive solutions of $(\mathcal{P}_\varepsilon)$ found in Theorem 2.6, $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \mathbb{R}^N$ be sequences such that $\varepsilon_n \to 0$ and $x_n = (x_n', x_n'') \to \bar{x} = (\bar{x}', \bar{x}'') \in \bar{\Lambda}$ as $n \to \infty$. Set*

$$\Omega_n := \mathbb{R}^{N-k-1} \times \left]-\frac{|x_n''|}{\varepsilon_n}, +\infty\right[$$

and let $v_n : \Omega_n \to \mathbb{R}$ be defined by

$$v_n(y, z) := \tilde{u}_{\varepsilon_n}(x_n' + \varepsilon_n y, \left|x_n''\right| + \varepsilon_n z), \tag{2.20}$$

where $\tilde{u}_{\varepsilon_n} : \mathbb{R}^{N-k}_+ \to \mathbb{R}$ is such that $u_{\varepsilon_n}(x', x'') = \tilde{u}_{\varepsilon_n}(x', |x''|)$. Then, there exists $v \in H^1(\mathbb{R}^{N-k})$ such that, along a subsequence that we still denote by $(v_n)_n$,

$$v_n \overset{C^1_{\mathrm{loc}}(\mathbb{R}^{N-k})}{\longrightarrow} v.$$

Proof. First observe that each v_n solves the equation

$$\begin{aligned} -\Delta v_n - \frac{\varepsilon_n k}{z}\frac{\partial v_n}{\partial z} + V(x_n' + \varepsilon_n y, \left|x_n''\right| + \varepsilon_n z) v_n \\ = g_{\varepsilon_n}(x_n' + \varepsilon_n y, \left|x_n''\right| + \varepsilon_n z, v_n), \end{aligned} \tag{2.21}$$

in Ω_n. We infer from Proposition 2.7 that for all $n \in \mathbb{N}$,

$$\int_{\Omega_n} \Big(\left|\nabla v_n(y,z)\right|^2 + V(x_n' + \varepsilon_n y, \left|x_n''\right| + \varepsilon_n z) \left|v_n(y,z)\right|^2\Big)\, dy\, dz \leqslant C,$$

with $C > 0$ independent of n.

Define a cut-off function $\eta_R \in \mathcal{D}(\mathbb{R}^{N-k})$ such that $0 \leqslant \eta_R \leqslant 1$, $\eta_R(x) = 1$ if $|x| \leqslant R/2$, $\eta_R(x) = 0$ if $|x| \geqslant R$ and $\left\|\nabla \eta_R\right\|_\infty \leqslant C/R$ for some $C > 0$. Choose $(R_n)_n$ such that $R_n \to \infty$ and $\varepsilon_n R_n \to 0$. Since $\bar{x} \in \Lambda$ and $\bar{\Lambda} \cap \mathcal{H} = \emptyset$, one has $\varepsilon_n R_n \leqslant |x_n''|$ if n is large enough. Define $w_n \in H^1_{\mathrm{loc}}(\mathbb{R}^{N-k})$ by

$$w_n(y) := \eta_{R_n}(y) v_n(y).$$

On the one hand, we notice that

$$\begin{aligned}\int_{\mathbb{R}^{N-k}} w_n^2 &\leqslant \int_{B(0,R_n)} v_n^2 \\ &\leqslant \frac{1}{\inf_{B(x_n,\varepsilon_n R_n)} V} \int_{\Omega_n} V(x_n' + \varepsilon_n y, |x_n''| + \varepsilon_n z) \, |v_n(y,z)|^2 \; dydz.\end{aligned}$$

Since V is positive on $\bar{\Lambda}$ and continuous on $\mathbb{R}^N$, the convergence of x_n to a point in $\bar{\Lambda}$ implies that

$$\int_{\mathbb{R}^{N-k}} w_n^2 \leqslant C. \tag{2.22}$$

On the other hand, we compute that

$$\begin{aligned}\int_{\mathbb{R}^{N-k}} |\nabla w_n|^2 = \int_{\mathbb{R}^{N-k}} |\nabla \eta_{R_n}|^2 \, |v_n|^2 + \int_{\mathbb{R}^{N-k}} |\nabla v_n|^2 \, |\eta_{R_n}|^2 \\ + 2 \int_{\mathbb{R}^{N-k}} \nabla v_n \cdot \nabla \eta_{R_n} \eta_{R_n} v_n.\end{aligned}$$

For the first term in the right-hand side, we have the estimate

$$\int_{\mathbb{R}^{N-k}} |\nabla \eta_{R_n}|^2 \, |v_n|^2 \leqslant \frac{C}{R_n^2} \|v_n\|_{L^2(B(0,R_n))} \, ,$$

while for the last one, we infer from the Cauchy-Schwarz inequality that

$$\left| \int_{\mathbb{R}^{N-k}} \nabla v_n \cdot \nabla \eta_{R_n} \eta_{R_n} v_n \right| \leqslant \frac{C}{R_n} \|\nabla v_n\|_{L^2(B(0,R_n))} \|v_n\|_{L^2(B(0,R_n))} \, .$$

Therefore,

$$\int_{\mathbb{R}^{N-k}} |\nabla w_n|^2 \leqslant C \, \|v_n\|^2_{H^1(B(0,R_n))} \, . \tag{2.23}$$

Since

$$\|v_n\|_{H^1(B(0,R_n))} \leqslant C \varepsilon_n^{-(N-k)/2} \|u_{\varepsilon_n}\|_{\varepsilon} \, ,$$

we deduce from (2.22), (2.23) and Proposition 2.7 that $(w_n)_n$ is bounded in $H^1(\mathbb{R}^{N-k})$. Since w_n solves equation (2.21) on $B(0, R_n)$ for all n, classical regularity estimates yield that for every $R > 0$ and every $q > 1$,

$$\sup_{n \in \mathbb{N}} \|v_n\|_{W^{2,q}(B(0,R))} < \infty. \tag{2.24}$$

Up to a subsequence, we can now assume that $(w_n)_n$ converges weakly in $H^1(\mathbb{R}^{N-k})$ to some function $v \in H^1(\mathbb{R}^{N-k})$. By (2.24), for every compact $K \subset \mathbb{R}^{N-k}$, w_n converges to v in $C^1(K)$. Moreover, for n large enough, $w_n = v_n$ in K so that $v_n \to v$ in $C^1(K)$. □

In the next two lemmas, we will estimate from below the action of u_ε inside and outside neighbourhoods of points. Since we expect the concentration set to be a k-sphere in $\mathbb{R}^N$, the following distance will be useful. For $x, y \in \mathbb{R}^N$, let

$$d_{\mathcal{H}}(x,y) := \sqrt{|x'-y'|^2 + (|x''|-|y''|)^2}.$$

Thus $d_{\mathcal{H}}(x,y)$ represents the distance between the k-spheres centered at x' and y', and of radius $|x''|$ and $|y''|$ respectively. We denote by $B_{\mathcal{H}}$ the balls for the distance $d_{\mathcal{H}}$, i.e.,

$$B_{\mathcal{H}}(x,r) = \{y \in \mathbb{R}^N : d_{\mathcal{H}}(x,y) < r\}.$$

Lemma 2.10. *Suppose that the assumptions of Theorem 2.6 are satisfied. Let $u_\varepsilon \in H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ be positive solutions of $(\mathcal{P}_\varepsilon)$ found in Theorem 2.6, $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \mathbb{R}^n$ be sequences such that $\varepsilon_n \to 0$ and $x_n = (x_n', x_n'') \to \bar{x} = (\bar{x}', \bar{x}'') \in \bar{\Lambda}$ as $n \to \infty$. If*

$$\liminf_{n\to\infty} u_{\varepsilon_n}(x_n) > 0, \tag{2.25}$$

then we have, up to a subsequence,

$$\liminf_{R\to\infty} \liminf_{n\to\infty} \varepsilon_n^{-(N-k)} \left(\int_{T_n(R)} \frac{1}{2}\left(\varepsilon_n^2 |\nabla u_{\varepsilon_n}|^2 + V u_{\varepsilon_n}^2\right) - G_{\varepsilon_n}(., u_{\varepsilon_n}) \right) \geqslant \omega_k \mathcal{M}(\bar{x}),$$

where $T_n(R) := B_{\mathcal{H}}(x_n, \varepsilon_n R)$.

Proof. Let v_n be defined by (2.20). Passing to a subsequence if necessary, we may assume that there exists $v \in H^1(\mathbb{R}^{N-k})$ such that $v_n \to v$ in $C^1_{\mathrm{loc}}(\mathbb{R}^{N-k})$. Since Λ is smooth, we can also assume that the sequence of characteristic functions $\chi_n(y,z) = \chi_\Lambda(x_n' + \varepsilon_n y, |x_n''| + \varepsilon_n z)$ converges almost everywhere to a measurable function χ satisfying $0 \leqslant \chi \leqslant 1$. We then deduce that v solves the limiting equation

$$-\Delta v + V(\bar{x})v = \tilde{g}(y,v) \qquad \text{in } \mathbb{R}^{N-k},$$

where

$$\tilde{g}(y,s) := \chi(y)K(\bar{x})f(s) + (1-\chi(y)) \min\{\mu V(\bar{x})s, K(\bar{x})f(s)\}.$$

By (2.25), we know that $v(0) = \lim_{n\to\infty} v_n(0) > 0$, so that v is not identically zero.

It was shown in Lemma 1.11 in Chapter I that

$$\liminf_{R\to\infty}\liminf_{n\to\infty}\int_{B(0,R)}\Bigl(\frac{1}{2}\Bigl(|\nabla v_n|^2+V(x_n'+\varepsilon_n y,\left|x_n''\right|+\varepsilon_n z)\,|v_n|^2\Bigr)$$
$$-G_{\varepsilon_n}\left(x_n'+\varepsilon_n y,\left|x_n''\right|+\varepsilon_n z,v_n\right)\Bigr)\,dz\,dy$$
$$\geqslant\frac{1}{2}\int_{\mathbb{R}^{N-k}}\Bigl(|\nabla v|^2+V(\bar x)v^2\Bigr)-\int_{\mathbb{R}^{N-k}}\tilde G(y,v(y))\,dy,$$

where $\tilde G(x,s):=\int_0^s\tilde g(x,\sigma)\,d\sigma$.

Set $B_n(R):=B((x',|x''|),\varepsilon_n R)\subset\mathbb{R}^{N-k}$. By a computation similar to the one leading to (2.19), we have

$$\int_{T_n(R)}\Bigl(\frac{1}{2}\Bigl(\varepsilon_n^2\,|\nabla u_{\varepsilon_n}(x)|^2+V(x)\,|u_{\varepsilon_n}(x)|^2\Bigr)-G_{\varepsilon_n}(x,u_{\varepsilon_n}(x))\Bigr)\,dx$$
$$=\omega_k\int_{B_n(R)}\Bigl(\frac{1}{2}\Bigl(\varepsilon_n^2\left|\nabla\tilde u_{\varepsilon_n}(x',r)\right|^2+V(x',r)\left|\tilde u_{\varepsilon_n}(x',r)\right|^2\Bigr)$$
$$-G_{\varepsilon_n}(x',r,\tilde u_{\varepsilon_n}(x',r))\Bigr)r^k\,dr\,dx'$$
$$=\omega_k\left|\bar x''\right|^k\varepsilon_n^{N-k}\int_{B(0,R)}\Bigl(\frac{1}{2}\Bigl(|\nabla v_n|^2+V(x_n'+\varepsilon_n y,\left|x_n''\right|+\varepsilon_n z)\,|v_n|^2\Bigr)$$
$$-G_{\varepsilon_n}(x_n'+\varepsilon_n y,\left|x_n''\right|+\varepsilon_n z,v_n)\Bigr)\,dz\,dy+o(1).$$

The conclusion follows. □

Lemma 2.11. *Suppose that the assumptions of Theorem 2.6 are satisfied. Let $u_\varepsilon\in H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ be positive solutions of $(\mathcal{P}_\varepsilon)$ found in Theorem 2.6, $(\varepsilon_n)_n\subset\mathbb{R}^+$ and $(x_n^i)_n\subset\mathbb{R}^N$ be sequences such that $\varepsilon_n\to 0$ and for $1\leqslant i\leqslant M$, $x_n^i\to\bar x^i\in\bar\Lambda$ as $n\to\infty$. Then, up to a subsequence, we have*

$$\liminf_{R\to\infty}\liminf_{n\to\infty}\varepsilon_n^{-(N-k)}\left(\int_{\mathbb{R}^N\setminus\mathcal{T}_n(R)}\frac{1}{2}\Bigl(\varepsilon_n^2\,|\nabla u_{\varepsilon_n}|^2+Vu_{\varepsilon_n}^2\Bigr)-G_{\varepsilon_n}(.,u_{\varepsilon_n})\right)\geqslant 0,$$

where $\mathcal{T}_n(R):=\bigcup_{i=1}^M B_{\mathcal{H}}(x_n^i,\varepsilon_n R)$.

Proof. See Lemma 1.12 in Chapter I. □

Proposition 2.12 (Lower estimate of the critical value). *Suppose that the assumptions of Theorem 2.6 are satisfied. Let $u_\varepsilon\in H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ be positive solutions of $(\mathcal{P}_\varepsilon)$ found in Theorem 2.6, $(\varepsilon_n)_n\subset\mathbb{R}^+$ and*

$(x_n^i)_n \subset \mathbb{R}^N$ *be sequences such that* $\varepsilon_n \to 0$ *and for* $1 \leqslant i \leqslant M$, $x_n^i \to \bar{x}^i \in \bar{\Lambda}$ *as* $n \to \infty$. *If for every* $1 \leqslant i < j \leqslant M$, *we have*

$$\limsup_{n\to\infty} \frac{d_{\mathcal{H}}(x_n^i, x_n^j)}{\varepsilon_n} = \infty$$

and if for every $1 \leqslant i \leqslant M$,

$$\liminf_{n\to\infty} u_{\varepsilon_n}(x_n^i) > 0,$$

then the critical value c_ε *defined in* (2.18) *satisfies*

$$\liminf_{n\to\infty} \varepsilon_n^{-(N-k)} c_{\varepsilon_n} \geqslant \omega_k \sum_{i=1}^{M} \mathcal{M}(\bar{x}^i).$$

Proof. This is a consequence of the two previous lemmas, see Lemma 1.13 in Chapter I for the details. □

The following proposition is the key result for the next section.

Proposition 2.13 (Uniform convergence to 0 outside small balls). *Suppose that the assumptions of Theorem 2.6 are satisfied and that* Λ *satisfies the assumptions of Section 1.4. Let* $(u_\varepsilon)_\varepsilon \subset H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ *be positive solutions of* $(\mathcal{P}_\varepsilon)$ *obtained in Theorem 2.6. If* $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ *is such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0,$$

then

(i) $\lim_{\varepsilon\to 0} \mathcal{M}(x_\varepsilon) = \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M}$,
(ii) $\limsup_{\varepsilon\to 0} \frac{\operatorname{dist}(x_\varepsilon, \mathcal{H}^\perp)}{\varepsilon} < \infty$,
(iii) $\liminf_{\varepsilon\to 0} d_{\mathcal{H}}(x_\varepsilon, \partial\Lambda) > 0$,
(iv) for every $\delta > 0$, *there exist* $\varepsilon_0 > 0$ *and* $R > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$,

$$\|u_\varepsilon\|_{L^\infty(\Lambda\setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon R))} \leqslant \delta.$$

Proof. The first assertion is a direct consequence of Propositions 2.7 and 2.12, see Lemma 1.15 in Chapter I for the details.

For the second assertion, since $\bar{\Lambda}$ is compact, we can assume by contradiction that there exist sequences $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \Lambda$ such that $\varepsilon_n \to 0$, $x_n \to \bar{x} \in \bar{\Lambda}$,

$$\liminf_{n\to\infty} u_{\varepsilon_n}(x_n) > 0,$$

and

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \mathcal{H}^\perp)}{\varepsilon_n} = \infty.$$

If $k = N-2$, let $R \in \mathbf{O}(N)$ denote the reflexion with respect to $\mathcal{H}^\perp$. By definition of $H^1_{V,\mathcal{H}}(\mathbb{R}^N)$, $u \circ R = u$, and thus

$$\liminf_{n\to\infty} u_{\varepsilon_n}(R(x_n)) > 0.$$

The preceding assumption implies that $\limsup_{n\to\infty} \frac{d_\mathcal{H}(x_n,R(x_n))}{\varepsilon_n} = \infty$. By Proposition 2.12, we obtain

$$\liminf_{n\to\infty} \varepsilon_n^{-(N-k)} c_{\varepsilon_n} \geqslant \omega_k \left(\mathcal{M}(\bar{x}) + \mathcal{M}(R(\bar{x}))\right) \geqslant 2\omega_k \inf_\Lambda \mathcal{M}.$$

which, together with Proposition 2.7

$$\liminf_{n\to\infty} \varepsilon_n^{-(N-k)} c_{\varepsilon_n} \leqslant \omega_k \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M}$$

is in contradiction with (2.7).

In the case where $k < N-2$, since $\inf_\Lambda \mathcal{M} > 0$, choose $\ell \in \mathbb{N}$ such that

$$\inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M} < \ell \inf_\Lambda \mathcal{M}. \tag{2.26}$$

There exist isometries $R_1, \dots, R_l$ of $\mathbb{R}^N$ such that $R_i(\mathcal{H}) = \mathcal{H}$ and $R_i(\bar{x}) \neq R_j(\bar{x})$, for every $i, j \in \{1, \dots, \ell\}$ with $i \neq j$. One has hence

$$\limsup_{n\to\infty} \frac{d_\mathcal{H}(R_i(x_n), R_j(x_n))}{\varepsilon_n} = \infty.$$

By Proposition 2.12, we get

$$\liminf_{n\to\infty} \varepsilon_n^{-(N-k)} c_{\varepsilon_n} \geqslant \omega_k \sum_{i=1}^{\ell} \mathcal{M}(R_i(\bar{x})) \geqslant l\omega_k \inf_\Lambda \mathcal{M},$$

so that, in view of the upper estimate of Proposition 2.7, we have a contradiction with (2.26).

For the third assertion, suppose by contradiction that there exist sequences $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \mathbb{R}^N$ such that $\varepsilon_n \to 0$,

$$\liminf_{n\to\infty} u_{\varepsilon_n}(x_n) > 0,$$

and, $x_n \to \bar{x} \in \partial\Lambda$. We have just proven that $\bar{x} \in \mathcal{H}^\perp$. By Proposition 2.12, we have

$$\liminf_{n\to\infty} \varepsilon_n^{-(N-k)} c_{\varepsilon_n} \geqslant \omega_k \mathcal{M}(\bar{x}) \geqslant \omega_k \inf_{\partial\Lambda\cap\mathcal{H}^\perp} \mathcal{M}.$$

This inequality, along with Proposition 2.7, contradicts (2.6).

In order to obtain the last assertion, suppose by contradiction that there exist sequences $(\varepsilon_n)_n \subset \mathbb{R}^+$, $(x_n)_n$ and $(y_n)_n \subset \Lambda$ such that $\varepsilon_n \to 0$,

$$u_{\varepsilon_n}(y_n) \geqslant \delta,$$

and

$$\lim_{n\to\infty} \frac{d_{\mathcal{H}}(x_n, y_n)}{\varepsilon_n} = \infty.$$

Up to a subsequence, we can assume that $x_n \to \bar{x} \in \Lambda$ and $y_n \to \bar{y} \in \Lambda$. In view of the second assertion, one has $\bar{x} \in \mathcal{H}^\perp$ and $\bar{y} \in \mathcal{H}^\perp$. Therefore, by Proposition 2.12,

$$\liminf_{n\to\infty} \varepsilon_n^{-(N-k)} c_{\varepsilon_n} \geqslant \omega_k \left(\mathcal{M}(\bar{x}) + \mathcal{M}(\bar{y})\right) \geqslant 2\omega_k \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M}.$$

In view of the assumption of (2.6), this would contradict Proposition 2.7. □

4. Barrier functions

4.1. Linear inequation outside small balls

In this section we prove that for ε small enough, the solutions of the penalized problem $(\mathcal{P}_\varepsilon)$ are also solutions of the initial problem (NLS). We follow the arguments of [113]. First we notice that the solutions of $(\mathcal{P}_\varepsilon)$ satisfy a linear inequation outside small balls.

Lemma 2.14. *Suppose that the assumptions of Proposition 2.13 are satisfied and let $(u_\varepsilon)_{\varepsilon>0} \subset H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ be positive solutions of $(\mathcal{P}_\varepsilon)$ found in Theorem 2.6 and $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ be such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Then there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$-\varepsilon^2 \left(\Delta u_\varepsilon + H u_\varepsilon\right) + (1-\mu) V u_\varepsilon \leqslant 0 \quad \textit{in } \mathbb{R}^N \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho). \tag{2.27}$$

Proof. Set

$$\eta := \inf_{x\in\Lambda} \frac{\mu V(x)}{K(x)}.$$

Since V and K are bounded positive continuous functions on $\bar{\Lambda}$, $\eta > 0$. By (f_1), there exists $\delta > 0$ such that

$$\frac{f(s)}{s} \leqslant \eta \quad \text{for all } s \leqslant \delta.$$

By Proposition 2.13, we can find $\varepsilon_0 > 0$ and $\rho > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, one has

$$u_\varepsilon(x) \leqslant \delta \quad \text{for all } x \in \Lambda \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho).$$

Hence

$$K(x)f(u_\varepsilon(x)) \leqslant \mu V(x)u_\varepsilon(x) \quad \text{in } \Lambda \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho).$$

We conclude that

$$-\varepsilon^2\Delta u_\varepsilon + (1-\mu)Vu_\varepsilon \leqslant -\varepsilon^2\Delta u_\varepsilon + Vu_\varepsilon - Kf(u_\varepsilon) = 0$$

in $\Lambda \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho)$. The fact that u_ε satisfies (2.27) in $\mathbb{R}^N \setminus \Lambda$ follows directly from the definition of the penalized nonlinearity. □

This lemma suggests that we can compare the solution u_ε with supersolutions of the operator $-\varepsilon^2(\Delta + H) + (1-\mu)V$ in order to obtain decay estimates of u_ε.

4.2. Comparison functions

The next lemma provides a minimal positive solution of the operator $-\Delta - H$ in $\mathbb{R}^N \setminus \bar{\Lambda}$.

Lemma 2.15. *For every $\varepsilon > 0$, there exists $\Psi_\varepsilon \in C^2\big((\mathbb{R}^N \setminus \{0\}) \setminus \Lambda\big)$ such that*

$$\begin{cases} -\varepsilon^2(\Delta\Psi_\varepsilon + H\Psi_\varepsilon) + (1-\mu)V\Psi_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Lambda}, \\ \Psi_\varepsilon = 1 & \text{on } \partial\Lambda, \end{cases}$$

and

$$\int_{\mathbb{R}^N\setminus\Lambda} \left(|\nabla\Psi_\varepsilon(x)|^2 + \frac{|\Psi_\varepsilon(x)|^2}{|x|^2}\right) dx < \infty. \tag{2.28}$$

Moreover, there exists $C > 0$ such that, for every $x \in \mathbb{R}^N \setminus \Lambda$ and every $\varepsilon > 0$,

$$0 < \Psi_\varepsilon(x) \leqslant \frac{C}{(1+|x|)^{N-2}}. \tag{2.29}$$

Proof. The function Ψ_ε is obtained by minimimizing

$$\int_{\mathbb{R}^N\setminus\Lambda} \left(\varepsilon^2\left(|\nabla u|^2 - Hu^2\right) + (1-\mu)Vu^2\right) dx$$

on the set

$$\{u \in H^1_V(\mathbb{R}^N) \,:\, u = 1 \text{ on } \partial\Lambda\}.$$

By classical elliptic regularity theory, $\Psi_\varepsilon \in C^2\big((\mathbb{R}^N \setminus \{0\}) \setminus \Lambda\big)$. The estimate (2.28) follows from (2.8).

In order to obtain the estimate (2.29) consider the problem

$$\begin{cases} -\Delta\Psi - H\Psi = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Lambda}, \\ \Psi = 1 & \text{on } \partial\Lambda. \end{cases}$$

We have just proved that this problem has a solution $\Psi \in C^2((\mathbb{R}^N \setminus \{0\}) \setminus \Lambda)$ such that

$$\int_{\mathbb{R}^N\setminus\Lambda} \left(|\nabla\Psi(x)|^2 + \frac{|\Psi(x)|^2}{|x|^2} \right) dx < \infty. \tag{2.30}$$

Now set for $\rho \in (0,1)$ and $x \in B(0,\rho)$,

$$W(x) := (N-2)\beta - \kappa \left(\log \frac{1}{|x|} \right)^{-\beta},$$

We compute that

$$-\Delta W(x) = \frac{\kappa\beta}{|x|^2} \left[(N-2) \left(\log \frac{1}{|x|} \right)^{-(1+\beta)} + (1+\beta) \left(\log \frac{1}{|x|} \right)^{-(2+\beta)} \right].$$

Since for $|x| \leqslant 1$,

$$H(x) \leqslant \frac{\kappa}{(|x|^2 \log \frac{1}{|x|})^{1+\beta}}$$

the function W is a supersolution of $-\Delta - H$ in $B(0,1)$. Moreover, if one takes $\rho < 1$ such that

$$(N-2)\beta \left(\log \frac{1}{\rho} \right)^{\beta} > \kappa,$$

W is positive on $\partial B(0,\rho)$. In view of (2.30) Proposition 2.3 implies that Ψ is bounded from above by a positive multiple of W in $B(0,\rho)$. Since Ψ is continuous and W is bounded in $B(0,1)$, we obtain that Ψ is bounded in $B(0,1)$. By similarly considering

$$W(x) := \frac{1}{|x|^{N-2}} \left((N-2)\beta - \kappa \left(\log |x| \right)^{-\beta} \right)$$

(see Lemma 3.4 of [113]), we obtain that $\Psi(x) \sim |x|^{N-2}$. We have thus proven that

$$\Psi(x) \leqslant \frac{C}{(1+|x|)^{N-2}}.$$

Now, note that since V is nonnegative,

$$-\Delta\Psi_\varepsilon - H\Psi_\varepsilon \leqslant 0.$$

In view of (2.30) and (2.28), Proposition 2.3 is applicable, and for every $x \in \mathbb{R}^N \setminus \Lambda$,

$$\Psi_\varepsilon(x) \leqslant \Psi(x) \leqslant \frac{C}{(1+|x|)^{N-2}}. \qquad \square$$

As explained in [113], the estimate (2.29) is the best one can hope for if V decays rapidly at infinity or is compactly supported. However, if V decays quadratically or subquadratically at infinity, we can improve (2.29).

Lemma 2.16. *Let Ψ_ε be given by Lemma 2.15.*

(1) If $\liminf_{|x|\to\infty} V(x)\,|x|^2 > 0$, then there exist $\lambda > 0$, $R > 0$ and $C > 0$ such that for every $\varepsilon > 0$ and $x \in \mathbb{R}^N \setminus B(0,R)$,

$$\Psi_\varepsilon(x) \leqslant C\left(\frac{R}{|x|}\right)^{\frac{N-2}{2}+\sqrt{\left(\frac{N-2}{2}\right)^2-\kappa+\frac{\lambda^2}{\varepsilon^2}}}.$$

(2) If $\liminf_{|x|\to\infty} V(x)\,|x|^\alpha > 0$ with $\alpha < 2$, then there exist $\lambda > 0$, $R > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ and $x \in \mathbb{R}^N \setminus B(0,R)$,

$$\Psi_\varepsilon(x) \leqslant C\exp\left(-\frac{\lambda}{\varepsilon}\left(|x|^{\frac{2-\alpha}{2}} - R^{\frac{2-\alpha}{2}}\right)\right).$$

(3) If $\liminf_{|x|\to 0} V(x)\,|x|^2 > 0$, then there exist $\lambda > 0$, $r > 0$ and $C > 0$ such that for every $\varepsilon > 0$ and $x \in B(0,r)$,

$$\Psi_\varepsilon(x) \leqslant C\left(\frac{|x|}{r}\right)^{\sqrt{\left(\frac{N-2}{2}\right)^2-\kappa+\frac{\lambda^2}{\varepsilon^2}}-\frac{N-2}{2}}.$$

(4) If $\liminf_{|x|\to 0} V(x)\,|x|^\alpha > 0$ with $\alpha < 2$, then there exist $\lambda > 0$, $R > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ and $x \in B(0,r)$,

$$\Psi_\varepsilon(x) \leqslant C\exp\left(-\frac{\lambda}{\varepsilon}\left(|x|^{-\frac{\alpha-2}{2}} - r^{-\frac{\alpha-2}{2}}\right)\right).$$

Proof. For (1), there exist $R > 0$ and $\lambda > 0$ such that for $x \in \mathbb{R}^N \setminus B(0,R)$

$$(1-\mu)V(x) \geqslant \frac{\lambda^2}{|x|^2}.$$

One then checks that

$$W(x) = \left(\frac{R}{|x|}\right)^{\frac{N-2}{2}+\sqrt{\left(\frac{N-2}{2}\right)^2-\kappa+\frac{\lambda^2}{\varepsilon^2}}}$$

is a supersolution in $\mathbb{R}^N \setminus B(0,R)$.

For (2), there exist $R > 0$ and $\eta > 0$ such that for $x \in \mathbb{R}^N \setminus B(0,R)$

$$(1-\mu)V(x) \geqslant \frac{\eta}{|x|^\alpha}.$$

One then checks that

$$W(x) = \exp\left(-\frac{\lambda}{\varepsilon}\left(|x|^{\frac{2-\alpha}{2}} - R^{\frac{2-\alpha}{2}}\right)\right)$$

is a supersolution in $\mathbb{R}^N \setminus B(0,R)$ with $\lambda^2 < (\frac{2}{2-\alpha})^2\nu$ and ε small enough.

The proofs of the other assertions are similar. □

Another important tool is a function that describes the exponential decay of u_ε inside Λ.

Lemma 2.17. *Let $\bar{x} \in \Lambda$ and $R > 0$ be such that*

$$B_{\mathcal{H}}(\bar{x}, R) \subset \Lambda. \tag{2.31}$$

Define

$$\Phi_\varepsilon^{\bar{x}}(x) := \cosh\left(\lambda \frac{R - d_{\mathcal{H}}(x,\bar{x})}{\varepsilon}\right). \tag{2.32}$$

There exist $\lambda > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, one has

$$-\varepsilon^2 \Delta \Phi_\varepsilon^{\bar{x}} + (1-\mu)V\Phi_\varepsilon^{\bar{x}} \geqslant 0 \text{ in } B_{\mathcal{H}}(\bar{x}, R).$$

Proof. First one computes

$$\begin{aligned} -\varepsilon^2 \Delta \Phi_\varepsilon^{\bar{x}}(x) &= -\lambda^2 \cosh\left(\frac{\lambda}{\varepsilon}\left(R - d_{\mathcal{H}}(x,\bar{x})\right)\right) \\ &\quad + \frac{\varepsilon\lambda}{d_{\mathcal{H}}(x,\bar{x})}\left(N - 1 - k\frac{|\bar{x}''|}{|x''|}\right)\sinh\left(\frac{\lambda}{\varepsilon}\left(R - d_{\mathcal{H}}(x,\bar{x})\right)\right). \end{aligned}$$

Let us choose $\lambda > 0$ such that $\lambda^2 < (1-\mu)\inf_\Lambda V$. In view of (2.31), one has for $x \in B_{\mathcal{H}}(\bar{x}, R)$,

$$\begin{aligned} &-\varepsilon^2 \Delta \Phi_\varepsilon^{\bar{x}}(x) + (1-\mu)V\Phi_\varepsilon^{\bar{x}}(x) \\ &\qquad \geqslant \frac{\varepsilon\lambda}{d_{\mathcal{H}}(x,\bar{x})}\left(N - 1 - k\frac{|\bar{x}''|}{|x''|}\right)\sinh\left(\frac{\lambda}{\varepsilon}\left(R - d_{\mathcal{H}}(x,\bar{x})\right)\right) \\ &\qquad\qquad + \left((1-\mu)\inf_\Lambda V - \lambda^2\right)\cosh\left(\frac{\lambda}{\varepsilon}\left(R - d_{\mathcal{H}}(x,\bar{x})\right)\right). \end{aligned}$$

This last expression is positive if ε is sufficiently small. □

Lemma 2.18. *Let $(x_\varepsilon)_\varepsilon \subset \Lambda$ be such that*

$$\liminf_{\varepsilon \to 0} d_{\mathcal{H}}(x_\varepsilon, \partial\Lambda) > 0$$

and let $\rho > 0$. Then, there exist $\varepsilon_0 > 0$ and a family of functions $(W_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subset C^{1,1}_{\mathrm{loc}}((\mathbb{R}^N \setminus \{0\}) \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho))$ such that for all $\varepsilon \in (0, \varepsilon_0)$, one has

(i) W_ε satisfies the inequation

$$-\varepsilon^2 (\Delta + H) W_\varepsilon + (1-\mu) V W_\varepsilon \geqslant 0 \text{ in } \mathbb{R}^N \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho),$$

(ii) $\nabla W_\varepsilon \in L^2(\mathbb{R}^N \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho))$ and $\frac{W_\varepsilon}{|x|} \in L^2(\mathbb{R}^N \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho))$,

(iii) $W_\varepsilon \geqslant 1$ on $\partial B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho)$,

(iv) for every $x \in B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho)$,

$$W_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_{\mathcal{H}}(x, x_\varepsilon)}{1 + d_{\mathcal{H}}(x, x_\varepsilon)}\right) (1 + |x|)^{-(N-2)}, \qquad x \in \mathbb{R}^N.$$

Moreover,

(1) If $\liminf_{|x|\to\infty} V(x)\,|x|^2 > 0$, then there exist $\lambda > 0$, $\nu > 0$ and $C > 0$ such that for $\varepsilon > 0$ small enough,

$$W_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_{\mathcal{H}}(x, x_\varepsilon)}{1 + d_{\mathcal{H}}(x, x_\varepsilon)}\right) (1 + |x|)^{-\frac{\nu}{\varepsilon}}.$$

(2) If $\liminf_{|x|\to\infty} V(x)\,|x|^\alpha > 0$ with $\alpha > 2$, then there exist $\lambda > 0$ and $C > 0$ such that for $\varepsilon > 0$ small enough,

$$W_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_{\mathcal{H}}(x, x_\varepsilon)}{1 + d_{\mathcal{H}}(x, x_\varepsilon)} (1 + |x|)^{\frac{2-\alpha}{2}}\right).$$

(3) If $\liminf_{|x|\to 0} V(x)\,|x|^2 > 0$, then there exist $\lambda > 0$, $\nu > 0$ and $C > 0$ such that for $\varepsilon > 0$ small enough,

$$W_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_{\mathcal{H}}(x, x_\varepsilon)}{1 + d_{\mathcal{H}}(x, x_\varepsilon)}\right) \left(\frac{|x|}{1 + |x|}\right)^{\frac{\nu}{\varepsilon}}.$$

(4) If $\liminf_{|x|\to 0} V(x)\,|x|^\alpha > 0$ with $\alpha > 2$, then there exist $\lambda > 0$ and $C > 0$ such that for $\varepsilon > 0$ small enough,

$$W_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_{\mathcal{H}}(x, x_\varepsilon)}{1 + d_{\mathcal{H}}(x, x_\varepsilon)} \left(\frac{|x|}{1 + |x|}\right)^{\frac{\alpha-2}{2}}\right).$$

Proof. Let Ψ_ε be given by Lemma 2.15. Choose a set $U \subset \mathbb{R}^N$ such that $\bar{\Lambda} \subset U$, $0 \notin \bar{U}$ and $\bar{U}$ is compact. Choose $\tilde{\Psi}_\varepsilon \in C^2(\mathbb{R}^N \setminus \{0\}) \cap H^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\tilde{\Psi}_\varepsilon = \Psi_\varepsilon$ in $\mathbb{R}^N \setminus U$ and $\tilde{\Psi}_\varepsilon = 1$ in Λ. In view of the estimate of

Lemma 2.15, one can also ensure that $\sup_{\varepsilon>0} \left\| \tilde{\Psi}_\varepsilon \right\|_{L^\infty(U)} < \infty$. Choose $R > 0$ such that

$$R < \liminf_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \partial\Lambda). \tag{2.33}$$

Let $\Phi_\varepsilon^{x_\varepsilon}$ be given by (2.32) and set

$$w_\varepsilon(x) := \begin{cases} \Phi_\varepsilon^{x_\varepsilon}(x) & \text{if } x \in B_{\mathcal{H}}(x_\varepsilon, R), \\ \tilde{\Psi}_\varepsilon(x) & \text{if } x \in \mathbb{R}^N \setminus B_{\mathcal{H}}(x_\varepsilon, R). \end{cases}$$

By (2.33), for ε small enough, $B_{\mathcal{H}}(x_\varepsilon, R) \subset \Lambda$ so that $w_\varepsilon \in C^{1,1}(\mathbb{R}^N)$. Besides, if ε is small enough, Lemma 2.17 is applicable and in $B_{\mathcal{H}}(x_\varepsilon, R) \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho)$, we have

$$-\varepsilon^2 (\Delta + H)\, w_\varepsilon + (1-\mu)V w_\varepsilon \geqslant -\varepsilon^2 \Delta \Phi_\varepsilon^{x_\varepsilon} + (1-\mu)V\Phi_\varepsilon^{x_\varepsilon} \geqslant 0.$$

In $\Lambda \setminus B_{\mathcal{H}}(x_\varepsilon, R)$, one has

$$-\varepsilon^2 (\Delta + H)\, w_\varepsilon + (1-\mu)V w_\varepsilon = -\varepsilon^2 H + (1-\mu)\left(\inf_\Lambda V\right) \geqslant 0,$$

for ε small enough. In $U \setminus \Lambda$, one has

$$-\varepsilon^2 (\Delta + H)\, w_\varepsilon + (1-\mu)V w_\varepsilon = -\varepsilon^2 (\Delta + H)\, \tilde{\Psi}_\varepsilon + (1-\mu)V\tilde{\Psi}_\varepsilon \geqslant 0,$$

for ε small enough since $V\tilde{\Psi}_\varepsilon$ is positive on $\overline{U}$. Finally, in $\mathbb{R}^N \setminus U$, one has

$$-\varepsilon^2 (\Delta + H)\, w_\varepsilon + (1-\mu)V w_\varepsilon = -\varepsilon^2 (\Delta + H)\, \Psi_\varepsilon + (1-\mu)V\Psi_\varepsilon = 0.$$

We set

$$W_\varepsilon(x) := \frac{w_\varepsilon(x)}{\cosh \lambda \left(\frac{R}{\varepsilon} - \rho\right)},$$

where λ is chosen as in the previous lemma. It is standard to see that W_ε satisfies properties (ii) and (iii). Statement (iv) follows from Lemma 2.15. The other conclusions follow from Lemma 2.16. $\square$

Thanks to the previous lemma, we obtain an upper bound on the solutions $(u_\varepsilon)_{\varepsilon>0}$ of $(\mathcal{P}_\varepsilon)$.

Proposition 2.19. *Suppose that the assumptions of Proposition 2.13 are satisfied. Let $(u_\varepsilon)_{\varepsilon>0} \subset H^1_{V,\mathcal{H}}(\mathbb{R}^N)$ be the positive solutions of $(\mathcal{P}_\varepsilon)$ found in Theorem 2.6 and $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ be such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Then there exist $C > 0$, $\lambda > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$u_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon}\frac{d(x, S_\varepsilon^k)}{1 + d(x, S_\varepsilon^k)}\right)(1 + |x|)^{-(N-2)}, \qquad x \in \mathbb{R}^N. \tag{2.34}$$

Moreover, (1), (2), (3) *and* (4) *in Lemma 2.18 hold with u_ε in place of W_ε.*

Proof. By Lemma 2.14, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, u_ε satisfies inequation (2.27). Further, $\|u_\varepsilon\|_{L^\infty(B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho))}$ is bounded as $\varepsilon \to 0$ in view of Lemma 2.9. Let $(W_\varepsilon)_\varepsilon$ be the family of barrier functions given by Lemma 2.18. By Proposition 2.3, we have

$$u_\varepsilon(x) \leqslant \|u_\varepsilon\|_{L^\infty(B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho))} W_\varepsilon(x) \quad \text{ in } \mathbb{R}^N \setminus B_{\mathcal{H}}(x_\varepsilon, \varepsilon\rho),$$

and the conclusion comes from Lemma 2.18. □

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. We know from Theorem 2.6 that the modified equation $(\mathcal{P}_\varepsilon)$ possesses a positive solution $u_\varepsilon \in H^1_{V,\mathcal{H}}(\mathbb{R}^N)$. In order to prove that for ε small enough, this solution actually solves (NLS), it suffices to show that, for every $x \in (\mathbb{R}^N \setminus \{0\}) \setminus \Lambda$, one has

$$K(x)\frac{f(u_\varepsilon(x))}{u_\varepsilon(x)} \leqslant \varepsilon^2 H(x) + \mu V(x).$$

Assume that V and K satisfy $(\mathcal{G}^1_\infty)$ and $(\mathcal{G}^1_0)$, by Proposition 2.19 and assumptions (f_4) and (f_1), if $\varepsilon > 0$ is small enough, we have for all $x \in \mathbb{R}^N \setminus \Lambda$,

$$\begin{aligned}
K(x)\frac{f(u_\varepsilon(x))}{u_\varepsilon(x)} &\leqslant K(x)\frac{f\left(Ce^{-\frac{\lambda}{\varepsilon}}(1 + |x|)^{-(N-2)}\right)}{Ce^{-\frac{\lambda}{\varepsilon}}(1 + |x|)^{-(N-2)}} \\
&\leqslant Ce^{-\frac{\lambda}{\varepsilon}(q-1)}(1 + |x|)^{\sigma-(N-2)(q-1)} \\
&\leqslant \frac{\varepsilon^2\kappa}{|x|^2\left((\log|x|)^2 + 1\right)^{\frac{1+\beta}{2}}} = \varepsilon^2 H(x).
\end{aligned}$$

The other cases can be treated in a similar way. □

In some settings, it is interesting to determine whether the solutions are in L^2. We obtain as a byproduct the following

Corollary 2.20. *Let u_ε be the solution of* (NLS) *found in Theorem 2.2. If $N \geqslant 5$ or $\liminf_{|x|\to\infty} |x|^2\, V(x) > 0$, then, for ε small enough, $u_\varepsilon \in L^2(\mathbb{R}^N)$.*

Proof. This follows immediately from Proposition 2.19. □

5. The two-dimensional case

In dimension $N = 2$, the method has to be modified because the classical Hardy inequality fails on unbounded domains of $\mathbb{R}^2$. Let us recall the Hardy-type inequality that was proved in [113, Lemma 6.1]:

Lemma 2.21. *Let $R > r$. Then there exists $C > 0$ such that for every $u \in \mathcal{D}(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} |\nabla u|^2 + C \int_{B(0,R)\setminus B(0,r)} u^2 \geqslant \frac{1}{4} \int_{\mathbb{R}^2 \setminus B(0,R)} \frac{u^2(x)}{|x|^2 \left(\log \frac{|x|}{r}\right)^2}\, dx.$$

We deduce therefrom

Lemma 2.22. *If $V \in C(\mathbb{R}^2 \setminus \{0\})$ is nonnnegative and non identically 0, then there exists $\kappa_0 > 0$ such that for $\varepsilon > 0$ sufficiently small, for every $u \in \mathcal{D}(\mathbb{R}^2)$,*

$$\kappa_0 \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2 \left(1 + (\log |x|)^2\right)}\, dx \leqslant \int_{\mathbb{R}^2} \varepsilon^2 |\nabla u|^2 + V u^2.$$

Proof. By the conformal transformation $x \mapsto \frac{x}{|x|^2}$, Lemma 2.21 becomes

$$\int_{\mathbb{R}^2} |\nabla u|^2 + C \int_{B(0,R)\setminus B(0,r)} u^2 \geqslant \frac{1}{4} \int_{B(0,r)} \frac{u^2(x)}{|x|^2 \,(\log |x|)^2}\, dx.$$

Therefore, one has

$$\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2 \left(1 + (\log |x|)^2\right)}\, dx \leqslant C\Big(\int_{\mathbb{R}^2} |\nabla u|^2 + \int_{B(0,2)\setminus B(0,1/2)} u^2\Big).$$

Since V is continuous and does not vanish identically, there exists $\bar{x} \in \mathbb{R}^2$ and $\bar{r} > 0$, such that $\inf_{B(\bar{x},\bar{r})} V > 0$. Hence, there exists $C > 0$ such that

$$\int_{B(0,2)\setminus B(0,1/2)} u^2 \leqslant C\Big(\int_{\mathbb{R}^2} |\nabla u|^2 + V\, |u|^2\Big).$$

Bringing the inequalities together, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2 \left(1 + (\log |x|)^2\right)}\, dx \leqslant C \int_{\mathbb{R}^2} (|\nabla u|^2 + V u^2).$$

This brings the conclusion when $\varepsilon > 0$ is small enough. $\square$

The space $H^1_V(\mathbb{R}^2)$ can thus be defined as in the case $N > 2$ as the closure of $\mathcal{D}(\mathbb{R}^2)$ with respect to the norm defined by (2.10).

The penalization potential $H : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$H(x) := \frac{\kappa}{|x|^2 \left(1 + (\log |x|)^2\right)^{\frac{2+\beta}{2}}}.$$

where $\beta > 0$ and $\kappa \in (0, \kappa_0)$ We see that

$$H(x) \leqslant \frac{\kappa}{|x|^2 \left(1 + (\log |x|)^2\right)}.$$

Together with Lemma (2.22), this ensures the positivity of the quadratic form associated to $-\varepsilon^2(\Delta + H) + V$.

As in the case $N > 2$, this inequality implies the following comparison principle.

Proposition 2.23. *Let $\Omega \subset \mathbb{R}^2$ be a smooth domain. Let $v, w \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be such that $\nabla(w - v)_- \in L^2(\Omega)$, $(w - v)_-/(|x| \,(1 + |\log |x||)) \in L^2(\Omega)$ and*

$$-\varepsilon^2(\Delta + H)w + Vw \geqslant -\varepsilon^2(\Delta + H)v + Vv, \qquad \text{in } \Omega.$$

If $\partial\Omega \neq \emptyset$, assume also that $w \geqslant v$ on $\partial\Omega$. Then $w \geqslant v$ in Ω.

One continues the proof of Theorem 2.2 as in the case $N > 2$. In Proposition 2.5, one takes $A_\lambda := \overline{B(0, e^{e^\lambda})} \setminus B(0, e^{-e^\lambda})$ and

$$\eta_\lambda(x) := \zeta\left(\frac{\log |\log |x||}{\lambda}\right).$$

One then obtains estimate (2.16) by using Lemma 2.22 instead of Hardy's inequality. The only other notable difference lies in the choice of the function W in the proof of Lemma 2.15, where one follows the construction of [113, Lemma 6.3], i.e.

$$W(x) = \beta(\beta + 1) - \kappa \,|\log |x||^{-\beta}.$$

CHAPTER III

Concentration on circles for the Schrödinger-Poisson system

The aim of the present chapter is to study the behavior of a certain class of solutions for the following nonlinear Schrödinger-Poisson system

$$\text{(NLSP)} \qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u + \rho(x)\phi u = K(x)u^p, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \rho(x)u^2, \end{cases}$$

in the semiclassical limit, namely for $\varepsilon \to 0$, where ε stands for the ratio $\hbar/(2m)^{1/2}$, involving the reduced Planck constant $\hbar$ and the elementary matter-field mass m, in a non-relativistic regime. In particular, we focus on solutions (u, ϕ) such that u concentrates around a circle.

Let us choose any 1-dimensional linear subspace $d \subset \mathbb{R}^3$. We denote by π the orthogonal complement of d. If $x \in \mathbb{R}^3$, we will write $x = (x', x'')$ with $x' \in d$ and $x'' \in \pi$.

As a particular case of our main result, we have the following theorem:

Theorem. *Let $p > 3$ and $V \in C(\mathbb{R}^3\setminus\{0\}, \mathbb{R}^+)$ be a radial potential. Write $V(x) = \tilde{V}(x', |x''|)$. If there exists $r^* > 0$ such that the function*

$$\mathcal{M}(r) := r\left[\tilde{V}(0, r)\right]^{\frac{2}{p-1}}$$

has an isolated local minimum at $r = r^*$ such that $\mathcal{M}(r^*) > 0$, then for ε small enough, the system

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = u^p, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \end{cases}$$

has a positive cylindrically symmetric solution u_ε that concentrates on the circle of radius r^* centered at the origin and contained in the plane π.

We point out that we have no assumption about the decay of V at infinity. In particular, V could be compactly supported. This is an improvement on previous works, see e.g. [5].

A fundamental physical problem arises from the correspondence principle, according to which quantum mechanics contains classical mechanics as $\hbar \to 0$. In the framework of the Schrödinger equation with Coulomb potential one can construct solutions which are localized around classical Keplerian elliptic orbits by superposition of states of minimal quantum fluctuation (coherent states), see [84, 114]. Due to the dispersive nature of the Schrödinger equation, a rigorous reduction to classical mechanics cannot in general be performed. By introducing a local nonlinear homogeneous term u^p, in [23], the authors prove the existence of solutions for the $2D$ nonlinear Schrödinger equation with radial potential, concentrating on a circle. In the case of radial potentials, due to the invariance by rotations, the classical and quantum angular momentum are conserved as indicated by Noether's Theorem. This suggests that the solutions concentrating on Keplerian orbits are suitable candidates in order to mimic, in the semi-classical limit, the classical dynamics described by Newton's equations. In [52], the existence of solutions concentrating on circles has been obtained for the $3D$ nonlinear Schrödinger equation with cylindrically symmetric potential. Under the assumption of a radial potential, solutions concentrating on spheres were found in [19] by a variational argument, and in [10] by an approach based on the Lyapunov-Schmidt finite dimensional reduction scheme. In both [23] and [52], the underlying idea is to find solutions with nonzero angular momentum. By a different method, in [30] and [27] the existence of solutions concentrating respectively on points and on $k-$dimensional spheres has been obtained for the nonlinear Schrödinger equation. In particular, in [27] the existence of solutions concentrating on a circle has been obtained when radial symmetry occurs, as in the preceding theorem. Our aim is to extend [27, 30] to the nonlinear Schrödinger-Poisson system.

1. Assumptions and main result

Now we describe our assumptions.

1.1. The potentials

We consider a nonnegative potential $V \in C(\mathbb{R}^3 \setminus \{0\})$, a nonnegative competing function $K \in C(\mathbb{R}^3 \setminus \{0\})$, $K \not\equiv 0$, and a weight $\rho \in L^{3/2}_{\text{loc}}(\mathbb{R}^3) \cap L^{\infty}_{\text{loc}}\left(\mathbb{R}^3 \setminus \{0\}\right)$. We assume that the functions V, K and ρ are cylindrically symmetric with respect to d, i.e. for every $R \in \mathbf{O}(3)$ such that $R(d) = d$, we have $V \circ R = V$, $K \circ R = K$ and $\rho \circ R = \rho$. This will be the case if for example V, K and ρ are radial functions. Notice that if we write as above $x = (x', x'')$ with $x' \in d$ and $x'' \in \pi$, these assumptions imply that the functions V, K and ρ are even in x' and depend only on x' and $|x''|$.

1.2. The nonlinearity

We consider, for simplicity, a homogeneous nonlinear term u^p with $3 < p < \infty$. The condition $p > 3$ will be used in order to ensure the boundedness of Palais-Smale sequences.

1.3. The growth conditions

Let

$$W(x) := V(x) + \frac{\rho(x)}{1+|x|}.$$

Following [30, 113] we impose one of the three sets of growth conditions at infinity :

($\mathcal{G}^1_\infty$) there exists $\sigma < p-3$ such that

$$\limsup_{|x|\to\infty} \frac{K(x)}{|x|^\sigma} < \infty;$$

($\mathcal{G}^2_\infty$) there exists $\sigma \in \mathbb{R}$ such that

$$\liminf_{|x|\to\infty} W(x)\,|x|^2 > 0 \qquad \text{and} \qquad \limsup_{|x|\to\infty} \frac{K(x)}{|x|^\sigma} < \infty;$$

($\mathcal{G}^3_\infty$) there exist $\alpha < 2$ and $\sigma \in \mathbb{R}$ such that

$$\liminf_{|x|\to\infty} W(x)\,|x|^\alpha > 0 \qquad \text{and} \qquad \limsup_{|x|\to\infty} \frac{K(x)}{\exp(\sigma\,|x|^{\frac{2-\alpha}{2}})} < \infty.$$

Note that in comparison with [30], in ($\mathcal{G}^2_\infty$) and ($\mathcal{G}^3_\infty$), V might vanish somewhere. We also impose one of the three sets of growth conditions at the origin, which mirror those at infinity :

($\mathcal{G}^1_0$) there exists $\tau > -2$, such that

$$\limsup_{|x|\to 0} \frac{K(x)}{|x|^\tau} < \infty,$$

($\mathcal{G}^2_0$) there exists $\tau \in \mathbb{R}$ such that

$$\liminf_{|x|\to 0} W(x)\,|x|^2 > 0 \qquad \text{and} \qquad \limsup_{|x|\to 0} \frac{K(x)}{|x|^\tau} < \infty;$$

($\mathcal{G}^3_0$) there exist $\gamma > 2$ and $\tau \in \mathbb{R}$ such that

$$\liminf_{|x|\to 0} W(x)\,|x|^\gamma > 0 \qquad \text{and} \qquad \limsup_{|x|\to 0} \frac{K(x)}{\exp(\tau\,|x|^{-\frac{\gamma-2}{2}})} < \infty.$$

1.4. The auxiliary potential

Before we can state our last assumption, we need a few preliminaries. Let $a, b > 0$. We consider the *limit equation*

$$-\Delta u + au = bu^p \qquad \text{in } \mathbb{R}^2. \tag{3.1}$$

The weak solutions of (3.1) are critical points of the functional $\mathcal{I}_{a,b} : H^1(\mathbb{R}^2) \to \mathbb{R}$ defined by

$$\mathcal{I}_{a,b}(u) := \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + au^2\right) dx - \frac{b}{p+1}\int_{\mathbb{R}^2} u^{p+1}\, dx. \tag{3.2}$$

Any nontrivial critical point $u \in H^1(\mathbb{R}^2)$ of $\mathcal{I}_{a,b}$, belongs to the Nehari manifold

$$\mathcal{N}_{a,b} := \left\{u \in H^1(\mathbb{R}^2) \mid u \not\equiv 0 \text{ and } \langle \mathcal{I}'_{a,b}(u), u\rangle = 0\right\}.$$

A solution $u \in H^1(\mathbb{R}^2)$ is a *least-energy solution* of (3.1) if

$$\mathcal{I}_{a,b}(u) = \inf_{v \in \mathcal{N}_{a,b}} \mathcal{I}_{a,b}(v).$$

The *ground-energy function* is defined by

$$\mathcal{E} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ : (a, b) \mapsto \mathcal{E}(a, b) := \inf_{u \in \mathcal{N}_{a,b}} \mathcal{I}_{a,b}(u).$$

It is standard to show that

$$\mathcal{E}(a, b) = \inf_{\gamma \in \Gamma_{a,b}} \max_{t \in [0,1]} \mathcal{I}_{a,b}(\gamma(t)), \tag{3.3}$$

where

$$\Gamma_{a,b} := \left\{\gamma \in C([0, 1], H^1(\mathbb{R}^2)) \mid \gamma(0) = 0,\ \mathcal{I}_{a,b}(\gamma(1)) < 0\right\}.$$

The *auxiliary potential* $\mathcal{M} :\ \mathbb{R}^3 \to (0, +\infty]$ is defined by

$$(x', x'') \mapsto \mathcal{M}(x', x'') := \begin{cases} |x''|\, \mathcal{E}\left(V(x), K(x)\right) & \text{if } K(x) > 0, \\ +\infty & \text{if } K(x) = 0. \end{cases}$$

We can rewrite Lemma 2.1 in the particular case $N = 3$ and $k = 1$ as follows.

Lemma 3.1. *For every $(a, b) \in \mathbb{R}^+_0 \times \mathbb{R}^+_0$, $\mathcal{E}(a, b)$ is a critical value of $\mathcal{I}_{a,b}$ and we have*

$$\mathcal{E}(a, b) = \inf_{\substack{u \in H^1(\mathbb{R}^2)\\ u \neq 0}} \max_{t \geqslant 0} \mathcal{I}_{a,b}(tu).$$

If $u \in \mathcal{N}_{a,b}$ and $\mathcal{E}(a, b) = \mathcal{I}_{a,b}(u)$, then $u \in C^1(\mathbb{R}^2)$ and up to a translation, u is a radial function such that $\nabla u(x) \cdot x < 0$ for every $x \in \mathbb{R}^2 \setminus \{0\}$. Moreover, the following properties hold:

(i) *$\mathcal{E}$ is continuous in $\mathbb{R}_0^+ \times \mathbb{R}_0^+$;*
(ii) *for every $b^* \in \mathbb{R}_0^+$, $a \to \mathcal{E}(a, b^*)$ is strictly increasing;*
(iii) *for every $a^* \in \mathbb{R}_0^+$, $b \to \mathcal{E}(a^*, b)$ is strictly decreasing;*
(iv) *for every $\lambda > 0$, $\mathcal{E}(\lambda a, \lambda b) = \lambda^{-1/2}\mathcal{E}(a, b)$;*
(v) *the ground-energy function satisfies*

$$\mathcal{E}(a, b) = \mathcal{E}(1, 1)a^{\frac{p+1}{p-1}-1}b^{-\frac{2}{p-1}}.$$

The last property of the preceding lemma implies the following explicit form of the auxiliary potential:

$$\mathcal{M}(x', x'') = \mathcal{E}(1, 1)\left|x''\right| [V(x)]^{\frac{p+1}{p-1}-1} [K(x)]^{\frac{-2}{p-1}}.$$

Due to the symmetry that we shall impose on the solution (see (3.11)), the concentration can only occur in the plane π. We assume that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$\bar{\Lambda} \cap d = \emptyset, \ \Lambda \cap \pi \neq \emptyset, \tag{3.4}$$

for every $R \in \mathbf{O}(3)$ such that $R(d) = d$,

$$R(\Lambda) = \Lambda \tag{3.5}$$

and the following inequalities hold

$$0 < \inf_{\Lambda\cap\pi} \mathcal{M} < \inf_{\partial\Lambda\cap\pi} \mathcal{M}, \tag{3.6}$$

$$\inf_{\Lambda\cap\pi} \mathcal{M} < 2 \inf_{\Lambda} \mathcal{M}. \tag{3.7}$$

By continuity of $\mathcal{M}$ in Λ, this last condition is not restrictive. Similarly, we can also assume that $V > 0$ on $\overline{\Lambda}$ and that $\mathcal{M}$ is continuous on $\overline{\Lambda}$.

Our main result is the following.

Theorem 3.2. *Let $p > 3$ and V, K and ρ be functions satisfying the assumptions in Section 1.1. Assume that one set $(\mathcal{G}_\infty^i)$ of growth conditions at infinity and one set $(\mathcal{G}_0^j)$ of growth conditions at the origin hold. Assume also that there exists an open bounded set $\Lambda \subset \mathbb{R}^3$ such that* (3.4), (3.5), (3.6) *and* (3.7) *hold. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, problem* (NLSP) *has at least one positive solution u_ε. Moreover, for every $0 < \varepsilon < \varepsilon_0$, there exists $x_\varepsilon \in \Lambda \cap \pi$ such that u_ε attains its maximum at x_ε,*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0, \quad \lim_{\varepsilon\to 0} \mathcal{M}(x_\varepsilon) = \inf_{\Lambda\cap\pi} \mathcal{M},$$

and there exist $C > 0$ and $\lambda > 0$ such that

$$u_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon}\frac{d(x, S_\varepsilon^1)}{1 + d(x, S_\varepsilon^1)}\right)\left(1 + |x|^2\right)^{\frac{-1}{2}}, \qquad \forall x \in \mathbb{R}^3,$$

where S^1_ε is the circle centered at the origin, contained in the plane π and of radius $|x''_\varepsilon|$.

We notice that the auxiliary potential $\mathcal{M}$ does not depend on the weight ρ. Hence, ρ does not play a role in the concentration phenomenon. Heuristically, this is due to the fact that the Poisson term is a higher-order term in the asymptotic development of the energy of the solutions (see Section 3). The first term of this asymptotic development is related to the auxiliary potential $\mathcal{M}$.

In Section 2, we deal with an auxiliary penalized problem. This by now classical penalization argument goes back to del Pino and Felmer [61]. The method has then been adapted in [30, 113] in the frame of vanishing or compactly supported potentials. Section 3 is devoted to the asymptotic analysis of the solutions of the penalized problem while in Section 4, we show how to go back to the original problem. At last, we give some final comments in Section 5.

We use the notations $u_+ := \max(u, 0)$ and $u_- := \max(-u, 0)$

2. Existence for the penalized problem

Following [113] and [27], we define the penalization potential $H : \mathbb{R}^3 \to \mathbb{R}$ by

$$H(x) := \frac{\kappa}{|x|^2 \left(\left(\log |x| \right)^2 + 1 \right)^{\frac{1+\beta}{2}}}$$

where $\beta > 0$ and $0 < \kappa < \frac{1}{8}$. Notice that for all $x \in \mathbb{R}^3$, we have

$$H(x) \leqslant \frac{\kappa}{|x|^2},$$

so that, as in Chapter II, we can deduce from Hardy's inequality that the operator $-\Delta - H$ is positive and satisfies the comparison principle (Lemma 2.3 in Chapter II).

Fix $\mu \in (0, \frac{1}{2})$. We define the penalized nonlinearity $g_\varepsilon : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} g_\varepsilon(x, s) := {} & \chi_\Lambda(x) K(x) s_+^p \\ & + (1 - \chi_\Lambda(x)) \min \left\{ \varepsilon^2 H(x) + \mu V(x), K(x)\, |s|^{p-1} \right\} s_+. \end{aligned}$$

Let $G_\varepsilon(x, s) := \int_0^s g_\varepsilon(x, \sigma) d\sigma$. One can check that g_ε is a Carathéodory function with the following properties :

(g_1) $g_\varepsilon(x, s) = o(s),\ s \to 0^+$, uniformly in compact subsets of $\mathbb{R}^3$.

(g_2) there exists $p > 3$ such that

$$\lim_{s\to\infty} \frac{g_\varepsilon(x,s)}{s^p} = 0,$$

(g_3) one has

$$\begin{array}{ll} 0 < (p+1)G_\varepsilon(x,s) \leqslant g_\varepsilon(x,s)s & \forall x \in \Lambda,\ \forall s > 0, \\ 0 < 2G_\varepsilon(x,s) \leqslant g_\varepsilon(x,s)s \leqslant \left(\varepsilon^2 H(x) + \mu V(x)\right) s^2 & \forall x \notin \Lambda,\ \forall s > 0, \end{array}$$

(g_4) the function

$$s \mapsto \frac{g_\varepsilon(x,s)}{s}$$

is nondecreasing for all $x \in \mathbb{R}^3$.

Now we use Critical Point Theory in order to find solutions to the penalized problem

$$\left\{ \begin{array}{ll} -\varepsilon^2 \Delta u + V(x)u + \rho(x)\phi u = g_\varepsilon(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \rho(x)u^2. & \end{array} \right. \tag{3.8}$$

For any $u^2\rho \in L^1_{loc}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)\rho(x)\rho(y)}{|x-y|}\, dx\, dy < \infty,$$

the standard distributional solution

$$\phi_u := \frac{1}{4\pi\,|x|} \star u^2\rho \tag{3.9}$$

belongs to $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and is a weak solution in $\mathcal{D}^{1,2}$ (e.g. [132]). Since we consider $u \in \mathcal{D}^{1,2}$, we will assume $\rho \in L^{3/2}_{loc}$. A suitable choice of the space X of admissible functions is given in the work of Ruiz [132], in the case $V \equiv 0$ and $\rho \equiv 1$. Inspired by [132], we define, for measurable $V, \rho \geqslant 0$,

$$\|u\|_X^2 := \int_{\mathbb{R}^3} \left(\varepsilon^2\,|\nabla u|^2 + Vu^2\right) + \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)\rho(x)\rho(y)}{|x-y|}\, dx\, dy\right)^{1/2}$$

and

$$X := \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^3)\ :\ \|u\|_X < \infty\right\}. \tag{3.10}$$

As pointed out in [132], the space X is a uniformly convex Banach space, hence it is reflexive. Precisely, we look for solutions $(u, \phi_u) \in X \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.

We also define $H^1_{V,\varepsilon}$ to be the closure of $\mathcal{D}(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_\varepsilon^2 := \int_{\mathbb{R}^3} \varepsilon^2 |\nabla u|^2 + V u^2.$$

We will focus on the closed subspace $E \subset X$ of functions which are radial in π, namely

$$E := \{u \in X \;:\; \forall R \in \mathbf{O}(3) \text{ s.t. } R(d) = d, u \circ R = u\}. \tag{3.11}$$

Solutions of (3.8) are the critical points of the functional

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(\varepsilon^2 |\nabla u|^2 + V u^2\right) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \rho - \int_{\mathbb{R}^3} G_\varepsilon(x, u(x))\, dx,$$

which is $C^1(X;\mathbb{R})$.

In the present section we find a critical point for J_ε through the mountain-pass theorem. The main result of this section is the following theorem.

Theorem 3.3. *Assume V, K, ρ satisfy the assumptions of Theorem 3.2. Then, for any $\varepsilon > 0$ and $\kappa, \mu > 0$ small enough, there exists a critical point for J_ε at level*

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} J_\varepsilon(\gamma(t)), \tag{3.12}$$

where

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], E) \;:\; \gamma(0) = 0,\; J_\varepsilon(\gamma(1)) < 0\}, \tag{3.13}$$

corresponding to a nontrivial solution $(u, \phi) \in E \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ for (3.8). Moreover u is positive.

Remark 3.4. *Due to the invariance of the Lebesgue measure by rotations, by the symmetric criticality principle* [123], *if $u \in E$ is critical for $J_\varepsilon|_E$, then u is also critical for $J_\varepsilon|_X$.*

The functional J_ε has the mountain pass geometry, as it is shown in the following lemma.

Lemma 3.5. *For any $p > 3$, the functional J_ε satisfies the mountain pass geometry. Furthermore, there exists a Palais-Smale (P-S) sequence at the minimax level c_ε. In particular, defining*

$$S := \{u \in E \;:\; u \geqslant 0\},$$

$$S_{1/n} := \left\{u \in E \;:\; \inf_{y \in S} \|u - y\|_X < 1/n\right\},$$

it is possible to select the P-S sequence $(u_n)_n$ in such a way that $u_k \in S_{1/k}$ for all $k \in \mathbb{N}$.

Proof. We first prove that, for any $p > 3$, the origin is a local minimum for J_ε in E. We follow the arguments in [28]. By the Sobolev inequality, there exists $C > 0$ such that

$$\frac{1}{p+1}\int_\Lambda K\,|u|^{p+1} \leqslant C\,\|u\|_X^{p+1}.$$

By the same estimate as in Lemma 2.4 in Chapter II, there exists $c > 0$ such that

$$J_\varepsilon(u) \geqslant c\int_{\mathbb{R}^3}\left(\varepsilon^2\,|\nabla u|^2 + Vu^2\right) + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\rho - C\|u\|_X^{p+1}.$$

Since, by definition, $\int_{\mathbb{R}^3}\phi_u u^2\rho = \left(\|u\|_X^2 - \|u\|_\varepsilon^2\right)^2$, one has

$$\begin{aligned} J_\varepsilon(u) &\geqslant c\,\|u\|_\varepsilon^2 + \frac{1}{4}\left[\|u\|_X^2 - \|u\|_\varepsilon^2\right]^2 - C\,\|u\|_X^{p+1} \\ &= c\,\|u\|_\varepsilon^2 + \frac{1}{4}\,\|u\|_X^4 - \frac{1}{2}\,\|u\|_\varepsilon^2\,\|u\|_X^2 + \frac{1}{4}\,\|u\|_\varepsilon^4 - C\,\|u\|_X^{p+1}. \end{aligned}$$

Let $\alpha > 1$. Using the inequality $AB \leqslant \alpha^2A^2 + B^2/(4\alpha^2)$, we get

$$J_\varepsilon(u) \geqslant c\,\|u\|_\varepsilon^2 - \frac{\alpha^2-1}{4}\,\|u\|_\varepsilon^4 + \frac{\alpha^2-1}{4\alpha^2}\,\|u\|_X^4 - C\,\|u\|_X^{p+1}.$$

For $\|u\|_X \leqslant \delta$, we have

$$J_\varepsilon(u) \geqslant \left[c - \frac{\alpha^2-1}{4}\delta^2\right]\|u\|_\varepsilon^2 + \left[\frac{\alpha^2-1}{4\alpha^2} - C\delta^{p-3}\right]\|u\|_X^4.$$

This yields, for δ small enough,

$$J_\varepsilon(u) \geqslant \left[\frac{\alpha^2-1}{4\alpha^2} - C\delta^{p-3}\right]\|u\|_X^4.$$

Hence, the origin is a strict local minimum point for J_ε in X (and thus in E).

Moreover, J_ε attains negative values along curves of the form $u_t := tu$, with $u \in E$ such that $u^+ \not\equiv 0$ and $t > 0$. Hence J_ε has the mountain pass geometry.

By the general minimax principle [145, p.41], there exists a P-S sequence $(u_n)_{n\in\mathbb{N}}$ such that, if for $\gamma_n \in \Gamma_\varepsilon$,

$$\max_{t\in[0,1]} J_\varepsilon(\gamma_n(t)) \leqslant c_\varepsilon + \frac{1}{n},$$

then

$$\operatorname{dist}(u_n, \gamma_n([0,1])) < \frac{1}{n}. \tag{3.14}$$

Finally, since $J_\varepsilon(u) = J_\varepsilon(|u|)$, the conclusion follows from (3.14). $\square$

We now study some properties of the P-S sequences.

Lemma 3.6. *Let $p > 3$ and let $(u_n)_n$ be a P-S sequence for J_ε. Then $(u_n)_n$ is bounded in X.*

Proof. Define

$$\Delta_n := \int_{\mathbb{R}^3} \big(g_\varepsilon(x, u_n(x))u_n(x) - (p+1)G_\varepsilon(x, u_n(x))\big)\, dx.$$

By (g_3) and Hardy's inequality, we have

$$\begin{aligned}
\Delta_n &\geqslant -(p-1)\int_{\mathbb{R}^3\setminus\Lambda} G_\varepsilon(x, u_n(x))\, dx \\
&\geqslant -\frac{p-1}{2}\int_{\mathbb{R}^3\setminus\Lambda} (\varepsilon^2 H + \mu V)u_n^2 \\
&\geqslant -\frac{p-1}{4}\, \|u_n\|_\varepsilon^2 .
\end{aligned}$$

Since $(u_n)_n$ is a P-S sequence, the above estimate yields

$$\begin{aligned}
C &\geqslant (p+1)J_\varepsilon(u_n) - \big\langle J_\varepsilon'(u_n), u_n\big\rangle \\
&= \frac{p-1}{2}\, \|u_n\|_\varepsilon^2 + \frac{p-3}{4}\int_{\mathbb{R}^3} \phi_{u_n}(x)u_n^2(x)\rho(x)\, dx + \Delta_n \\
&\geqslant \frac{p-1}{4}\, \|u_n\|_\varepsilon^2 + \frac{p-3}{4}\int_{\mathbb{R}^3} \phi_{u_n}(x)u_n^2(x)\rho(x)\, dx. \qquad (3.15)
\end{aligned}$$

As a consequence, the conclusion follows. □

Lemma 3.7. *Let $(u_n)_n$ be as in Lemma 3.5 such that $u_k \in S_{1/k}$ for all $k \in \mathbb{N}$. Then*

$$(a) \qquad \lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{(u_n)_-}(x)(u_n)_-^2\rho(x)\, dx = 0;$$

$$(b) \qquad \lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n}(x)(u_n)_-^2\rho(x)\, dx = 0.$$

Proof. Since $u_n \in S_{1/n}$, there exists a sequence $(y_n)_n \subset S$ such that

$$\|u_n - y_n\|_X \to 0.$$

Hence (a) follows:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \phi_{(u_n)_-}(x)(u_n)_-^2(x)\rho(x)\,dx \\
&\qquad \leqslant \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u_n)_-^2(x)\rho(x)(u_n)_-^2(z)\rho(z)}{|x-z|}\,dx\,dz \\
&\qquad \leqslant \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u_n-y_n)^2(x)\rho(x)(u_n-y_n)^2(z)\rho(z)}{|x-z|}\,dx\,dz \\
&\qquad \leqslant \|u_n-y_n\|_X^4 \to 0.
\end{aligned}$$

To prove (b), define, for measurable and nonnegative functions f and g, the quantity

$$D(f,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x)|x-y|^{-1}g(y)\,dx\,dy.$$

From [103, p.250], we have

$$|D(f,g)|^2 \leqslant D(f,f)D(g,g). \tag{3.16}$$

Since $(u_n)_n$ is bounded in X, by the inequality above with $f := u_n^2\rho$ and $g := (u_n)_-^2\rho$ and by (a) we have

$$\int_{\mathbb{R}^3} \phi_{u_n}(x)(u_n)_-^2\rho(x)\,dx \leqslant C\int_{\mathbb{R}^3} \phi_{(u_n)_-}(x)(u_n)_-^2\rho(x)\,dx \to 0,$$

as $n \to \infty$. □

In the following we shall need a family of cut-off functions. Consider a smooth function ζ such that $\zeta \equiv 1$ on $[2,\infty)$ and $\zeta \equiv 0$ on $[0,1]$. Then define

$$\eta_R(x) := \zeta\left(\frac{\log(1+|x|)}{R}\right).$$

One has

$$\||x|\;\nabla\eta_R\|_\infty \leqslant \frac{C}{R}. \tag{3.17}$$

Lemma 3.8. *Let $(u_n)_n \subset E$ be a P-S sequence as in Lemma 3.5 and assume that $u_n \rightharpoonup u \geqslant 0$ in E. Then, for all $\delta > 0$, there exists a ball*

$B \subset \mathbb{R}^3$ *such that*

$$a) \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B} \phi_{u_n}(x)u_n^2(x)\rho(x)\,dx < \delta,$$

$$b) \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B} V u_n^2 < \delta,$$

$$c) \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B} H u_n^2 < \delta,$$

$$d) \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B} \phi_{u_n}(u_n)_-\, u\, \rho < \delta,$$

$$e) \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B} \phi_{u_n}(u_n)_+\, u\, \rho < \delta.$$

Proof. Consider the above family of cut-off functions. We claim that, uniformly in n,

$$\int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u_n|^2 - H u_n^2\Big)\eta_R^2 \geqslant O\Big(\frac{1}{R}\Big), \quad \text{as } R \to \infty. \tag{3.18}$$

In order to prove this, compute

$$|\nabla(u_n\eta_R)|^2 = \eta_R^2\,|\nabla u_n|^2 + 2u_n\eta_R\nabla u_n\cdot\nabla\eta_R + u_n^2\,|\nabla\eta_R|^2\,.$$

We have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} u_n\eta_R\nabla u_n\cdot\nabla\eta_R\right| &= \left|\int_{\mathbb{R}^3} \frac{u_n}{|x|}\nabla u_n\cdot(|x|\,\nabla\eta_R)\right| \\ &\leqslant \||x|\,\nabla\eta_R\|_\infty \left|\int_{\mathbb{R}^3}\frac{u_n}{|x|}\nabla u_n\right|.\end{aligned}$$

By (3.17), Cauchy-Schwarz and Hardy inequalities, we obtain

$$\left|\int_{\mathbb{R}^3} u_n\eta_R\nabla u_n\cdot\nabla\eta_R\right| \leqslant \frac{C}{R}\|\nabla u_n\|_2^2 \leqslant \frac{C'}{R}. \tag{3.19}$$

Here we have taken into account that, since $u_n \rightharpoonup u$ in E, $\|\nabla u_n\|_2$ is bounded. In the same way, one can easily obtain

$$\int_{\mathbb{R}^3} u_n^2\,|\nabla\eta_R|^2 \leqslant \frac{C}{R^2}.$$

Hence, by Hardy's inequality and the above estimates, we have, as $R \to \infty$,

$$\begin{aligned}\int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla u_n|^2 - H u_n^2\right)\eta_R^2 &\geqslant \int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla(u_n\eta_R)|^2 - \kappa\frac{(u_n\eta_R)^2}{|x|^2}\right) + O\Big(\frac{1}{R}\Big) \\ &\geqslant O\Big(\frac{1}{R}\Big),\end{aligned}$$

and the claim follows. Furthermore, notice that

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n \eta_R^2) = \int_{\mathbb{R}^3} |\nabla u_n|^2 \eta_R^2 + O\Big(\frac{1}{R}\Big), \quad \text{as } R \to \infty. \tag{3.20}$$

Finally, for R large enough so that $\eta_R \equiv 0$ on Λ, we deduce from (3.18) and (3.20) that

$$\begin{aligned} o(1) &= \big\langle J_\varepsilon'(u_n), u_n \eta_R^2 \big\rangle \\ &\geqslant \frac{\varepsilon^2}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 \eta_R^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \eta_R^2 \rho + (1-\mu) \int_{\mathbb{R}^3} V u_n^2 \eta_R^2 + O\Big(\frac{1}{R}\Big), \end{aligned}$$

as $R \to \infty$. Since all the terms are nonnegative, taking $B := \{x \in \mathbb{R}^3 : |x| \leqslant e^{2R}\}$ with R large enough, the above estimate yields statements $(a), (b)$ and, using (3.18), statement (c).

In order to prove (d) we use Cauchy-Schwarz inequality and (3.16), obtaining

$$\int_{\mathbb{R}^3} \phi_{u_n} (u_n)_- \, u \, \rho \eta_R \leqslant (D(u_n^2, (u_n)_-^2))^{1/2} \Big(D(u_n^2, u_n^2) D(u^2, u^2) \Big)^{1/4} \to 0,$$

since $D(u_n^2, u_n^2)$ is bounded and $D(u_n^2, (u_n)_-^2) \to 0$ by Lemma 3.7.

Finally we prove (e). We have, for any $R > 0$,

$$\begin{aligned} o(1) &= \big\langle J_\varepsilon'(u_n), u \eta_R \big\rangle \\ &\geqslant (u_n | u \eta_R)_\varepsilon + \int_{\mathbb{R}^3} \phi_{u_n} (u_n)_+ \, u \eta_R \rho \\ &\quad - \int_{\mathbb{R}^3} \phi_{u_n} (u_n)_- \, u \eta_R \rho - \int_{\mathbb{R}^3} (\varepsilon^2 H + \mu V)(u_n)_+ u \eta_R, \end{aligned}$$

as $n \to \infty$. Notice that, by weak convergence, we have, for any $R > 0$,

$$(u_n | u \eta_R)_\varepsilon \to (u | u \eta_R)_\varepsilon,$$

and

$$\int_{\mathbb{R}^3} (\varepsilon^2 H + \mu V)(u_n)_+ u \eta_R \to \int_{\mathbb{R}^3} (\varepsilon^2 H + \mu V) u^2 \eta_R.$$

Now fix $\alpha > 0$ small and take R_α such that for all $R > R_\alpha$ we have

$$\int_{\mathbb{R}^3} (\varepsilon^2 H + \mu V) u^2 \eta_R \leqslant \alpha$$

and, by (d),

$$\int_{\mathbb{R}^3} \phi_{u_n} (u_n)_- \, u \eta_R \rho \leqslant \alpha.$$

Arguing as for the estimate (3.19), we can choose R_α large enough such that

$$\left|\int_{\mathbb{R}^3} u\nabla u\cdot\nabla\eta_R\right| \leqslant \frac{C}{R} < \alpha,$$

for every $R > R_\alpha$. Hence, writing $\nabla(\eta_R\, u) = \eta_R\nabla u + u\nabla\eta_R$, the term $(u|u\eta_R)_\varepsilon$ is the sum of a positive term plus a small term. Therefore, we obtain

$$o(1) + 3\alpha \geqslant \int_{\mathbb{R}^3} \phi_{u_n}(u_n)_+\, u\eta_R\rho$$

and claim (e) follows. This concludes the proof. $\square$

Arguing as in the above lemmas we have

Lemma 3.9. *Under the assumptions on ρ, V, K given in Theorem 3.2, let (u_n) be as in the above lemma. Then for all $\delta > 0$, there exists a ball $B(0) \subset \mathbb{R}^3$, such that*

$$(a)\qquad \limsup_{n\to\infty} \int_{B(0)} \phi_{u_n}(x)u_n^2\rho(x)\; dx < \delta,$$

$$(b)\qquad \limsup_{n\to\infty} \left|\int_{B(0)} \phi_{u_n}(x)(u_n)_- u\rho(x)\; dx\right| < \delta,$$

$$(c)\qquad \limsup_{n\to\infty} \left|\int_{B(0)} \phi_{u_n}(x)(u_n)_+ u\rho(x)\; dx\right| < \delta.$$

Lemma 3.10. *Let $(u_n)_n$ be as in Lemma 3.8. Then, passing if necessary to a subsequence, we have*

$$\|u_n\|_\varepsilon^2 \to \|u\|_\varepsilon^2\,.$$

Proof. Since $u_n \rightharpoonup u$ in $H^1_{V,\varepsilon}$, for some subsequence, we have

$$o(1) = \left\langle J'_\varepsilon(u_n), u_n - u\right\rangle = \|u_n\|_\varepsilon^2 - \|u\|_\varepsilon^2 + o(1)$$

$$+\int_{\mathbb{R}^3} \phi_{u_n}(x)u_n(u_n-u)\rho(x)dx + \int_{\mathbb{R}^3} g_\varepsilon(x,u_n)(u_n-u)dx. \tag{3.21}$$

We show that

$$A_n := \int_{\mathbb{R}^3} g_\varepsilon(x,u_n)(u_n-u)dx \to 0$$

and

$$B_n := \int_{\mathbb{R}^3} \phi_{u_n}(x)u_n(u_n-u)\rho(x)dx \to 0.$$

Observe that

$$A_n := \int_\Lambda ... + \int_{B\setminus\Lambda} ... + \int_{\mathbb{R}^3\setminus B} ...,$$

for some large ball B containing Λ. Since $(u_n)_n$ is bounded in E and E is compactly embedded in $L^q(\Lambda)$ for all $q > 1$, passing to a subsequence, we can assume $u_n \to u$ in $L^{p+1}(\Lambda)$. As a consequence, passing if necessary to a subsequence, we have $|g_\varepsilon(x, u_n)| < g(x)$ for some $g \in L^{\frac{p+1}{p}}(\Lambda)$. Using Hölder inequality and dominated convergence theorem, we have

$$\int_\Lambda g_\varepsilon(x, u_n)(u_n - u)dx \to 0.$$

In the same way, using the compact embedding $E \hookrightarrow L^2_{\varepsilon^2 H+\mu V}(B \setminus \Lambda)$, it follows that

$$\int_{B\setminus\Lambda} g_\varepsilon(x, u_n)(u_n - u)dx \to 0.$$

Finally, taking B large and using $(b), (c)$ in Lemma 3.8, we have

$$\int_{\mathbb{R}^3\setminus B} g_\varepsilon(x, u_n)(u_n - u)dx \to 0,$$

hence $A_n \to 0$.

In order to prove $B_n \to 0$, we use a similar splitting argument. Fix $\delta > 0$.

$$\begin{aligned} B_n &= \int_{B(0)} ... + \int_{B\setminus B(0)} ... + \int_{\mathbb{R}^3\setminus B} ... \\ &= I_{1,n} + I_{2,n} + I_{3,n}, \end{aligned}$$

Now we choose B such that, using Lemma 3.8, we have

$$|I_{3,n}| < \delta.$$

Shrinking the ball $B(0)$ if necessary, we infer from Lemma 3.9 that $|I_{1,n}| < \delta$.

Next, we estimate $I_{2,n}$ as follows. By Hölder and Sobolev inequalities, we have

$$\int_{B\setminus B(0)} \phi_{u_n}(x)|u_n(u_n - u)|\rho(x)dx \leqslant \\ C\|\rho\|_{L^\infty(B\setminus B(0))}\|\phi_{u_n}\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)}\|u_n(u_n - u)\|_{L^{6/5}(B\setminus B(0))}. \tag{3.22}$$

Due to the weak convergence in E, $\|\phi_{u_n}\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)}$ is bounded, and therefore, by compactness, we get

$$|I_{2,n}| < \delta.$$

This concludes the proof. □

Proof of Theorem 3.3. By Lemma 3.5 and Lemma 3.6, there exists a bounded P-S sequence $(u_n)_n$ at the minimax level c_ε, such that $u_n \rightharpoonup u$ in E. We are going to prove that, passing if necessary to a subsequence,

$$\begin{array}{ll} i) & J_\varepsilon(u_n) \to J_\varepsilon(u), \\ ii) & J'_\varepsilon(u) = 0. \end{array}$$

This will imply the existence of a nontrivial solution u.

In order to prove $i)$, notice that, by Lemma 3.10, it is enough to show that

$$\int_{\mathbb{R}^3} G_\varepsilon(x, u_n)\, dx \to \int_{\mathbb{R}^N} G_\varepsilon(x, u)\, dx$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 \rho(x)\, dx \to \int_{\mathbb{R}^3} \phi_u(x) u^2 \rho(x)\, dx.$$

In order to prove the former limit, we can argue as for the terms involving g_ε in Lemma 3.10, splitting the integral

$$\int_{\mathbb{R}^3} |G_\varepsilon(x, u_n) - G_\varepsilon(x, u)|\, dx = \int_\Lambda ... + \int_{B\setminus\Lambda} ... + \int_{\mathbb{R}^3\setminus B} ... .$$

We can assume $u_n \to u$ in $L^{p+1}(\Lambda)$ and almost everywhere. Fix $\delta > 0$. By using the compact embedding $E \hookrightarrow L^q(\Lambda)$ which holds for all $q > 1$, the dominated convergence theorem yields

$$\int_\Lambda |G_\varepsilon(x, u_n) - G_\varepsilon(x, u)| dx < \delta$$

for large n and, in the same fashion, using property (g_3) and the compact embedding of E in $L^2_{\varepsilon^2 H + \mu V}(B \setminus \Lambda)$, it follows that

$$\int_{B\setminus\Lambda} |G_\varepsilon(x, u_n) - G_\varepsilon(x, u)| dx < \delta$$

for some subsequence, taking n larger if necessary. Finally, observe that, by $(b), (c)$ of Lemma 3.8, there exists B large enough, such that for n large enough,

$$\int_{\mathbb{R}^3\setminus B} |G_\varepsilon(x, u_n) - G_\varepsilon(x, u)| dx < \delta.$$

Now we prove the second limit. We compute

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}[\phi_{u_n}(x)u_n^2\rho(x)-\phi_u(x)u^2\rho(x)]dx\right| \\
&\qquad\leqslant \Big|\int_{B(0)}...\Big|+\Big|\int_{B\setminus B(0)}...\Big|+\Big|\int_{\mathbb{R}^3\setminus B}...\Big| \\
&\qquad= J_{1,n}+J_{2,n}+J_{3,n}.
\end{aligned}
$$

Fix $\delta>0$. Arguing as in the proof of Lemma 3.10, we can take B large enough in such a way that

$$\int_{\mathbb{R}^3\setminus B}\phi_u(x)u^2\rho(x)dx<\delta,$$

yielding, with Lemma 3.8,

$$J_{3,n}\leqslant\Big|\int_{\mathbb{R}^3\setminus B}\phi_{u_n}(x)u_n^2\rho(x)dx\Big|+\Big|\int_{\mathbb{R}^3\setminus B}\phi_u(x)u^2\rho(x)dx\Big|<2\delta.$$

Since we can shrink $B(0)$ so that

$$\int_{B(0)}\phi_u(x)u^2\rho(x)dx<\delta,$$

we deduce from Lemma 3.9 that

$$J_{1,n}\leqslant\Big|\int_{B(0)}\phi_{u_n}(x)u_n^2\rho(x)dx\Big|+\Big|\int_{B(0)}\phi_u(x)u^2\rho(x)dx\Big|<2\delta.$$

Now, with by now familiar arguments, we can estimate $J_{2,n}$. Indeed, using Hölder and Sobolev inequalities, we have

$$
\begin{aligned}
&\int_{B\setminus B(0)}|\phi_{u_n}(x)u_n^2-\phi_u(x)u^2|\rho(x)dx \\
&\leqslant\int_{B\setminus B(0)}|\phi_{u_n}u_n^2-\phi_{u_n}u^2|\rho+\int_{B\setminus B(0)}|\phi_{u_n}u^2-\phi_u u^2|\rho \\
&\leqslant C\|\rho\|_{L^\infty(B\setminus B(0))}\|\phi_{u_n}\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)}\|u_n^2-u^2\|_{L^{6/5}(B\setminus B(0))} \\
&\qquad+\|\rho\|_{L^\infty(B\setminus B(0))}\int_{B\setminus B(0)}|\phi_{u_n}(x)u^2-\phi_u(x)u^2|dx.
\end{aligned}
$$

By compactness, we infer that $\|u_n^2-u^2\|_{L^{6/5}(B\setminus B(0))}\to 0$, while $(\phi_{u_n})_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, hence the first term in the last expression goes to zero. On the other hand, since $\phi_{u_n}\rightharpoonup\phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, we have $\phi_{u_n}\to\phi_u$ strongly in $L^d(B\setminus B(0))$ for any $d<2^*$. Hence, Hölder inequality implies the last term also goes to zero. As a consequence $J_{2,n}\leqslant\delta$ and this yields *i*).

The proof of *ii*) is rather standard, using the weak convergence in E and the same splitting arguments. The maximum principle implies $u > 0$ on $\mathbb{R}^3$. This completes the proof of the theorem. $\square$

3. Asymptotics of solutions

In order to show that the solution u_ε found in Theorem 3.3 satisfies, for ε small enough, the original problem and concentrates around a circle, we need to study the asymptotic behaviour of u_ε as $\varepsilon \to 0$. Since many arguments are similar to the ones in [27], we only stress the differences with these. We begin with an energy estimate.

Proposition 3.11 (Upper estimate of the critical value). *Suppose that the assumptions of Theorem 3.3 are satisfied. For ε small enough, the critical value c_ε defined in* (3.12) *satisfies*

$$c_\varepsilon \leqslant \varepsilon^2\big(\pi \inf_{\Lambda\cap\pi} \mathcal{M} + o(1)\big) \text{ as } \varepsilon \to 0.$$

Moreover, the solution u_ε of (3.8) *found in Theorem* 3.3 *satisfies, for some $C > 0$,*

$$\|u_\varepsilon\|^2_{H_{V,\varepsilon}} \leqslant C\varepsilon^2 \tag{3.23}$$

and

$$\int_{\mathbb{R}^3} |\nabla \phi_{u_\varepsilon}|^2 \, dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\varepsilon^2(x)u_\varepsilon^2(y)\rho(x)\rho(y)}{4\pi|x-y|} \, dx \, dy \leqslant C\varepsilon^2. \tag{3.24}$$

Proof. Take $x_0 = (0, x_0'') \in \Lambda \cap \pi$ such that $\mathcal{M}(x_0) = \inf_{\Lambda\cap\pi} \mathcal{M}$. Denote by I_0 the functional defined by (3.2) with $a = V(x_0)$ and $b = K(x_0)$ and let $c_0 := \mathcal{E}(V(x_0), K(x_0))$. From (3.3), we infer that for every $\delta > 0$, there exists a continuous path $\gamma_\delta : [0,1] \to H^1(\mathbb{R}^2)$ such that $\gamma_\delta(0) = 0$, $I_0(\gamma_\delta(1)) < 0$ and

$$c_0 \leqslant \max_{t\in[0,1]} I_0(\gamma_\delta(t)) \leqslant c_0 + \delta.$$

Let $\eta \in \mathcal{D}\left(\mathbb{R}^2_+\right)$ be a cut-off function with support in Λ such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in a neighbourhood of $(0, |x_0''|)$ and $\|\nabla\eta\|_\infty \leqslant C$. We consider the path

$$\bar{\gamma}_\delta(t) : x \to \eta(x', |x''|)\gamma_\delta(t)\left(\frac{x'}{\varepsilon}, \frac{|x''| - |x_0''|}{\varepsilon}\right).$$

Setting

$$\bar{\gamma}_\delta(t)(x', x'') =: v_t\left(\frac{x'}{\varepsilon}, \frac{|x''| - |x_0''|}{\varepsilon}\right),$$

we compute, by a change of variable,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2\left|\nabla\bar{\gamma}_\delta(t)\right|^2+V(x)\bar{\gamma}_\delta(t)^2)dx-\int_{\mathbb{R}^3}G_\varepsilon(x,\bar{\gamma}_\delta(t))dx\\
&=\frac{\pi}{2}\int_{\mathbb{R}}\int_{-\frac{|x_0''|}{\varepsilon}}^{\infty}\Big(\left|\nabla v_t\right|^2+V(\varepsilon y',\varepsilon\sigma+\left|x_0''\right|)v_t^2\Big)(\varepsilon\sigma+\left|x_0''\right|)\varepsilon\,d\sigma\,\varepsilon dy'\\
&\qquad-\pi\int_{\mathbb{R}}\int_{-\frac{|x_0''|}{\varepsilon}}^{\infty}G(\varepsilon y',\varepsilon\sigma+\left|x_0''\right|,v_t)(\varepsilon\sigma+\left|x_0''\right|)\varepsilon\,d\sigma\,\varepsilon dy'.
\end{aligned}$$

The boundedness of ρ in Λ and Hardy-Littlewood-Sobolev inequality lead to

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\bar{\gamma}_\delta(t)(x)^2\bar{\gamma}_\delta(t)(y)^2\rho(x)\rho(y)}{|x-y|}\,dx\,dy\\
&\leqslant\|\rho\|_{L^\infty(\Lambda)}^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{v_t^2(x',\sigma)v_t^2(y',\tau)}{\varepsilon\Big(|x'-y'|^2+|\sigma-\tau|^2\Big)^{\frac{1}{2}}}\varepsilon^4\,dx'\,dy'\,d\sigma\,d\tau\\
&\leqslant C\varepsilon^3\left\|v_t\right\|_{L^2(\mathbb{R}^2)}^4\\
&=o\left(\varepsilon^2\right).
\end{aligned}$$

For ε small enough, we obtain

$$\varepsilon^{-2}J_\varepsilon(\bar{\gamma}_\delta(t))\leqslant\omega_k\left|x_0''\right|^k I_0(\gamma_\delta(t))+o(1). \tag{3.25}$$

It follows that for ε small enough, $\bar{\gamma}_\delta$ belongs to the class of paths Γ_ε defined by (3.13). We deduce from (3.12) that

$$\begin{aligned}
\varepsilon^{-2}c_\varepsilon&\leqslant\max_{t\in[0,1]}\varepsilon^{-2}J_\varepsilon(\bar{\gamma}_\delta(t))\\
&\leqslant\pi\left|x_0''\right|\max_{t\in[0,1]}I_0(\gamma_\delta(t))+o(1)\\
&\leqslant\pi\left|x_0''\right|(c_0+\delta)+o(1).
\end{aligned}$$

Since $\delta>0$ is arbitrary and $|x_0''|\,c_0=\mathcal{M}(x_0)$, the first statement is established. The second statement is proved by a computation similar to (3.15). □

Proposition 3.12 (No uniform convergence to 0 in Λ). *Suppose that the assumptions of Theorem 3.3 are satisfied and let $(u_\varepsilon)_\varepsilon\subset E$ be positive solutions of (3.8) obtained in Theorem 3.3. Then there exists $\delta>0$ such that*

$$\|u_\varepsilon\|_{L^\infty(\Lambda)}\geqslant\delta.$$

Proof. Since the Poisson term is positive, we can do the same proof as in Proposition 2.8 in Chapter II. □

As in Chapter II, we write $u_\varepsilon(x', x'') = \tilde{u}_\varepsilon(x', |x''|)$ with $\tilde{u}_\varepsilon : \mathbb{R}^2_+ \to \mathbb{R}$. Notice that ϕ_{u_ε} has the same symmetry as u_ε, i.e. for all $R \in \mathbf{O}(3)$ such that $R(d) = d$, $\phi_{u_\varepsilon} \circ R = \phi_{u_\varepsilon}$. This follows easily from the representation formula (3.9).

Since the $H^1_{V,\varepsilon}$-norm of u_ε is of the order ε, it is natural to rescale $\tilde{u}_\varepsilon(x', |x''|)$ as $\tilde{u}_\varepsilon(x'_\varepsilon + \varepsilon y', |x''_\varepsilon| + \varepsilon\,|y''|)$ around a well-chosen family of points $x_\varepsilon = (x'_\varepsilon, x''_\varepsilon) \in \mathbb{R}^3$. The next lemma shows that the sequences of rescaled solutions converge, up to a subsequence, in $C^1_{\text{loc}}(\mathbb{R}^2)$ to a function $v \in H^1(\mathbb{R}^2)$.

Lemma 3.13. *Suppose that the assumptions of Theorem 3.3 are satisfied. Let $(u_\varepsilon)_\varepsilon \subset E$ be positive solutions of (3.8) found in Theorem 3.3, $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$ and $x_n = (x'_n, x''_n) \to \bar{x} = (\bar{x}', \bar{x}'') \in \bar{\Lambda}$ as $n \to \infty$. Set*

$$\Omega_n := \mathbb{R} \times \left]-\frac{|x''_n|}{\varepsilon_n}, +\infty\right[$$

and let $v_n : \Omega_n \to \mathbb{R}$ be defined by

$$v_n(y, z) := \tilde{u}_{\varepsilon_n}(x'_n + \varepsilon_n y, \left|x''_n\right| + \varepsilon_n z),$$

where $\tilde{u}_{\varepsilon_n} : \mathbb{R}^2_+ \to \mathbb{R}$ is such that $u_{\varepsilon_n}(x', x'') = \tilde{u}_{\varepsilon_n}(x', |x''|)$. Then, there exists $v \in H^1(\mathbb{R}^2)$ such that, along a subsequence that we still denote by $(v_n)_n$,

$$v_n \overset{C^1_{\text{loc}}(\mathbb{R}^2)}{\longrightarrow} v.$$

Moreover, v is a solution of the equation

$$-\Delta v + V(\bar{x})v = K(\bar{x})v^p, \;\; \bar{x} \in \mathbb{R}^2.$$

Proof. We infer from Proposition 3.11 that for all $n \in \mathbb{N}$,

(3.26)
$$\int_{\Omega_n} \Big(\left|\nabla v_n(y, z)\right|^2 + V(x'_n + \varepsilon_n y, \left|x''_n\right| + \varepsilon_n z)\left|v_n(y, z)\right|^2\Big)\, dy\, dz \leqslant C,$$

with $C > 0$ independent of n.

Observe also that each v_n solves the equation

$$(3.27) \quad -\Delta v_n - \frac{\varepsilon_n}{z}\frac{\partial v_n}{\partial z} + V(x'_n + \varepsilon_n y, \left|x''_n\right| + \varepsilon_n z)v_n$$
$$+\rho(x'_n + \varepsilon_n y, \left|x''_n\right| + \varepsilon_n z)\tilde{\phi}_n v_n = g_{\varepsilon_n}(x'_n + \varepsilon_n y, \left|x''_n\right| + \varepsilon_n z, v_n), \quad x \in \mathbb{R}^2,$$

where we have set $\tilde{\phi}_n(y,z) = \phi_{u_{\varepsilon_n}}(x'_n + \varepsilon_n y, |x''_n| + \varepsilon_n z)$. As a consequence of (3.24), the sequence $(\tilde{\phi}_n)_n$ converges to zero in $\mathcal{D}^{1,2}(\mathbb{R}^2)$. It follows then from Hölder inequality that the term $(\rho(x'_n + \varepsilon_n y, |x''_n| + \varepsilon_n z)\tilde{\phi}_n v_n)_n$ is bounded in $L^2_{\mathrm{loc}}\big(\mathbb{R}^2 \setminus \{0\}\big)$.

Define a cut-off function $\eta_R \in \mathcal{D}(\mathbb{R}^2)$ such that $0 \leqslant \eta_R \leqslant 1$, $\eta_R(x) = 1$ if $|x| \leqslant R/2$, $\eta_R(x) = 0$ if $|x| \geqslant R$ and $\|\nabla \eta_R\|_\infty \leqslant C/R$ for some $C > 0$. Choose $(R_n)_n$ such that $R_n \to \infty$ and $\varepsilon_n R_n \to 0$. Since $\bar{x} \in \Lambda$ and $\bar{\Lambda} \cap d = \emptyset$, one has $\varepsilon_n R_n \leqslant |x''_n|$ if n is large enough. Define $w_n \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ by

$$w_n(y) := \eta_{R_n}(y) v_n(y).$$

It was shown in Lemma 2.9 in Chapter II that (3.26) implies that $(w_n)_n$ is bounded in $H^1(\mathbb{R}^2)$. Since w_n solves equation (3.27) on $B(0, R_n)$ for all n, classical regularity estimates yield that for every $R > 0$ and every $q > 1$,

$$\sup_{n \in \mathbb{N}} \|v_n\|_{W^{2,q}(B(0,R))} < \infty. \tag{3.28}$$

Up to a subsequence, we can now assume that $(w_n)_n$ converges weakly in $H^1(\mathbb{R}^2)$ to some function $v \in H^1(\mathbb{R}^2)$. By (3.28), for every compact set $K \subset \mathbb{R}^2$, w_n converges to v in $C^1(K)$. Moreover, for n large enough, $w_n = v_n$ in K so that $v_n \to v$ in $C^1(K)$. □

For $x, y \in \mathbb{R}^3$, denote by

$$d_d(x,y) := \sqrt{|x' - y'|^2 + (|x''| - |y''|)^2}.$$

the distance between the circles centered at x' and y', and of radius $|x''|$ and $|y''|$ respectively. We denote by B_d the balls for the distance d_d, i.e.,

$$B_d(x,r) = \{y \in \mathbb{R}^3 : d_d(x,y) < r\}.$$

We are now going to estimate from below the critical value c_ε. In the next two lemmas we estimate the action respectively inside and outside neighbourhoods of points.

Lemma 3.14. *Suppose that the assumptions of Theorem 3.3 are satisfied. Let $u_\varepsilon \in E$ be positive solutions of* (3.8) *found in Theorem 3.3, $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \mathbb{R}^n$ be sequences such that $\varepsilon_n \to 0$ and $x_n = (x'_n, x''_n) \to \bar{x} = (\bar{x}', \bar{x}'') \in \bar{\Lambda}$ as $n \to \infty$. If*

$$\liminf_{n \to \infty} u_{\varepsilon_n}(x_n) > 0, \tag{3.29}$$

then we have, up to a subsequence,

$$\liminf_{R\to\infty}\liminf_{n\to\infty}\varepsilon_n^{-2}\left(\int_{T_n(R)}\frac{1}{2}\left(\varepsilon_n^2\,|\nabla u_{\varepsilon_n}|^2+Vu_{\varepsilon_n}^2\right)-G_{\varepsilon_n}(.,u_{\varepsilon_n})\right)\geqslant \pi\mathcal{M}(\bar{x}),$$

where $T_n(R):=B_d(x_n,\varepsilon_n R)$.

Proof. The proof is the same as the one of Lemma 2.10 in Chapter II. Indeed, the Poisson term is positive and the equation satisfied by the limit of the sequence of rescaled solutions is the same as in Chapter II. □

Lemma 3.15. *Suppose that the assumptions of Theorem 3.3 are satisfied. Let* $u_\varepsilon\in E$ *be positive solutions of* (3.8) *found in Theorem 3.3,* $(\varepsilon_n)_n\subset\mathbb{R}^+$ *and* $(x_n^i)_n\subset\mathbb{R}^3$ *be sequences such that* $\varepsilon_n\to 0$ *and for* $1\leqslant i\leqslant M$, $x_n^i\to\bar{x}^i\in\bar{\Lambda}$ *as* $n\to\infty$. *Then, up to a subsequence, we have*

$$\liminf_{R\to\infty}\liminf_{n\to\infty}\varepsilon_n^{-2}\left(\int_{\mathbb{R}^3\setminus\mathcal{T}_n(R)}\frac{1}{2}\left(\varepsilon_n^2\,|\nabla u_{\varepsilon_n}|^2+Vu_{\varepsilon_n}^2\right)-G_{\varepsilon_n}(.,u_{\varepsilon_n})\right)\geqslant 0,$$

where $\mathcal{T}_n(R):=\bigcup_{i=1}^M B_d(x_n^i,\varepsilon_n R)$.

Proof. Since the Poisson term is positive, the proof is the same as the one of Lemma 1.12 in Chapter I. □

Proposition 3.16 (Lower estimate of the critical value). *Suppose that the assumptions of Theorem 3.3 are satisfied. Let* $u_\varepsilon\in E$ *be positive solutions of* (3.8) *found in Theorem 3.3,* $(\varepsilon_n)_n\subset\mathbb{R}^+$ *and* $(x_n^i)_n\subset\mathbb{R}^3$ *be sequences such that* $\varepsilon_n\to 0$ *and for* $1\leqslant i\leqslant M$, $x_n^i\to\bar{x}^i\in\bar{\Lambda}$ *as* $n\to\infty$. *If for every* $1\leqslant i<j\leqslant M$, *we have*

$$\limsup_{n\to\infty}\frac{d_d(x_n^i,x_n^j)}{\varepsilon_n}=\infty$$

and if for every $1\leqslant i\leqslant M$,

$$\liminf_{n\to\infty}u_{\varepsilon_n}(x_n^i)>0,$$

then the critical value c_ε *defined in* (3.12) *satisfies*

$$\liminf_{n\to\infty}\varepsilon_n^{-2}c_{\varepsilon_n}\geqslant\pi\sum_{i=1}^M\mathcal{M}(\bar{x}^i).$$

Proof. This is a consequence of the two previous lemmas, see Lemma 1.13 in Chapter I for the details. □

Now we can state a first concentration result. It will be completed in the next section by a decay estimate.

Proposition 3.17 (Uniform convergence to 0 outside small balls). *Suppose that the assumptions of Theorem 3.3 are satisfied and that* Λ *satisfies the assumptions* (3.4)- (3.7). *Let* $(u_\varepsilon)_\varepsilon \subset E$ *be positive solutions of* (3.8) *obtained in Theorem 3.3. If* $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ *is such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0,$$

then

(i) $\lim_{\varepsilon\to 0} \mathcal{M}(x_\varepsilon) = \inf_{\Lambda\cap\pi} \mathcal{M}$,
(ii) $\limsup_{\varepsilon\to 0} \frac{\operatorname{dist}(x_\varepsilon,\mathcal{H}^\perp)}{\varepsilon} < \infty$,
(iii) $\liminf_{\varepsilon\to 0} d_d(x_\varepsilon, \partial\Lambda) > 0$,
(iv) for every $\delta > 0$, *there exist* $\varepsilon_0 > 0$ *and* $R > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$,
$$\|u_\varepsilon\|_{L^\infty(\Lambda\setminus B_d(x_\varepsilon,\varepsilon R))} \leqslant \delta.$$

Proof. All the assertions are proved by energy comparisons, using only Propositions 3.11 and 3.16, as in Proposition 2.13 in Chapter II. □

4. Solution of the initial problem

4.1. Linear inequation outside small balls

In this section we prove that for ε small enough, the solutions of the penalized problem (3.8) are also solutions of the initial problem (NLSP). We follow the arguments of [113] and [27]. First we notice that the solutions of (3.8) satisfy a linear inequation outside small balls. As observed in [28, Theorem 5], the function ϕ_{u_ε} satisfies the estimate

$$\phi_{u_\varepsilon}(x) \geqslant \frac{C_\varepsilon}{C'_\varepsilon + |x|},$$

for some constants $C_\varepsilon, C'_\varepsilon > 0$. Set

$$W_\varepsilon(x) := (1-\mu)V(x) + \frac{C_\varepsilon \rho(x)}{C'_\varepsilon + |x|}.$$

Lemma 3.18. *Suppose that the assumptions of Proposition 3.17 are satisfied and let* $(u_\varepsilon)_\varepsilon \subset E$ *be positive solutions of* (3.8) *found in Theorem 3.3 and* $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ *be such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Then there exist $R > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$,

$$-\varepsilon^2 (\Delta u_\varepsilon + H u_\varepsilon) + W_\varepsilon u_\varepsilon \leqslant 0 \quad \text{in } \mathbb{R}^3 \setminus B_d(x_\varepsilon, \varepsilon R). \tag{3.30}$$

Proof. Set

$$\eta := \inf_{x\in\Lambda}\left(\frac{\mu V(x)}{K(x)}\right)^{\frac{1}{p-1}}.$$

Since V and K are bounded positive continuous functions on $\bar\Lambda$, $\eta > 0$. By Proposition 3.17, we can find $\varepsilon_0 > 0$ and $R > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, one has

$$u_\varepsilon(x) \leqslant \eta \quad \text{for all } x \in \Lambda \setminus B_d(x_\varepsilon, \varepsilon R).$$

We conclude that

$$-\varepsilon^2\Delta u_\varepsilon + (1-\mu)V u_\varepsilon + \rho\phi_{u_\varepsilon}u_\varepsilon \leqslant -\varepsilon^2\Delta u_\varepsilon + V u_\varepsilon + \rho\phi_{u_\varepsilon}u_\varepsilon - K u_\varepsilon^p = 0$$

in $\Lambda \setminus B_d(x_\varepsilon, \varepsilon R)$. The fact that u_ε satisfies (3.30) in $\mathbb{R}^3 \setminus \Lambda$ follows directly from the definition of the penalized nonlinearity. □

This lemma suggests that we can compare the solution u_ε with supersolutions of the operator $-\varepsilon^2(\Delta + H) + W_\varepsilon$ in order to obtain decay estimates of u_ε.

4.2. Comparison functions

In this section we recall results from Chapter II about the comparison functions. The next lemma provides a minimal positive solution of the operator $-\Delta - H$ in $\mathbb{R}^3 \setminus \bar\Lambda$.

Lemma 3.19. *For every $\varepsilon > 0$, there exists $\Psi_\varepsilon \in C^2\big((\mathbb{R}^3 \setminus \{0\}) \setminus \Lambda\big)$ such that*

$$\begin{cases} -\varepsilon^2(\Delta\Psi_\varepsilon + H\Psi_\varepsilon) + W_\varepsilon\Psi_\varepsilon = 0 & \text{in } \mathbb{R}^3 \setminus \bar\Lambda, \\ \Psi_\varepsilon = 1 & \text{on } \partial\Lambda, \end{cases}$$

and

$$\int_{\mathbb{R}^3\setminus\Lambda}\left(|\nabla\Psi_\varepsilon(x)|^2 + \frac{|\Psi_\varepsilon(x)|^2}{|x|^2}\right)dx < \infty. \tag{3.31}$$

Moreover, there exists $C > 0$ such that, for every $x \in \mathbb{R}^3 \setminus \Lambda$ and every $\varepsilon > 0$,

$$0 < \Psi_\varepsilon(x) \leqslant \frac{C}{1+|x|}. \tag{3.32}$$

Proof. See Lemma 2.15 in Chapter II. □

As explained in [113], the estimate (3.32) is the best one can hope for, at least if W_ε decays rapidly at infinity. However, if W_ε decays quadratically or subquadratically at infinity, we can improve (3.32).

Lemma 3.20. *Let Ψ_ε be given by Lemma 3.19.*

(1) If $\liminf_{|x|\to\infty} W_\varepsilon(x)\,|x|^2 > 0$, then there exist $\lambda > 0$, $R > 0$ and $C > 0$ such that for every $\varepsilon > 0$ and $x \in \mathbb{R}^3 \setminus B(0,R)$,

$$\Psi_\varepsilon(x) \leqslant C\left(\frac{R}{|x|}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}-\kappa+\frac{\lambda^2}{\varepsilon^2}}}.$$

(2) If $\liminf_{|x|\to\infty} W_\varepsilon(x)\,|x|^\alpha > 0$ with $\alpha < 2$, then there exist $\lambda > 0$, $R > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ and $x \in \mathbb{R}^3 \setminus B(0,R)$,

$$\Psi_\varepsilon(x) \leqslant C\exp\left(-\frac{\lambda}{\varepsilon}\left(|x|^{\frac{2-\alpha}{2}} - R^{\frac{2-\alpha}{2}}\right)\right).$$

(3) If $\liminf_{|x|\to 0} W_\varepsilon(x)\,|x|^2 > 0$, then there exist $\lambda > 0$, $r > 0$ and $C > 0$ such that for every $\varepsilon > 0$ and $x \in B(0,r)$,

$$\Psi_\varepsilon(x) \leqslant C\left(\frac{|x|}{r}\right)^{\sqrt{\frac{1}{4}-\kappa+\frac{\lambda^2}{\varepsilon^2}}-\frac{1}{2}}.$$

(4) If $\liminf_{|x|\to 0} W_\varepsilon(x)\,|x|^\alpha > 0$ with $\alpha < 2$, then there exist $\lambda > 0$, $R > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ and $x \in B(0,r)$,

$$\Psi_\varepsilon(x) \leqslant C\exp\left(-\frac{\lambda}{\varepsilon}\left(|x|^{-\frac{\alpha-2}{2}} - r^{-\frac{\alpha-2}{2}}\right)\right).$$

Proof. See Lemma 2.16 in Chapter II. □

Now we provide a comparison function that describes the exponential decay of u_ε inside Λ.

Lemma 3.21. *Let $\bar{x} \in \Lambda$ and $R > 0$ be such that*

$$B_d(\bar{x}, R) \subset \Lambda.$$

Define

$$\Phi_\varepsilon^{\bar{x}}(x) := \cosh\left(\lambda\frac{R - d_d(x,\bar{x})}{\varepsilon}\right).$$

There exist $\lambda > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$, one has

$$-\varepsilon^2\Delta\Phi_\varepsilon^{\bar{x}} + W_\varepsilon\Phi_\varepsilon^{\bar{x}} \geqslant 0 \text{ in } B_d(\bar{x}, R).$$

Proof. See Lemma 2.17 in Chapter II. □

Lemma 3.22. *Let $(x_\varepsilon)_\varepsilon \subset \Lambda$ be such that*

$$\liminf_{\varepsilon \to 0} d_d(x_\varepsilon, \partial\Lambda) > 0$$

and $R > 0$. Then, there exist $\varepsilon_0 > 0$ and a family of functions $(w_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subset C^{1,1}_{\text{loc}}((\mathbb{R}^3 \setminus \{0\}) \setminus B_d(x_\varepsilon, \varepsilon R))$ such that for all $\varepsilon \in (0, \varepsilon_0)$, one has

(i) w_ε satisfies the inequation

$$-\varepsilon^2 (\Delta + H)\, w_\varepsilon + W_\varepsilon w_\varepsilon \geqslant 0 \text{ in } \mathbb{R}^3 \setminus B_d(x_\varepsilon, \varepsilon R),$$

(ii) $\nabla w_\varepsilon \in L^2(\mathbb{R}^3 \setminus B_d(x_\varepsilon, \varepsilon R))$ and $\frac{w_\varepsilon}{|x|} \in L^2(\mathbb{R}^3 \setminus B_d(x_\varepsilon, \varepsilon R))$,

(iii) $w_\varepsilon \geqslant 1$ on $\partial B_d(x_\varepsilon, \varepsilon R)$,

(iv) for every $x \in B_d(x_\varepsilon, \varepsilon R)$,

$$w_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_d(x, x_\varepsilon)}{1 + d_d(x, x_\varepsilon)}\right) (1 + |x|)^{-1}, \qquad x \in \mathbb{R}^3.$$

Moreover,

(1) If $\liminf_{|x|\to\infty} W_\varepsilon(x)\,|x|^2 > 0$, then there exist $\lambda > 0$, $\nu > 0$ and $C > 0$ such that for $\varepsilon > 0$ small enough,

$$w_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_d(x, x_\varepsilon)}{1 + d_d(x, x_\varepsilon)}\right) (1 + |x|)^{-\frac{\nu}{\varepsilon}}.$$

(2) If $\liminf_{|x|\to\infty} W_\varepsilon(x)\,|x|^\alpha > 0$ with $\alpha > 2$, then there exist $\lambda > 0$ and $C > 0$ such that for $\varepsilon > 0$ small enough,

$$w_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_d(x, x_\varepsilon)}{1 + d_d(x, x_\varepsilon)} (1 + |x|)^{\frac{2-\alpha}{2}}\right).$$

(3) If $\liminf_{|x|\to 0} W_\varepsilon(x)\,|x|^2 > 0$, then there exist $\lambda > 0$, $\nu > 0$ and $C > 0$ such that for $\varepsilon > 0$ small enough,

$$w_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_d(x, x_\varepsilon)}{1 + d_d(x, x_\varepsilon)}\right) \left(\frac{|x|}{1 + |x|}\right)^{\frac{\nu}{\varepsilon}}.$$

(4) If $\liminf_{|x|\to 0} W_\varepsilon(x)\,|x|^\alpha > 0$ with $\alpha > 2$, then there exist $\lambda > 0$ and $C > 0$ such that for $\varepsilon > 0$ small enough,

$$w_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon} \frac{d_d(x, x_\varepsilon)}{1 + d_d(x, x_\varepsilon)} \left(\frac{|x|}{1 + |x|}\right)^{\frac{\alpha-2}{2}}\right).$$

Proof. See Lemma 2.18 in Chapter II. □

Thanks to the previous lemma, we obtain an upper bound on the solutions $(u_\varepsilon)_{\varepsilon>0}$ of (3.8).

Proposition 3.23. *Suppose that the assumptions of Proposition 3.17 are satisfied. Let $(u_\varepsilon)_{\varepsilon>0} \subset E$ be the positive solutions of* (3.8) *found in Theorem 3.3 and $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ be such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Then there exist $C > 0$, $\lambda > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$u_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon}\frac{d(x, S^1_\varepsilon)}{1 + d(x, S^1_\varepsilon)}\right)(1 + |x|)^{-1}, \qquad x \in \mathbb{R}^3.$$

Moreover, (1), (2), (3) *and* (4) *in Lemma 3.22 hold with u_ε in place of w_ε.*

Proof. See Proposition 2.19 in Chapter II. □

4.3. Solution of the original problem

Proposition 3.24. *Let $(u_\varepsilon)_{\varepsilon>0} \subset E$ be the positive solutions of* (3.8) *found in Theorem 3.3. Assume that one set $(\mathcal{G}^i_\infty)$ of growth conditions at infinity and one set $(\mathcal{G}^j_0)$ of growth conditions at the origin hold. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, u_ε solves the original problem* (NLSP).

Proof. We infer from Lemma 3.12 that there exists a family of points $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ such that

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Assume that the assumptions $(\mathcal{G}^1_\infty)$ and $(\mathcal{G}^1_0)$ are satisfied. By Proposition 3.23, we obtain for $\varepsilon > 0$ small enough and $x \in \mathbb{R}^3 \setminus \Lambda$,

$$\begin{aligned} K(x)u_\varepsilon^{p-1} &\leqslant M(1 + |x|)^\sigma \left(Ce^{\frac{-\lambda}{\varepsilon}}\frac{1}{|x|}\right)^{p-1} \\ &\leqslant Ce^{\frac{-\lambda}{\varepsilon}(p-1)}(1 + |x|)^{-(p-1)+\sigma} \\ &\leqslant \frac{\varepsilon^2\kappa}{|x|^2\left((\log|x|)^2 + 1\right)^{\frac{1+\beta}{2}}} = \varepsilon^2 H(x). \end{aligned}$$

We conclude by definition of g_ε that $g_\varepsilon(x, u_\varepsilon(x)) = K(x)u_\varepsilon^p(x)$, hence u_ε solves the original problem (NLSP). The other cases can be treated in a similar way. □

Proof of Theorem 3.2. We proved in Theorem 3.3 that, for any $\varepsilon > 0$, the penalized problem (3.8) possesses a solution u_ε. By Proposition 3.24, for $\varepsilon > 0$ small enough, u_ε is a solution of the initial problem (NLSP). The existence of a sequence $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ such that

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0$$

follows from Lemma 3.12 and the concentration result follows from Proposition 3.17. Finally, Proposition 3.23 yields the decay estimate. □

5. Remarks and further results

5.1. Concentration at points

We can also obtain a result about solutions concentrating at points. In this case, the concentration function is given by

$$\mathcal{A}(x) = [V(x)]^{\frac{p+1}{p-1}-\frac{3}{2}}\,[K(x)]^{\frac{-2}{p-1}}\,.$$

Theorem 3.25. *Let $3 < p < 5$, $V, K \in C(\mathbb{R}^3\setminus\{0\}, \mathbb{R}^+)$, $K \not\equiv 0$ and $\rho \in L^{3/2}_{loc}(\mathbb{R}^3) \cap L^\infty_{loc}\left(\mathbb{R}^3 \setminus \{0\}\right)$. Assume that one set $(\mathcal{G}^i_\infty)$ of growth conditions at infinity and one set $(\mathcal{G}^j_0)$ of growth conditions at the origin hold. Assume also that there exists a smooth open bounded set $\Lambda \subset \mathbb{R}^3$ such that*

$$0 < \inf_\Lambda \mathcal{A} < \inf_{\partial\Lambda} \mathcal{A}. \tag{3.33}$$

Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, problem (NLSP) *has at least one positive solution u_ε. Moreover, for every $0 < \varepsilon < \varepsilon_0$, there exists $x_\varepsilon \in \Lambda$ such that u_ε attains its maximum at x_ε,*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0,$$
$$\lim_{\varepsilon\to 0} \mathcal{A}(x_\varepsilon) = \inf_\Lambda \mathcal{A},$$

and there exist $C > 0$ and $\lambda > 0$ such that

$$u_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon}\frac{|x-x_\varepsilon|}{1+|x-x_\varepsilon|}\right)\left(1+|x-x_\varepsilon|^2\right)^{\frac{-1}{2}}, \qquad \forall x \in \mathbb{R}^3.$$

The proof of this theorem is similar to the proof of Theorem 3.2, but simpler. Let us only sketch the proof. First of all, we impose no symmetry neither on the potentials nor on the solution. This makes the critical Sobolev exponent to appear, in spite of what happens in the preceding results. We modify the problem in the same way as before and we search for a critical point of the functional J_ε in the space X defined by (3.10). Theorem 3.3 remains true with the same proof.

The limiting problem associated to concentration at points is the problem

$$-\Delta u + au = bu^p \qquad \text{in } \mathbb{R}^3.$$

Let $x_0 \in \Lambda$ be a point such that $\mathcal{A}(x_0) = \inf_\Lambda \mathcal{A}$. We denote by c_0 the least energy critical value of the limiting problem with $a = V(x_0)$ and $b = K(x_0)$. As in Proposition 3.11, we prove that the critical value c_ε defined in (3.12) satisfies

$$c_\varepsilon \leqslant \varepsilon^3 \left(c_0 + o(1)\right) \text{ as } \varepsilon \to 0.$$

Then we prove as before that the L^∞-norm of u_ε does not converge to 0 in Λ and that the sequence of rescaled solutions converges in C^1_{loc} to a solution of the limiting equation. The analogous of Proposition 3.16 is the following one.

Proposition 3.26. *Let $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n^i)_n \subset \mathbb{R}^3$ be sequences such that $\varepsilon_n \to 0$ and for $1 \leqslant i \leqslant M$, $x_n^i \to \bar{x}^i \in \bar{\Lambda}$ as $n \to \infty$. If for every $1 \leqslant i < j \leqslant M$, we have*

$$\limsup_{n\to\infty} \frac{\left|x_n^i - x_n^j\right|}{\varepsilon_n} = \infty$$

and if for every $1 \leqslant i \leqslant M$,

$$\liminf_{n\to\infty} u_{\varepsilon_n}(x_n^i) > 0,$$

then

$$\liminf_{n\to\infty} \varepsilon_n^{-3} c_{\varepsilon_n} \geqslant \sum_{i=1}^{M} \mathcal{C}(\bar{x}^i),$$

where $\mathcal{C}(x)$ is the least energy critical value of the limiting problem with $a = V(\bar{x})$ and $b = K(\bar{x})$.

The concentration result can be stated as follows.

Proposition 3.27. *Suppose that Λ satisfies (3.33). If $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ is such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0,$$

then

(i) $\lim_{\varepsilon\to 0} \mathcal{A}(x_\varepsilon) = \inf_\Lambda \mathcal{A}$,
(ii) $\liminf_{\varepsilon\to 0} d(x_\varepsilon, \partial\Lambda) > 0$,
(iii) for every $\delta > 0$, there exist $\varepsilon_0 > 0$ and $R > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$,

$$\|u_\varepsilon\|_{L^\infty(\Lambda\setminus B(x_\varepsilon,\varepsilon R))} \leqslant \delta.$$

Finally the comparison arguments in order to get back to the original problem are the same as in section 4.

5.2. Concentration on spheres

Using the same method, we can prove the existence of solutions concentrating on a sphere for the following problem.

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \rho(x)\phi u = K(x)u^p, \quad x \in \mathbb{R}^3, \\ \\ -\Delta\phi = \varepsilon\rho(x)u^2. \end{cases}$$

However we are not sure whether this problem has a physical meaning.

5.3. Concentration on Keplerian orbits

An interesting question related to [84, 114], concerns the existence of solutions concentrating on Kepler orbits, assuming radial potentials. For the reasons described in the Introduction, this might be a typical situation where the correspondence principle can be checked using solutions localized on classical planar orbits. We wonder if this result could be obtained for the $3D$ nonlinear Schrödinger and Schrödinger-Poisson equations with radial potentials.

5.4. Concentration at local maxima of the auxiliary potential

We believe that there exist solutions concentrating on circles whose radii are local maxima of the auxiliary potential $\mathcal{M}$. Concentration at local maxima of the potential V has been obtained by del Pino and Felmer [63, 64] for the nonlinear Schrödinger equation in the case $\inf_{\mathbb{R}^N} V > 0$ and for a decaying potential in Chapter IV.

5.5. Concentration on curves

Solutions concentrating on curves were obtained for the nonlinear Schrödinger equation without any symmetry assumption in [70, 106]. This problem is completely open for the Schrödinger-Poisson equation.

5.6. Concentration driven by ρ

If $V \equiv K \equiv 1$, it is natural to ask whether there still exist solutions with a concentration behaviour. In this case, we expect the location of the concentration points to be governed by the weight ρ. The asymptotic analysis seems more delicate since it requires higher order estimates.

CHAPTER IV

Concentration at local maxima

We study positive bound states for the stationary nonlinear Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = u^p, \qquad x \in \mathbb{R}^N, \tag{NLS}$$

where $\varepsilon > 0$ is a real parameter, $N \geqslant 3$, $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and $V \in C(\mathbb{R}^N, \mathbb{R}^+)$. Solutions concentrating at local minima of V were found for example in [7, 30, 113]. We are interested in solutions which concentrate at local maxima of V. Such solutions were found under more restrictive assumptions on the potential V. In [6], Ambrosetti, Badiale and Cingolani use the perturbation method and can treat local maxima in the same way as local minima. When using purely variational methods, the situation is more delicate because the mountain-pass theorem can no longer be applied. del Pino and Felmer [63, 64] developed an approach based on the general minimax principle. In all these works, it was assumed that $\inf_{\mathbb{R}^N} V > 0$. By contrast, we can treat fastly decaying potentials or even compactly supported potentials. To do so, we have combined the del Pino-Felmer approach with the penalization by a Hardy-type potential. The main difficulty lies in the proof of a strict inequality between two energy levels (see inequality (4.23)).

1. Assumptions and main result

We assume that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$\sup_{\Lambda} V > \inf_{\Lambda} V = \sup_{\partial\Lambda} V \tag{4.1}$$

and

$$\sup_{\Lambda} V^{\frac{p+1}{p-1}-\frac{N}{2}} < 2 \inf_{\overline{\Lambda}} V^{\frac{p+1}{p-1}-\frac{N}{2}} . \tag{4.2}$$

Note that (4.1) implies that

$$\sup_{\partial\Lambda} V = \inf_{\partial\Lambda} V ,$$

that is, $\partial\Lambda$ is a level line of V. We prove the following result.

Theorem 4.1. *Let $N \geqslant 3$, $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and $V \in C(\mathbb{R}^N, \mathbb{R}^+)$. Assume that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ satisfying* (4.1), (4.2). *Then for $\varepsilon > 0$ small enough, equation* (NLS) *possesses a positive solution $u_\varepsilon \in C^{1,\alpha}(\mathbb{R}^N)$ such that u_ε attains its maximum at $x_\varepsilon \in \Lambda$,*

$$\liminf_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) > 0, \tag{4.3}$$

$$\liminf_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathbb{R}^N \setminus \Lambda) > 0 \tag{4.4}$$

and, if $V \in C^N(\mathbb{R}^N, \mathbb{R}^+)$,

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = \sup_{x \in \Lambda} V(x). \tag{4.5}$$

For example, we can find a solution concentrating at the origin for equation (NLS) with

$$V(x) = \frac{1}{1 + |x|^4}.$$

This result is new because $\inf_{\mathbb{R}^N} V = 0$ and thus the results in [6, 63, 64] cannot be applied.

As mentioned earlier, if V is compactly supported and $p \leqslant \frac{N}{N-2}$, then equation (NLS) has no positive solution in the neighborhood of infinity, in accordance with a result of Gidas and Spruck [85].

2. The penalized problem

Following del Pino and Felmer [63], we first solve a penalized problem. Bonheure and Van Schaftingen [29, 30] have introduced a penalized problem for decaying potentials. The penalization for fast decay potentials is due to Moroz and Van Schaftingen [112, 113]. It was used by Bonheure together with the authors to study solutions concentrating around spheres [27]. Another penalized problem was defined by Yin Huicheng and Zhang Pingzheng [146] (see also Fei Mingwen and Yin Huicheng [82] and Ba Na, Deng Yinbin and Peng Shuangjie [18]). In these papers, only concentration at local minima of the concentration function was investigated.

2.1. The penalization potential

Let $x_0 \in \Lambda$ and $\rho > 0$ be such that $\overline{B(x_0,\rho)} \subset \Lambda$, and let χ_Λ denote the characteristic function of the set Λ. The penalization potential $H : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$H(x) := \bigl(1-\chi_\Lambda(x)\bigr)\frac{(N-2)^2}{4\,|x-x_0|^2}\left(\frac{\log\frac{\rho}{\rho_0}}{\log\frac{|x-x_0|}{\rho_0}}\right)^{1+\beta}$$

for some fixed $\beta > 0$ and $\rho_0 \in (0,\rho)$.

Let us recall the main properties of the operator $-\Delta - H$ [113, §3.1].

Lemma 4.2 (Positivity [113, Lemma 3.1]**).** *For every $u \in H^1_V(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N}\bigl(|\nabla u|^2 - H\,|u|^2\bigr) \geqslant 0\,.$$

Sketch of the proof. This is a consequence of the classical Hardy inequality since for every $x \in \mathbb{R}^N \setminus B(x_0,\rho)$,

$$H(x) \le \frac{(N-2)^2}{4\,|x-x_0|^2}\,. \qquad \square$$

Lemma 4.3 (Comparison Principle [113, Lemma 3.2]**).** *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain. Let $v, w \in H^1_{loc}(\Omega) \cap C(\overline{\Omega})$ be such that*

$$-\Delta w - Hw \geqslant -\Delta v - Hv, \qquad \text{weakly in } \Omega.$$

If $w \geqslant v$ on $\partial\Omega$ and

$$\int_\Omega |\nabla (w-v)_-|^2 + \int_\Omega \frac{|(w(x)-v(x))_-|^2}{1+|x|^2}\,dx < \infty\,,$$

then $w \geqslant v$ in Ω.

Proof. Take $\psi \in \mathcal{D}(\mathbb{R}^N)$ such that $\psi \equiv 1$ on B_1 and define $\psi_n \in \mathcal{D}(\mathbb{R}^N)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$ by $\psi_n(x) = \psi(x/n)$. Take $\psi_n(w-v)_-$ as a test function in the equation. One has then

$$\int_{\mathbb{R}^N}\psi_n\bigl(|\nabla(w-v)_-|^2 - H\,|(w-v)_-|^2\bigr) = -\int_{\mathbb{R}^N} 2(w-v)_-\nabla\psi_n\cdot\nabla(w-v).$$

One concludes by letting $n \to \infty$. $\square$

2.2. The penalization nonlinearity

Let $\mu \in (0,1)$. The penalized nonlinearity $g_\varepsilon : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is defined for $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$ by

$$g_\varepsilon(x,s) := \chi_\Lambda(x){s_+}^p + \bigl(1-\chi_\Lambda(x)\bigr)\min\bigl(\mu\bigl(\varepsilon^2 H(x) + V(x)\bigr), |s|^{p-1}\bigr)s_+\,.$$

Also set $G_\varepsilon(x,s) := \int_0^s g_\varepsilon(x,\sigma)\,d\sigma$. The function g_ε is a Carathéodory function with the following properties :

(g_1) $g_\varepsilon(x,s) = o(s)$, $s \to 0$, uniformly on compact subsets of $\mathbb{R}^N$.

(g_2) one has

$$\limsup_{s\to\infty} \frac{g_\varepsilon(x,s)}{|s|^p} < \infty,$$

(g_3) one has

$$\begin{aligned} (p+1)G_\varepsilon(x,s) &\leqslant g_\varepsilon(x,s)s && \forall x \in \Lambda,\ \forall s \in \mathbb{R},\\ 2G_\varepsilon(x,s) &\leqslant g_\varepsilon(x,s)s \leqslant \mu\bigl(\varepsilon^2 H(x) + V(x)\bigr)s^2 && \forall x \notin \Lambda,\ \forall s \in \mathbb{R}, \end{aligned}$$

(g_4) the function

$$t \in \mathbb{R}^+_* \mapsto \frac{g_\varepsilon(x,ts)s}{t}$$

is nondecreasing for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$.

2.3. The penalized functional

Let us introduce some functional spaces naturally associated to our settings. We define the space

$$H^1_V(\mathbb{R}^N) := \Bigl\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\ :\ \int_{\mathbb{R}^N} V\,|u|^2 < \infty\Bigr\}\,.$$

This space is a Hilbert space endowed with any of the equivalent norms

$$\|u\|_\varepsilon^2 := \int_{\mathbb{R}^N} \bigl(\varepsilon^2\,|\nabla u|^2 + V\,|u|^2\bigr)$$

defined for $\varepsilon > 0$.

We look for a solution $u \in H^1_V$ of the penalized equation

$$(\mathcal{P}_\varepsilon) \qquad -\varepsilon^2\Delta u(x) + V(x)\,u(x) = g_\varepsilon\bigl(x,u(x)\bigr) \qquad \text{for } x \in \mathbb{R}^N.$$

The associated functional is

$$\mathcal{J}_\varepsilon : H^1_V \to \mathbb{R} : \mathcal{J}_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} \bigl(\varepsilon^2\,|\nabla u|^2 + V\,|u|^2\bigr) - \int_{\mathbb{R}^N} G_\varepsilon\bigl(x,u(x)\bigr)\,dx\,.$$

It is standard that $\mathcal{J}_\varepsilon$ is well-defined and continuously differentiable and that its critical points are weak solutions of the penalized equation $(\mathcal{P}_\varepsilon)$.

2.4. The Nehari manifold

The Nehari manifold associated to the functional $\mathcal{J}_\varepsilon$ is defined by

$$\mathcal{N}_\varepsilon := \left\{ u \in H^1_V(\mathbb{R}^N) \setminus \{0\} \ : \ \langle \mathcal{J}_\varepsilon'(u), u \rangle = 0 \right\} .$$

It is well-known that $u \in H^1_V(\mathbb{R}^N) \setminus \{0\}$ is a critical point of $\mathcal{J}_\varepsilon$ if and only if $u \in \mathcal{N}_\varepsilon$ and u is a critical point of $\mathcal{J}_\varepsilon$ restricted to $\mathcal{N}_\varepsilon$.

We point out that $\mathcal{N}_\varepsilon$ is bounded away from 0. We first have an integral estimate.

Lemma 4.4. *Let $\varepsilon > 0$ and $u \in \mathcal{N}_\varepsilon$. Then*

$$\int_\Lambda (u)_+^{p+1} \geq (1-\mu) \int_{\mathbb{R}^N} \varepsilon^2 \left|\nabla u\right|^2 + V \left|u\right|^2 \geq c\varepsilon^N$$

where $c > 0$ is independent of ε and u.

Proof. By (g_3), one has

$$\begin{aligned} \int_{\mathbb{R}^N} \varepsilon^2 \left|\nabla u\right|^2 + V \left|u\right|^2 &= \int_{\mathbb{R}^N} g_\varepsilon(x, u(x)) u(x)\, dx \\ &\leq \int_\Lambda \left|u\right|^{p+1} + \mu \int_{\mathbb{R}^N \setminus \Lambda} (V + \varepsilon^2 H) \left|u\right|^2 . \end{aligned}$$

We deduce from Lemma 4.2 that

$$(1-\mu) \int_{\mathbb{R}^N} \varepsilon^2 \left|\nabla u\right|^2 + V \left|u\right|^2 \leq \int_\Lambda \left|u\right|^{p+1} .$$

The Sobolev and Hölder inequalities imply, since $\inf_\Lambda V > 0$,

$$\begin{aligned} \int_\Lambda \left|u\right|^{p+1} &\leq C \Bigl(\int_{\mathbb{R}^N} \left|\nabla u\right|^2 \Bigr)^{\frac{p-1}{4} N} \Bigl(\int_\Lambda \left|u\right|^2 \Bigr)^{\frac{p+1}{2} - \frac{p-1}{4} N} \\ &\leq \frac{C}{\varepsilon^{\frac{p-1}{2} N}} \Bigl(\int_\Lambda \varepsilon^2 \left|\nabla u\right|^2 + V \left|u\right|^2 \Bigr)^{\frac{p+1}{2}} . \end{aligned}$$

The conclusion follows. □

We also have an estimate on the maximum as $\varepsilon \to 0$.

Lemma 4.5. *Let $\varepsilon > 0$ and $u \in \mathcal{N}_\varepsilon$. Then*

$$\sup_\Lambda \frac{u_+^{p-1}}{V} \geq 1 .$$

This was proved for solutions by Moroz and J. Van Schaftingen [113, Lemma 4.2] (see also [30, Lemma 17]).

Proof. One has by (g_3),

$$\begin{aligned}\int_{\mathbb{R}^N} \varepsilon^2 \lvert\nabla u\rvert^2 + V \lvert u\rvert^2 &\le \int_\Lambda (u)_+^{p+1} + \mu \int_{\mathbb{R}^N\setminus\Lambda} (V+\varepsilon^2 H)\lvert u\rvert^2\\ &\le \sup_\Lambda \frac{u_+^{p-1}}{V} \int_\Lambda V\lvert u\rvert^2 + \mu \int_{\mathbb{R}^N\setminus\Lambda} (V+\varepsilon^2 H)\lvert u\rvert^2\ ,\end{aligned}$$

and thus by Lemma 4.2,

$$\sup_\Lambda \frac{u_+^{p-1}}{V} \int_\Lambda V\lvert u\rvert^2 \ge \int_\Lambda V\lvert u\rvert^2\ .$$

By Lemma 4.4, $\int_\Lambda V\lvert u\rvert^2 > 0$, and the conclusion follows. □

We also note the following coercivity estimate.

Lemma 4.6. *There exists $C > 0$ such that for every $\varepsilon > 0$ and $u \in \mathcal{N}_\varepsilon$,*

$$\Bigl(\frac{1}{2} - \frac{1}{p+1}\Bigr)(1-\mu)\int_{\mathbb{R}^N} \bigl(\varepsilon^2\lvert\nabla u\rvert^2 + V\lvert u\rvert^2\bigr) \leqslant C\mathcal{J}_\varepsilon(u)\,.$$

Proof. Since $u \in \mathcal{N}_\varepsilon$, one has

$$\begin{aligned}\mathcal{J}_\varepsilon(u) = \Bigl(\frac{1}{2} - \frac{1}{p+1}\Bigr)&\int_{\mathbb{R}^N} \bigl(\varepsilon^2\lvert\nabla u\rvert^2 + V\lvert u\rvert^2\bigr)\\ &+ \frac{1}{p+1}\int_{\mathbb{R}^N} g_\varepsilon\bigl(x,u(x)\bigr)u(x) - (p+1)G_\varepsilon\bigl(x,u(x)\bigr)\,dx.\end{aligned}$$

In view of (g_3),

$$\begin{aligned}\frac{1}{p+1}\int_{\mathbb{R}^N} &g_\varepsilon\bigl(x,u(x)\bigr)u(x) - (p+1)G_\varepsilon(x,u(x))\,dx\\ &\geqslant -\frac{p-1}{p+1}\int_{\mathbb{R}^N\setminus\Lambda} G_\varepsilon\bigl(x,u(x)\bigr)\,dx\\ &\geqslant -\Bigl(\frac{1}{2} - \frac{1}{p+1}\Bigr)\mu\int_{\mathbb{R}^N\setminus\Lambda} (\varepsilon^2 H + V)\lvert u\rvert^2\,.\end{aligned}$$

Thanks to Lemma 4.2, we reach the conclusion. □

2.5. The Palais-Smale condition

For every $\varepsilon > 0$, the functional $\mathcal{J}_\varepsilon$ satisfies the Palais-Smale compactness condition:

Lemma 4.7. *For every $\varepsilon > 0$, if $(u_n)_{n\in\mathbb{N}}$ is a sequence such that $(\mathcal{J}_\varepsilon(u_n))_{n\in\mathbb{N}}$ converges and $(\mathcal{J}_\varepsilon'(u_n))_{n\in\mathbb{N}}$ converges to 0 in $(H^1_V(\mathbb{R}^N))'$, then, up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges in $H^1_V(\mathbb{R}^N)$.*

The proof of Lemma 4.7 is a combination of the arguments for the penalization without H [30, Lemma 6] and without V [113, Lemma 3.5] whose main lines originate in the proof for nondecaying potentials [61, Lemma 1.1]. It was already proved with the present penalization for the functional restricted to a subspace of symmetric functions [27].

2.6. Minimizers on the Nehari manifold

Proposition 4.8. *For every $\varepsilon > 0$, there exists $u \in \mathcal{N}_\varepsilon$ such that*

$$\mathcal{J}_\varepsilon(u) = \inf_{\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon \,.$$

Proposition 4.8 was proved for the penalization for nondecaying potentials [61, Lemma 2.1], the penalization without H [30, Proposition 9] the penalization without V [30, Proposition 3.7] and the present penalization under symmetry constraints [27].

Proof of Proposition 4.8. The proof is standard: by (g_4) one has the equality [126, Proposition 3.11]

$$\inf_{\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon = \inf_{\substack{u\in H^1_V(\mathbb{R}^N)\\ u_+|_\Lambda\neq 0}} \sup_{t>0} \mathcal{J}_\varepsilon(tu) = \inf_{\substack{\gamma\in C([0,1],H^1_V(\mathbb{R}^N))\\ \gamma(0)=0\\ \mathcal{J}_\varepsilon(\gamma(1))<0}} \sup_{t\in[0,1]} \mathcal{J}_\varepsilon(\gamma(t)) \,;$$

by Lemmas 4.4 and 4.6, $\mathcal{J}_\varepsilon$ is bounded from 0 on $\mathcal{N}_\varepsilon$. Since $\mathcal{J}_\varepsilon$ satisfies the Palais-Smale compactness condition by Lemma 4.7, the existence of u follows. ☐

3. Limiting problems

3.1. The limit problem

For $\nu > 0$ let U_ν be the unique positive solution of the problem

$$\begin{cases} -\Delta u + \nu u = u^p & \text{in } \mathbb{R}^N,\\ u > 0, & \\ u(0) = \max_{\mathbb{R}^N} u. & \end{cases} \tag{4.6}$$

The function U_ν is radial around the origin [102]. The functional associated to (4.6) is $\mathcal{I}_\nu : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined for $u \in H^1(\mathbb{R}^N)$ by

$$\mathcal{I}_\nu(u) := \frac{1}{2}\int_{\mathbb{R}^N} \bigl(|\nabla u|^2 + \nu\,|u|^2\bigr) - \frac{1}{p+1}\int_{\mathbb{R}^N} u_+^{p+1}. \tag{4.7}$$

One has

$$b_\nu := \inf_{\mathcal{M}_\nu} \mathcal{I}_\nu = \mathcal{I}_\nu(U_\nu)$$

where

$$\mathcal{M}_\nu = \Big\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \ : \ \int_{\mathbb{R}^N} |\nabla u|^2 + \nu\, |u|^2 = \int_{\mathbb{R}^N} u_+^{p+1} \Big\} .$$

We also set

$$\mathcal{C}(y) := b_{V(y)} = \frac{S_{p+1}^r}{r} V(y)^{\frac{p+1}{p-1} - \frac{N}{2}}, \tag{4.8}$$

where $\frac{1}{r} = \frac{1}{2} - \frac{1}{p+1}$ and

$$S_{p+1}^2 := \inf \Big\{ \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \ : \ u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u_+^{p+1} = 1 \Big\} .$$

We also recall the following classical result

Lemma 4.9. *Let $\nu > 0$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_\nu \subset H^1(\mathbb{R}^N)$. If*

$$\lim_{n\to\infty} \mathcal{I}_\nu(v_n) = b_\nu,$$

then there exists a sequence of points $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that $v_n(\cdot - y_n) \to U_\nu$ in $H^1(\mathbb{R}^N)$.

Proof. Let $\mathcal{F}_\nu : H^1(\mathbb{R}^N) \to \mathbb{R}$ be defined for $v \in H^1(\mathbb{R}^N)$ by

$$\mathcal{F}_\nu(v) = \int_{\mathbb{R}^N} |\nabla v|^2 + \nu\, |v|^2 - \int_{\mathbb{R}^N} v_+^{p+1}$$

By a standard application of the Ekeland variational principle on the manifold $\mathcal{M}_\nu$ (see for example [109, Theorem 4.1]), there exist sequences $(\tilde{v}_n)_{n\in\mathbb{N}} \subset \mathcal{M}_\nu$ and $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\mathcal{I}_\nu(\tilde{v}_n) \to b_\nu$, $\mathcal{I}_\nu'(\tilde{v}_n) + \lambda_n \mathcal{F}_\nu'(\tilde{v}_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ and $v_n - \tilde{v}_n \to 0$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. The sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence for the unconstrained functional $\mathcal{I}_\nu$, that is $\mathcal{I}_\nu'(\tilde{v}_n) \to 0$. Indeed one has

$$\begin{aligned} \lambda_n (1-p) \int_{\mathbb{R}^N} |\nabla v_n|^2 + |v_n|^2 &= \lambda_n \langle \mathcal{F}_\nu'(\tilde{v}_n), \tilde{v}_n \rangle \\ &= \langle \mathcal{I}_\nu'(\tilde{v}_n), \tilde{v}_n \rangle + o(\|\tilde{v}_n\|_{H^1}) \\ &= o(\|\tilde{v}_n\|_{H^1}). \end{aligned}$$

Since there exists a constant $c > 0$ such that $\|v\|_{H^1(\mathbb{R}^N)} \geqslant c$ for every $v \in \mathcal{M}_\nu$ we deduce that $\lim_{n\to\infty} \lambda_n = 0$.

We compute that

$$2\mathcal{I}_\nu(\tilde{v}_n) - \langle \mathcal{I}_\nu'(\tilde{v}_n), \tilde{v}_n \rangle = \Big(1 - \frac{2}{p+1}\Big) \int_{\mathbb{R}^N} |\tilde{v}_n|^{p+1} \, dx \to 2b_\nu.$$

Hence, there exists a constant C such that

$$\int_{\mathbb{R}^N} (\tilde{v}_n)_+^{p+1} \geqslant C > 0.$$

Since $(\tilde{v}_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, we deduce from [105, Part II, Lemma I.1] (see also [145, Lemma 1.21]) that

$$\liminf_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |\tilde{v}_n|^2 > 0\,.$$

Consequently, there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that, if we set $\bar{v}_n := \tilde{v}_n(\cdot - y_n)$, we have

$$\int_{B(0,1)} \bar{v}_n^2 \geqslant C > 0. \tag{4.9}$$

Since $(\bar{v}_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, we can assume that $\bar{v}_n \rightharpoonup \bar{v}$ in $H^1(\mathbb{R}^N)$, $\bar{v}_n \to \bar{v}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\bar{v}_n \to \bar{v}$ almost everywhere. By (4.9), $\bar{v} \not\equiv 0$. For all $v \in H^1(\mathbb{R}^N)$, we have $\langle \mathcal{I}'_\nu(\bar{v}_n), v\rangle \to 0$ because $\bar{v}_n$ is a Palais-Smale sequence, and $\langle \mathcal{I}'_\nu(\bar{v}_n), v\rangle \to \langle \mathcal{I}'_\nu(\bar{v}), v\rangle$ because $\bar{v}_n \rightharpoonup \bar{v}$. We conclude that $\langle \mathcal{I}'_\nu(\bar{v}), v\rangle = 0$ and so $\bar{v}$ is a solution of (4.6). We compute that

$$\frac{p-1}{p+1}\int_{\mathbb{R}^N} (\bar{v}_n)_+^{p+1} = 2\mathcal{I}_\nu(\bar{v}_n) - \langle \mathcal{I}'_\nu(\bar{v}_n), \bar{v}_n\rangle \to 2b_\nu$$

and

$$\frac{p-1}{p+1}\int_{\mathbb{R}^N} (\bar{v})_+^{p+1} = 2\mathcal{I}_\nu(\bar{v}) = 2b_\nu.$$

Therefore $\|\bar{v}_n\|_{L^{p+1}(\mathbb{R}^N)} \to \|\bar{v}\|_{L^{p+1}(\mathbb{R}^N)}$. We infer that $\bar{v}_n$ converges to $\bar{v}$ in $L^{p+1}(\mathbb{R}^N)$. Finally we can write

$$\begin{aligned}\|\bar{v}_n - \bar{v}\|^2_{H^1(\mathbb{R}^N)} &= \langle \mathcal{I}'_\nu(\bar{v}_n) - \mathcal{I}'_\nu(\bar{v}), \bar{v}_n - \bar{v}\rangle \\ &\quad + \int_{\mathbb{R}^N} \left((\bar{v}_n)_+^p - (\bar{v})_+^p\right)(\bar{v}_n - \bar{v})\ .\end{aligned}$$

Since $\mathcal{I}'_\nu(\bar{v}) = 0$, $\mathcal{I}'_\nu(\bar{v}_n) \to 0$ as $n \to \infty$ and the last term goes to 0 by Hölder's inequality, we conclude that $\bar{v}_n \to \bar{v}$ in $H^1(\mathbb{R}^N)$. The conclusion follows. □

3.2. Penalized limit problems

The two following lemmas will provide information about the limit of sequences of rescaled solutions. The first lemma is due to del Pino and Felmer [65, Lemma 2.3]. Let $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \ :\ x_N > 0\}$.

Lemma 4.10. *Let $\nu \geqslant 0$ and $\mu \in [0, \nu]$. If $u \in H^1(\mathbb{R}^N)$ is a solution of*

$$\begin{cases} -\Delta u + \nu u = u_+^p & \text{in } \mathbb{R}^N_+, \\ -\Delta u + \nu u = \min\bigl(\mu, |u|^{p-1}\bigr) u_+ & \text{in } \mathbb{R}^N_-, \end{cases} \tag{4.10}$$

then $|u|^{p-1} \leqslant \mu$ in $\mathbb{R}^N_-$.

Proof. For completeness, we repeat the argument of del Pino and Felmer [65, Lemma 2.3]. By elliptic regularity, $u \in H^2(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$. Thus we can use $\frac{\partial u}{\partial x_N}$ as a test function in (4.10). Writing $g(s) := \min(\mu, |s|^{p-1}) s_+$ and $G(s) := \int_0^s g(\sigma)\, d\sigma$, we obtain

$$\frac{1}{2} \int_{\mathbb{R}^N} \partial_N \bigl(|\nabla u|^2 + \nu\, |u|^2\bigr) = \frac{1}{p+1} \int_{\mathbb{R}^N_+} \partial_N \bigl(u_+^{p+1}\bigr) + \int_{\mathbb{R}^N_-} \partial_N (G \circ u).$$

This reduces to

$$\int_{\mathbb{R}^{N-1}} \Bigl(G\bigl(u(x',0)\bigr) - \frac{1}{p+1} \bigl(u(x',0)\bigr)_+^{p+1}\Bigr)\, dx' = 0.$$

Since $G(u) \leqslant \frac{u_+^{p+1}}{p+1}$ on $\mathbb{R}^N$, we have $G\left(u(x',0)\right) = \frac{1}{p+1}(u(x',0))_+^{p+1}$ for all $x' \in \mathbb{R}^{N-1}$ and hence, for every $x' \in \mathbb{R}^{N-1}$, $u(x',0) \leqslant \mu^{\frac{1}{p-1}}$. One has on $\mathbb{R}^N_-$

$$-\Delta u + (\nu - \mu) u \leq 0.$$

Since $\nu \geq \mu$, we deduce by the maximum principle that $u \leqslant \mu^{\frac{1}{p-1}}$ in $\mathbb{R}^N_-$. □

The second lemma is an application of the maximum principle.

Lemma 4.11. *Let $\nu \geqslant 0$ and $\mu \in [0, \nu]$. If $u \in H^1(\mathbb{R}^N)$, $u \geqslant 0$, is a solution of*

$$- \Delta u + \nu u = \min\bigl(\mu, u^{p-1}\bigr) u_+ \qquad \text{in } \mathbb{R}^N, \tag{4.11}$$

then $u \equiv 0$.

Proof. If u is a solution of (4.11), we have

$$-\Delta u + (\nu - \mu) u \leqslant 0 \qquad \text{in } \mathbb{R}^N.$$

Taking u as a test function, we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 + (\nu - \mu)\, |u|^2 \leqslant 0.$$

Since $\nu - \mu \geqslant 0$, this implies that $u \equiv 0$. □

4. Asymptotics of families of critical points

In this section we refine the asymptotic analysis in [30, Section 5] in order to obtain an estimate of the energy of a critical point u_ε of $\mathcal{J}_\varepsilon$ depending on the number and the location of its local maxima. The corresponding lower estimate was proved in [30].

4.1. Asymptotics on small balls

The next lemma states that the sequences of rescaled solutions converge in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ to a function in $H^1(\mathbb{R}^N)$. It was essentially proved in [30].

Lemma 4.12. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ such that $\varepsilon_n \to 0$ as $n \to \infty$, let $(u_n)_{n\in\mathbb{N}}$ be a family of solutions of $(\mathcal{P}_\varepsilon)$ such that*

$$\liminf_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) < \infty$$

and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N$ such that $x_n \to \bar{x}$ as $n \to \infty$. Denote by $(v_n)_{n\in\mathbb{N}}$ the sequence defined by $v_n(x) = u_{\varepsilon_n}(x_n + \varepsilon_n x)$. If $V(\bar{x}) > 0$, then there exists $v \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$v_n \overset{C^1_{\mathrm{loc}}(\mathbb{R}^N)}{\longrightarrow} v$$

as $n \to \infty$.

Proof. Take $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on $B(0,1)$. Set for $R > 0$ and $x \in \mathbb{R}^N$ $\varphi_R(x) = \varphi(\frac{x}{R})$. The sequence $(\varphi_R v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ for every $R > 0$. By a diagonal argument, there exists $v \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $v_n \to v$ along a subsequence.

Now note that for every $R > 0$,

$$\int_{B(0,R)} |\nabla v|^2 \leqslant \liminf_{n\to\infty} \int_{B(0,R)} |\nabla v_n|^2 \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2$$

and

$$V(\bar{x}) \int_{B(0,R)} |v|^2 \leqslant \liminf_{n\to\infty} \int_{B(0,R)} V\,|v_n|^2 \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} V\,|v_n|^2 \;,$$

so that $v \in H^1(\mathbb{R}^N)$.

The remainder follows from classical regularity and compactness arguments. □

Lemma 4.13. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ such that $\varepsilon_n \to 0$ as $n \to \infty$, let $(u_n)_{n\in\mathbb{N}}$ be a family of solutions of $(\mathcal{P}_\varepsilon)$ such that*

$$\liminf_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) < \infty$$

and let $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}^N$ *such that* $x_n \to \bar{x}$ *as* $n \to \infty$. *If* $V(\bar{x}) > 0$ *and*

$$\liminf_{n\to\infty} u_n(x_n) > 0 \,, \tag{4.12}$$

then, $\bar{x} \in \bar{\Lambda}$,

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \Lambda)}{\varepsilon_n} < \infty \,,$$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \left| \varepsilon_n^{-N} \int_{B_n(R)} \Bigl(\frac{1}{2} \bigl(\varepsilon_n^2 \, |\nabla u_n|^2 + V \, |u_n|^2 \bigr) - G_{\varepsilon_n}(.,u_n) \Bigr) - \mathcal{C}(\bar{x}) \right| = 0 \,,$$

where $B_n(R) := B(x_n, \varepsilon_n R)$, *and*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \varepsilon_n^{-N} \int_{B_n(2R)\setminus B_n(R)} |\nabla u_n|^2 + V\,|u_n|^2 = 0 \,.$$

Proof. Set $v_n(x) := u_n(x_n + \varepsilon_n x)$. By Lemma 4.12 up to a subsequence, there exists a $v \in H^1(\mathbb{R}^N)$ such that $v_n \to v$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$. By (4.12), $v(0) = \lim_{n\to\infty} v_n(0) > 0$ so that $v \not\equiv 0$.

Let us now prove by contradiction that

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \Lambda)}{\varepsilon_n} < \infty \,. \tag{4.13}$$

Up to a subsequence, we can assume that $\lim_{n\to\infty} \operatorname{dist}(x_n, \Lambda)/\varepsilon_n = \infty$. Then, since the sequence of characteristic functions $\chi_n(x) := \chi_\Lambda(x_n + \varepsilon_n x)$ converges pointwise to 0, we have as $n \to \infty$

$$\begin{aligned} &g_{\varepsilon_n}(x_n + \varepsilon_n x, v_n(x)) \\ &\quad = \min\bigl(\mu\bigl(\varepsilon_n^2 H(x_n + \varepsilon_n x) v_n(x) + V(x_n + \varepsilon_n x)\bigr), |v_n(x)|^{p-1}\bigr) v_n(x)_+ \\ &\quad \to \min\bigl(\mu V(\bar{x}), |v(x)|^{p-1}\bigr) v(x)_+ \,, \end{aligned}$$

in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $1 \le q < \frac{2N}{p(N-2)}$, and thus v solves the limiting equation

$$-\Delta v + V(\bar{x}) v = \min\bigl(\mu V(\bar{x}), |v|^{p-1}\bigr) v_+, \qquad \text{in } \mathbb{R}^N.$$

By Lemma 4.11, $v \equiv 0$, which is a contradiction. Thus (4.13) holds.

Now, let us assume that

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \mathbb{R}^N \setminus \Lambda)}{\varepsilon_n} = \infty \,. \tag{4.14}$$

Since $\chi_n(x)$ converges pointwise to 1, we have, up to a subsequence, for n large enough,

$$g_{\varepsilon_n}(x_n + \varepsilon_n x, v_n(x)) = v_n^p(x) \to v^p(x) \,,$$

in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $1 \le q < \frac{2N}{p(N-2)}$. Hence v solves the limiting equation

$$-\Delta v + V(\bar{x})v = v^p \qquad \text{in } \mathbb{R}^N. \tag{4.15}$$

If (4.13) holds but (4.14) does not, then

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \partial\Lambda)}{\varepsilon_n} < \infty\,.$$

Since Λ is smooth, $\chi_n \to \chi_E$, almost everywhere as $n \to \infty$, where E is a half-space. By Lemma 4.10, v is again a solution of (4.15).

In any case, v is thus a nontrivial solution of (4.15). Now we claim that

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty} \varepsilon_n^{-N}\frac{1}{2}\int_{B(x_n,\varepsilon_n R)} &\left(\varepsilon_n^2\left|\nabla u_n\right|^2 + V\left|u_n\right|^2\right)\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left|\nabla U_{V(\bar{x})}\right|^2 + V(\bar{x})\left|U_{V(\bar{x})}\right|^2\,,\end{aligned}\tag{4.16}$$

where $U_{V(\bar{x})}$ is the solution of (4.6) with $\nu = V(\bar{x})$. For every $R > 0$, the convergence of v_n to v in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ implies that

$$\lim_{n\to\infty} \varepsilon_n^{-N}\frac{1}{2}\int_{B(x_n,\varepsilon_n R)}\left(\varepsilon_n^2\left|\nabla u_n\right|^2 + V\left|u_n\right|^2\right) = \frac{1}{2}\int_{B(0,R)}\left|\nabla v\right|^2 + V(\bar{x})\left|v\right|^2.$$

Since $v \in H^1(\mathbb{R}^N)$, and v and $U_{V(\bar{x})}$ are equal up to a translation, we conclude that (4.16) holds. The argument for the other limit is similar. $\square$

4.2. Asymptotics outside small balls

The solutions decay outside a neighborhood of Λ:

Lemma 4.14. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of solutions of $(\mathcal{P}_\varepsilon)$ such that*

$$\limsup_{n\to\infty} \varepsilon_n^{-N}\mathcal{J}_{\varepsilon_n}(u_n) < \infty\,.$$

For every open set U such that $\bar{\Lambda} \subset U$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus U}\left(\varepsilon_n^2\left|\nabla u_n\right|^2 + (V + \varepsilon_n^2 H)\left|u_n\right|^2\right) = 0\,.$$

Proof. Since V is continuous and $\inf_\Lambda V > 0$, we can assume without loss of generality that $\inf_U V > 0$. Take $\psi \in \mathcal{D}(\mathbb{R}^N)$ such that $\psi = 0$ on

$\bar{\Lambda}$ and $\psi = 1$ on $\mathbb{R}^N \setminus U$. By taking $\psi^2 u_n$ as a test function in $(\mathcal{P}_\varepsilon)$, we obtain

$$\int_{\mathbb{R}^N} \left(\varepsilon_n^2 \left|\nabla(\psi u_n)\right|^2 + V \left|\psi u_n\right|^2\right) = \int_{\mathbb{R}^N} g_{\varepsilon_n}(x, u_n(x))\psi^2(x) u_n(x)\, dx + \int_{\mathbb{R}^N} \varepsilon_n^2 \left|\nabla\psi\right|^2 \left|u_n\right|^2 .$$

Since $\psi = 0$ in Λ, we deduce from (g_3) and Lemma 4.2 that

$$\begin{aligned}\int_{\mathbb{R}^N} g_{\varepsilon_n}(x, u_n(x))\psi^2(x)u_n(x)\, dx &\leqslant \mu \int_{\mathbb{R}^N} (V + \varepsilon_n^2 H) \left|\psi u_n\right|^2 \\ &\leqslant \mu \int_{\mathbb{R}^N} \left(\varepsilon_n^2 \left|\nabla(\psi u_n)\right|^2 + V \left|\psi u_n\right|^2\right) .\end{aligned}$$

Therefore, since $\operatorname{supp} \nabla\psi \subset U \setminus \bar{\Lambda}$ and $\inf_U V > 0$, we have

$$\begin{aligned}\frac{1-\mu}{2} \int_{\mathbb{R}^N} &\left(\varepsilon_n^2 \left|\nabla(\psi u_n)\right|^2 + (V + \varepsilon_n^2 H) \left|\psi u_n\right|^2\right) \\ &\leqslant \int_{\mathbb{R}^N} \varepsilon_n^2 \left|\nabla\psi\right|^2 \left|u_n\right|^2 \\ &\leqslant C\varepsilon_n^2 \int_{U\setminus\bar{\Lambda}} V \left|u_n\right|^2 \\ &\leqslant C\varepsilon_n^2 \int_{\mathbb{R}^N} \left(\varepsilon_n^2 \left|\nabla u_n\right|^2 + V \left|u_n\right|^2\right) .\end{aligned}$$

The conclusion follows from our assumption and from Lemma 4.6. $\square$

Now we have an estimate outside small balls.

Lemma 4.15. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ such that $\varepsilon_n \to 0$ as $n \to \infty$, let $(u_n)_{n\in\mathbb{N}}$ be a sequence of solutions of $(\mathcal{P}_\varepsilon)$ such that*

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) < \infty ,$$

and let $(x_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^N$, $1 \leqslant i \leqslant M$, be sequences such that $x_n^i \to \bar{x}^i \in \mathbb{R}^N$ as $n \to \infty$. If for every $i \in \{1, \dots, M\}$, $V(\bar{x}^i) > 0$ and

$$\liminf_{n\to\infty} u_n(x_n^i) > 0 ,$$

then

$$\liminf_{R\to\infty} \liminf_{n\to\infty} \varepsilon_n^{-N} \Big(\int_{\mathbb{R}^N\setminus\mathcal{B}_n(R)} \frac{1}{2}\left(\varepsilon_n^2 \left|\nabla u_n\right|^2 + V \left|u_n\right|^2\right) - G_{\varepsilon_n}(., u_n) \Big) \geqslant 0,$$

where $\mathcal{B}_n(R) := \bigcup_{i=1}^M B(x_n^i, \varepsilon_n R)$. Furthermore, if

$$\inf\big\{\limsup_{n\to\infty} \|u_n\|_{L^\infty(U\setminus\mathcal{B}_n(R))} \;:\; R > 0,\ U \text{ is open and } \bar{\Lambda} \subset U\big\} = 0,$$

then

$$\lim_{R\to\infty}\limsup_{n\to\infty}\varepsilon_n^{-N}\left|\int_{\mathbb{R}^N\setminus\mathcal{B}_n(R)}\frac{1}{2}\bigl(\varepsilon_n^2\,|\nabla u_n|^2+V\,|u_n|^2\bigr)-G_{\varepsilon_n}(.,u_n)\right|=0.$$

The first assertion was proved in [30, Lemma 15].

Proof. First we claim that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\varepsilon_n^{-N}\left|\int_{\mathbb{R}^N\setminus\mathcal{B}_n(R)}\bigl(\varepsilon_n^2\,|\nabla u_n|^2+V\,|u_n|^2-g_{\varepsilon_n}(.,u_n)u_n\bigr)\right|=0\,.$$

This is proved in [30, Lemma 15] by taking a suitable family of test functions and using Lemma 4.12. We do not need to go to a subsequence since by Lemma 4.13,

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{B(x_n^i,\varepsilon_n R)\setminus B(x_n^i,\varepsilon_n R/2)}\varepsilon_n^2\,|\nabla u_n|^2+V\,|u_n|^2=0\,.$$

The first assertion follows, as in [30, Lemma 15] from the inequality

$$\begin{aligned}\frac{1}{2}\Bigl(\int_{\mathbb{R}^N\setminus\mathcal{B}_n(R)}&\bigl(\varepsilon_n^2\,|\nabla u_n|^2+V\,|u_n|^2\bigr)-\int_{\mathbb{R}^N\setminus\mathcal{B}_n(R)}g_{\varepsilon_n}(x,u_n(x))u_n(x)\,dx\Bigr)\\&\le\frac{1}{2}\int_{\mathbb{R}^N\setminus\mathcal{B}_n(R)}\bigl(\varepsilon_n^2\,|\nabla u_n|^2+V\,|u_n|^2\bigr)-\int_{\mathbb{R}^N\setminus\mathcal{B}_n(R)}G_{\varepsilon_n}(x,u_n(x))\,dx\end{aligned}$$

For the second assertion, we have by (g_2) and (g_3),

$$\begin{aligned}\left|\int_{\mathbb{R}^N\setminus\mathcal{B}_n(R)}G_{\varepsilon_n}(x,u_n(x))-\frac{1}{2}g_{\varepsilon_n}(x,u_n(x))u_n(x)\,dx\right|&\\\leqslant C\int_{U\setminus\mathcal{B}_n(R)}(u_n)_+^{p+1}+\frac{\mu}{2}\int_{\mathbb{R}^N\setminus U}\bigl(\varepsilon_n^2H+V\bigr)\,|u_n|^2\ .&\end{aligned}$$

We compute that

$$\begin{aligned}\int_{U\setminus\mathcal{B}_n(R)}|u_n|^{p+1}&\leqslant C\,\|u_n\|_{L^\infty(U\setminus\mathcal{B}_n(R))}^{p-1}\int_{U\setminus\mathcal{B}_n(R)}|u_n|^2\\&\leqslant\frac{C}{(\inf_U V)^2}\,\|u_n\|_{L^\infty(U\setminus\mathcal{B}_n(R))}^{p-1}\int_{\mathbb{R}^N}\bigl(\varepsilon_n^2\,|\nabla u_n|^2+V\,|u_n|^2\bigr)\,.\end{aligned}$$

We conclude by taking $U\supset\bar\Lambda$ small enough and R and n large enough, in view of the hypothesis and Lemma 4.6 together with Lemma 4.14. $\square$

4.3. Conclusion

The main result of this section is

Proposition 4.16. *Let* $(\varepsilon_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}^+$ *such that* $\varepsilon_n \to 0$ *as* $n \to \infty$*, let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of solutions of* $(\mathcal{P}_\varepsilon)$ *such that*

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) < \infty ,$$

and let $(x_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^N$*,* $1 \leqslant i \leqslant M$*, be sequences such that* $x_n^i \to \bar{x}^i \in \mathbb{R}^N$ *as* $n \to \infty$*. If for every* $i \in \{1, \dots, M\}$*,* $V(\bar{x}^i) > 0$ *and*

$$\liminf_{n\to\infty} u_n(x_n^i) > 0 ,$$

and if for every $i, j \in \{1, \dots, M\}$ *such that* $i \neq j$*,*

$$\lim_{n\to\infty} \frac{\left|x_n^i - x_n^j\right|}{\varepsilon_n} = +\infty ,$$

then for every $i \in \{1, \dots, M\}$*,* $\bar{x}^i \in \bar{\Lambda}$*,*

$$\lim_{n\to\infty} \frac{\operatorname{dist}(x_n^i, \Lambda)}{\varepsilon_n} < +\infty ,$$

and

$$\liminf_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) \geqslant \sum_{i=1}^{M} \mathcal{C}(\bar{x}^i) .$$

Furthermore, if

$$\inf\bigl\{\limsup_{n\to\infty} \|u_n\|_{L^\infty(U\setminus\mathcal{B}_n(R))} \ :\ R > 0,\ U \text{ is open and } \bar{\Lambda} \subset U\bigr\} = 0,$$

then

$$\lim_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) = \sum_{i=1}^{M} \mathcal{C}(\bar{x}^i) .$$

Proof. This follows from Lemmas 4.13 and 4.15 [30, Proposition 16] (see also [113, Lemma 4.3]). □

5. Asymptotics of families of almost minimizers

5.1. Families of minimizers

Let us recall how the results of Section 4 allow to study the asymptotics of $\inf_{\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon$.

Proposition 4.17. *If* $\inf_\Lambda V > 0$*, then*

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} \inf_{\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon = \inf_\Lambda \mathcal{C} .$$

This has been proved by del Pino and Felmer [61, (2.4) and Lemma 2.2] when $\inf_{\mathbb{R}^N} V > 0$, and has been extended to decaying potentials [30, Lemma 12 and proof of Proposition 21; 113, Lemmas 4.1 and 4.3].

Sketch of the proof. First one shows that for every $x \in \Lambda$,

$$\limsup_{\varepsilon \to 0} \varepsilon^{-N} \inf_{\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon \le \mathcal{C}(x)$$

by taking suitable multiples of cutoffs of $U_{V(x)}(\frac{\cdot - x}{\varepsilon})$.

By Proposition 4.8, for every $\varepsilon > 0$, there exists $u_\varepsilon \in \mathcal{N}_\varepsilon$ such that $\mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon$. By classical regularity theory, u_ε is continuous. Choose $x_\varepsilon \in \bar{\Lambda}$ such that $u_\varepsilon(x_\varepsilon) = \sup_\Lambda u_\varepsilon$. By Lemma 4.5, $\liminf_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) > 0$. By Proposition 4.16,

$$\liminf_{\varepsilon \to 0} \varepsilon^{-N} \mathcal{J}_\varepsilon(u_\varepsilon) \ge \inf_\Lambda \mathcal{C} \,. \qquad \square$$

5.2. Decay of almost minimizers

The next ingredient is a decay estimate that will allow to control the functional outside Λ in the proof of the strict inequality (4.23).

Lemma 4.18. *Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$ and let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$. If $\inf_\Lambda V > 0$ and*

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) \leqslant \inf_\Lambda \mathcal{C},$$

then, for every open set $U \subset \mathbb{R}^N$ such that $\bar{\Lambda} \subset U$,

$$\lim_{n\to\infty} \varepsilon_n^{-N} \int_{\mathbb{R}^N \setminus U} \varepsilon_n^2 \, |\nabla u_n|^2 + V \, |u_n|^2 = 0 \,.$$

This lemma is proved by del Pino and Felmer [63, (1.19)] when $\inf_{\mathbb{R}^N} V > 0$.

The proof of Lemma 4.18 relies on the following lemma

Lemma 4.19. *Let $U \subset \mathbb{R}^N$ be such that $\bar{\Lambda} \subset U$ and $\inf_U V > 0$. Let $\psi \in C^1(\mathbb{R}^N)$ and $\varphi \in C^1(\mathbb{R}^N)$ be such that $\psi = 0$ on Λ, $\varphi = 0$ on $\mathbb{R}^N \setminus U$, and $\psi^2 + \varphi^2 = 1$ on $\mathbb{R}^N$. There exists $C > 0$ such that for every $\varepsilon > 0$,*

$$\mathcal{J}_\varepsilon(\varphi u) + (1-\mu) \int_{\mathbb{R}^N} \bigl(\varepsilon^2 \, |\nabla(\psi u)|^2 + V \, |\psi u|^2\bigr) \le \mathcal{J}_\varepsilon(u) + C\varepsilon^2 \int_{U \setminus \bar{\Lambda}} V \, |u|^2 \,.$$

Proof. One has

$$\mathcal{J}_\varepsilon(u) = \mathcal{J}_\varepsilon(\varphi u) + \frac{1}{2} \int_{\mathbb{R}^N} \bigl(\varepsilon^2 \, |\nabla(\psi u)|^2 + V \, |\psi u|^2\bigr)$$
$$- \frac{1}{2} \int_{U \setminus \bar{\Lambda}} \varepsilon^2 (|\nabla \psi|^2 + |\nabla \varphi|^2) \, |u|^2 - \int_{\mathbb{R}^N} G_\varepsilon(x, u(x)) - G_\varepsilon(x, \varphi(x) u(x)) \, dx.$$

One has for every $x \in \mathbb{R}^N \setminus \Lambda$, by (g_3),

$$\begin{aligned} G_\varepsilon(x, u(x)) - G_\varepsilon(x, \varphi(x)u(x)) &= \int_{\varphi(x)u(x)}^{u(x)} g_\varepsilon(x, \sigma)\, d\sigma \\ &\leqslant \mu \int_{\varphi(x)u(x)}^{u(x)} \big(V(x) + \varepsilon^2 H(x)\big)\sigma\, d\sigma \\ &= \frac{\mu}{2}\big(V(x) + \varepsilon^2 H(x)\big) \left|\psi(x)u(x)\right|^2, \end{aligned}$$

so that

$$\int_{\mathbb{R}^N} G_\varepsilon(x, u(x)) - G_\varepsilon(x, \varphi(x)u(x))\, dx \leqslant \frac{\mu}{2} \int_{\mathbb{R}^N} \big(V + \varepsilon^2 H\big) \left|\psi u\right|^2 .$$

On the other hand,

$$\int_{U \setminus \bar{\Lambda}} \varepsilon^2(\left|\nabla \psi\right|^2 + \left|\nabla \varphi\right|^2) \left|u\right|^2 \leqslant C\varepsilon^2 \int_{U \setminus \bar{\Lambda}} V \left|u\right|^2 .$$

The conclusion follows. □

Proof of Lemma 4.18. By Lemma 4.6,

$$\limsup_{n \to \infty} \varepsilon_n^{-N} \int_{\mathbb{R}^N} \varepsilon^2 \left|\nabla u_n\right|^2 + V \left|u_n\right|^2 < \infty .$$

Without loss of generality, assume that $\inf_U V > 0$. Let $\psi \in C^1(\mathbb{R}^N)$ and $\varphi \in C^1(\mathbb{R}^N)$ be such that $\psi = 0$ on Λ, $\varphi = 0$ on $\mathbb{R}^N \setminus U$, and $\psi^2 + \varphi^2 = 1$ on $\mathbb{R}^N$. Define t_n so that $t_n \varphi u_n \in \mathcal{N}_{\varepsilon_n}$. One has

$$\limsup_{n \to \infty} \varepsilon_n^{-N} \int_{\mathbb{R}^N} \varepsilon_n^2 \left|\nabla(\varphi u_n)\right|^2 + V \left|\varphi u_n\right|^2 < \infty .$$

By the choice of t_n,

$$\begin{aligned} &t_n^2 \int_{\mathbb{R}^N} \varepsilon_n^2 \left|\nabla(\varphi u_n)\right|^2 + V \left|\varphi u_n\right|^2 \\ &\qquad = \int_{\mathbb{R}^N} g_{\varepsilon_n}(x, t_n \varphi(x) u_n(x)) t_n \varphi(x) u_n(x)\, dx \geqslant t_n^{p+1} \int_\Lambda \left|u_n\right|^{p+1} . \end{aligned}$$

We infer from Lemma 4.4 that

$$\liminf_{n \to \infty} \varepsilon_n^{-N} \int_\Lambda \left|u_n\right|^{p+1} > 0 .$$

Therefore, $\limsup_{n \to \infty} t_n < \infty$ and

$$\limsup_{n \to \infty} \varepsilon_n^{-N} \int_{U \setminus \bar{\Lambda}} V \left|t_n u_n\right|^2 < \infty .$$

By Lemma 4.19,

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(t_n u_n) \geqslant \liminf_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(t_n \varphi u_n) \\ + \limsup_{n\to\infty} \varepsilon_n^{-N}(1-\mu)t_n^2 \int_{\mathbb{R}^N} \left(\varepsilon_n^2 \left|\nabla(\psi u_n)\right|^2 + V\left|\psi u_n\right|^2\right) .$$

By assumption, we have

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(t_n u_n) \leq \limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) \leq \inf_{\Lambda} \mathcal{C} ,$$

and since $t_n \varphi u_n \in \mathcal{N}_{\varepsilon_n}$, we deduce from Proposition 4.17 that

$$\liminf_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(t_n \varphi u_n) \geq \inf_{\Lambda} \mathcal{C} .$$

Combining the last three inequalities, we obtain the conclusion. □

5.3. Asymptotics of the barycenters

As in [63], we introduce a barycenter map in order to localize functions. Let $\psi \in C^1(\mathbb{R}^N)$ be such that $\operatorname{supp}\psi$ is compact, $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^N : V(x) > 0\}$ and $\psi = 1$ on a neighborhood of Λ. For every $u \in \mathcal{N}_\varepsilon$,

$$\int_{\mathbb{R}^N} |\psi u|^2 > 0 .$$

The barycenter of a function $u \in L^2(\mathbb{R}^N)$ is defined by

$$\beta(u) := \frac{\displaystyle\int_{\mathbb{R}^N} x\left|\psi(x)u(x)\right|^2\, dx}{\displaystyle\int_{\mathbb{R}^N} \left|\psi(x)u(x)\right|^2\, dx}.$$

The map β is well-defined on the set $\{u \in H^1_V(\mathbb{R}^N) \; : \; \psi u \neq 0\}$, which contains $\mathcal{N}_\varepsilon$ for each $\varepsilon > 0$ by Lemma 4.4.

Proposition 4.20. *Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$ and let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$. If $\inf_\Lambda V > 0$ and*

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) \leqslant \inf_{\Lambda} \mathcal{C} ,$$

then

$$\lim_{n\to\infty} V(\beta(u_n)) = \inf_{\Lambda} V .$$

Proof. Set $V_0 = \inf_\Lambda V$. Let $s_n > 0$ be such that

$$\int_{\mathbb{R}^N} \varepsilon_n^2 \left|\nabla s_n \psi u_n\right|^2 + V_0 \left|s_n \psi u_n\right|^2 = \int_{\mathbb{R}^N} (s_n \psi u_n)_+^{p+1} . \tag{4.17}$$

Define $v_n : \mathbb{R}^N \to \mathbb{R}$ for $y \in \mathbb{R}^N$ by

$$v_n(y) := s_n \psi(\beta(u_n) + \varepsilon_n y) u_n(\beta(u_n) + \varepsilon_n y).$$

Claim 1. *The sequence $(s_n)_{n\in\mathbb{N}}$ is bounded.* In view of (g_3), we have

$$\int_{\mathbb{R}^N} \left(\varepsilon_n^2 \left|\nabla u_n\right|^2 + V \left|u_n\right|^2\right) \leqslant \int_\Lambda (u_n)_+^{p+1} + \mu \int_{\mathbb{R}^N\setminus\Lambda} (\varepsilon_n^2 H + V) \left|u_n\right|^2,$$

and therefore by Lemma 4.2,

$$\int_\Lambda (u_n)_+^{p+1} \geqslant (1-\mu) \int_{\mathbb{R}^N} \varepsilon_n^2 \left|\nabla u_n\right|^2 + V \left|u_n\right|^2.$$

Since by the Sobolev estimate

$$\int_\Lambda \left|u_n\right|^{p+1} \leqslant \frac{C}{\varepsilon_n^{\frac{p-1}{2}N}} \Big(\int_{\mathbb{R}^N} \varepsilon_n^2 \left|\nabla u_n\right|^2 + V \left|u_n\right|^2\Big)^{\frac{p+1}{2}},$$

we conclude that

$$C \int_\Lambda (u_n)_+^{p+1} \geqslant c\varepsilon_n^N. \tag{4.18}$$

Hence, we deduce from (4.17) that the sequence $(s_n)_{n\in\mathbb{N}}$ is bounded.

Claim 2. *One has*

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(s_n \psi u_n) \leqslant \inf_\Lambda \mathcal{C}.$$

We can write

$$\begin{aligned}\mathcal{J}_{\varepsilon_n}(s_n\psi u_n) = \mathcal{J}_{\varepsilon_n}(s_n u_n) &+ \frac{\varepsilon_n^2}{2} s_n^2 \int_{\mathbb{R}^N} \left|\nabla(\psi u_n)\right|^2 - \left|\nabla u_n\right|^2 \\ &+ \frac{s_n^2}{2} \int_{\mathbb{R}^N} (\psi^2 - 1) V \left|u_n\right|^2 \\ &+ \int_{\mathbb{R}^N} G_{\varepsilon_n}(x, s_n u_n(x)) - G_{\varepsilon_n}(x, s_n \psi(x) u_n(x))\, dx\end{aligned}$$

Now, in view of Lemma 4.18, since $\psi = 1$ in a neighborhood of Λ and $(s_n)_{n\in\mathbb{N}}$ is bounded,

$$\begin{aligned}&\frac{\varepsilon_n^2}{2} s_n^2 \int_{\mathbb{R}^N} \left|\nabla(\psi u_n)\right|^2 - \left|\nabla u_n\right|^2 \\ &= \frac{\varepsilon_n^2}{2} s_n^2 \int_{\mathbb{R}^N} (\psi^2 - 1) \left|\nabla u_n\right|^2 + 2\psi u_n \nabla\psi \cdot \nabla u_n + \left|\nabla\psi\right|^2 \left|u_n\right|^2 = o(\varepsilon_n^N).\end{aligned}$$

Lemma 4.18 also implies that

$$\int_{\mathbb{R}^N} (\psi^2 - 1) V \left|u_n\right|^2 = o(\varepsilon_n^N).$$

Finally, since $\psi = 1$ on a neighborhood U of Λ, we deduce from (g_3) that

$$\int_{\mathbb{R}^N} G_{\varepsilon_n}\big(x, s_n u_n(x)\big) - G_{\varepsilon_n}(x, s_n \psi(x) u_n(x))\, dx \\ \leq \frac{\mu}{2} s_n^2 \int_{\mathbb{R}^N \setminus U} \big(\varepsilon_n^2 H + V\big) \lvert u_n \rvert^2 = o(\varepsilon_n^N)\,,$$

where we have used Lemma 4.18 again. It follows from the hypothesis that

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(s_n u_n) \leq \limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_n) \leq \inf_{\Lambda} \mathcal{C}\,;$$

the claim follows.

Claim 3. *There holds*

$$\limsup_{n\to\infty} \mathcal{I}_{V_0}(v_n) \leqslant \inf_{\Lambda} \mathcal{C}\,.$$

One computes that

$$\begin{aligned} \varepsilon_n^N \mathcal{I}_{V_0}(v_n) &= \mathcal{J}_{\varepsilon_n}(s_n \psi u_n) + \frac{s_n^2}{2} \int_{\mathbb{R}^N} (V_0 - V)\psi^2 \lvert u_n \rvert^2 \\ &\quad + \int_{\mathbb{R}^N} G_{\varepsilon_n}(x, s_n \psi(x) u_n(x)) - \frac{s_n^{p+1}}{p+1} (\psi(x) u_n(x))_+^{p+1}\, dx \\ &\leqslant \mathcal{J}_{\varepsilon_n}(s_n \psi u_n) + \frac{s_n^2}{2} \int_{\mathbb{R}^N} (V_0 - V)\psi^2 \lvert u_n \rvert^2\,. \end{aligned}$$

For $\kappa \in (0,1)$, define

$$U_\kappa = \big\{ x \in \mathbb{R}^N \ :\ (V_0 - V(x))\psi^2(x) < \kappa V(x) \big\}.$$

Since V is continuous, U_κ is open and by (4.1), $\bar{\Lambda} \subset U_\kappa$. By Lemma 4.18,

$$\begin{aligned} \int_{\mathbb{R}^N} (V_0 - V)\psi^2 \lvert u_n \rvert^2 &\leq \kappa \int_{U_\kappa} V \lvert u_n \rvert^2 + \int_{\mathbb{R}^N \setminus U_\kappa} (V_0 - V)\psi^2 \lvert u_n \rvert^2 \\ &\leq \kappa \int_{\mathbb{R}^N} V \lvert u_n \rvert^2 + o(\varepsilon_n^N)\,. \end{aligned}$$

Since $\kappa > 0$ is arbitrary, and $(s_n)_{n\in\mathbb{N}}$ and $(\lVert u_n \rVert_\varepsilon)_{n\in\mathbb{N}}$ are bounded,

$$\frac{s_n^2}{2} \int_{\mathbb{R}^N} (V_0 - V)\psi^2 \lvert u_n \rvert^2 \leq o(\varepsilon_n^N)\,.$$

The claim now follows from Claim 2.

Conclusion. We know from Claim 3 that $(v_n)_{n\in\mathbb{N}}$ is a minimizing sequence of $\mathcal{I}_{V_0}$ on its associated Nehari manifold $\mathcal{M}_{V_0}$. By Lemma 4.9, there exists a sequence of points $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n(\cdot - y_n)$ converges in $H^1(\mathbb{R}^N)$ to the positive solution U_{V_0} of problem (4.6). Let $x_n := \varepsilon_n y_n$. Since

$$\begin{aligned}\beta(u_n) &= x_n + \frac{\displaystyle\int_{\mathbb{R}^N} (x - x_n)\,|\psi(x)u_n(x)|^2\ dx}{\displaystyle\int_{\mathbb{R}^N} |\psi(x)u_n(x)|^2\ dx}\\ &= x_n + \frac{\displaystyle\varepsilon_n \int_{\mathbb{R}^N} y\,|v_n(y - y_n)|^2\ dy}{\displaystyle\int_{\mathbb{R}^N} |v_n(y - y_n)|^2\ dy},\end{aligned}$$

we have $\lim_{n\to\infty} |\beta(u_n) - x_n| = 0$.

Now, note that for n large enough, by (4.18),

$$\liminf_{n\to\infty} \varepsilon_n^{-N} \int_\Lambda |u_n|^2 > 0.$$

Since $v_n(\cdot - y_n) \to U_{V_0}$ in $L^2(\mathbb{R}^N)$, we must have $\operatorname{dist}(x_n, \Lambda) = O(\varepsilon_n)$. Let

$$\bar V := \limsup_{n\to\infty} V(x_n) = \limsup_{n\to\infty} V(\beta(u_n)).$$

Since V is continuous on the compact set $\operatorname{supp}\psi$, one has $\lim_{n\to\infty} V(x_n + \varepsilon y) \geqslant V_0$.

By Claim 2,

$$\begin{aligned}b_{V_0} &\geqslant \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(s_n \psi u_n) + o(1)\\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x_n + \varepsilon_n y)\,|v_n|^2\\ &\qquad - \frac{1}{p+1}\int_{\mathbb{R}^N} (v_n)_+^{p+1} + o(1),\end{aligned}$$

and thus we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N} |\nabla U_{V_0}|^2 + V_0\,|U_{V_0}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} (U_{V_0})_+^{p+1} &= b_{V_0}\\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla U_{V_0}|^2 + \bar V\,|U_{V_0}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} (U_{V_0})_+^{p+1}.\end{aligned}$$

This implies that $\bar V \leqslant V_0$. The conclusion follows. $\square$

6. The minimax level

6.1. Definition of the minimax level

Following del Pino and Felmer [63, 64], we define a minimax value for $\mathcal{J}_\varepsilon$.

Let $\eta \in \mathcal{D}(\mathbb{R}^N)$ be a cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ on a neighborhood of Λ and $\operatorname{supp} \eta \subset \{x \in \mathbb{R}^N : V(x) > 0\}$. We define $w_{\varepsilon,y} \in \mathcal{D}(\mathbb{R}^N)$ by

$$w_{\varepsilon,y}(x) := t_{\varepsilon,y}\eta(x)U_{V(y)}\Big(\frac{y-x}{\varepsilon}\Big), \tag{4.19}$$

where $t_{\varepsilon,y} > 0$ is such that $w_{\varepsilon,y} \in \mathcal{N}_\varepsilon$. Let $\Lambda_\varepsilon \subset \Lambda$ be such that

$$\lim_{\varepsilon\to 0} \sup_{x\in\partial\Lambda_\varepsilon} \operatorname{dist}(x, \partial\Lambda) = 0 \tag{4.20}$$

and

$$\lim_{\varepsilon\to 0} \inf_{x\in\partial\Lambda_\varepsilon} \frac{\operatorname{dist}(x,\partial\Lambda)}{\varepsilon} = \infty\,. \tag{4.21}$$

We define the family of paths

$$\Gamma_\varepsilon := \big\{\gamma \in C(\overline{\Lambda}_\varepsilon, \mathcal{N}_\varepsilon) \;:\; \text{for every } y \in \partial\Lambda_\varepsilon,\, \gamma(y) = w_{\varepsilon,y}\big\}$$

and the minimax value

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \sup_{y\in\Lambda_\varepsilon} \mathcal{J}_\varepsilon(\gamma(y)). \tag{4.22}$$

We want to apply the following theorem.

Theorem 4.21 (General Minimax Principle [145, Theorem 2.9]**).** *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma := \big\{\gamma \in C(M, X) \;:\; \gamma_{|M_0} \in \Gamma_0\big\}\,.$$

If $\varphi \in C^1(X, \mathbb{R})$ satisfies

$$\infty > c := \inf_{\gamma\in\Gamma} \sup_{z\in M} \varphi\big(\gamma(z)\big) > a := \sup_{\gamma_0\in\Gamma_0} \sup_{z\in M_0} \varphi\big(\gamma_0(z)\big)$$

and if φ satisfies the Palais-Smale condition at the level c, then c is a critical value of φ.

Since $\mathcal{J}_\varepsilon$ satisfies the Palais-Smale condition (Lemma 4.7), we have to show that for $\varepsilon > 0$ small enough,

$$c_\varepsilon > \sup_{y\in\partial\Lambda_\varepsilon} \mathcal{J}_\varepsilon(w_{\varepsilon,y}) =: a_\varepsilon. \tag{4.23}$$

6.2. Estimates on the levels

6.2.1. *Estimate of a_ε.* We begin with an estimate of a_ε.

Lemma 4.22. *We have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} a_\varepsilon = \sup_{\partial\Lambda} \mathcal{C}\ .$$

Proof. By a straightforward computation, we find in view of (4.21)

$$\mathcal{J}_\varepsilon(w_{\varepsilon,y}) = \varepsilon^N \mathcal{I}_{V(y)}\left(U_{V(y)}\right) + o(\varepsilon^N)\ .$$

as $\varepsilon \to 0$, uniformly in $y \in \Lambda$.

Thus

$$a_\varepsilon = \sup_{y\in\partial\Lambda_\varepsilon} \mathcal{J}_\varepsilon(w_{\varepsilon,y}) = \varepsilon^N \sup_{y\in\partial\Lambda_\varepsilon} b_{V(y)} + o(\varepsilon^N)\ .$$

The estimate follows from (4.8), (4.20) and the continuity of V. $\square$

6.2.2. *Upper estimate of the critical level c_ε.* The same method gives an upper estimate on c_ε.

Lemma 4.23. *We have*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} c_\varepsilon \le \sup_{\Lambda} \mathcal{C}\ .$$

Proof. As a test path in (4.22), we take $w_{\varepsilon,y}$ defined by (4.19) for every $y \in \Lambda_\varepsilon$. We obtain the first estimate after a straightforward computation in view of (4.21). $\square$

6.2.3. *Lower estimate of the critical level c_ε.* A more delicate construction gives a lower estimate of the critical level c_ε.

Lemma 4.24. *If*

$$\sup_{\Lambda} \mathcal{C} > \inf_{\Lambda} \mathcal{C},$$

then

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N} c_\varepsilon > \inf_{\Lambda} \mathcal{C}.$$

We do not know whether the natural stronger conclusion

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N} c_\varepsilon \geqslant \sup_{\Lambda} \mathcal{C}$$

holds.

Lemma 4.25. *Let $x \in \Lambda$. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and every $\gamma \in \Gamma_\varepsilon$, there exists $z \in \Lambda_\varepsilon$ such that $\beta(\gamma(z)) = x$.*

Proof. For every $y \in \partial\Lambda_\varepsilon$, one has in view of the definition of $w_{\varepsilon,y}$,

$$\beta(w_{\varepsilon,y}) = y + o(1),$$

as $\varepsilon \to 0$, uniformly in y.

Let $\gamma \in \Gamma_\varepsilon$. If ε is small enough, one has for every $y \in \partial\Lambda_\varepsilon$, by (4.20), $x \in \Lambda_\varepsilon$ for ε small enough.

When ε is small enough, we have

$$\sup_{y\in\partial\Lambda_\varepsilon} |\beta(w_{\varepsilon,y}) - y| < \inf_{y\in\partial\Lambda_\varepsilon} |y - x| .$$

Therefore, by the properties of the topological degree, if $x \in \Lambda_\varepsilon$ there exists $z \in \Lambda_\varepsilon$ such that $\beta(\gamma(z)) = x$. □

We follow the arguments of [63]. Heuristically, the idea is to show that a sequence of functions violating the strict inequality (4.23) cannot have enough energy to stay concentrated inside Λ and must thus concentrate around a point of $\partial\Lambda$. But this would in fact contradict the continuity of the paths in Γ_ε.

Proof of Lemma 4.24. Assume by contradiction that there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} \varepsilon_n^{-N} c_{\varepsilon_n} \le \inf_\Lambda \mathcal{C}$. By definition of c_ε, there exists $\gamma_n \in \Gamma_{\varepsilon_n}$ such that

$$\lim_{n\to\infty} \sup_{x\in\Lambda_{\varepsilon_n}} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(\gamma_n(x)) \le \inf_\Lambda \mathcal{C} .$$

Choose $x \in \Lambda$ such that $V(x) > \inf_\Lambda V$. For each $n \in \mathbb{N}$ large enough, let x_n be given by Lemma 4.25 so that $\beta(\gamma_n(x_n)) = x$. One has

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(\gamma_n(x_n)) \leqslant \inf_\Lambda \mathcal{C} .$$

Proposition 4.20 brings then a contradiction. □

6.3. Existence of a solution

We are now in a position to prove the strict inequality (4.23).

Lemma 4.26. *If*

$$\sup_\Lambda \mathcal{C} > \inf_\Lambda \mathcal{C} = \sup_{\partial\Lambda} \mathcal{C} ,$$

then

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N}(c_\varepsilon - a_\varepsilon) > 0 .$$

Proof. This follows directly from Lemmas 4.22 and 4.24. □

As a consequence of the General Minimax Principle, we have thus proved the following existence result for the penalized problem $(\mathcal{P}_\varepsilon)$.

Proposition 4.27. *Let $N \geqslant 3$, $1 < p < \frac{N+2}{N-2}$ and let $g_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying assumptions (g_1)- (g_4). For $\varepsilon > 0$ small enough, there exists $u_\varepsilon \in \mathcal{N}_\varepsilon$ such that $\mathcal{J}_\varepsilon(u_\varepsilon) = c_\varepsilon$ and $\mathcal{J}_\varepsilon'(u_\varepsilon) = 0$.*

Proof. This follows from the general minimax principle (Theorem 4.21), the Palais-Smale condition coming from Lemma 4.7 and the strict inequality from Lemma 4.26. □

7. Back to the original problem

7.1. Asymptotics of solutions

Thanks to the asymptotics of solutions of Section 4 and the estimates on the critical level of Section 6.2, we prove that the solution u_ε is single-peaked.

Lemma 4.28. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a family such that for $\varepsilon > 0$ small enough, $\mathcal{J}_\varepsilon(u_\varepsilon) = c_\varepsilon$ and $\mathcal{J}_\varepsilon'(u_\varepsilon) = 0$. Let $(x_\varepsilon)_{\varepsilon>0}$ in Λ be such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0 \,.$$

If

$$\sup_\Lambda \mathcal{C} < 2 \inf_\Lambda \mathcal{C} \,,$$

then for every $U \subset \mathbb{R}^N$ such that $\bar{U}$ is compact and $\inf_U V > 0$,

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \|u_\varepsilon\|_{L^\infty(U\setminus B(x_\varepsilon,\varepsilon R))} = 0 \,.$$

If moreover

$$\sup_\Lambda \mathcal{C} > \inf_\Lambda \mathcal{C} = \sup_{\partial\Lambda} \mathcal{C} \,,$$

then

$$\liminf_{\varepsilon\to 0} d(x_\varepsilon, \mathbb{R}^N \setminus \Lambda) > 0 \,.$$

Proof. First we prove that

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \|u_\varepsilon\|_{L^\infty(U\setminus B(x_\varepsilon,\varepsilon R))} = 0 \,.$$

Assume by contradiction that there exist sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $y_n \in U$,

$$\lim_{n\to\infty} \varepsilon_n = 0 \,, \quad \liminf_{n\to\infty} u_{\varepsilon_n}(y_n) > 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{|x_{\varepsilon_n} - y_n|}{\varepsilon_n} = +\infty \,.$$

Then, by Lemma 4.23,

$$\limsup_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) \le \sup_\Lambda \mathcal{C} \,, \tag{4.24}$$

while by Proposition 4.16
$$\liminf_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) \geqslant \liminf_{n\to\infty} \big(\mathcal{C}(x_{\varepsilon_n}) + \mathcal{C}(y_n)\big) \geqslant 2 \inf_{\Lambda} \mathcal{C} .$$
This is a contradiction with our assumption.

Now we turn to the second assertion. Assume by contradiction that there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ and $\lim_{n\to\infty} \operatorname{dist}(x_{\varepsilon_n}, \mathbb{R}^N \setminus \Lambda) = 0$. Then, by the first assertion, the second part of Proposition 4.16 and Lemma 4.24,
$$\inf_{\Lambda} \mathcal{C} < \liminf_{n\to\infty} \varepsilon_n^{-N} \mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) = \liminf_{n\to\infty} \mathcal{C}(x_{\varepsilon_n}) = \sup_{\partial\Lambda} \mathcal{C} .$$
But this contradicts our assumption. □

7.2. Barrier functions and solution of the original problem

Since we know that the solutions do not concentrate on the boundary of Λ, we are in the same situation as in [113, Section 5]. Therefore we only recall the results of that paper.

Lemma 4.29. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a family of positive solutions of $(\mathcal{P}_\varepsilon)$ at the level c_ε and let $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ be such that*
$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$
Then there exists $\varepsilon_0 > 0$ and $R > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,
$$-\varepsilon^2(\Delta + \mu H)u_\varepsilon + (1-\mu)V u_\varepsilon \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R).$$

Proof. This follows from Lemma 4.28, see [113, Lemma 5.1] for the details. □

Lemma 4.30. *Let $(x_\varepsilon)_\varepsilon \subset \Lambda$ be such that $\liminf_{\varepsilon\to 0} d(x_\varepsilon, \partial\Lambda) > 0$, let $\mu \in (0,1)$ and let $R > 0$. Then, there exists $\varepsilon_0 > 0$ and a family of functions $(W_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ in $C^{1,1}(\mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R))$ such that, for $\varepsilon \in (0, \varepsilon_0)$,*

(i) W_ε satisfies the inequation
$$-\varepsilon^2(\Delta + \mu H)W_\varepsilon + (1-\mu)V W_\varepsilon \geqslant 0 \quad \text{in } \mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R),$$
(ii) $\nabla W_\varepsilon \in L^2(\mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R))$,
(iii) $W_\varepsilon = 1$ on $\partial B(x_\varepsilon, \varepsilon R)$,
(iv) for every $x \in \mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R)$,
$$W_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon}\frac{|x-x_\varepsilon|}{1+|x-x_\varepsilon|}\right)\left(1+|x|^2\right)^{-\frac{N-2}{2}} .$$

Proof. The arguments are the same as those of Moroz and Van Schaftingen [113, Lemma 5.2], since the penalization potential H is the same. □

Proposition 4.31. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a family of positive solutions of $(\mathcal{P}_\varepsilon)$ at the level c_ε and let $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ be such that*

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

Then there exists $C, \lambda > 0$ and $\varepsilon_0 > 0$ and $R > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

(4.25)
$$u_\varepsilon(x) \leqslant C \exp\left(-\frac{\lambda}{\varepsilon}\frac{|x-x_\varepsilon|}{1+|x-x_\varepsilon|}\right)\left(1+|x|^2\right)^{-\frac{N-2}{2}}, \qquad x \in \mathbb{R}^N.$$

Proof. This is a consequence of the two previous lemmas and of the comparison principle. □

The estimate (4.25) implies the following one

$$u_\varepsilon(x) \leqslant Ce^{-\frac{\lambda}{\varepsilon}}|x-x_\varepsilon|^{-(N-2)}, \qquad x \in \mathbb{R}^N \setminus \Lambda. \tag{4.26}$$

Proposition 4.32. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a family of positive solutions of $(\mathcal{P}_\varepsilon)$ at the level c_ε. If*

$$p > \frac{N}{N-2},$$

then there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, u_ε solves the original problem (NLS).

Proof. The proof is the same as in [113, Proposition 5.4]. We give the proof for the sake of completeness. By Lemma 4.5, there exists a family of points $(x_\varepsilon)_{\varepsilon>0} \subset \Lambda$ such that

$$\liminf_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) > 0.$$

By Lemma 4.28, $d_0 := \inf d(x_\varepsilon, \partial\Lambda) > 0$. Hence, by estimate (4.26), we have for $\varepsilon > 0$ small enough and for $x \in \mathbb{R}^N \setminus \Lambda$,

$$\begin{aligned}(u_\varepsilon(x))^{p-1} &\leqslant \left(Ce^{-\frac{\lambda}{\varepsilon}}|x-x_\varepsilon|^{-(N-2)}\right)^{p-1}\\ &\leqslant Ce^{-\frac{\lambda}{\varepsilon}(p-1)}(1+|x|)^{-(N-2)(p-1)}\\ &\leqslant \mu\varepsilon^2\frac{(N-2)^2}{4|x-x_0|^2}\left(\frac{\log\frac{\rho}{\rho_0}}{\log\frac{|x-x_0|}{\rho_0}}\right)^{1+\beta} = \mu\varepsilon^2 H(x).\end{aligned}$$

By definition of the penalized nonlinearity g_ε, one has then

$$g_\varepsilon(x, u_\varepsilon(x)) = (u_\varepsilon(x))^p, \qquad x \in \mathbb{R}^N \setminus \Lambda,$$

and therefore u_ε solves the original problem (NLS). □

Proof of Theorem 4.1. The existence of a solution u_ε has been established in the previous proposition. The assertion (4.3) has been proved in Lemma 4.5 and (4.4) has been proved in Lemma 4.28. To prove (4.5), we consider

$$\Lambda_\delta := \left\{ x \in \Lambda : V(x) > \sup_\Lambda V - \delta \right\},$$

By Sard's lemma, the set Λ_δ is smooth for almost every $\delta > 0$. Let us choose a sequence $(\delta_n)_{n\in\mathbb{N}}$ such that $\delta_n \to 0$ and Λ_{δ_n} is smooth. The first part of Theorem 4.1, applied to Λ_{δ_n}, ensures that, for every $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that, for every $\varepsilon \in (0, \varepsilon_n)$, problem (NLS) has a solution u_n which attains its maximum at $x_n \in \Lambda_{\delta_n}$. We may assume that the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ is decreasing. For every $\varepsilon \in (0, \varepsilon_1)$, we define $u_\varepsilon := u_n$, where $n \in \mathbb{N}$ is such that $\varepsilon_{n+1} \leqslant \varepsilon < \varepsilon_n$. If u_ε attains its maximum at x_ε, we deduce from the definition of Λ_δ that

$$\lim_{\varepsilon\to 0} V(x_\varepsilon) = \sup_\Lambda V.$$

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