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Weak representability of actions of non-associative algebras[☆]

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ABSTRACT

We study the categorical-algebraic condition that *internal actions are weakly representable* (WRA) in the context of varieties of (non-associative) algebras over a field.

Our first aim is to give a complete characterization of action accessible, operadic quadratic varieties of non-associative algebras which satisfy an identity of degree two and to study the representability of actions for them. Here we prove that the varieties of two-step nilpotent (anti-)commutative algebras and that of commutative associative algebras are weakly action representable, and we explain that the condition (WRA) is closely connected to the existence of a so-called *amalgam*.

Our second aim is to work towards the construction, still within the context of algebras over a field, of a weakly representing object $\mathcal{E}(X)$ for the actions on (or split extensions of) an object X . We actually obtain a *partial algebra* $\mathcal{E}(X)$, which we call *external weak actor* of X , together with a monomorphism of functors $\text{SplExt}(-, X) \hookrightarrow \text{Hom}(U(-), \mathcal{E}(X))$, which we study in detail in the case of quadratic varieties. Furthermore, the relations between the construction of the *universal strict general actor* $\text{USGA}(X)$ and that of $\mathcal{E}(X)$ are described in detail. We end with some open questions.

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Introduction

In the article [3], F. Borceux, G. Janelidze and G. M. Kelly introduce the concept of an *internal object action*, with the aim of extending the correspondence between actions and split extensions from the context of groups and Lie algebras to arbitrary semi-abelian categories [19]. In certain of those categories, internal actions are exceptionally well behaved, in the sense that the actions on each object X are *representable*: this means that there exists an object $[X]$, called the *actor* of X , such that the functor $\text{Act}(-, X) \cong \text{SplExt}(-, X)$, which sends an object B to the set of actions/split extensions of B on/by X , is naturally isomorphic to the functor $\text{Hom}(-, [X])$. The context of action representable semi-abelian categories is further studied in [4], where it is for instance explained that the category of commutative associative algebras over a field is not action representable. Later it was shown that the only action representable variety of non-associative algebras over an infinite field $\mathbb{F}$ of characteristic different from 2 is the variety of Lie algebras [14]. The relative strength of the notion naturally led to the definition of closely related weaker notions.

The first of these was the concept of an *action accessible* category due to D. Bourn and G. Janelidze [6]: it is weak enough to include all *Orzech categories of interest* [29], as proved by A. Montoli in [28].

Alternatively, the properties of the representing object $[X]$ may be weakened; this is the aim in [8], where it is shown that each Orzech category of interest admits a so-called *universal strict general actor* (USGA for short).

Our present article focuses on a concept which was more recently introduced, by G. Janelidze in [17]: *weak representability of actions* (WRA). Instead of asking that for each object X in a semi-abelian category $\mathcal{C}$ we have an object $[X]$ and a natural isomorphism $\text{SplExt}(-, X) \cong \text{Hom}_{\mathcal{C}}(-, [X])$, we require the existence of an object T and a monomorphism of functors

$$\tau: \text{SplExt}(-, X) \rightarrow \text{Hom}_{\mathcal{C}}(-, T).$$

Such an object T is then called a *weak actor* of X , and when each X admits a weak actor, $\mathcal{C}$ is said to be *weakly action representable*. For instance, if in an Orzech category of interest, each $\text{USGA}(X)$ is an object of the category, then this category is weakly action representable [10]. This is the case of the category **Assoc** of associative algebras [17] or the category **Leib** of Leibniz algebras [10] over a field.

J. R. A. Gray observed in [15] that an Orzech category of interest need not be weakly action representable. One of our aims in the present article is to study the condition (WRA) in the context of varieties of (non-associative) algebras over a field. (We recall basic definitions and results concerning this setting in Section 1.) It is known that such a variety is action accessible if and only if it is algebraically coherent [14], and it is also known [17] that action accessibility is implied by (WRA). In Section 2 we give a complete classification of the action accessible, operadic quadratic varieties of non-associative algebras with an identity of degree 2 (so commutative or anti-commutative algebras) and we study the representability of actions of each of them. Moreover, we prove that the variety of commutative associative algebras, the variety of two-step nilpotent commutative algebras and that of two-step nilpotent anti-commutative algebras are weakly action representable categories. For the variety of commutative associative algebras, we show that the existence of a weak representation is closely connected to the *amalgamation property* (AP) [20] which already appeared in [4] in relation to action representability. In Section 3 the study of (WRA) and its relations with the condition (AP) is extended to a general variety of algebras over a field.

Our second aim is to work towards the construction, still within the context of algebras over a field, of a weakly representing object $\mathcal{E}(X)$ for the actions on/split extensions of an object X of a variety of non-associative algebras $\mathcal{V}$. We believe that in certain settings, this object may be easier to work with than the more abstract weak actor. In Definition 3.3 we actually obtain a *partial* algebra $\mathcal{E}(X)$, which we call *external weak actor* of X , together with a monomorphism of functors $\text{SplExt}(-, X) \rightarrow \text{Hom}(U(-), \mathcal{E}(X))$, where U is the forgetful functor from $\mathcal{V}$ to the category of partial algebras, which we study in detail in the case of quadratic varieties of algebras (Section 4).

We end the article with some open questions (Section 5).

1. Preliminaries

The present work takes place in *semi-abelian categories* which were introduced in [19] in order to capture categorical-algebraic properties of non-abelian algebraic structures. A category is *semi-abelian* if it is pointed, admits binary coproducts, is protomodular and Barr-exact. Well-known examples are the category **Grp** of groups, the category **Rng** of not necessarily unitary rings, any variety $\mathcal{V}$ of non-associative algebras over a field $\mathbb{F}$, as well as all abelian categories. Throughout the remainder of the paper, when we consider a category $\mathcal{C}$, we assume it to be semi-abelian; when we consider a variety $\mathcal{V}$, we assume that $\mathcal{V}$ is a variety of non-associative algebra over a field $\mathbb{F}$. We fix the field $\mathbb{F}$, so that we may drop it from our notation.

1.1. Internal actions and their representability

A central notion which appears in the semi-abelian context is that of *split extensions*. Let X, B be objects of a semi-abelian category $\mathcal{C}$; a *split extension* of B by X is a diagram

$$0 \longrightarrow X \xrightarrow{k} A \xrightleftharpoons[\beta]{\alpha} B \longrightarrow 0 \quad (1.1)$$

in $\mathcal{C}$ such that $\alpha \circ \beta = 1_B$ and (X, k) is a kernel of α . We observe that since protomodularity implies that the pair (k, β) is jointly strongly epic, the morphism α is indeed the cokernel of k and diagram (1.1) represents an extension of B by X in the usual sense. Morphisms of split extensions are morphisms of extensions that commute with the sections. Let us observe that, again by protomodularity, a morphism of split extensions fixing X and B is necessarily an isomorphism. For an object X of $\mathcal{C}$, we define the functor

$$\mathrm{SplExt}_{\mathcal{C}}(-, X): \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Set}$$

which assigns to any object B of $\mathcal{C}$, the set $\mathrm{SplExt}_{\mathcal{C}}(B, X)$ of isomorphism classes of split extensions of B by X in $\mathcal{C}$, and to any arrow $f: B' \rightarrow B$ the *change of base function* $f^*: \mathrm{SplExt}_{\mathcal{C}}(B, X) \rightarrow \mathrm{SplExt}_{\mathcal{C}}(B', X)$ given by pulling back along f . When there is no ambiguity on the category $\mathcal{C}$, we will use the notation $\mathrm{SplExt}(-, X)$.

A feature of semi-abelian categories is that one can define a notion of *internal action*. Internal actions correspond to split extensions via a semidirect product construction; it turns out that, as a result, for our purposes we need no explicit description of what is an internal action. We refer the interested reader to [5, 4], where the equivalence between the two concepts is described in detail. For us here, it suffices to note that if we fix an object X , internal actions on X in $\mathcal{C}$ give rise to a functor

$$\mathrm{Act}(-, X) = \mathrm{Act}_{\mathcal{C}}(-, X): \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Set}$$

and a natural isomorphism of functors $\text{Act}(-, X) \cong \text{SplExt}(-, X)$. This justifies the terminology in the definition that follows.

Definition 1.1 ([4]). A semi-abelian category $\mathcal{C}$ is said to be **action representable** if for every object X in it the functor $\text{Act}(-, X)$ is representable. In other words, there exists an object $[X]$ in $\mathcal{C}$, called the **actor** of X , and a natural isomorphism

$$\text{SplExt}(-, X) \cong \text{Hom}_{\mathcal{C}}(-, [X]).$$

Basic examples of semi-abelian categories which satisfy action representability are the category **Grp** of groups with the actor of X being the group of automorphisms $\text{Aut}(X)$, the category **Lie** of Lie algebras with the actor of X being the Lie algebra of derivations $\text{Der}(X)$, and any abelian category with the actor of X being the zero object. For the categories **Assoc** of associative algebras and **CAssoc** of commutative associative algebras, representability of actions was studied in [4], where the authors proved that they are not action representable.

It is explained in [3] that action representability is equivalent to the condition that for every object X in $\mathcal{C}$ the category $\text{SplExt}(X)$ of split extensions in $\mathcal{C}$ with kernel X has a terminal object

$$0 \longrightarrow X \longrightarrow [X] \ltimes X \rightleftarrows [X] \longrightarrow 0.$$

We can weaken this condition assuming instead that for any X , every object in $\text{SplExt}(X)$ is accessible (i.e. it has a morphism into a subterminal or so-called *faithful* object, see [6]). In this way, we encompass a wider class of examples that did not satisfy representability of actions such as the category **Pois** of (non-commutative) Poisson algebras, the category **Assoc** of associative algebras or the category **CAssoc** of commutative associative algebras. This notion called **action accessibility** was introduced by D. Bourn and G. Janelidze [6] in order to calculate centralisers of normal subobjects or of equivalence relations. It was then shown by A. Montoli that any Orzech category of interest is an action accessible category [28]. This explains why all of the varieties of non-associative algebras mentioned above are action accessible.

Since by definition the existence of a terminal object in $\text{SplExt}(X)$ is stronger than every object being accessible, it is immediate that

$$\text{action representability} \Rightarrow \text{action accessibility}.$$

Recently, in [17], G. Janelidze introduced an intermediate notion: *weak representability of actions*.

Definition 1.2. A semi-abelian category $\mathcal{C}$ is said to be **weakly action representable (WRA)** if for every object X in it, there exists an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau: \text{SplExt}(-, X) \rightarrow \text{Hom}_{\mathcal{C}}(-, T).$$

We call such an object T a **weak actor** of X , and a morphism $\varphi: B \rightarrow T$ in the image of τ_B an **acting morphism**.

It is clear from the definitions that every action representable category is weakly action representable. Also in [17], it is proven that the category **Assoc** is weakly action representable with a weak actor of X given by the associative algebra

$$\begin{aligned} \text{Bim}(X) = \{ (f * -, - * f) \in \text{End}(X) \times \text{End}(X)^{\text{op}} \mid f * (xy) = (f * x)y, \\ (xy) * f = x(y * f), x(f * y) = (x * f)y, \forall x, y \in X \} \end{aligned}$$

of **bimultipliers** of X (see [25]). The case of the category **Leib** of Leibniz algebras was studied in [10]. There the authors showed that a weak actor of a Leibniz algebra X is the Leibniz algebra

$$\begin{aligned} \text{Bider}(X) = \{ (d, D) \in \text{End}(X)^2 \mid d(xy) = d(x)y + xd(y), \\ D(xy) = D(x)y - D(y)x, xd(y) = xD(y), \forall x, y \in X \} \end{aligned}$$

of **biderivations** of X (see [24] and [26]), where the bilinear operation is defined by

$$[(d, D), (d', D')] = (d \circ d' - d' \circ d, D \circ d' - d' \circ D).$$

In the same paper, the representability of actions in the categories **Pois** and **CPois** of (commutative) Poisson algebras was studied.

Another important observation made by G. Janelidze is that every weakly action representable category is action accessible. We thus have that

$$\text{action representability} \Rightarrow \text{weak action representability} \Rightarrow \text{action accessibility}.$$

J. R. A. Gray proved in [15] that the varieties of n -solvable groups where $n \geq 3$ are action accessible but not weakly action representable. This partially answers a question asked by G. Janelidze in [17], whether reasonably mild conditions may be found on a semi-abelian category under which the second implication may be reversed: already it makes clear that not all action accessible semi-abelian varieties are weakly action representable. Our aim here is to study what happens for a different class of categories, namely varieties of not necessarily associative algebras over a field.

1.2. Varieties of non-associative algebras

We now recall the algebraic setting we are working in: *varieties of non-associative algebras* over a field $\mathbb{F}$. We think of those as collections of algebras satisfying a chosen

set of polynomial equations. The interested reader can find a more detailed presentation of the subject in [33].

By a **(non-associative) algebra** A we mean a vector space A equipped with a bilinear operation $A \times A \rightarrow A: (x, y) \mapsto xy$ which we call the **multiplication**. The existence of a unit element is not assumed, nor are any other conditions on the multiplication besides its bilinearity. Let **Alg** denote the category of non-associative algebras, where morphisms are linear maps which preserve the multiplication.

We consider the *free algebra* functor $\mathbf{Set} \rightarrow \mathbf{Alg}$ which sends a set S to the *free algebra* generated by elements of S . This functor has the forgetful functor as a right adjoint. Moreover, it factorises through the *free magma functor* $\mathbf{FM}: \mathbf{Set} \rightarrow \mathbf{Mag}$, which sends a set S to the magma $\mathbf{FM}(S)$ of non-associative words in S , and the *magma algebra functor* $\mathbb{F}[-]: \mathbf{Mag} \rightarrow \mathbf{Alg}$.

Let S be a set. An element φ of $\mathbb{F}[\mathbf{FM}(S)]$ is called a **non-associative polynomial** on S . We say that such a polynomial is a **monomial** when it is a scalar multiple of an element in $\mathbf{FM}(S)$. For example, if $S = \{x, y, z, t\}$, then $(xy)t + (zy)x$, $xx + yz$ and $(xt)(yz)$ are polynomials in S and only the last one is a monomial. For a monomial φ on a set $\{x_1, \dots, x_n\}$, we define its **type** as the n -tuple $(k_1, \dots, k_n) \in \mathbb{N}^n$ where k_i is the number of times x_i appears in φ , and its **degree** as the natural number $k_1 + \dots + k_n$. A polynomial is said to be **multilinear** if all monomials composing it have the same type of the form $(1, \dots, 1)$. Among the examples above, only the last one is multilinear.

Definition 1.3. An **identity** of an algebra A is a non-associative polynomial $\varphi = \varphi(x_1, \dots, x_n)$ such that $\varphi(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$. We say that the algebra A *satisfies* the identity φ .

Let I be a subset of $\mathbb{F}[\mathbf{FM}(S)]$ with S being a set of variables. The **variety of algebras** determined by I is the class of all algebras which satisfy all the identities in I . We say that a variety **satisfies the identities in** I if all algebras in this variety satisfy the given identities. In particular, if the variety is determined by a set of multilinear polynomials, then we say that the variety is **operadic**. If there exists a set of identities of degree 2 or 3 that generate all the identities of $\mathcal{V}$, we say that the variety is **quadratic**. Recall—see for instance [11] where this is explained in detail—that an operadic, quadratic variety of algebras over a field can be viewed as a variety determined by a quadratic operad.

Any variety of non-associative algebras can, of course, be seen as a category where the morphisms are the same as in **Alg**. In particular, any such variety is a semi-abelian category.

Remark 1.4. Whenever the characteristic of the field $\mathbb{F}$ is zero, any variety of non-associative algebras over $\mathbb{F}$ is operadic. This is due to the well-known multilinearisation process, see [30, Corollary 3.7]. The reason behind the name “operadic” is explained in [31, Section 2].

Examples 1.5.

- (1) We write **AbAlg** for the variety of *abelian* algebras determined by the identity $xy = 0$. Seen as a category, this variety is isomorphic to the category **Vec** of vector spaces over $\mathbb{F}$. It is the only non-trivial variety of non-associative algebras which is an abelian category; this explains the terminology.
- (2) We write **Assoc** for the variety of *associative* algebras determined by the identity of *associativity* which is $x(yz) - (xy)z = 0$, or equivalently $x(yz) = (xy)z$.
- (3) We write **AAssoc** for the variety of *anti-associative* algebras, determined by the *anti-associative* identity $x(yz) = -(xy)z$.
- (4) We write **Com** for the variety of *commutative* algebras determined by the identity of *commutativity* which is $xy - yx = 0$, or equivalently $xy = yx$.
- (5) We write **ACom** for the variety of *anti-commutative* algebras determined by *anti-commutativity* which is $xy + yx = 0$, or equivalently $xy = -yx$.
- (6) We write **CAssoc** for the variety of commutative associative algebras.
- (7) We write **ACAAssoc** for the variety of anti-commutative anti-associative algebras.
- (8) We write **Lie** for the variety of *Lie algebras* determined by *anti-commutativity* and the *Jacobi identity*, which respectively are $xy + yx = 0$ and $x(yz) + y(zx) + z(xy) = 0$.
- (9) One can see that all the previous examples are operadic varieties. Let us provide a non-operadic example: the variety **Bool** of *Boolean rings*, which may be seen as associative $\mathbb{Z}_2$ -algebras satisfying $xx = x$. This variety is action representable.
- (10) We write **JJord** for the variety of *Jacobi–Jordan algebras* which is determined by commutativity and the Jacobi identity. Jacobi–Jordan algebras, also known as *mock-Lie* algebras, are the commutative counterpart of Lie algebras. The name of Jordan in the definition is justified by the fact that every Jacobi–Jordan algebra is a Jordan algebra (see [7]).
- (11) We write **Leib** for the variety of (*right*) *Leibniz algebras* determined by the (*right*) *Leibniz identity* which is $(xy)z - (xz)y - x(yz) = 0$.
- (12) We write **Alt** for the variety of *alternative algebras*, which is determined by the identities $(yx)x - yx^2 = 0$ and $x(xy) - x^2y = 0$. Every associative algebra is obviously alternative and an example of an alternative algebra which is not associative is given by the *octonions* $\mathbb{O}$, that is the eight-dimensional algebra with basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ and multiplication table

$$e_i e_j = \begin{cases} e_j, & \text{if } i = 1 \\ e_i, & \text{if } j = 1 \\ -\delta_{ij}e_1 + \varepsilon_{ijk}e_k, & \text{otherwise,} \end{cases}$$

where δ_{ij} is the *Kronecker delta* and ε_{ijk} a *completely antisymmetric tensor* with value 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$. Notice that e_1 is the unit of the algebra $\mathbb{O}$.

When $\text{char}(\mathbb{F}) \neq 2$, the multilinearisation process shows that **Alt** is equivalent to the variety defined by

$$(xy)z + (xz)y - x(yz) - x(zy) = 0$$

and

$$(xy)z + (yx)z - x(yz) - y(xz) = 0.$$

- (13) Taking any variety $\mathcal{V}$, one can look at a subvariety of it by adding further identities to be satisfied. For example, let $\mathcal{V}$ be a variety determined by a set of identities I and let k be any positive natural number, then we write $\mathbf{Nil}_k(\mathcal{V})$ for the variety of *k-step nilpotent algebras in $\mathcal{V}$* determined by the identities in I and the identities of the form $x_1 \cdots x_{k+1} = 0$ with all possible choices of parentheses.

We now want to explain how we may describe actions in a variety of non-associative algebras. As we already mentioned before, in a semi-abelian category, actions are split extensions.

Definition 1.6. Let

$$0 \longrightarrow X \xrightarrow{i} A \xrightleftharpoons[s]{\pi} B \longrightarrow 0 \quad (1.2)$$

be a split extension in the variety $\mathcal{V}$. The pair of bilinear maps

$$l: B \times X \rightarrow X, \quad r: X \times B \rightarrow X$$

defined by

$$b * x = s(b)i(x), \quad x * b = i(x)s(b), \quad \forall b \in B, \forall x \in X$$

where $b * - = l(b, -)$ and $- * b = r(-, b)$, is called the *derived action* of B on X associated with (1.2).

Given a pair of bilinear maps

$$l: B \times X \rightarrow X, \quad r: X \times B \rightarrow X$$

with B, X objects of $\mathcal{V}$, we may define a multiplication on the direct sum of vector spaces $B \oplus X$ by

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + x * b') \quad (1.3)$$

with $b * x' := l(b, x')$ and $x * b' := r(x, b')$. This construction allows us to build the split extension in **Alg**

$$0 \longrightarrow X \xrightarrow{i_2} B \oplus X \xrightleftharpoons[i_1]{\pi_1} B \longrightarrow 0 \quad (1.4)$$

with $i_2(x) = (0, x)$, $i_1(b) = (b, 0)$ and $\pi_1(b, x) = b$. This is a split extension in $\mathcal{V}$ if and only if $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$, i.e. it satisfies the identities which determine $\mathcal{V}$. In other words, we have the following result analogous to [29, Theorem 2.4] and [14, Lemma 1.8]:

Lemma 1.7. *In a variety of non-associative algebras $\mathcal{V}$, given a pair of bilinear maps*

$$l: B \times X \rightarrow X, \quad r: X \times B \rightarrow X,$$

*we define the multiplication on $B \oplus X$ as above in (1.3). Then, the pair (l, r) is a derived action of B on X if and only if $(B \oplus X, \cdot)$ is in $\mathcal{V}$. In this case, we call $B \oplus X$ the **semidirect product** of B and X (with respect to the derived action) and we denote it by $B \ltimes X$.*

Remark 1.8. Notice that, for any split extension (1.2) and the corresponding derived action (l, r) , there is an isomorphism of split extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{i_2} & B \ltimes X & \xrightleftharpoons[i_1]{\pi_1} & B \longrightarrow 0 \\ & & \downarrow 1_X & & \downarrow \theta & & \downarrow 1_B \\ 0 & \longrightarrow & X & \xrightarrow{i} & A & \xrightleftharpoons[s]{\pi} & B \longrightarrow 0 \end{array}$$

where $\theta: B \ltimes X \rightarrow A: (b, x) \mapsto s(b) + i(x)$. Thus, when we write $b * x$ (resp. $x * b$), one can think of it as the multiplication $(b, 0) \cdot (0, x)$ (resp. $(0, x) \cdot (b, 0)$) in $B \ltimes X$.

1.3. Categorical consequences

Let $\mathcal{V}$ be an operadic variety of non-associative algebras. We recall two results which will be useful for understanding the rest of the paper.

Theorem 1.9 ([12, 13]). *The following conditions are equivalent:*

- (i) $\mathcal{V}$ is **algebraically coherent** [9];
- (ii) $\mathcal{V}$ is an Orzech category of interest;
- (iii) $\mathcal{V}$ is action accessible;

(iv) there exist $\lambda_1, \dots, \lambda_8, \mu_1, \dots, \mu_8$ in $\mathbb{F}$ such that

$$\begin{aligned} x(yz) &= \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3z(xy) + \lambda_4z(yx) \\ &\quad + \lambda_5(xz)y + \lambda_6(zx)y + \lambda_7y(xz) + \lambda_8y(zx) \end{aligned}$$

and

$$\begin{aligned} (yz)x &= \mu_1(xy)z + \mu_2(yx)z + \mu_3z(xy) + \mu_4z(yx) \\ &\quad + \mu_5(xz)y + \mu_6(zx)y + \mu_7y(xz) + \mu_8y(zx) \end{aligned}$$

are identities in $\mathcal{V}$. $\square$

We call the two previous identities together the λ/μ -rules. Since (WRA) implies action accessibility in general, the existence of the λ/μ -rules is a necessary condition for the variety $\mathcal{V}$ to be weakly action representable.

Theorem 1.10 ([14]). *The following conditions are equivalent:*

- (i) $\mathcal{V}$ is action representable;
- (ii) $\mathcal{V}$ is either the variety **Lie** or the variety **AbAlg**. $\square$

Theorem 1.10 helps motivating our interest in the condition (WRA). In fact, in our context, there is only one non-trivial example of a variety which is action representable. This suggests to study a generalisation of the notion of representability of actions. On the other hand, action accessibility may not be enough to study some kind of (weak) actor. The next result, which is closely related to [17, Proposition 4.5], explains one way of understanding weak action representability for any variety of non-associative algebras over a field.

Proposition 1.11. *A variety of non-associative algebras $\mathcal{V}$ is weakly action representable if and only if for any object X in it, there exists an object T of $\mathcal{V}$ such that for every derived action of an object B of $\mathcal{V}$ on X*

$$l: B \times X \rightarrow X, \quad r: X \times B \rightarrow X,$$

there exists a unique morphism $\varphi \in \text{Hom}_{\mathcal{V}}(B, T)$ and a derived action (l', r') of $\varphi(B)$ on X such that

$$l'(\varphi(b), x) = l(b, x), \quad r'(x, \varphi(b)) = r(x, b),$$

for every $b \in B$ and for every $x \in X$.

Proof. ($\Rightarrow$) If $\mathcal{V}$ is weakly action representable, then for any object X in it there exists a weak representation (T, τ) . Let B be an object of $\mathcal{V}$ which acts on X and let $\varphi: B \rightarrow T$ be the corresponding acting morphism. Consider the split extension diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & B \ltimes X & \xrightleftharpoons[s]{\pi} & B \longrightarrow 0 \\ & & \downarrow 1_X & & \downarrow \exists! f & & \downarrow \tilde{\varphi} \\ 0 & \longrightarrow & X & \xrightarrow{i'} & \varphi(B) \ltimes X & \xrightleftharpoons[s']{\pi'} & \varphi(B) \longrightarrow 0 \end{array}$$

where $\tilde{\varphi}$ is the corestriction of φ to its image, $i'(x) = (0, x)$, $s'(\varphi(b)) = (\varphi(c), 0)$, where $(c, 0) = s(b)$, and $f(b, x) = (\varphi(b), x)$. Then the action of $\varphi(B)$ on X is defined by the pair of bilinear maps

$$l': \varphi(B) \times X \rightarrow X, \quad r': X \times \varphi(B) \rightarrow X$$

where

$$l'(\varphi(b), x) = s'(\varphi(b))i'(x) = s(b)i(x) = l(b, x)$$

and

$$r'(\varphi(b), x) = i(x)s'(\varphi(b)) = i(x)s(b) = r(b, x),$$

for every $b \in B$ and for every $x \in X$ (we use [17, Proposition 4.5] to see that l' and r' are well defined).

($\Leftarrow$) Conversely, given an object X of $\mathcal{V}$, a weak representation of $\text{SplExt}(-, X)$ is given by (T, τ) , where the component

$$\tau_B: \text{SplExt}(B, X) \rightarrow \text{Hom}_{\mathcal{V}}(B, T)$$

sends every action of B on X to the corresponding morphism φ . Moreover, τ_B is an injection since the morphism φ is uniquely determined by the action of B on X . Thus τ is a monomorphism of functors. $\square$

1.4. Partial algebras

We end this chapter with a notion we shall use throughout the text.

Definition 1.12. Let X be an $\mathbb{F}$ -vector space. A *bilinear partial operation* on X is a map

$$\cdot: \Omega \rightarrow X,$$

where Ω is a vector subspace of $X \times X$, which is bilinear on Ω , i.e.

$$(\alpha_1 x_1 + \alpha_2 x_2) \cdot y = \alpha_1 x_1 \cdot y + \alpha_2 x_2 \cdot y$$

for any $\alpha_1, \alpha_2 \in \mathbb{F}$ and $x_1, x_2, y \in X$ such that $(x_1, y), (x_2, y) \in \Omega$ and

$$x \cdot (\beta_1 y_1 + \beta_2 y_2) = \beta_1 x \cdot y_1 + \beta_2 x \cdot y_2$$

for any $\beta_1, \beta_2 \in \mathbb{F}$ and $x, y_1, y_2 \in X$ such that $(x, y_1), (x, y_2) \in \Omega$.

Definition 1.13. A *partial algebra* over $\mathbb{F}$ is an $\mathbb{F}$ -vector space X endowed with a bilinear partial operation

$$\cdot : \Omega \rightarrow X.$$

We denote it by $(X, \cdot, \Omega)$. When $\Omega = X \times X$ we say that the algebra is *total*.

Let $(X, \cdot, \Omega)$ and $(X', *, \Omega')$ be partial algebras over $\mathbb{F}$. A homomorphism of partial algebras is an $\mathbb{F}$ -linear map $f: X \rightarrow X'$ such that $f(x \cdot y) = f(x) * f(y)$ whenever $(x, y) \in \Omega$, which tacitly implies that $(f(x), f(y)) \in \Omega'$ (i.e. both $x \cdot y$ and $f(x) * f(y)$ are defined). We denote by **PA**lg the category whose objects are partial algebras and whose morphisms are partial algebra homomorphisms.

Definition 1.14. We say that a partial algebra $(X, \cdot, \Omega)$ *satisfies an identity* when that identity holds wherever the bilinear partial operation is well defined.

For instance, a partial algebra $(X, \cdot, \Omega)$ is associative if

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for every $x, y, z \in X$ such that $(x, y), (y, z), (x, yz), (xy, z) \in \Omega$.

Remark 1.15. We observe that any variety of non-associative algebras $\mathcal{V}$ has an obvious forgetful functor $U: \mathcal{V} \rightarrow \mathbf{PA}lg$.

2. Commutative and anti-commutative algebras

In this section we aim to study the (weak) representability of actions of some varieties of non-associative algebras which satisfy the commutative law or the anti-commutative law. As explained in Section 1, we may assume our variety satisfies the λ/μ -rules, or equivalently is action accessible.

When $\mathcal{V}$ is either a variety of commutative or anti-commutative algebras, i.e. $xy = \varepsilon yx$ is an identity of $\mathcal{V}$, with $\varepsilon = \pm 1$, the λ/μ -rules reduce to

$$x(yz) = \alpha(xy)z + \beta(xz)y,$$

for some $\alpha, \beta \in \mathbb{F}$. The following proposition is a representation theory exercise:

Proposition 2.1. *Let $\mathcal{V}$ be a non-abelian, action accessible, operadic variety of non-associative algebras.*

- (1) *If $\mathcal{V}$ is a variety of commutative algebras, then $\mathcal{V}$ is either a subvariety of **CAssoc** or a subvariety of **JJord**.*
- (2) *If $\mathcal{V}$ is a variety of anti-commutative algebras, then $\mathcal{V}$ is either a subvariety of **Lie** or a subvariety of **ACAAssoc**. $\square$*

Remark 2.2. We observe that $\mathbf{Nil}_2(\mathbf{Com})$ is a subvariety of both **CAssoc** and **JJord**: in fact, from $x(yz) = (xy)z = 0$ we may deduce that associativity holds and the Jacobi identity is satisfied:

$$x(yz) + y(zx) + z(xy) = 0 + 0 + 0 = 0.$$

If $\text{char}(\mathbb{F}) \neq 3$, then $\mathbf{Nil}_2(\mathbf{Com})$ is precisely the intersection of the varieties **CAssoc** and **JJord**. Indeed, let $\mathcal{V}$ be a subvariety of both **CAssoc** and **JJord**. Since commutativity, associativity and the Jacobi identity hold in $\mathcal{V}$, we have

$$(xy)z = x(yz) = -y(zx) - z(xy) = -x(yz) - (xy)z = -2(xy)z$$

and thus $3(xy)z = 3x(yz) = 0$.

An example of an algebra which lies in the intersection of **CAssoc** and **JJord** but which is not two-step nilpotent is the two-dimensional $\mathbb{F}_3$ -algebra with basis $\{e_1, e_2\}$ and bilinear multiplication determined by

$$e_1^2 = e_1e_2 = e_2e_1 = e_2^2 = e_2.$$

Likewise, $\mathbf{Nil}_2(\mathbf{ACom})$ is a subvariety of both **Lie** and **ACAAssoc**: from $x(yz) = (xy)z = 0$ we may deduce anti-associativity and the Jacobi identity. If $\text{char}(\mathbb{F}) \neq 3$, then $\mathbf{Nil}_2(\mathbf{ACom})$ coincides with the intersection of the varieties **Lie** and **ACAAssoc**. Indeed, let $\mathcal{V}$ be a subvariety of both **Lie** and **ACAAssoc**. Since anti-commutativity, anti-associativity and the Jacobi identity hold in $\mathcal{V}$, we have

$$(xy)z = -x(yz) = -(xy)z - y(xz) = -(xy)z + (yx)z = -2(xy)z$$

and thus $3(xy)z = -3x(yz) = 0$.

When $\text{char}(\mathbb{F}) = 3$, it is possible to construct an algebra that lies in the intersection of **Lie** and **ACAAssoc** but which is not two-step nilpotent. Let X be the algebra of dimension 7 over $\mathbb{F}_3$ with basis

$$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

and bilinear multiplication determined by

$$e_1e_2 = -e_2e_1 = e_4, e_1e_3 = -e_3e_1 = -e_6, e_2e_3 = -e_3e_2 = e_5$$

and

$$e_1e_5 = -e_5e_1 = e_2e_6 = -e_6e_2 = e_3, e_4 = -e_4e_3 = e_7.$$

Then X is a Lie algebra such that

$$x(xy) = 0$$

for any $x, y \in X$ and, using the multi-linearisation process, one can check this identity is equivalent to anti-associativity if the characteristic of the field is different from 2. This X is not two-step nilpotent, since

$$e_1(e_2e_3) = e_1e_5 = e_7.$$

Corollary 2.3. *Let $\mathcal{V}$ be an action accessible, operadic, quadratic variety of non-associative algebras and suppose that $\mathcal{V}$ is not the variety **AbAlg** of abelian algebras.*

- (1) *If $\mathcal{V}$ is commutative, then it has to be one of the following varieties: **JJord**, **CAssoc**, their intersection, or $\mathbf{Nil}_2(\mathbf{Com})$.*
- (2) *If $\mathcal{V}$ is anti-commutative, then it has to be one of the following varieties: **Lie**, **ACAAssoc**, their intersection, or $\mathbf{Nil}_2(\mathbf{ACom})$. $\square$*

We already know that **Lie** is action representable and that the actor of a Lie algebra X is the Lie algebra $\text{Der}(X)$ of derivations of X . Therefore, we shall study the representability of actions of the varieties **CAssoc**, **JJord**, $\mathbf{Nil}_2(\mathbf{Com})$, **ACAAssoc** and $\mathbf{Nil}_2(\mathbf{ACom})$.

2.1. Commutative associative algebras

The representability of actions of the variety of commutative associative algebras over a field was studied in [4], where F. Borceux, G. Janelidze and G. M. Kelly proved that it is not action representable. We want to extend this result proving that the variety **CAssoc** is weakly action representable. In Section 3 this is further extended to general algebras over a field. We start by recalling the following result, where $U: \mathbf{CAssoc} \rightarrow \mathbf{Assoc}$ denotes the forgetful functor.

Lemma 2.4 ([4], proof of Theorem 2.6). *Let X be a commutative associative algebra. There exists a natural isomorphism of functors from $\mathbf{CAssoc}^{\text{op}} \rightarrow \mathbf{Set}$ which we denote*

$$\rho: \text{SplExt}(-, X) \cong \text{Hom}_{\mathbf{Assoc}}(U(-), \text{M}(X)),$$

where $\text{SplExt}(-, X) = \text{SplExt}_{\mathbf{CAssoc}}(-, X)$ and

$$M(X) = \{f \in \text{End}(X) \mid f(xy) = f(x)y, \quad \forall x, y \in X\}$$

is the associative algebra of multipliers of X , endowed with the product induced by the usual composition of functions (see [8, 25]). $\square$

We recall that $M(X)$ in general does not need to be a commutative algebra. For instance, let $X = \mathbb{F}^2$ be the abelian two-dimensional algebra, then $M(X) = \text{End}(X)$ which is not commutative. However there are special cases where $M(X)$ is an object of $\mathbf{CAssoc}$, such as when the *annihilator* of X (which coincides with the categorical notion of centre)

$$\text{Ann}(X) = \{x \in X \mid xy = 0, \quad \forall y \in X\}$$

is trivial or when $X^2 = X$, where X^2 denotes the subalgebra of X generated by the products xy where $x, y \in X$. We refer the reader to [8] for further details.

Theorem 2.5 ([4], Theorem 2.6). *Let X be a commutative associative algebra. The following statements are equivalent:*

- (i) $M(X)$ is a commutative associative algebra;
- (ii) the functor $\text{SplExt}(-, X)$ is representable. $\square$

Since we have examples where $M(X)$ is not commutative, we conclude that $\mathbf{CAssoc}$ is not action representable. We now want to prove that it is a weakly action representable category. We analyse what this means and then prove that the category does indeed fulfil these requirements.

For any commutative associative algebra T , the fully faithful embedding U of the category $\mathbf{CAssoc}$ into $\mathbf{Assoc}$ induces a natural isomorphism

$$i: \text{Hom}_{\mathbf{CAssoc}}(-, T) \cong \text{Hom}_{\mathbf{Assoc}}(U(-), U(T)): \mathbf{CAssoc}^{\text{op}} \rightarrow \mathbf{Set}.$$

Lemma 2.6. *If the functor $\text{SplExt}(-, X)$ admits a weak representation (T, τ) , then there exists an injective function $j: M(X) \rightarrow T$ such that for each commutative associative algebra B , the square*

$$\begin{array}{ccc} \text{SplExt}(B, X) & \xrightarrow{\tau_B} & \text{Hom}_{\mathbf{CAssoc}}(B, T) \\ \rho_B \downarrow & & \cong \downarrow i_B \\ \text{Hom}_{\mathbf{Assoc}}(U(B), M(X)) & \xrightarrow{j \circ (-)} & \text{Hom}_{\mathbf{Assoc}}(U(B), U(T)) \end{array}$$

commutes.

Proof. The free associative $\mathbb{F}$ -algebra on a single generator is the algebra $\mathbb{F}[x]$ of non-constant polynomials in a single variable x , which since it is commutative is also the free algebra on a single generator in **CAssoc**. We find j as the injective function

$$V(M(X)) \cong \text{Hom}_{\mathbf{Assoc}}(\mathbb{F}[x], M(X)) \rightarrow \text{Hom}_{\mathbf{Assoc}}(\mathbb{F}[x], U(T)) \cong V(U(T))$$

where $V(A)$ denotes the underlying set of an algebra A and the function in the middle is the $\mathbb{F}[x]$ -component of the monomorphism of functors

$$i \circ \tau \circ \rho^{-1}: \text{Hom}_{\mathbf{Assoc}}(U(-), M(X)) \rightarrow \text{Hom}_{\mathbf{Assoc}}(U(-), U(T)).$$

Now each $b \in B$ induces a morphism $b: \mathbb{F}[x] \rightarrow B$, and the collection of morphisms $(b: \mathbb{F}[x] \rightarrow B)_{b \in B}$ is jointly epic. Hence its image

$$(\text{Hom}_{\mathbf{Assoc}}(U(B), U(T)) \rightarrow \text{Hom}_{\mathbf{Assoc}}(U(\mathbb{F}[x]), U(T)))_{b \in B}$$

through the contravariant functor $\text{Hom}_{\mathbf{Assoc}}(U(-), U(T))$ is a jointly monic collection of arrows. It thus suffices that for each $b \in B$, the outer rectangle in the diagram

$$\begin{array}{ccc} \text{SplExt}(B, X) & \xrightarrow{\tau_B} & \text{Hom}_{\mathbf{CAssoc}}(B, T) \\ \rho_B \downarrow & & \downarrow \cong i_B \\ \text{Hom}_{\mathbf{Assoc}}(U(B), M(X)) & \xrightarrow{j \circ (-)} & \text{Hom}_{\mathbf{Assoc}}(U(B), U(T)) \\ (-) \circ U(b) \downarrow & & \downarrow (-) \circ U(b) \\ \text{Hom}_{\mathbf{Assoc}}(U(\mathbb{F}[x]), M(X)) & \xrightarrow{i_{\mathbb{F}[x]} \circ \tau_{\mathbb{F}[x]} \circ \rho_{\mathbb{F}[x]}^{-1}} & \text{Hom}_{\mathbf{Assoc}}(U(\mathbb{F}[x]), U(T)) \\ \cong \downarrow & & \downarrow \cong \\ V(M(X)) & \xrightarrow{j} & V(U(T)) \end{array}$$

commutes in **Set**. This is an immediate consequence of the naturality of the transformations involved. $\square$

Remark 2.7. The above proof can be modified to show that the function j is in fact a vector space monomorphism. If it were moreover an algebra monomorphism, then this would yield a proof that all $M(X)$ are commutative, which is false by the above-mentioned example. Thus we would be able to conclude that **CAssoc** is not weakly action representable. Theorem 2.11 below proves that this is wrong. Hence j cannot preserve the algebra multiplication in general.

Each action ξ of a commutative associative algebra B on X gives rise to a morphism $\rho_B(\xi): U(B) \rightarrow M(X)$ in **Assoc**. If the actions in **CAssoc** are weakly representable, then

ξ is also determined by a morphism of commutative associative algebras $\tau_B(\xi): B \rightarrow T$. The above lemma tells us that $j \circ \rho_B(\xi) = \tau_B(\xi)$. Note that here we drop the i_B for the sake of clarity.

Each $\rho_B(\xi): U(B) \rightarrow M(X)$ is the composite of a surjective commutative associative algebra map $\rho'_B(\xi): U(B) \rightarrow M_\xi$ and the canonical inclusion of a subalgebra M_ξ of $M(X)$. We find a diagram of subalgebras of $M(X)$ indexed over the commutative associative algebra actions on X . Note that since trivial actions exist, the image in **Assoc** of the diagram $(M_\xi)_\xi$ actually consists of all commutative subalgebras of $M(X)$, with the canonical inclusions between them. We may re-index and view $(M_\xi)_\xi$ as a diagram in **Assoc** over the thin category of commutative subalgebras of $M(X)$.

By the above, the M_ξ further include into T via j . For each ξ , an image factorisation of $\tau_B(\xi): B \rightarrow T$ is given by the surjective algebra map $\rho'_B(\xi): U(B) \rightarrow M_\xi$ followed by the inclusion of M_ξ into $M(X)$ composed with j . We denote this function $\mu_{M_\xi}: M_\xi \rightarrow T$ and note that it only depends on the object M_ξ . (That is to say, if ξ and ψ are two B -actions such that $M_\xi = M_\psi$, then the induced inclusions into T coincide as well.) *A priori* this μ_{M_ξ} is only an injective map, but since $\tau_B(\xi)$ and $\rho'_B(\xi)$ are morphisms of algebras and $\rho'_B(\xi)$ is a surjection, that injection is a monomorphism of commutative associative algebras. Furthermore, the μ_{M_ξ} form a cocone on the diagram of all commutative subalgebras of $M(X)$ with vertex T .

Recall that for a diagram in a category, an *amalgam* is a monic cocone, i.e. a cocone which is a monomorphic natural transformation. This means that each component of that cocone is a monomorphism, which implies that all the morphisms of the given diagram were monomorphisms to begin with. Note that in a category with colimits, an amalgam for a diagram exists if and only if its colimit cocone is such an amalgam. A category is said to have the *amalgamation property* (AP) when each span of monomorphisms admits an amalgam; equivalently, for each pushout square

$$\begin{array}{ccc} I & \xrightarrow{t} & T \\ s \downarrow & & \downarrow \iota_T \\ S & \xrightarrow{\iota_S} & S +_I T \end{array}$$

if s and t are monomorphisms then so are ι_S and ι_T . It is known that neither the category of associative algebras, nor the category of commutative associative algebras satisfies the condition (AP)—see [20] for an overview of examples and references to the rich literature on the subject.

This is as follows related to the problem at hand. The associative algebra $M(X)$ is an amalgam in **Assoc** of the diagram consisting of the commutative subalgebras M_ξ of $M(X)$. So if the functor $\text{SplExt}(-, X)$ admits a weak representation (T, τ) , then the natural transformation τ factors through the diagram $(M_\xi)_\xi$ as explained above, and we see that (T, τ) restricts to an amalgam of that diagram in the category **CAssoc**.

Thus we find a necessary condition for weak representability of actions in the category **CAssoc**: we need that for each commutative associative algebra X , the diagram $(M_\xi)_\xi$ of commutative subalgebras of the associative algebra $M(X)$ not only admits the amalgam $M(X)$ in the category **Assoc**; it should also admit an amalgam T in **CAssoc**. Actually, the converse also holds:

Proposition 2.8. *For a commutative associative algebra X , a weak representation (T, τ) of $\text{SplExt}(-, X)$ exists if and only if an amalgam in **CAssoc** exists for the diagram of commutative subalgebras of $M(X)$.*

Proof. We already explained that any weak representation of $\text{SplExt}(-, X)$ restricts to such an amalgam. So let us assume that a commutative amalgam for the diagram of commutative subalgebras of $M(X)$ exists. For each commutative associative algebra action ξ of an object B on X , we let $\tau_B(\xi): B \rightarrow T$ be the composite of $\rho'_B(\xi): U(B) \rightarrow M_\xi$ with the inclusion $\mu_{M_\xi}: M_\xi \rightarrow T$ of M_ξ into the amalgam T .

The thus defined τ is a natural transformation by the naturality of both ρ' and the cocone components in the amalgam. Note that if two maps, say $\rho_B(\xi)$ and $\rho_C(\psi)$, to $M(X)$ have the same image subalgebra $M_\xi = M_\psi$ of $M(X)$, then by naturality of ρ and the fact that the inclusion of M_ξ into $M(X)$ is a monomorphism, for any equivariant map $f: B \rightarrow C$ we have that the square on the left

$$\begin{array}{ccccc} B & \xrightarrow{\rho'_B(\xi)} & M_\xi & \xrightarrow{\mu_{M_\xi}} & T \\ f \downarrow & & \parallel & & \parallel \\ C & \xrightarrow{\rho'_C(\psi)} & M_\psi & \xrightarrow{\mu_{M_\psi}} & T \end{array}$$

commutes. The commutativity of the entire diagram proves naturality of τ .

We still have to prove that the components of τ are monomorphisms: two different actions ξ and ψ of B on X give rise to two different maps $\tau_B(\xi), \tau_B(\psi): B \rightarrow T$. Suppose, on the contrary, that $\tau_B(\xi) = \tau_B(\psi)$. Then by uniqueness of image factorisations, the images of $\mu_{M_\xi}: M_\xi \rightarrow T$ and $\mu_{M_\psi}: M_\psi \rightarrow T$ are isomorphic subobjects of T . Now the image in **Assoc** of the diagram $(M_\xi)_\xi$ is a thin category, so that $M_\xi = M_\psi$. Hence $\mu_{M_\xi} \circ \rho'_B(\xi) = \tau_B(\xi) = \tau_B(\psi) = \mu_{M_\psi} \circ \rho'_B(\psi) = \mu_{M_\xi} \circ \rho'_B(\psi)$, which implies $\rho'_B(\xi) = \rho'_B(\psi)$. But then the actions ξ and ψ are equal, since ρ is a natural isomorphism by Lemma 2.4. $\square$

Thus we see that the problem of weak representability of actions of **CAssoc** amounts to proving that an amalgam in **CAssoc** exists for the diagram of commutative subalgebras of $M(X)$ for any object X . We are actually going to prove something a bit stronger: namely, that an amalgam in **CAssoc** exists for any diagram of commutative associative

algebras for which an amalgam exists in **Assoc**. The essence of the proof is contained in the following special case.

Theorem 2.9. *If $S \leftarrow I \rightarrow T$ is a span of commutative associative algebras for which an amalgam exists in **Assoc**, then it has an amalgam in **CAssoc**.*

The proof depends on the following lemma.

Lemma 2.10. *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be morphisms in a semi-abelian category. The composite $g \circ f: X \rightarrow Z$ is a monomorphism if and only if f is a monomorphism and $\text{Im}(f) \cap \text{Ker}(g)$ is trivial.*

Proof. Note that if $g \circ f$ is a monomorphism, then so is f . So we may assume that f is a monomorphism in either case. The composite $g \circ f$ is a monomorphism precisely when $\text{Ker}(g \circ f)$ is trivial. Now this kernel is a pullback of $\text{Ker}(g)$ along f . Since f is a monomorphism, that pullback is $\text{Im}(f) \cap \text{Ker}(g)$. So $\text{Ker}(g \circ f)$ is zero if and only if $\text{Im}(f) \cap \text{Ker}(g)$ is zero. $\square$

Proof of Theorem 2.9. Let $S \leftarrow I \rightarrow T$ be such a span. Recall that an amalgam in either category exists if and only if the S and T components of the induced pushout cocone in either category are monic. We focus on the case $\iota_S: S \rightarrow S +_I T = S +_I^{\mathbf{Assoc}} T$ which we assume to be monic. The question is whether its composite with the reflection unit $\eta_{S+IT}: S +_I^{\mathbf{Assoc}} T \rightarrow S +_I^{\mathbf{CAssoc}} T$ is still monic. We are going to prove that the answer is yes, indeed it is.

Consider the following morphism of short exact sequences in **Assoc**:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & S + T & \longrightarrow & S +_I T & \longrightarrow & 0 \\ & & \downarrow \eta_{S+T}|_J & & \downarrow \eta_{S+T} & & \downarrow \eta_{S+IT} & & \\ 0 & \longrightarrow & K & \longrightarrow & S +^{\mathbf{CAssoc}} T & \longrightarrow & S +_I^{\mathbf{CAssoc}} T & \longrightarrow & 0 \end{array}$$

As a vector space, the coproduct $S + T$ of S and T in **Assoc** is $S \oplus T \oplus U$ where $U = (S \otimes T) \oplus (T \otimes S) \oplus (S \otimes T \otimes S) \oplus \cdots$ contains all the tensors. The coproduct $S +^{\mathbf{CAssoc}} T$ of S and T in **CAssoc** is $S \oplus T \oplus (S \otimes T)$, so that η_{S+T} admits a canonical splitting $\sigma: S +^{\mathbf{CAssoc}} T \rightarrow S + T$ in **Vec** which commutes with the inclusions of S , T and $S \otimes T$.

We note that K is the ideal of $S +^{\mathbf{CAssoc}} T = S \oplus T \oplus (S \otimes T)$ generated by the elements of the form $\underline{i} - \bar{i}$, where $\underline{i}$ is $i \in I$ viewed as an element of S , while $\bar{i}$ is i viewed as an element of T . Let G denote the set of generators $\{\underline{i} - \bar{i} \mid i \in I\}$. The algebra J is generated by G as well, but now as an ideal of $S + T = S \oplus T \oplus U$. It follows that $\sigma(K) \subseteq J$ —even though σ is not a morphism of algebras. We give a detailed proof of this claim. We know that K consists of all elements of the form $x_1 g_1 + \cdots + x_n g_n$

where $x_1, \dots, x_n \in S + {}^{\mathbf{CAssoc}} T$ and $g_1, \dots, g_n \in G$. Since σ is a morphism of abelian groups, it suffices that $\sigma(xg)$ belongs to J for all $x \in S + {}^{\mathbf{CAssoc}} T$ and $g \in G$. Now each $x \in S + {}^{\mathbf{CAssoc}} T$ is of the form $(s_1, t_1, s'_1 \otimes t'_1) + \dots + (s_n, t_n, s'_n \otimes t'_n)$ with $s_1, \dots, s_n, s'_1, \dots, s'_n \in S$ and $t_1, \dots, t_n, t'_1, \dots, t'_n \in T$. Hence it suffices to prove that $\sigma((s, t, s' \otimes t')g)$ belongs to J for all $s, s' \in S, t, t' \in T$ and $g \in G$. Next, we see that $(s, t, s' \otimes t') = (s, 0, 0) + (0, t, 0) + (0, 0, s' \otimes t')$. As a consequence, it suffices to prove that $\sigma((s, 0, 0)g)$, $\sigma((0, t, 0)g)$ and $\sigma((0, 0, s \otimes t)g)$ belong to J for all $s \in S$ and $t \in T$. Writing $g = (i, -i, 0) \in S + {}^{\mathbf{CAssoc}} T = S \oplus T \oplus (S \otimes T)$ and $S + T = S \oplus T \oplus U$, we calculate what happens in each of these three cases:

$$\begin{aligned} \sigma((s, 0, 0)g) &= \sigma((s, 0, 0)(i, -i, 0)) = \sigma(si, 0, -s \otimes i) = (si, 0, -s \otimes i) \\ &= (s, 0, 0)(i, -i, 0) \in J, \\ \sigma((0, t, 0)g) &= \sigma((0, t, 0)(i, -i, 0)) = \sigma(0, -it, i \otimes t) = (0, -it, i \otimes t) \\ &= (i, -i, 0)(0, t, 0) \in J, \\ \sigma((0, 0, s \otimes t)g) &= \sigma((0, 0, s \otimes t)(i, -i, 0)) = \sigma(0, 0, si \otimes t - s \otimes it) \\ &= (0, 0, si \otimes t - s \otimes it) = (si, 0, -s \otimes i)(0, t, 0) \\ &= (s, 0, 0)(i, -i, 0)(0, t, 0) \in J. \end{aligned}$$

Note that in the above calculations, we used commutativity twice: in the second equality of the second and third cases.

Thanks to Lemma 2.10, inside the vector space $S + T$, the intersection $S \cap J$ is trivial, by the assumption that the composite $\iota_S: S \rightarrow S + T \rightarrow S +_I T$ is monic. But then the smaller space $S \cap \sigma(K)$ is trivial as well, so that the composite

$$S \rightarrow S + {}^{\mathbf{CAssoc}} T \rightarrow S +_I {}^{\mathbf{CAssoc}} T$$

is a monomorphism by Lemma 2.10. $\square$

For arbitrary diagrams of monomorphisms of commutative associative algebras, the proof stays essentially the same. This allows us to conclude:

Theorem 2.11. *The category $\mathbf{CAssoc}$ of commutative associative algebras is weakly action representable.* $\square$

Remark 2.12. For a given diagram of commutative associative algebras as above, the amalgam T in $\mathbf{CAssoc}$ is also an amalgam in $\mathbf{Assoc}$.

Remark 2.13. Note that by its construction as a colimit, the weak representation (T, τ) is automatically an *initial weak representation* (see [17, Section 5]). As explained in [17, Corollary 5.3, Corollary 5.4], the existence of the initial weak representation also follows from the existence of a weak representation and the fact that $\mathbf{CAssoc}$, as a semi-abelian variety of *universal* algebras, is a total category [32].

2.2. Jacobi–Jordan algebras

We now want to study the representability of actions of the variety **JJord** of Jacobi–Jordan algebras. As already mentioned in Section 1, every split extension of B by X in **Lie** is represented by a homomorphism $B \rightarrow \text{Der}(X)$. For Jacobi–Jordan algebras, the role the derivations have in **Lie** is played by the so-called *anti-derivations*.

Definition 2.14. Let X be a Jacobi–Jordan algebra. An *anti-derivation* is a linear map $d: X \rightarrow X$ such that

$$d(xy) = -d(x)y - d(y)x, \quad \forall x, y \in X.$$

The (left) multiplications L_x for $x \in X$ are particular anti-derivations, called *inner anti-derivations*. We denote by $\text{ADer}(X)$ the space of anti-derivations of X and by $\text{Inn}(X)$ the subspace of the inner anti-derivations. Anti-derivations play a significant role in the study of cohomology of Jacobi–Jordan algebras: see [1] for further details.

We now want to make explicit what are the derived actions in the category **JJord** and how they are related with the anti-derivations. The following is an easy application of Lemma 1.7.

Proposition 2.15. Let X and B be two Jacobi–Jordan algebras. Given a pair of bilinear maps

$$l: B \times X \rightarrow X, \quad r: X \times B \rightarrow X$$

defined by

$$b * x = l(b, x), \quad x * b = r(x, b),$$

we construct $(B \oplus X, \cdot)$ as in (1.3). Then $(B \oplus X, \cdot)$ is a Jacobi–Jordan algebra if and only if

- (1) $b * x = x * b$;
- (2) $b * (xx') = -(b * x)x' - (b * x) * x'$;
- (3) $(bb') * x = -b * (b' * x) - b' * (b * x)$;

for all $b, b' \in B$ and $x, x' \in X$. $\square$

In an equivalent way, a derived action of B on X in the variety **JJord** is given by a linear map

$$B \rightarrow \text{ADer}(X): b \mapsto b * -$$

which satisfies

$$(bb') * x = -b * (b' * x) - b' * (b * x), \quad \forall b, b' \in B, \forall x \in X. \quad (2.1)$$

Remark 2.16. The vector space $\text{ADer}(X)$ endowed with the *anti-commutator*

$$\langle -, - \rangle: \text{ADer}(X) \times \text{ADer}(X) \rightarrow \text{End}(X), \quad \langle f, f' \rangle = -f \circ f' - f' \circ f$$

is not in general an algebra, since the anti-commutator of two anti-derivations is not in general an anti-derivation: in [2, Remark 2.2], the authors proved that $\langle f, f' \rangle \in \text{ADer}(X)$ if and only if

$$\langle f, f' \rangle(xy) = -f(x)f'(y) - f'(x)f(y), \quad \forall x, y \in X.$$

Moreover, it can happen that the anti-commutator $\langle -, - \rangle$ is a well defined bilinear operation on the space $\text{ADer}(X)$ but it does not define a Jacobi–Jordan algebra structure: for instance, if $X = \mathbb{F}$ is the abelian one-dimensional algebra, then $\text{ADer}(X) = \text{End}(X) \cong \mathbb{F}$ (every linear endomorphism of X is of the form $\varphi_\alpha: x \mapsto \alpha x$, for some $\alpha \in \mathbb{F}$) and the Jacobi identity is not satisfied. Nevertheless, there are some subspaces of $\text{ADer}(X)$ that are Jacobi–Jordan algebras. For instance, the subspace $\text{Inn}(X)$ of all inner anti-derivations of X . Indeed, the linear map

$$X \rightarrow \text{ADer}(X): x \mapsto L_x,$$

restricts to a Jacobi–Jordan algebra homomorphism $X \rightarrow \text{Inn}(X)$. This is true in general for the image of any linear map $B \rightarrow \text{ADer}(X)$ satisfying equation (2.1).

Thus we need to use an algebraic structure which includes the space of anti-derivations endowed with the anti-commutator and which allows us to describe categorically the representability of actions of the variety **JJord**. One possible solution is given by *partial algebras*.

Indeed, the vector space $\text{ADer}(X)$ endowed with the anti-commutator $\langle -, - \rangle$ is a commutative partial algebra. In this case Ω is the preimage

$$\langle -, - \rangle^{-1}(\text{ADer}(X))$$

of the inclusion $\text{ADer}(X) \hookrightarrow \text{End}(X)$.

Theorem 2.17. *Let X be a Jacobi–Jordan algebra and let $U: \mathbf{JJord}^{\text{op}} \rightarrow \mathbf{PAlg}$ denote the forgetful functor.*

(1) *There exists a natural isomorphism of functors from $\mathbf{JJord}^{\text{op}} \rightarrow \mathbf{Set}$*

$$\rho: \text{SplExt}(-, X) \cong \text{Hom}_{\mathbf{PAlg}}(U(-), \text{ADer}(X)),$$

where $\text{SplExt}(-, X) = \text{SplExt}_{\mathbf{JJord}}(-, X)$;

- (2) if $\text{ADer}(X)$ is a Jacobi–Jordan algebra, then the functor $\text{SplExt}(-, X)$ is representable and $\text{ADer}(X)$ is the actor of X .

Proof. (1) For a Jacobi–Jordan algebra B , we define the component

$$\rho_B: \text{SplExt}(B, X) \rightarrow \text{Hom}_{\mathbf{PAlg}}(U(B), \text{ADer}(X))$$

as the functor which sends any split extension

$$0 \longrightarrow X \xrightarrow{i} A \xrightleftharpoons[s]{\pi} B \longrightarrow 0$$

to the morphism $B \rightarrow \text{ADer}(X): b \mapsto b * -$. The transformation ρ is natural. Indeed, for any Jacobi–Jordan algebra homomorphism $f: B' \rightarrow B$, it is easy to check that the diagram in **Set**

$$\begin{array}{ccc} \text{SplExt}(B, X) & \xrightarrow{\rho_B} & \text{Hom}(U(B), \text{ADer}(X)) \\ \downarrow \text{SplExt}(f, X) & & \downarrow \text{Hom}(U(f), \text{ADer}(X)) \\ \text{SplExt}(B', X) & \xrightarrow{\rho_{B'}} & \text{Hom}(U(B'), \text{ADer}(X)) \end{array}$$

where $\text{Hom}(U(-), -) = \text{Hom}_{\mathbf{PAlg}}(U(-), -)$, is commutative. Moreover, for any Jacobi–Jordan algebra B , the morphism ρ_B is an injection, as each element of $\text{SplExt}(B, X)$ is uniquely determined by the corresponding action of B on X . Thus ρ is a monomorphism of functors. Finally ρ is a natural isomorphism since, given any Jacobi–Jordan algebra B and any homomorphism of partial algebras $\varphi: B \rightarrow \text{ADer}(X)$, the bilinear maps $l_\varphi: B \times X \rightarrow X: (b, x) \mapsto \varphi(b)(x)$, $r_\varphi = l_\varphi$ define a (unique) derived action of B on X such that $\rho_B(l_\varphi, r_\varphi) = \varphi$.

- (2) If $\text{ADer}(X)$ is a Jacobi–Jordan algebra, then by (1) we have a natural isomorphism

$$\text{SplExt}(-, X) \cong \text{Hom}_{\mathbf{JJord}}(-, \text{ADer}(X)),$$

hence $\text{ADer}(X)$ is the actor of X . $\square$

2.3. Two-step nilpotent commutative algebras

We now analyse the case where $\mathcal{V}$ is a subvariety of both **CAssoc** and **JJord**, i.e. $\mathcal{V}$ is the variety $\mathbf{Nil}_2(\mathbf{Com})$ of two-step nilpotent commutative algebras. We recall this means that $xyz = 0$ is an identity of $\mathcal{V}$. An example of such an algebra is the *Kronecker algebra* $\mathfrak{k}_1$ (see [21]), which is the three-dimensional algebra with basis $\{e_1, e_2, e_3\}$ and multiplication determined by $e_1e_2 = e_2e_1 = e_3$.

We shall show that $\mathbf{Nil}_2(\mathbf{Com})$ is an example of a weakly action representable, operadic, quadratic variety of commutative algebras.

Proposition 2.18. *Let X and B be two algebras in $\mathbf{Nil}_2(\mathbf{Com})$. Given a pair of bilinear maps*

$$l: B \times X \rightarrow X, \quad r: X \times B \rightarrow X,$$

we construct $(B \oplus X, \cdot)$ as in (1.3). Then $(B \oplus X, \cdot)$ is in $\mathbf{Nil}_2(\mathbf{Com})$ if and only if

- (1) $b * x = x * b$;
- (2) $b * (xx') = (b * x)x' = 0$;
- (3) $(bb') * x = b * (b' * x) = 0$;

for any $b, b' \in B$ and $x, x' \in X$. $\square$

The second equation of Proposition 2.18 states that, for every $b \in B$, the linear map $b * -$ belongs to the vector space

$$[X]_2 = \{f \in \text{End}(X) \mid f(xy) = f(x)y = 0, \forall x \in X\}.$$

Moreover, seeing $[X]_2$ as an abelian algebra (i.e. $\langle f, g \rangle = 0_{\text{End}(X)}$, for every $f, g \in [X]_2$), from the third equation we deduce that the linear map

$$B \rightarrow [X]_2: b \mapsto b * -$$

is an algebra homomorphism.

On the other hand, given a morphism of algebras

$$\varphi: B \rightarrow [X]_2, \quad \varphi(b) = b * -$$

satisfying

$$b * (b' * x) = 0, \quad \forall b, b' \in B, \forall x \in X,$$

we can consider the split extension

$$0 \longrightarrow X \xrightarrow{i} (B \oplus X, *_\varphi) \xrightleftharpoons[s]{\pi} B \longrightarrow 0$$

where the two-step nilpotent commutative algebra structure of $B \oplus X$ is given by

$$(b, x) *_\varphi (b', x') = (bb', xx' + b * x' + b' * x), \quad \forall (b, x), (b', x') \in B \oplus X.$$

We can now claim the following result.

Theorem 2.19.

- (1) Let B and X be two-step nilpotent commutative algebras. The isomorphism classes of split extensions of B by X are in bijection with the algebra homomorphisms

$$B \rightarrow [X]_2: b \mapsto b * -$$

satisfying

$$b * (b' * x) = 0, \quad \forall b, b' \in B, \forall x \in X. \quad (2.2)$$

- (2) The variety $\mathbf{Nil}_2(\mathbf{Com})$ is weakly action representable. A weak representation of the functor $\text{SplExt}(-, X) = \text{SplExt}_{\mathbf{Nil}_2(\mathbf{Com})}(-, X)$ is given by

$$\tau: \text{SplExt}(-, X) \rightarrow \text{Hom}_{\mathbf{Nil}_2(\mathbf{Com})}(-, [X]_2),$$

where τ_B is the injection which sends any split extension of B by X to the corresponding homomorphism $B \rightarrow [X]_2$, defined by $b \mapsto b * -$ as above.

- (3) A homomorphism $B \rightarrow [X]_2$ is an acting morphism if and only if it satisfies Equation (2.2).

Proof. (1) It follows from the analysis above.

(2) We observe that τ is a natural transformation. Indeed, for every morphism $f: B' \rightarrow B$ in $\mathbf{Nil}_2(\mathbf{Com})$, we can check that the diagram in **Set**

$$\begin{array}{ccc} \text{SplExt}(B, X) & \xrightarrow{\tau_B} & \text{Hom}(B, [X]_2) \\ \downarrow \text{SplExt}(f, X) & & \downarrow \text{Hom}(f, [X]_2) \\ \text{SplExt}(B', X) & \xrightarrow{\tau_{B'}} & \text{Hom}(B', [X]_2) \end{array}$$

is commutative. Moreover τ_B is an injection since every isomorphism class of split extensions of B by X is uniquely determined by the corresponding derived action. Thus τ is a monomorphism of functors and $\mathbf{Nil}_2(\mathbf{Com})$ is a weakly action representable category.

(3) Finally, $\varphi: B \rightarrow [X]_2$ is an acting morphism if and only if it defines a split extension of B by X in $\mathbf{Nil}_2(\mathbf{Com})$, i.e. it satisfies equation (2.2). $\square$

Let us observe that not every morphism $B \rightarrow [X]_2$ defines a split extension of B by X . For instance, if $B = \mathbb{F}\{b, b'\}$ and $X = \mathbb{F}\{x\} \cong \mathbb{F}$ are abelian algebras, then $[X]_2 = \text{End}(X)$ and the homomorphism $\varphi: B \rightarrow [X]_2$, defined by

$$\varphi(b) = \varphi(b') = 1_X$$

is not an acting morphism. Indeed,

$$\varphi(b)(\varphi(b')(x)) = 1_X(1_X(x)) = x \neq 0.$$

2.4. Anti-commutative anti-associative algebras

For the variety **ACAAssoc** of anti-commutative anti-associative algebras, a similar description of split extensions and derived actions can be made as for the variety **JJord**. The role of the anti-derivations is played here by the endomorphisms in the associative partial algebra

$$[X] := \{f \in \text{End}(X) \mid f(xy) = -f(x)y, \forall x \in X\},$$

whose bilinear partial operation is given by

$$\langle f, g \rangle = -f \circ g.$$

It is easy to see that $\langle -, - \rangle$ does not define, in general, a total algebra structure on $[X]$, nor need it be anti-commutative or anti-associative. An example is given by the abelian two-dimensional algebra $X = \mathbb{F}^2$, where $[X] = \text{End}(X)$.

We may check that a derived action of B by X in the variety **ACAAssoc** is the same thing as a partial algebra homomorphism

$$B \rightarrow [X]: b \mapsto b * -$$

which satisfies

$$(bb') * - = -b * (b' * -), \quad \forall b, b' \in B.$$

Moreover, we obtain the following result whose proof is similar to the one of Theorem 2.17.

Theorem 2.20. *Let X be an object of **ACAAssoc** and let $U: \mathbf{ACAAssoc} \rightarrow \mathbf{PAlg}$ denote the forgetful functor.*

(1) *There exists a natural isomorphism*

$$\text{SplExt}(-, X) \cong \text{Hom}_{\mathbf{PAlg}}(U(-), [X]),$$

where $\text{SplExt}(-, X) = \text{SplExt}_{\mathbf{ACAAssoc}}(-, X)$;

(2) *if $[X]$ is an anti-commutative anti-associative algebra, then the functor $\text{SplExt}(-, X)$ is representable and $[X]$ is the actor of X . $\square$*

2.5. Two-step nilpotent anti-commutative algebras

We conclude this section by studying the representability of actions of the variety $\mathbf{Nil}_2(\mathbf{ACom})$. An important example of a two-step nilpotent anti-commutative algebra is the $(2n+1)$ -dimensional *Heisenberg algebra* $\mathfrak{h}_{2n+1}$, that is the algebra with basis

$$\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$$

and non-trivial products $e_i f_j = -f_j e_i = \delta_{ij} h$, for all $i, j = 1, \dots, n$, where δ_{ij} is the Kronecker delta.

An analysis, similar to the case of two-step nilpotent commutative algebras can be made, so we simply state the following theorem:

Theorem 2.21.

- (1) *Let B and X be two-step nilpotent anti-commutative algebras. The isomorphism classes of split extensions of B by X are in bijection with the algebra homomorphisms*

$$B \rightarrow [X]_2: b \mapsto b * -$$

where $[X]_2$ is defined as in the commutative case, which satisfy the condition

$$b * (b' * x) = 0, \quad \forall b, b' \in B, \forall x \in X. \quad (2.3)$$

- (2) *The variety $\mathbf{Nil}_2(\mathbf{ACom})$ is weakly action representable. A weak representation of $\mathbf{SplExt}(-, X) = \mathbf{SplExt}_{\mathbf{Nil}_2(\mathbf{ACom})}$ is given by*

$$\tau: \mathbf{SplExt}(-, X) \rightarrow \mathbf{Hom}_{\mathbf{Nil}_2(\mathbf{ACom})}(-, [X]_2),$$

where τ_B is the injection which associates with any split extension of B by X , the corresponding homomorphism $B \rightarrow [X]_2: b \mapsto b * -$ as in (1).

- (3) *A homomorphism $B \rightarrow [X]_2$ is an acting morphism if and only if it satisfies Equation (2.3). $\square$*

Again, if $B = \mathbb{F}\{b, b'\}$ is the abelian two-dimensional algebra and $X = \mathbb{F}$ is the abelian one-dimensional algebra, the linear map $\varphi: B \rightarrow [X]_2 = \text{End}(X)$, defined by $\varphi(b) = \varphi(b') = 1_X$ is an example of a morphism in $\mathbf{Nil}_2(\mathbf{ACom})$ which is not an acting morphism.

3. Representability of actions of non-associative algebras

We want to extend the results obtained in the previous section by studying the (weak) representability of actions of a general variety of non-associative algebras over a field $\mathbb{F}$. Again, we assume that $\mathcal{V}$ is an action accessible, operadic variety of non-associative algebras over $\mathbb{F}$. Thus $\mathcal{V}$ satisfies a set of multilinear identities

$$\Phi_{k,i}(x_1, \dots, x_k) = 0, \quad i = 1, \dots, n,$$

where k is the degree of the polynomial $\Phi_{k,i}$. We fix $\lambda_1, \dots, \lambda_8, \mu_1, \dots, \mu_8 \in \mathbb{F}$ which determine a choice of λ/μ rules, i.e.

$$\begin{aligned} x(yz) &= \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3z(xy) + \lambda_4z(yx) \\ &\quad + \lambda_5(xz)y + \lambda_6(zx)y + \lambda_7y(xz) + \lambda_8y(zx) \end{aligned}$$

and

$$\begin{aligned} (yz)x &= \mu_1(xy)z + \mu_2(yx)z + \mu_3z(xy) + \mu_4z(yx) \\ &\quad + \mu_5(xz)y + \mu_6(zx)y + \mu_7y(xz) + \mu_8y(zx) \end{aligned}$$

which are identities in $\mathcal{V}$. Note that these are not unique, but fixed for our purposes.

For any object X of $\mathcal{V}$, we want to define a vector space $\mathcal{E}(X)$ such that

$$\text{Inn}(X) \leq \mathcal{E}(X) \leq \text{End}(X)^2,$$

where $\text{Inn}(X) = \{(L_x, R_x) \mid x \in X\}$ is the vector space of left and right multiplications of X , and we want to endow it with a bilinear partial operation

$$\langle -, - \rangle: \Omega \subseteq X \times X \rightarrow X,$$

such that we can associate in a natural way a homomorphism of partial algebras $B \rightarrow \mathcal{E}(X)$, with every split extension of B by X in $\mathcal{V}$. To do this, we describe derived actions in $\mathcal{V}$ in a similar fashion as in the previous section.

Proposition 3.1. *Let X and B be two algebras in $\mathcal{V}$. Given a pair of bilinear maps*

$$l: B \times X \rightarrow X, \quad r: X \times B \rightarrow X,$$

we construct $(B \oplus X, \cdot)$ as in (1.3). Then $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$ if and only if

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad \forall i = 1, \dots, n,$$

where at least one of the $\alpha_1, \dots, \alpha_k$ is an element of the form $(0, x)$, with $x \in X$, and the others are of the form $(b, 0)$, with $b \in B$. The resulting algebra is the semidirect product of B and X , denoted by $B \ltimes X$. $\square$

Using the same notation of Remark 1.8, we obtain the following:

Corollary 3.2. *When every identity of $\mathcal{V}$ can be deduced from the λ/μ rules, $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$ if and only if*

- (1) $b * (xx') = \lambda_1(b * x)x' + \dots + \lambda_8x(x' * b);$
- (2) $(xx') * b = \mu_1(b * x)x' + \dots + \mu_8x(x' * b);$
- (3) $x(x' * b) = \lambda_1(xx') * b + \dots + \lambda_8x'(b * x);$
- (4) $(x' * b)x = \mu_1(xx') * b + \dots + \mu_8x'(b * x);$

- (5) $x(b * x') = \lambda_1(x * b)x' + \cdots + \lambda_8 b * (x'x);$
- (6) $(b * x')x = \mu_1(x * b)x' + \cdots + \mu_8 b * (x'x);$
- (7) $x * (bb') = \lambda_1(x * b) * b' + \cdots + \lambda_8 b * (b' * x);$
- (8) $(bb') * x = \mu_1(x * b) * b' + \cdots + \mu_8 b * (b' * x);$
- (9) $b * (b' * x) = \lambda_1(bb') * x + \cdots + \lambda_8 b' * (x * b);$
- (10) $(b' * x) * b = \mu_1(bb') * x + \cdots + \mu_8 b' * (x * b);$
- (11) $b * (x * b') = \lambda_1(b * x) * b' + \cdots + \lambda_8 x * (b'b);$
- (12) $(x * b) * b' = \mu_1(b * x) * b' + \cdots + \mu_8 x * (b'b),$

for all $b, b' \in B$ and $x, x' \in X$. $\square$

Definition 3.3. For every object X of $\mathcal{V}$, we define $\mathcal{E}(X)$ as the subspace of all pairs $(f * -, - * f) \in \text{End}(X)^2$ satisfying

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad \forall i = 1, \dots, n,$$

for each choice of $\alpha_j = f$ and $\alpha_t \in X$, where $t \neq j \in \{1, \dots, k\}$ and $fx := f * x$, $xf := x * f$. We endow it with the bilinear map $\langle -, - \rangle: \mathcal{E}(X) \times \mathcal{E}(X) \rightarrow \text{End}(X)^2$

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h),$$

where

$$\begin{aligned} x * h &= \lambda_1(x * f) * g + \lambda_2(f * x) * g + \lambda_3 g * (x * f) + \lambda_4 g * (f * x) \\ &\quad + \lambda_5(x * g) * f + \lambda_6(g * x) * f + \lambda_7 f * (x * g) + \lambda_8 f * (g * x) \end{aligned}$$

and

$$\begin{aligned} h * x &= \mu_1(x * f) * g + \mu_2(f * x) * g + \mu_3 g * (x * f) + \mu_4 g * (f * x) \\ &\quad + \mu_5(x * g) * f + \mu_6(g * x) * f + \mu_7 f * (x * g) + \mu_8 f * (g * x). \end{aligned}$$

When every identity of $\mathcal{V}$ is a consequence of the λ/μ rules, $\mathcal{E}(X)$ becomes the subspace of all pairs $(f * -, - * f) \in \text{End}(X)^2$ satisfying

- (1) $f * (xx') = \lambda_1(f * x)x' + \cdots + \lambda_8 x(x' * f);$
- (2) $(xx') * f = \mu_1(f * x)x' + \cdots + \mu_8 x(x' * f);$
- (3) $x(x' * f) = \lambda_1(xx') * f + \cdots + \lambda_8 x'(f * x);$
- (4) $(x' * f)x = \mu_1(xx') * f + \cdots + \mu_8 x'(f * x);$
- (5) $x(f * x') = \lambda_1(x * f)x' + \cdots + \lambda_8 f * (x'x);$
- (6) $(f * x')x = \mu_1(x * f)x' + \cdots + \mu_8 f * (x'x),$

for every $x, x' \in X$.

Note that the choice of λ/μ rules does not affect to the definition of the underlying vector space of $\mathcal{E}(X)$, but it does play an important role in the bilinear map $\langle -, - \rangle$. In general, the vector space $\mathcal{E}(X)$ endowed with the bilinear map $\langle -, - \rangle$ is not an object of $\mathcal{V}$. It may happen that $\langle -, - \rangle$ does not even define a bilinear operation on $\mathcal{E}(X)$, i.e. there exist $(f * -, - * f), (g * -, - * g) \in \mathcal{E}(X)$ such that

$$\langle (f * -, - * f), (g * -, - * g) \rangle \notin \mathcal{E}(X)$$

or that $(\mathcal{E}(X), \langle -, - \rangle)$ is a non-associative algebra which does not satisfy some identity of $\mathcal{V}$.

Example 3.4. We may check that, if $\mathcal{V} = \mathbf{Assoc}$, then $\mathcal{E}(X) \cong \mathbf{Bim}(X)$ as vector spaces. Moreover, with the standard choice of λ/μ rules $\lambda_1 = \mu_8 = 1$ and the rest equal to zero, it is also an isomorphism of associative algebras.

Example 3.5. Let $\mathcal{V} = \mathbf{Leib}$, it is easy to see that $\mathcal{E}(X) \cong \mathbf{Bider}(X)$ as vector spaces. Choosing the λ/μ rules as

$$\begin{aligned} x(yz) &= (xy)z - (xz)y, \\ (yz)x &= (yx)z - y(xz), \end{aligned}$$

we get the standard multiplication defined in $\mathbf{Bider}(X)$ as in [24], that defines a weak actor in $\mathbf{Leib}$. On the other hand, choosing the λ/μ rules as

$$\begin{aligned} x(yz) &= (xy)z - (xz)y, \\ (yz)x &= (yx)z + y(zx), \end{aligned}$$

we get the non-associative algebra structure defined in [8, Definition 5.2], which, in general, is not a Leibniz algebra.

Example 3.6. If $\mathcal{V} = \mathbf{Nil}_k(\mathbf{Assoc})$, with $k \geq 3$, then

$$\mathcal{E}(X) = \{(f * -, - * f) \in \mathbf{Bim}(X) \mid f * (x_1 \cdots x_k) = (x_1 \cdots x_k) * f = 0\}.$$

With the same choice of λ/μ rules as in Example 3.4, the bilinear operation $\langle -, - \rangle$ becomes

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (f * (g * -), (- * f) * g)$$

which makes $\mathcal{E}(X)$ an associative algebra, but not a k -step nilpotent algebra. For instance, let X be the abelian one-dimensional algebra, then

$$\mathcal{E}(X) = \mathbf{End}(X) \times \mathbf{End}(X)^{\text{op}} \cong \mathbb{F}^2$$

which is not nilpotent. Every linear endomorphism of X is of the form $\varphi_\alpha: x \mapsto \alpha x$, for some $\alpha \in \mathbb{F}$ and

$$\langle (\varphi_\alpha, \varphi_\beta), (\varphi_{\alpha'}, \varphi_{\beta'}) \rangle = (\varphi_\alpha \circ \varphi_{\alpha'}, \varphi_{\beta'} \circ \varphi_\beta) = (\varphi_{\alpha\alpha'}, \varphi_{\beta\beta'}).$$

Example 3.7. If $\mathcal{V} = \mathbf{Alt}$ is the variety of alternative algebras over a field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$, then $\mathcal{E}(X)$ consists of the pairs $(f * -, - * f) \in \text{End}(X)^2$ satisfying

$$\begin{aligned} f * (xy) &= (x * f)y + (f * x)y - x(f * y), \\ (xy) * f &= x(f * y) + x(y * f) - (x * f)y, \\ x(y * f) &= (yx) * f + (xy) * f - y(x * f) \end{aligned}$$

and

$$(f * x)y = f * (yx) + f * (xy) - (f * y)x$$

for any $x, y \in X$, and the bilinear map

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h)$$

is given by

$$h * x = -(f * x) * g + f * (g * x) + f * (x * g)$$

and

$$x * h = (x * f) * g + (f * x) * g - f * (x * g).$$

One can check that $\langle -, - \rangle$ does not define an algebra structure. Nevertheless, it is possible to find examples where $\mathcal{E}(X)$ is an alternative algebra.

For instance, if X is a *unitary*, or *unital*, alternative algebra (i.e. there exists an element $e \in X$ such that $xe = ex = x$, for any $x \in X$), such as the algebra of octonions $\mathbb{O}$, then the elements of $\mathcal{E}(X)$ satisfy the following set of equations

$$\begin{aligned} f * x &= (x * f)e + (f * x)e - x(f * e), \\ x * f &= e(f * x) + e(x * f) - (e * f)x, \\ x * f &= x * f + x * f - x(e * f), \\ f * x &= f * x + f * x - (f * e)x, \end{aligned}$$

for any $x \in X$. Thus, if $\alpha := f * e$ and $\beta := e * f$, one has

$$f * x = \alpha x = \beta x, \quad x * f = x\alpha = x\beta$$

and, for $x = e$, one obtains $\alpha = \beta$. In other words, an element of $\mathcal{E}(X)$ is uniquely determined by an element $\alpha = f * e = e * f$ of X , i.e.

$$\mathcal{E}(X) \cong \{(\alpha, \alpha) \mid \alpha \in X\} \cong X$$

is an object of **Alt**.

Remark 3.8. The same result can be obtained for unitary algebras in the variety **Assoc**. In fact, let X be a unitary associative algebra and let $(f * -, - * f) \in \text{Bim}(X)$. Thus

$$\begin{aligned} f * x &= f * (ex) = \alpha x, \\ x * f &= (xe) * f = x\beta \end{aligned}$$

and

$$x\alpha = (x * f)e = x\beta,$$

where $\alpha := f * e$ and $\beta := e * f$. For $x = e$, we obtain $\alpha = \beta$ and

$$\text{Bim}(X) \cong \{(\alpha, \alpha) \mid \alpha \in X\} \cong X.$$

Since unitary algebras are perfect and have trivial centre, from [8] we have a natural isomorphism

$$\text{SplExt}(-, X) \cong \text{Hom}_{\mathbf{Assoc}}(-, X)$$

for any unitary associative algebra X , i.e. X is its own actor.

The construction of $\mathcal{E}(X)$ gives rise to an alternative characterisation of the split extensions in $\mathcal{V}$. In fact, a split extension of B by X in $\mathcal{V}$ is the same as a linear map

$$B \rightarrow \mathcal{E}(X): b \mapsto (b * -, - * b),$$

such that $((bb') * -, - * (bb')) = \langle (b * -, - * b), (b' * -, - * b') \rangle$ and

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad i = 1, \dots, n,$$

where $\alpha_1, \dots, \alpha_k$ are as in Proposition 3.1.

We remark also that the bilinear map

$$\langle -, - \rangle: \mathcal{E}(X) \times \mathcal{E}(X) \rightarrow \text{End}(X)^2$$

defines a partial operation $\langle -, - \rangle: \Omega \rightarrow \mathcal{E}(X)$, where Ω is the preimage

$$\langle -, - \rangle^{-1}(\mathcal{E}(X))$$

of the inclusion $\mathcal{E}(X) \hookrightarrow \text{End}(X)^2$.

Now we are ready to announce and prove our main result about the weak representability of actions of non-associative algebras.

Theorem 3.9. *Let $\mathcal{V}$ be an action accessible, operadic variety of non-associative algebras over a field $\mathbb{F}$ and let $U: \mathcal{V} \rightarrow \mathbf{PAlg}$ denote the forgetful functor.*

- (1) *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau: \text{SplExt}(-, X) \hookrightarrow \text{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X)),$$

where $\text{SplExt}(-, X) = \text{SplExt}_{\mathcal{V}}(-, X)$ and, for every object B of $\mathcal{V}$, τ_B is the injection which sends an element of $\text{SplExt}(B, X)$ to the corresponding partial algebra homomorphism

$$B \rightarrow \mathcal{E}(X): b \mapsto (b * -, - * b).$$

- (2) *Let B, X be objects of $\mathcal{V}$. The homomorphism of partial algebras*

$$B \rightarrow \mathcal{E}(X): b \mapsto (b * -, - * b)$$

belongs to $\text{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0$, as in Proposition 3.1.

- (3) *If $(\mathcal{E}(X), \langle -, - \rangle)$ is an object of $\mathcal{V}$, then $(\mathcal{E}(X), \tau)$ becomes a weak representation of $\text{SplExt}(-, X)$.*
- (4) *If the diagram in $\mathcal{V}$ given by those subalgebras of $\mathcal{E}(X)$ which occur as a codomain of a morphism $(B \rightarrow \mathcal{E}(X)) \in \text{Im}(\tau_B)$ admits an amalgam in $\mathcal{V}$, then the colimit of that diagram determines an (initial) weak representation of $\text{SplExt}(-, X)$.*
- (5) *If $\mathcal{V}$ is a variety of commutative or anti-commutative algebras, then $\mathcal{E}(X)$ is isomorphic to the partial algebra*

$$\{f \in \text{End}(X) \mid \Phi_{k,i}(f, x_2, \dots, x_k) = 0, \forall x_2, \dots, x_k \in X\}$$

endowed with the bilinear partial operation $\langle f, g \rangle = \alpha(f \circ g) + \beta(g \circ f)$, where $\alpha, \beta \in \mathbb{F}$ are given by the λ/μ rules.

Because of these results, we can give the following definitions.

Definition 3.10. Let X be an object of an action accessible, operadic variety of non-associative algebras $\mathcal{V}$ with a choice of λ/μ rules. The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X . The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of the functor $\text{SplExt}(-, X)$. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X .

Proof. (1) The collection $\{\tau_B\}_B$ gives rise to a natural transformation since, for every algebra homomorphism $f: B' \rightarrow B$, the diagram in **Set**

$$\begin{array}{ccc} \text{SplExt}(B, X) & \xrightarrow{\tau_B} & \text{Hom}(U(B), \mathcal{E}(X)) \\ \downarrow \text{SplExt}(f, X) & & \downarrow \text{Hom}(U(f), \mathcal{E}(X)) \\ \text{SplExt}(B', X) & \xrightarrow{\tau_{B'}} & \text{Hom}(U(B'), \mathcal{E}(X)) \end{array}$$

where $\text{Hom}(U(-), -) = \text{Hom}_{\mathbf{PAIlg}}(U(-), -)$, is commutative. Moreover, for every object B of $\mathcal{V}$, the map τ_B is an injection, since every element of $\text{SplExt}(B, X)$ is uniquely determined by the corresponding derived action of B on X , i.e. by the pair of bilinear maps

$$l: B \times X \rightarrow X, \quad r: X \times B \rightarrow X$$

defined as in Definition 1.6. Thus τ is a monomorphism of functors.

(2) Let B, X be objects of $\mathcal{V}$. A homomorphism of partial algebras $B \rightarrow \mathcal{E}(X)$ belongs to $\text{Im}(\tau_B)$ if and only if it defines a split extension of B by X in $\mathcal{V}$. This is equivalent to saying that

$$\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0, \quad \forall i = 1, \dots, n,$$

where $\alpha_1, \dots, \alpha_k$ are as in Proposition 3.1

(3) If $(\mathcal{E}(X), \langle -, - \rangle)$ is an object of $\mathcal{V}$, then we have a monomorphism of functors

$$\tau: \text{SplExt}(-, X) \hookrightarrow \text{Hom}_{\mathcal{V}}(-, \mathcal{E}(X)),$$

and $(\mathcal{E}(X), \tau)$ is a weak representation of $\text{SplExt}(-, X)$.

(4) We may copy the “if” part of the proof of Proposition 2.8, replacing the subalgebras of $M(X)$ in **CAssoc** with those subalgebras of $\mathcal{E}(X)$ in $\mathcal{V}$ which occur as codomain of a morphism $(B \rightarrow \mathcal{E}(X)) \in \text{Im}(\tau_B)$. As in Remark 2.13, by its construction as a colimit, the weak representation thus obtained is automatically an initial weak representation [17].

(5) If $\mathcal{V}$ is a variety of commutative (resp. anti-commutative) algebras, then for every object X of $\mathcal{V}$, $\mathcal{E}(X)$ consists of pairs of the form $(f * -, - * f)$ with $x * f = f * x$ (resp. $x * f = -f * x$), for every $x \in X$. Thus, an explicit isomorphism

$$\{f \in \text{End}(X) \mid \Phi_{k,i}(f, x_2, \dots, x_k) = 0\} \rightarrow \mathcal{E}(X)$$

is given by $f \mapsto (f, \pm f)$. $\square$

Example 3.11. We may check that, with the *obvious* choices of the λ/μ rules,

(1) if $\mathcal{V} = \mathbf{AbAlg}$, then $\mathcal{E}(X) = 0$ is the actor of X ;

- (2) if $\mathcal{V} = \mathbf{CAssoc}$, then $\mathcal{E}(X) \cong \mathbf{M}(X)$ is an external actor of X (see Lemma 2.4);
- (3) if $\mathcal{V} = \mathbf{JJord}$, then as observed in Theorem 2.17, the external actor $\mathcal{E}(X)$ is isomorphic to the partial algebra $\mathbf{ADer}(X)$ of anti-derivations of X ;
- (4) if $\mathcal{V} = \mathbf{Lie}$, then $\mathcal{E}(X) \cong \mathbf{Der}(X)$ is the actor of X ;
- (5) if $\mathcal{V} = \mathbf{ACAAssoc}$, then as observed in Theorem 2.20, the external actor $\mathcal{E}(X)$ is isomorphic to the partial algebra $[X]$;
- (6) if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{Com})$ or $\mathcal{V} = \mathbf{Nil}_2(\mathbf{ACom})$, then $\mathcal{E}(X) \cong [X]_2$ is a weak actor of X ;
- (7) if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{Alg})$, then $\mathcal{E}(X)$ is an abelian algebra and it is a weak actor of X ;
- (8) if $\mathcal{V} = \mathbf{Alt}$ over a field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$ and X is a unitary alternative algebra, then $\mathcal{E}(X) \cong X$ is an alternative algebra and we have a natural isomorphism

$$\text{SplExt}(-, X) \cong \text{Hom}_{\mathbf{Alt}}(-, X)$$

i.e. X is the actor of itself. In particular, the algebra of octonions $\mathbb{O}$ has representable actions.

Remark 3.12. The construction of the vector space $\mathcal{E}(X)$ can be done also in a variety of non-associative algebras $\mathcal{V}$ which is not action accessible. However, there is no canonical way to endow $\mathcal{E}(X)$ with a bilinear map $\langle -, - \rangle$ as in Definition 3.3 so we only have a monomorphism of functors

$$\tau: \text{SplExt}(-, X) \rightarrow \text{Hom}_{\mathbf{Vec}}(U(-), \mathcal{E}(X)),$$

where $U: \mathcal{V} \rightarrow \mathbf{Vec}$ denotes the forgetful functor.

Remark 3.13. As described in [10, Section 3], for every Orzech category of interest $\mathcal{C}$ and for every object X of $\mathcal{C}$, it is possible to define a monomorphism of functors

$$\mu: \text{SplExt}(-, X) \rightarrow \text{Hom}_{\mathcal{C}'}(V(-), \text{USGA}(X)),$$

where $\mathcal{C}'$ is a category which contains $\mathcal{C}$ as a full subcategory, $\text{USGA}(X)$ is an object of $\mathcal{C}'$ called the *universal strict general actor* of X [8] and $V: \mathcal{C} \rightarrow \mathcal{C}'$ denotes the forgetful functor. We further recall that $\text{USGA}(X)$ is unique up to isomorphism, once the presentation of the Orzech category of interest $\mathcal{C}$ is fixed.

For a variety of non-associative algebras $\mathcal{V}$, a presentation is given by a choice of constants $\lambda_1, \dots, \lambda_8, \mu_1, \dots, \mu_8 \in \mathbb{F}$ which determine the λ/μ rules. In this case, it turns out that $\mathcal{V}' = \mathbf{Alg}$. Thus we have monomorphism of functors

$$\mu: \text{SplExt}(-, X) \rightarrow \text{Hom}_{\mathbf{Alg}}(V(-), \text{USGA}(X))$$

and, by Theorem 3.9, another monomorphism of functors

$$\tau: \text{SplExt}(-, X) \rightarrow \text{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X)).$$

As explained at the beginning of [8, Section 4], $\text{USGA}(X)$ is the algebraic closure of the external weak actor $\mathcal{E}(X)$ with respect to the bilinear partial operation $\langle -, - \rangle$. When $\langle -, - \rangle$ is well defined on $\mathcal{E}(X) \times \mathcal{E}(X)$, then $\text{USGA}(X) = \mathcal{E}(X)$ and $\mu = \tau$.

However, it is often more convenient to work with the external weak actor $\mathcal{E}(X)$, since it is easier to construct than the universal strict general actor $\text{USGA}(X)$. In fact, in the next section we shall present the construction of $\mathcal{E}(X)$ in different varieties of non-associative algebras.

4. The quadratic case

In this section we introduce a systematic approach to finding the explicit structure of $\mathcal{E}(X)$ in the setting of operadic, quadratic varieties of algebras. Here we shall denote an element $(f * -, - * f)$ of $\mathcal{E}(X)$ by the symbol f ; this means that $fx := f * x$ and $xf := x * f$.

Let $\mathcal{V}$ be an action accessible, operadic, quadratic variety of non-associative algebras with no identities of degree 2. Let us consider the free non-associative algebra generated by the symbols f , x and y , and let us focus on its multilinear component of degree 3. There are 12 possible monomials which we order as follows:

$$\begin{aligned} f(xy) &> f(yx) > (xy)f > (yx)f > (fy)x > (fx)y \\ &> (yf)x > (xf)y > x(fy) > y(fx) > x(yf) > y(xf). \end{aligned}$$

Permuting the variables determines an action of the symmetric group S_3 on this space. For a given variety of algebras $\mathcal{V}$, we can write the orbit under the S_3 -action of its defining equations in matrix form, where each row corresponds to an equation and each column corresponds to a monomial, ordered as above. Let us denote this matrix by M_3 , and its reduced row echelon form by RM_3 . Action accessibility implies the following:

Lemma 4.1. *The rank of M_3 is at least 4. Moreover, the 4×4 minor located on the top left of RM_3 is the identity matrix. $\square$*

The vector space $\mathcal{E}(X)$ will be the subspace of $\text{End}(X)^2$ formed by the pairs that satisfy the identities coming from RM_3 .

Our task now is to endow this vector space with a partial multiplication, induced by action accessibility, and to provide strategies to check

- (1) when this multiplication is total;
- (2) when it induces a $\mathcal{V}$ -algebra structure on $\mathcal{E}(X)$.

Let us rename the tags on the columns of M_3 by the following rule: $f \mapsto x$, $x \mapsto f$ and $y \mapsto g$. Then, the third and first columns of RM_3 will give us equations of the form

$$(fg)x = \lambda_1(fg)x + \lambda_2(fx)g + \lambda_3(gf)x + \lambda_4(xf)g \\ + \lambda_5x(fg) + \lambda_6y(fx) + \lambda_7x(gf) + \lambda_8g(xf)$$

and

$$x(fg) = \mu_1(fg)x + \mu_2(fx)g + \mu_3(gf)x + \mu_4(xf)g \\ + \mu_5x(fg) + \mu_6g(fx) + \mu_7x(gf) + \mu_8g(xf).$$

At a first glance, these rules seem to yield a way of multiplying two elements f and g belonging to $\mathcal{E}(X)$. However, this choice might not be unique. If the rank of M_3 is strictly larger than 4, the lower rows will have zeroes in the first four positions, so adding any linear combination of them will produce a new bracket in $\mathcal{E}(X)$. Let us exemplify this with a concrete variety:

Example 4.2. The most common presentation of the variety of right Leibniz algebras is given by the identity $(xy)z - (xz)y - x(yz) = 0$. Then, M_3 will be the matrix

$$\begin{pmatrix} f(xy) & f(yx) & (xy)f & (yx)f & (fy)x & (fx)y & (yf)x & (xf)y & x(fy) & y(fx) & x(yf) & y(xf) \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

while its reduced row echelon form is

$$\begin{pmatrix} f(xy) & f(yx) & (xy)f & (yx)f & (fy)x & (fx)y & (yf)x & (xf)y & x(fy) & y(fx) & x(yf) & y(xf) \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Removing the rows in odd position—which we are entitled to, thanks to the obvious symmetry—we obtain that $\mathcal{E}(X)$ is formed by the elements of $\text{End}(X)^2$ satisfying the following identities:

$$\begin{aligned} f(xy) &= (fx)y - (fy)x, \\ (xy)f &= (xf)y + x(yf), \\ x(fy) &= x(yf). \end{aligned} \tag{4.1}$$

These are exactly the identities satisfied by biderivations. With the change of tag in the columns described before, we obtain

$$\begin{pmatrix} x(fg) & x(gf) & (fg)x & (gf)x & (xg)f & (xf)g & (gx)f & (fx)g & f(xg) & g(xf) & f(gx) & g(fx) \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Therefore, the multiplication

$$\begin{aligned} (fg)x &= (fx)g + f(gx) + \alpha_1(f(xg) + f(gx)) + \alpha_2(g(xf) + g(fx)) \\ x(fg) &= (xf)g - (xg)f + \beta_1(f(xg) + f(gx)) + \beta_2(g(xf) + g(fx)) \end{aligned} \quad (4.2)$$

induces a partial algebra structure on $\mathcal{E}(X)$, for any choice of $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$.

Now that we have a partial algebra structure induced on a general $\mathcal{E}(X)$, the next step is to verify when it is total. To do so, we have to focus on a partial subset of the set of consequences of degree 4. Let us consider the 120-dimensional space formed by the multilinear monomials of degree 4 in the free non-associative algebra generated by the symbols f, g, x, y . To gather all the consequences of the identities in degree 3, we have three different ways of operating. Let us take any identity from RM_3 . The first way is to multiply it from the right or from the left by g . The second way, is to substitute x by (gx) or (xg) . Finally, we can substitute y by (gy) or by (yg) . Doing all these substitutions together with the permutations of f and g , we obtain all the consequences. Note that in none of these identities the terms (fg) or (gf) will appear.

Now we need to check if the defining bracket satisfies the identities of $\mathcal{E}(X)$. To do so, we take again the identities from RM_3 , substitute f by (fg) and expand it by the already defined product. The bracket will be closed if and only if these new obtained equations are linear combination of the previously obtained consequences.

To conclude, we shall check when the bracket satisfies the identities of the variety. This can be done just by directly substituting elements of $\mathcal{E}(X)$ in the defining equations of the variety. After applying them to a generic element x , once on the left and once on the right, it is a matter of substituting the bracket on $\mathcal{E}(X)$ when necessary.

Example 4.3. Continuing with the Leibniz algebras Example 4.2, applying the procedure described before to the first equation in (4.1) yields:

$$\begin{aligned} g(f(xy)) &= g((fx)y) - g((fy)x), \\ (f(xy))g &= ((fx)y)g - ((fy)x)g, \\ f((gx)y) &= (f(gx))y - (fy)(gx), \\ f((xg)y) &= (f(xg))y - (fy)(xg), \\ f(x(gy)) &= (fx)(gy) - (f(gy))x, \\ f(x(yg)) &= (fx)(yg) - (f(yg))x. \end{aligned}$$

It is a straightforward computation to check that the rank of the matrix formed by all the consequences is 72. Then, we have to compare it with the multiplication defined in Equation (4.2). For instance, taking again the first equation in Equation (4.1) we have to expand the identity

$$(fg)(xy) = ((fg)x)y - ((fg)y)x,$$

which gives us

$$\begin{aligned} & f(g(xy)) + (f(xy))g + \alpha_1(f(g(xy)) + f((xy)g)) + \alpha_2(g(f(xy)) + g((xy)f)) \\ &= (f(gx))y + ((fx)g)y + \alpha_1((f(gx))y + (f(xg))y) + \alpha_2((g(fx))y + (g(xf))y) \\ & - (f(gy))x + ((fy)g)x + \alpha_1((f(gy))x + (f(yg))x) + \alpha_2((g(fy))x + (g(yf))x). \end{aligned}$$

After a linear algebra computation it can be checked that no matter which α_1 and α_2 we choose, it belongs to the subspace generated by the consequences. In fact, this will be true for all the identities (4.1), so any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$ will produce a total multiplication on $\mathcal{E}(X)$.

To check whether the induced bracket endows $\mathcal{E}(X)$ with a Leibniz algebra structure, we just need to check when the following identities hold

$$\begin{aligned} (f(gh))x &= ((fg)h)x - ((fh)g)x, \\ x(f(gh)) &= x((fg)h) - x((fh)g). \end{aligned}$$

A quick computation tells us that this is only true when $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1, 0, 0, 0)$, so that we recover exactly the multiplication defined in [8, Definition 5.1].

We consider some further examples.

Example 4.4. In the case of associative algebras any choice of bracket will induce a total algebra structure, but only the already known example of bimultipliers will be an associative algebra.

Example 4.5. The variety of *symmetric Leibniz algebras* is formed by the intersection between the varieties of right and left Leibniz algebras, i.e. the variety determined by $(xy)z - (xz)y - x(yz) = 0$ (right Leibniz identity) and $z(xy) - (xz)y - x(zx) = 0$ (left Leibniz identity). The space generated by its bilinear identities of degree 3 has dimension 10, which means that there are 12 parameters to define a product in $\mathcal{E}(X)$. With the help of a computer algebra system such as Macaulay2 [16] we check that any choice will give us a total algebra structure, and the set of variables that induces a symmetric Leibniz algebra structure on $\mathcal{E}(X)$ forms an affine variety of dimension 2.

Example 4.6. Following the algorithm proposed before, it can be checked easily that the variety of two-step nilpotent (non-commutative) algebras is weakly action representable. In fact, a weak actor may be given by the expected structure

$$\mathcal{E}(X) = \{f \in \text{End}(X)^2 \mid f(xy) = (xy)f = 0 = (fx)y = x(yf)\}$$

with product $fg = 0 = gf$. Nevertheless, this is not the only product that can be induced. Since the space generated by its bilinear identities of degree 3 has maximum dimension 12, there are 16 parameters that can be taken into account to define a product in $\mathcal{E}(X)$. All of them induce a total multiplication on it, and the set of parameters which induce a two-step nilpotent algebra on $\mathcal{E}(X)$ forms an affine variety of dimension 3. Note that these algebras were studied and classified in [21–23].

Example 4.7. Although commutative Poisson algebras are usually defined as a variety with two operations, in [27] it was shown that with the depolarisation technique they can be seen as a quadratic variety (with one operation), so they fit in the scope of this section. The algorithmic approach presented before shows that it is possible to induce several total algebra structures on $\mathcal{E}(X)$, more precisely it gives rise to a 3-parametric family.

Example 4.8. The varieties of Novikov algebras or anti-associative algebras do not allow a total algebra structure on their respective $\mathcal{E}(X)$ induced by action accessibility, but it is still an open problem whether or not these varieties are weakly action representable.

5. Open questions and further directions

5.1. Converse of the implication “weakly action representable category $\Rightarrow$ action accessible category”

We studied the representability of actions of a general operadic variety of non-associative algebras over a field but we were not able to find an example of an action accessible variety which is not weakly action representable. Does the converse of the implication

$$\text{weakly action representable category} \Rightarrow \text{action accessible category}$$

hold in this context?

5.2. Subvarieties

We do not know how the condition (WRA) behaves under taking subvarieties (especially in the non-quadratic case, when the degree of the identities may be higher than 3).

For instance, we know that the variety **Assoc** is weakly action representable, but we do not know whether the subvariety $\mathbf{Nil}_k(\mathbf{Assoc})$ for $k \geq 3$ satisfies the same condition. We recall that in this case, $\mathcal{E}(X)$ is an associative algebra, but it is not k -step nilpotent in general (see Example 3.6).

5.3. Initial weak representation

As already mentioned in Remark 2.13, a variety of non-associative algebras $\mathcal{V}$ is weakly action representable if and only if it is *initially weakly action representable*, which means that for every object X , the functor $\mathrm{SplExt}(-, X)$ admits an initial weak representation. We do not know whether or not the weak representations that occur in this article when the external weak actor $\mathcal{E}(X)$ is an object of $\mathcal{V}$ (such as for Leibniz algebras or associative algebras) are initial, or how we would check this in practice.

5.4. Representability of actions of unitary algebras

In the recent article [18], G. Janelidze introduced the notion of *ideally exact* category, with the aim of generalising semi-abelian categories in a way which includes relevant examples of *non-pointed* categories, such as the categories **Ring** and **CRing** of (commutative) rings with unit.

A category $\mathcal{C}$ is *ideally exact* when it is Barr-exact and Bourn-protomodular with finite coproducts, such that the unique morphism $0 \rightarrow 1$ in $\mathcal{C}$ is a regular epimorphism. Thus, semi-abelian categories are precisely the pointed ideally exact categories.

G. Janelidze also extended the notions of action representability and weak action representability to ideally exact categories, showing **Ring** and **CRing** are action representable, with the actor of a (commutative) unitary ring X being isomorphic to X itself. We do not recall the construction here, since it is essentially the same as the one for alternative algebras with unit given in Example 3.7.

We recall that a variety of non-associative algebras $\mathcal{V}$ is said to be *unitary closed* if for any object X in it, the algebra $\tilde{X}$ spanned by X and the element 1, equipped with the multiplication $x \cdot 1 = 1 \cdot x = x$ for any $x \in X$, is still an object of $\mathcal{V}$. For instance, **Assoc** and **Alt** are unitary closed, and the category **Leib**, or any variety of anti-commutative algebras over a field of characteristic different from 2, such as **Lie**, are examples of varieties which are not. Thus, the condition of being unitary closed depends on the set of identities which determine the variety $\mathcal{V}$.

When a variety of algebras $\mathcal{V}$ is unitary closed, one can consider the subcategory $\mathcal{V}_1$ of unitary algebras of $\mathcal{V}$ with the arrows being the algebra morphisms of $\mathcal{V}$ that preserve the unit. Of course, $\mathcal{V}_1$ is an ideally-exact category and it is not pointed.

Examples 3.7 and 3.8 suggest that for a unitary closed variety $\mathcal{V}$, one may use the construction of the external weak actor $\mathcal{E}(X)$ to study the representability of actions of the subcategory $\mathcal{V}_1$. For instance, it follows easily that **Alt**₁ and **Assoc**₁ are action representable, with the actor of an object X in both cases being the object X itself.

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Data availability

No data was used for the research described in the article.

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