

How should retirement policy adjust to the baby bust?

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October 10, 2000

Abstract

To cope with the observed drop in fertility, four policy options are generally considered: lowering pension benefits, increasing social security contributions, postponing retirement, and reducing public debt in advance. To assess the respective merits of these options, we analyze the optimal allocation of resources in an overlapping generations economy where old agents care about leisure. We characterize the decentralization of the optimum both when the retirement age is compulsory and when it results from a private decision. We conclude that the policy recommendation of postponing retirement is not robust to a wide class of preferences and technologies. In contrast, policies aimed at increasing capital through a reduction in public debt are more robust.

JEL Classification numbers: E62, H55, O41.

Keywords: pensions, fertility, debt, first-best policy, decentralization.

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E-mail: rillaers@ires.ucl.ac.be. The financial support of the PAI program P4/01 and the ARC program “Growth and Incentive Design” is gratefully acknowledged. We are thankful to the participants to the T2M meeting, SED 2000 and to A. d’Autume, P. Dehez, O. Galor, L. Gevers, T. Kehoe, M. Marchand, O. Paddison, P. Pestieau, V. Rios-Rull and H. Sneessens for their comments.

Introduction

Over the next half-century, the demographic structure of most of the OECD countries is expected to change significantly due to the declining fertility observed since the end of the 60's (see for example Morrow and Roeger (1999)). Aging of the EU, US and Japanese populations is therefore an inescapable fact and will result in an increase in the financial burden of public pension schemes by augmenting the dependency ratio. Although opinions differ about the extent of the problem, it is a general consensus that some adjustment is necessary. The question is how the public pension schemes should be changed.

Different policies have been suggested to cope with this demographic change. Four options to reform the current pension systems are generally considered: lowering pension benefits, increasing contribution rates, increasing national savings through an accelerated reduction of public debt, and increasing the (effective and/or legal) retirement age. Most of the literature concerned with the aging of the population adopts a numerical approach. These papers provide evidence on the consequences of the demographic change and estimate the impact of various social security reforms. However, these empirical contributions are not very conclusive relative to the optimal strategy to follow, as none of them propose a clear and precise choice criteria to rank the various policy options. Some study the effects on the macroeconomic outcomes (output, savings, taxation level, etc.), others on the actuarial fairness (ratio of present value of lifetime benefits over present value of lifetime contributions), and the others on utilities and welfare.

Our objective is to lead the debate from the design of parametric reforms to the optimal allocation of resources (consumption, leisure and capital). Endowed with the well-defined criterion of Ramsey (1928), we are able to theoretically assess the impact of a fertility drop on the optimal allocation and to determine the required policy reaction.

Moreover, in contrast with the theoretical contributions in this field, our model includes endogenous retirement decisions and is therefore suitable to assess the respective merits of all policy options mentioned above. The adequate starting point for the determination of the optimal adjustment to a demographic change is the second-welfare theorem applied to an overlapping generations model: assuming both that an optimum policy exists and that it converges to a steady state, Atkinson and Sandmo (1980) show that a first-best allocation can be achieved if the government can use lump-sum taxes that redistribute wealth among generations. Various contributions analyze the capacity of intergenerational transfers to induce optimal allocations,¹ but all these papers assume an exogenous retirement age. Endogenous retirement decisions are analyzed by Hu (1979) and Michel and Pestieau (1999). Hu (1979) assumes that the agents optimally select the share of their time in the second period of life devoted to retirement. If the transfers are tied to the individual retirement decision, the pension system introduces some distur-

¹Marchand, Michel and Pestieau (1990) study the divergence between the market and the optimal solutions in an overlapping generations model with changing productivity and fertility. They show that appropriate social security contributions, which might fluctuate over time, can induce a market allocation to be optimal. Marchand, Michel and Pestieau (1996) extend this analysis to an endogenous growth setting. When the size of the population unexpectedly declines, optimality implies some smoothing of consumptions and investment and makes more likely ascending intergenerational transfers such as unfunded social security systems. Boadway, Marchand and Pestieau (1990), Blanchet and Kessler (1991), Peters (1991) and Meijdam and Verbon (1997) also study the capacity of a pay-as-you-go social security system to reallocate resources across generations when fertility and productivity vary over time.

tions on the labor supply choice. Michel and Pestieau (1999) compare the decentralized equilibrium with the Golden Rule and show that in order to achieve the steady-state first best optimum, one needs to control both an unrestricted pay-as-you-go transfer and the retirement age.

The short-term perspective is a critical issue in this debate since the new steady state may be far away from the initial one and may take long to reach. In addition to determining the effect of a reduced fertility on the first-best long-run allocation of resources, our paper also analyzes the optimal dynamics. We provide insights concerning the effects of the decline in fertility on the welfare of the first generations and show that they crucially depend on the preferences of the agents.

The paper is organized as follows. A first section reviews the empirical literature concerning the impact of the demographic transition in various countries. The second section details the economic model and characterizes the problem of a benevolent planner. In section 3, we analyze the long-run effect of a drop in fertility. We find that the effects of a decrease in fertility on the optimal capital stock depends on the degree of substitution between the productive factors. When production factors are relatively good substitutes, the steady state capital stock will be higher after the reduction in fertility. The optimal adjustment of the retirement age is also a function of the characteristics of the production technology and of the preferences of the agents. Section 4 focuses on the dynamics along the transition path to the new steady-state and highlights the short-run effect of the demographic shock. In section 5 we investigate how the optimal allocation can be decentralized both when the retirement age is mandatory and when it results from an individual decision. In section 6 we consider the effect of an increase in longevity on the optimal allocation of resources. We conclude in section 7 that the policy recommendation of postponing the retirement age is not robust to a wide class of preferences and technologies. In contrast, policies aimed at increasing capital through a reduction in public debt are more robust.

1 Empirical evaluations of parametric pension reforms

Various empirical evaluations have been carried out to highlight the impact of demographic change and the effect of potential reforms. The methodology, the reforms considered as well as the conclusions differ substantially.

Various authors have estimated the effect of three types of corrective measures: raising contribution rates, reducing pension benefits or switching to a fully-funded scheme. Miles (1999) calibrates an overlapping generations model on European data to assess the impact of the aging of the population. Maintaining the balance of public pension schemes would require a sharp increase in the tax rate or a reduction in the replacement rate by 2050, both leading to a significant fall in savings. The impact of a complete phasing out of state pensions is also considered. This more radical option generates higher savings and benefits all future generations once the switch from unfunded pensions is completed. However, the scale of losses for the cohorts during the transition could be substantial. Chauveau and Loufir (1996) find that for Japan, the smoothing of the profile of the contribution rate by an active Fund policy outclasses a constant replacement rate policy and a constant contribution rate policy due to the low Japanese tax pressure. Kotlikoff

(1996) analyzes the macroeconomic and efficiency effects of a radical solution to the expected demographic changes: the privatization of social security. This is done in an overlapping generations model calibrated on the US economy. He concludes that efficiency gains may be substantial. Their precise size depends however on the method chosen to finance benefits during the transition and on the existing tax structure and social security system.

Other studies have considered an additional reform: a change in the retirement age. Cazes, Chauveau, Le Cacheux and Loufir (1994) study the future of the French public pension scheme using a computable general equilibrium model with overlapping generations. The analysis shows that both in terms of macroeconomic outcomes and welfare criteria, raising the legal age of retirement would clearly dominate other alternatives such as curtailing benefits or switching to a pure capitalization scheme. This option would however hurt the current working populations significantly. A computable general equilibrium model with overlapping generations has also been used by Docquier (1994) for the Belgian case. He shows that under “laissez-faire”, the reduced number of working agents would generate substantial welfare losses for some cohorts. Evaluating the impact of corrective measures, he finds that a rapid creation of a social security fund is successful in stabilizing the macroeconomic performances. However, an increase in the retirement age seems to be more desirable when intergenerational fairness is on the agenda. The financial implications of aging for the Belgian pension system are computed by de Callatay and Turtelboom (1997). In order to put pensions on a sustainable path, they recommend a combination of reform options (benefit cuts, increase in legal and effective retirement age and a harmonization of public and wage-earners schemes) that should be borne rather by relatively favored groups (such as existing pensioners and civil servants). Auerbach, Kotlikoff, Hagemann and Nicoletti (1989) calibrate an overlapping generations model for Japan, the Federal Republic of Germany, Sweden and the US to study anticipated demographic changes. They find that these changes could have major impacts on the rate of national saving, on the real wage rate, and on the current account. In the long term population aging would result in a general decline in the national saving rates of all four countries, although at different paces and with various intensities. They further study three types of policies (smoothing government expenditures, reducing benefits, and increasing the retirement age) and show that the welfare effects differ across cohorts but also across countries.²

There do not seem to exist any empirical studies investigating a fourth policy option: a reduction in national debt. This measure is however considered as a potential reform by some economists, as is the case in the US. As Greenspan advocated in March 2000 in testimony to the Congressional Special Committee on Aging, using budget surpluses to repay national debt would be best for the U.S. economy and a good way to prepare for the baby boomers’ retirement.

²A partial equilibrium approach has also been followed by Heller (1989) seeking to evaluate the implications of demographic changes for saving rates, distribution of consumption across age groups, and the level of intergenerational support for each country of the G-7. He too concludes that a general decline in the aggregate private saving rate may be expected after 2000 in all G-7 countries, although the timing may differ among them. Larger government deficits will arise from increased social security expenditures and this will put further pressure on the overall saving rate.

2 The planner's problem

We consider an overlapping generations economy à la Diamond (1965) with one physical good that can be either consumed or stored in the form of capital. Time is discrete and goes from zero to infinity. In each period we take the good as the numeraire. The households live two periods, they consume and work during these two periods. The time endowment of both periods is one. They supply labor inelastically when young. When old they get utility from leisure and they work a share λ of their time endowment, as in Hu (1979). This share $\lambda \in (0, x)$, where $x \leq 1$ represents the maximum life span in which their physical and health condition allows them to work. λ is thus proportional to the retirement age.

Households have a utility function defined over consumption when young, consumption when old, and leisure when old. It is assumed separable and takes the following form:

$$U(.) = u(c_t) + \beta u(d_{t+1}) - v(\lambda_{t+1})$$

where $\beta \in (0, 1)$ is the psychological discount factor. The instantaneous utility $u(.)$ is a strictly increasing, concave function from $\mathbb{R}_+$ to $\mathbb{R}$. The derivatives are denoted $u_c > 0$ and $u_{cc} < 0$. It is smooth on the interior of $\mathbb{R}_+$ and

$$\lim_{c \rightarrow 0} u_c(c) = +\infty. \quad (1)$$

The disutility of work $v(.)$ is a strictly increasing, convex function from $[0, x]$ to $\mathbb{R}$. It is smooth on $(0, x)$ and

$$\lim_{\lambda \rightarrow 0} v_\lambda(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow x} v_\lambda(\lambda) = +\infty.$$

The *population* grows exogenously with

$$n_t = \frac{N_t}{N_{t-1}} > 0.$$

The variable n_t denotes the growth factor of the population. In our settings, the demographic change is a once-for-all reduction in n_t . This corresponds to a drop in the fertility rate at time t : at time t , young agents represent a smaller fraction of total population. In section 6, we also consider a lengthening of the period during which old agents are physically able to work (parameter x).

Firms have a production function displaying constant returns to scale: $f(K_t, L_t)$. It is an increasing and concave function from $\mathbb{R}_+^2$ to $\mathbb{R}_+$, homogeneous of degree 1. It is smooth on the interior of $\mathbb{R}_+$. The derivatives are denoted $f_k, f_l > 0$, $f_{kk} < 0$ and $f_{kl} > 0$. When the whole labor force is employed ($L_t = N_t + \lambda_t N_{t-1}$), production is given by³

$$f(K_t, N_t + \lambda_t N_{t-1}).$$

Denoting the capital stock per old agent $k_t = \frac{K_t}{N_{t-1}}$, output per old agent is given by $y_t = f(k_t, n_t + \lambda_t)$.

³Allowing for different productivities of young and old workers would only require a slight modification of the production function. This would be given by: $f(K_t, N_t + \lambda_t N_{t-1} \theta)$, with $\theta < 1$ ($\theta > 1$) implying a negative (positive) return to seniority.

The question of the choice of the planner's discount factor is an old debate. If we assume an exogenous discount factor smaller than one to ensure the convergence of the infinite sum, the ex-ante choice of the constant discount factor is equivalent to an ex-post choice of the long-run stationary state (the so-called modified Golden Rule). ? argues that, within an utilitarian set-up, one should choose the discount rate which allows the economy to converge to the Golden Rule. This discount rate is equal to the growth rate of population, and the corresponding social objective function is the un-discounted sum of Ramsey.

Therefore, the objective function we use is the one proposed by Ramsey (1928). According to Ramsey (1928), "it is assumed that we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of imagination"; thus the optimal growth problem should not be discounted. The social planner maximizes the sum of the life-cycle utility of all current and future generations. As this social welfare objective may not be defined, we consider, as Ramsey did,

$$\sum_{t=-1}^{\infty} (U(c_t, d_{t+1}, \lambda_{t+1}) - \hat{U})$$

where $\hat{U} = \sup\{U(c, d, \lambda) \text{ under the resource constraint}\}$ is the maximum stationary utility. Ramsey assumes that this maximum utility is finite, as "economic causes alone could never give us more than a certain finite rate of enjoyment". The resource constraint of the economy is

$$f(k_t, n_t + \lambda_t) = n_t k_{t+1} + n_t c_t + d_t, \quad (2)$$

with k_0 , and c_{-1} given. c_{-1} is the hypothetical youth consumption of the first old generation. As the life-cycle utility function is separable, we can rearrange the objective function in the following way (grouping the contemporaneous terms together and ignoring the constant term $u(c_{-1})$):

$$\mathcal{W} = \sum_{t=0}^{\infty} [u(c_t) + \beta u(d_t) - v(\lambda_t) - \hat{U}]. \quad (3)$$

The planner thus maximizes $\mathcal{W}$ given an initial capital stock k_0 and given the resource constraint (2).

Definition 1 (Optimal allocation)

Given an exogenous path $(n_t)_{t \geq 0}$ and an initial capital stock $k_0 > 0$, an optimal allocation is a sequence of strictly positive quantities $(c_t, d_t, \lambda_t, k_{t+1})_{t \geq 0}$ with $\lambda_t \in (0, x)$ such that the objective function (3) is maximized subject to the resource constraint (2).

The Lagrangean $\mathcal{L}_t$ for period t is the sum of the current utilities and of the increase in the shadow value of the capital stock, $q_{t+1}k_{t+1} - q_t k_t$, i.e.

$$\mathcal{L}_t = u(c_t) + \beta u(d_t) - v(\lambda_t) + \frac{q_{t+1}}{n_t} (f(k_t, n_t + \lambda_t) - n_t c_t - d_t) - q_t k_t.$$

For an interior optimal solution, the derivatives of $\mathcal{L}_t$ with respect to c_t , d_t , λ_t and k_t must be equal to zero. Hence:

$$u_c(c_t) = q_{t+1} \quad (4)$$

$$u_c(d_t) = \frac{q_{t+1}}{\beta n_t} \quad (5)$$

$$v_\lambda(\lambda_t) = \frac{q_{t+1}}{n_t} f_l(k_t, n_t + \lambda_t) \quad (6)$$

$$\frac{q_{t+1}}{n_t} = \frac{q_t}{f_k(k_t, n_t + \lambda_t)}. \quad (7)$$

We can characterize the first-best solution as follows: an optimal allocation and its supporting implicit price $(q_t)_{t \geq 0}$ are characterized by the two dynamic equations

$$k_{t+1} = \frac{1}{n_t} \left[f(k_t, n_t + \lambda_t) - \mathcal{C} \left(\frac{q_t}{f_k(k_t, n_t + \lambda_t)} \right) \right] \equiv \Phi(k_t, q_t, \lambda_t) \quad (8)$$

$$q_{t+1} = \frac{n_t q_t}{f_k(k_t, n_t + \lambda_t)} \equiv \Psi(k_t, q_t, \lambda_t) \quad (9)$$

where total consumption $\mathcal{C}(\cdot)$ is given by

$$\mathcal{C} \left(\frac{q_t}{f_k(k_t, n_t + \lambda_t)} \right) = n_t c_t + d_t = n_t u_c^{-1} \left(\frac{n_t q_t}{f_k(k_t, n_t + \lambda_t)} \right) + u_c^{-1} \left(\frac{q_t}{\beta f_k(k_t, n_t + \lambda_t)} \right),$$

the static equation

$$v_\lambda(\lambda_t) - q_t \frac{f_l(k_t, n_t + \lambda_t)}{f_k(k_t, n_t + \lambda_t)} \equiv h(k_t, q_t, \lambda_t) = 0, \quad (10)$$

and the transversality condition.⁴

3 Long-run effect of the baby bust

Consider that the growth factor of the population is constant, i.e. $n_t = n \forall t$. Then, a steady state is a stationary path $(c_t, d_t, q_t, \lambda_t, k_{t+1}) = (c, d, q, \lambda, k)$ with positive quantities verifying the optimality conditions:

$$u_c(c) = q \quad (11)$$

$$u_c(d) = \frac{q}{\beta n} \quad (12)$$

$$v_\lambda(\lambda) = \frac{q}{n} f_l(k, n + \lambda) \quad (13)$$

$$f_k(k, n + \lambda) = n \quad (14)$$

$$f(k, n + \lambda) = nk + nc + d \quad (15)$$

Proposition 1 (Existence and Uniqueness of the optimal solution)

Under the assumption

$$\lim_{k \rightarrow 0} f_k(k, n) > n > \lim_{k \rightarrow +\infty} f_k(k, n + x)$$

the steady state of the dynamics described by (8)-(10) is characterized by the Golden Rule (14) and it is stable in the saddle-point sense.

⁴The transversality condition of this problem states that the limit of the capital stock is the Golden Rule capital stock in the case where it exists (see Michel (1990)).

See appendix A for a proof. Note that the assumption made in proposition 1 is weaker than the usual Inada conditions.⁵

3.1 Optimal capital and retirement age

In order to study the effect of a drop in fertility on the optimal age of retirement at the steady state, we compute the total derivatives of the optimality conditions (11)-(14) and the resource constraint (15) with respect to c , d , k , λ , q and n . The total derivatives of (11) and (12) allow us to express dc/c and dd/d as follows

$$\frac{dc}{c} = -\sigma(c)\frac{dq}{q} \quad \text{and} \quad \frac{dd}{d} = -\sigma(d)\left(\frac{dq}{q} - \frac{dn}{n}\right) \quad \text{where} \quad \sigma(c) = -\frac{u_c(c)}{u_{cc}(c)c}. \quad (16)$$

The term $\sigma(c)$ is the elasticity of substitution between consumption at two points of time. We also define

$$\nu \equiv \frac{v_{\lambda\lambda}\lambda}{v_\lambda} \quad (17)$$

as the elasticity of the disutility of work, and

$$\eta \equiv \frac{f_{ll}l}{f_l}, \quad \epsilon \equiv \frac{f_{kl}l}{f_k} \quad (18)$$

as the elasticities of the marginal productivity of labor and capital with respect to labor.

When substituting (16) in the total derivatives of (13), (14), and (15), we obtain a system of three equations and three unknowns, dk/dn , dq/dn , and $d\lambda/dn$. Using (17) and (18), we obtain

$$\frac{dk}{dn} \frac{n}{k} = -\frac{1}{\epsilon} + \frac{c + k - c\sigma(c) + \left(\frac{\eta}{\epsilon} + \frac{\nu}{\lambda}\right)(n c \sigma(c) + d \sigma(d))}{c + \frac{d}{n} + \nu \left(\frac{1}{\lambda} + \frac{1}{n}\right)(n c \sigma(c) + d \sigma(d))}. \quad (19)$$

The sign of which is a priori indeterminate. However, with a CES production function,

$$f(k, l) = A [\alpha k^{-\rho} + (1 - \alpha)l^{-\rho}]^{-1/\rho}, \quad A \in \mathbb{R}_+, \quad \alpha \in (0, 1), \quad \rho \in (-1, \infty) \setminus \{0\} \quad (20)$$

a rise in n causes the new steady state to display a lower k when the degree of substitutability between production factors is high (ρ low):

Proposition 2 (Population growth and optimal capital)

With a CES production function, $\frac{dk}{dn} < 0$ if $\rho \leq 0$.

See appendix B for a proof. The parameter ρ determines the degree of substitutability between the production factors. The smaller ρ , the larger the degree of substitutability and the more a decrease in fertility and thereby in labor supply will be accompanied by a rise in capital: labor is replaced by capital: $dk/dn < 0$ (Note that proposition 2 applies to a Cobb-Douglas production function ($\rho \rightarrow 0$)).

When capital and labor are poor substitutes (when ρ is large) however, a decrease in n might be accompanied by a decrease in k :

⁵This will allow us to provide examples with CES production functions.

Corollary 1 *For large ρ and small $\sigma (\rightarrow 0)$ we have $dk/dn > 0$*

See appendix C for a proof. When ρ is very large, a decrease in n is likely to be accompanied by a decrease in k ($dk/dn > 0$). The opposite result may only arise if labor input has risen, which implies that λ has increased so much that it outweighs the drop in n . This only occurs if agents are ready to substitute consumption for leisure, which is the case when σ is large. When σ is small, agents do not accept consumption as a substitute for leisure and so k will move in the same direction as n .

We now use the total derivative of equation (14) in order to compute $d\lambda/dn$:

Proposition 3 (Population growth and optimal retirement age)

The effect of a drop in fertility on the optimal retirement age at the steady state is given by

$$\frac{d\lambda}{dn} = \left(-1 + \frac{1}{f_{kl}} \right) + \frac{n + \lambda}{n} \frac{dk}{dn} \frac{n}{k}, \quad (21)$$

The total effect of fertility drop consists of two different effects. The first term in (21) constitutes the direct effect of a change in n , while the second term represents the effect induced by a change in n through a subsequent change in k , as given by (19).

The intuition behind the direct term can be understood by looking at the Golden Rule (equation (14)). On the one hand, a drop in fertility reduces the optimal marginal productivity of capital. On the other hand, a drop in n directly diminishes the marginal productivity of capital through its effect on labor input. The higher f_{kl} , the larger this second effect. If $f_{kl} > 1$, the marginal productivity decreases too much and the retirement age must increase in order to keep it at a sufficiently high level. On the contrary, if a change in the labor supply only slightly affects the marginal productivity of capital ($f_{kl} < 1$), the retirement age should decrease to bring the marginal productivity to its optimal level. If $f_{kl} = 1$ and if a change in n is not accompanied by a change in k ($dk/dn = 0$), the retirement age should remain unchanged.

A change in n , however, will also induce a change in steady state capital as given by (19). The second term in (21) depends on the elasticity of capital per old to population growth. The effect on λ of a change in n through this elasticity always goes in the same direction as dk/dn . Indeed, as the marginal productivity of labor rises with capital, an increase in capital input gives an incentive (for the planner) to increase the labor input through a rise in the retirement age. The relative importance of these two effects will finally determine the sign of $d\lambda/dn$. The value of σ plays an important role in the balance of these two effects.

Corollary 2 *For large σ we have $d\lambda/dn < 0$*

See appendix D for a proof. As mentioned before a large σ implies that agents easily substitute consumption for leisure. They are thus ready to give up leisure in order to safeguard consumption when fertility declines.

3.2 Numerical examples

To further understand the role of some key parameters in the adjustment to a drop in fertility, we examine the following examples. Assume a utility function with constant elasticity of substitution:

$$u(c) = (1 - \sigma^{-1})^{-1} c^{1-\frac{1}{\sigma}}, \quad \sigma \in (0, \infty) \setminus \{1\}$$

and the following disutility:

$$v(\lambda) = -\frac{\ln(x - \lambda) + \lambda/x}{2}$$

which satisfies $v_\lambda(0) = 0$ and $v_\lambda(x) = \infty$. Assume a CES production function as (20). We choose the following values for the parameters: $\beta = 0.2727$, $\alpha = .3$, $A = 2.5$, $x = 1$ and we consider two cases: case 1: $\sigma = 2$ (high σ), $\rho = .001$ (Cobb-Douglas production); and case 2: $\sigma = .1$ (low σ), $\rho = .001$ (Cobb-Douglas production). Case 1 and 2 highlight the role of σ . They both illustrate proposition 2: the optimal capital stock increases when fertility declines.

In this section we concentrate on steady state comparative statics, namely the long-term adjustment to a fertility drop (n decreases from 1.4 to 1.1 – a large drop for the clarity of the graphical presentation). Dynamic adjustments are studied in the next section (with n going from 1.2 to 1.1).

Let us consider figure 1. In the space of commodities $\{c, d, 1 - \lambda\}$, we have represented indifference curves together with two resource constraints corresponding to two different values of n . The top chart corresponds to case 1 and the bottom chart to case 2.

Consider first the resource constraints (represented by the flat surfaces in figure 1): $f(k, n + \lambda) - n(k + c) - d$. They are evaluated with the capital stock set at its optimal level. Hence, they represent the feasible combinations of d, c and λ for this optimal level of k . The drop in n moves the feasibility surface to the right: the resource constraint is less stringent with fewer agents in the economy. Indeed, a slower population growth reduces the investment requirements to keep k , per capita capital of the old, constant, thus increasing the feasible consumption level for a given capital-output ratio. In addition, this “consumption dividend” has to be shared among a smaller number of young agents. This is called the “capital-thickening effect” (see Cutler, Poterba, Sheiner and Summers (1990) and Meijdam and Verbon (1997)). A closer look at the slope of the two surfaces allows us to notice that the shift favors consumption when young relative to consumption when old. Since the drop in n reduces the number of young agents, it is easier to increase per capita consumption of the young.⁶ Moreover, the shift is unfavorable to leisure. The productivity of workers is increased by the drop in n , and early retirement is therefore more costly.

Consider now the indifference curves. A large value of σ given ν implies that the three commodities are substitutes in the utility function (Case 1). The best way to benefit from the new feasibility constraint is to decrease leisure and consumption when old, and to increase consumption when young. It is thus optimal to increase the retirement age. On

⁶A fall in the population growth rate has the same effect as a decrease of the interest rate in a competitive economy: it reduces the social return to postpone consumption.

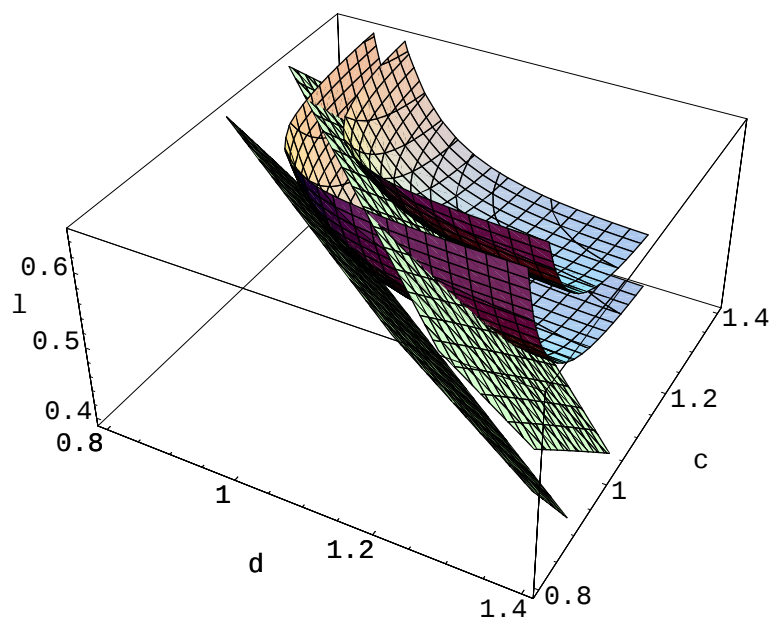
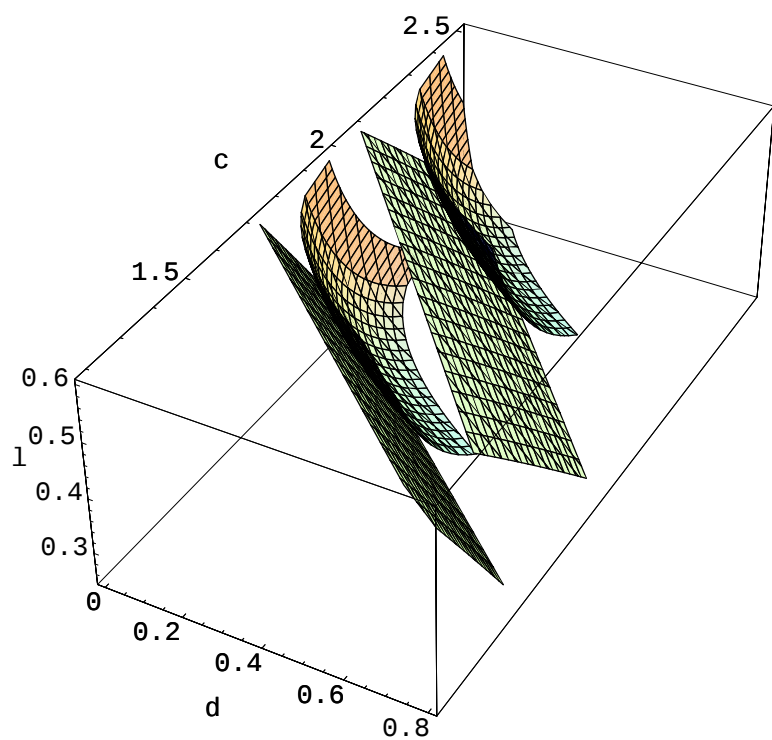


Figure 1: Cases 1 and 2

the contrary, when σ is low, the commodities are complements in the utility function (Case 2). It is no longer optimal for the planner to substitute young-age consumption for leisure and old-age consumption. The only way to benefit from the new feasibility constraint is to increase simultaneously the three commodities. In this case, the retirement age should decrease.

The conclusions of our analysis of the optimal allocation can be summarized as follows. First, except when parameters take extreme values, — this is when substitutability between capital and labour on the one hand, and between consumption when young and when old on the other one is small (Corollary 1)—, the optimal stock of capital increases after a drop in population growth. This enhances the production possibilities of the economy. A drop in fertility constitutes therefore a positive shock that allows the planner to increase per-capita consumption. Second, from Proposition 3 and Figure 1, the optimal response of the retirement age crucially depends on the parameters, and in particular on the concavity of the utility function. When consumption and leisure are highly substitutable, the optimal retirement age should increase. On the contrary, when they are low substitutes, the optimal retirement age should decrease.

4 Optimal dynamics with an anticipated baby bust

The previous section focused on the long-run optimal adjustment to the baby bust. However, the new steady-state may be far away from the initial one and long to reach. Section 3 does not provide insights concerning the effects of this shock during the transition and hence on the welfare of the first generations, which actually matters in the current policy debate. In the present section we study the optimal dynamics. For the sake of reality, the baby bust is supposed to be anticipated by one period. Indeed, in this theoretical framework individuals are only represented when they have reached the working age. It is therefore realistic to assume that the shock affecting the economy in period 2 is observed and anticipated today (period 1).

The dynamics of the system around the steady state can be described using a phase diagram. To build this diagram we use the two equations of the dynamic system (8)-(9):

$$\begin{aligned} k_{t+1} &= \Phi(k_t, q_t, \lambda_t) \\ q_{t+1} &= \Psi(k_t, q_t, \lambda_t) \end{aligned}$$

where λ_t is defined by a function Ω applying the implicit function theorem to (10):

$$\lambda_t = \Omega(k_t, q_t) \Leftrightarrow h(k_t, q_t, \lambda_t) = 0$$

The set of points (k_t, q_t) for which there is no change in k_t is characterized by $\Phi(k_t, q_t, \Omega(k_t, q_t)) = k_t$. By totally differentiating this equation, one can show that this phase line is downwards-sloping around the steady state. This result is proven in appendix E. To describe the direction of a change in k_t , we notice that $\Phi(\cdot)$ increases monotonically with q_t . Hence, $k_{t+1} > k_t$ above the curve and $k_{t+1} < k_t$ below. The set of points (k_t, q_t) for which there is no change in q_t is characterized by $\Psi(k_t, q_t, \Omega(k_t, q_t)) = q_t$. At the steady state the slope of this phase-line is positive. This is shown in appendix E. To describe the direction of a change in q_t , we notice that $\Psi(\cdot)$ increases unambiguously with k_t . Hence, $q_{t+1} < q_t$ at

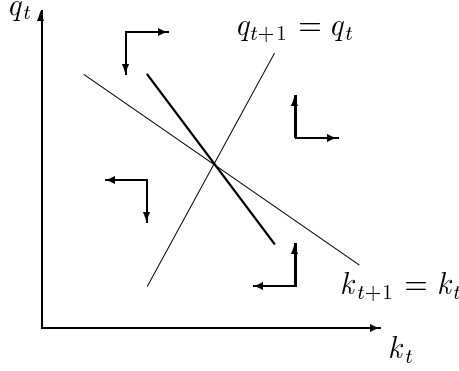


Figure 2: The phase diagram.

the left of the phase-line and $q_{t+1} > q_t$ at the right of it. We gather the information in figure 2. The intersection of the two phase-lines represents the steady-state. This figure illustrates that there is only one trajectory converging to the steady-state (represented by the bold line in figure 2).

The dynamic adjustments for case 1 and 2 are represented in figures 3 and 4. The solid and dotted lines represent the phaselines before and after the shock respectively. Bold dots represent the optimal path. Time 0 is the initial steady state. At time 1 the shock is anticipated and it takes place at time 2. The thin arrow indicates the new saddle-path. A planner anticipating (in period 1) a fertility drop for the next period (period 2) will change the allocation in period 1 so as to be on the new saddle-path in the next period.

The major difference between these two figures is the direction of the shift of the line $k_{t+1} = k_t$. This line shifts upwards in case 1 (σ high) and downwards in case 2 (σ small).⁷ The saddle-path is therefore located above the initial line $k_{t+1} = k_t$ in case 1 and below in case 2. This difference has major short-term consequences for the evolution of q , hence for the evolution of young- and old-age consumption in the first periods following the baby bust. Indeed, in the period preceding the drop in fertility (period 1), n is still unchanged, k is still at its initial steady-state level and an increase (decrease) in q decreases (increases) both consumption when young and old (see equations (4), (5) and (7)).

The economy may thus display different dynamics, depending on the value of σ . In case 1, the optimal long-run adjustment consists of reducing old-age consumption and leisure and increasing youth consumption. However, as can be noticed in figure 3, q is higher than its initial value in period 1 and 2. Hence, in period 1 (n still unchanged), c and d must be decreased. In period 2, the drop in fertility permits the planner to increase consumption of young, while old-age consumption is still lower than initially. However, this consumption squeeze fosters capital accumulation and allows for a rapid convergence to the higher long-term capital stock. Consequently, not only current old individuals but

⁷The line $k_{t+1} = k_t$ represents the resource constraint of the economy, where c and d have been substituted by their optimal value. The higher σ , the greater the willingness of households to accept a reduction in old-age consumption in compensation of an increase in young-age consumption, and the less stringent is the resource constraint.

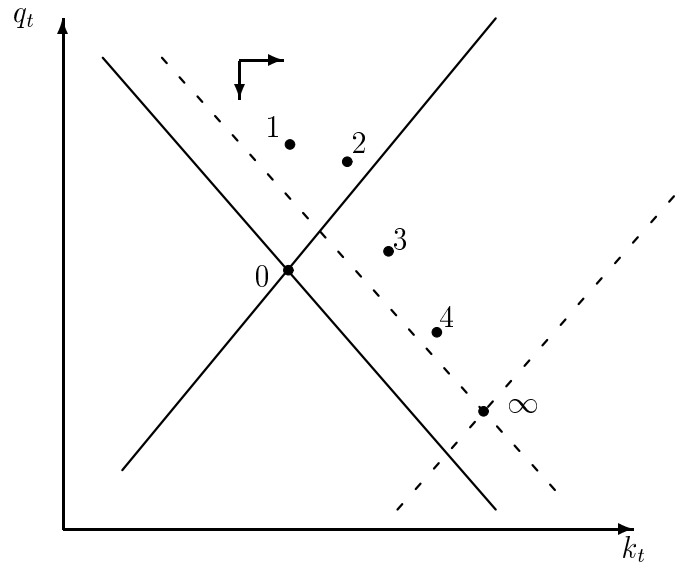


Figure 3: Dynamics - Case 1.

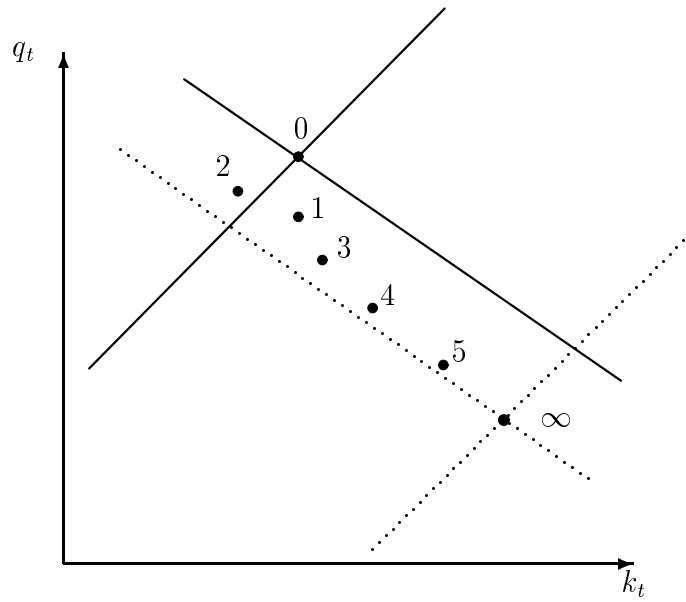


Figure 4: Dynamics - Case 2.

also currently young individuals (in the two periods of their life) will bear the cost of the rapid transition to the new steady state to the benefit of all future generations.

The story is different in case 2. It is now optimal for the planner to increase the long-run values of the three commodities. The planner therefore increases consumption of individuals in each period of their life so as to keep the ratio of consumptions optimal. In addition, the planner uses the capital freed by the “capital-thickening effect” to increase consumption in period 1 (reflected by the drop in q). This temporarily reduces the capital accumulation and puts the economy on the new saddle path. Along this path, the adjustment is slower than in the first case. Both periods consumption rise at the pace allowed by capital accumulation. In this case, no generation loses in the transition process.

The analysis of the optimal dynamics highlights that the welfare effects of a fertility drop on the initial generations crucially depend on the parameters, and in particular on the concavity of the utility function. When consumption and leisure are highly substitutable, the transition is rapid but induces a loss for the two initial generations. When they are poor substitutes, every generation is better off. In the next section we investigate how this optimal allocation can be decentralized by means of policy instruments.

5 Decentralization of the optimal trajectory

As stressed by Michel and Pestieau (1999), a steady-state first-best optimum can be decentralized if one controls the social security tax and the retirement age. In the following first sub-section, we generalize their result to non-steady-state allocations. In the next sub-section, we consider the case where the retirement age is not mandatory. The decentralization then requires another instrument, e.g. public debt.

5.1 Decentralization with mandatory retirement age

When the retirement age is mandatory, individuals do not choose the end of their working period. The decentralization of the first best solution aims at inducing the level of savings that corresponds to the optimal stock of capital. For this objective, one intergenerational lump-sum transfer is sufficient. We assume that the government taxes the wage income proportionally and provides the old with a lump-sum transfer. As the retirement age is mandatory, the proportional tax is not distortionary and amounts to a lump-sum transfer. The maximization program of the individual is:

$$\max_{c_t, d_{t+1}} u(c_t) + \beta u(d_{t+1}) - v(\lambda_{t+1})$$

$$\text{subject to} \quad c_t + s_t = w_t(1 - \tau_t) \quad (22)$$

$$d_{t+1} = R_{t+1}s_t + p_{t+1} + w_{t+1}(1 - \tau_{t+1})\lambda_{t+1} \quad (23)$$

where $\tau_t < 1$ is the tax rate, $p_{t+1} \in \mathbb{R}$ is the lump-sum transfer, s_t represents savings and R_{t+1} is the interest factor. w_t and w_{t+1} denote wages. The maximization problem has a

solution if the life-cycle income is positive, i.e.

$$w_t(1 - \tau_t) + \frac{p_{t+1} + w_{t+1}\lambda_{t+1}(1 - \tau_{t+1})}{R_{t+1}} > 0.$$

The first order condition of the maximization problem is

$$u_c(c_t) = \beta R_{t+1} u_c(d_{t+1})$$

which allows to define a saving function

$$s_t = s(w_t(1 - \tau_t), p_{t+1} + w_{t+1}\lambda_{t+1}(1 - \tau_{t+1}), R_{t+1}) \quad (24)$$

with its derivatives satisfying $s_1 \in (0, 1)$, $s_2 \in (-1, 0)$ and $s_3 \in \mathbb{R}$.

The competitive behavior of firms leads to the equalization of marginal productivities to marginal costs:

$$R_t = f_k(k_t, n_t + \lambda_t) \quad (25)$$

$$w_t = f_l(k_t, n_t + \lambda_t). \quad (26)$$

The budget constraint of the government is:

$$p_t = \tau_t w_t(n_t + \lambda_t) \quad (27)$$

and the equilibrium condition in the capital market implies

$$k_{t+1} = s(w_t(1 - \tau_t), p_{t+1} + w_{t+1}\lambda_{t+1}(1 - \tau_{t+1}), R_{t+1}). \quad (28)$$

Definition 2 (Competitive equilibrium with mandatory retirement)

Assume an exogenous path $(n_t)_{t \geq 0}$, an initial capital stock $k_0 > 0$, a mandatory retirement age $(\lambda_t)_{t \geq 0}$ and a transfer system $(\tau_t, p_t)_{t \geq 0}$ satisfying (27). A competitive, perfect-foresight, inter-temporal equilibrium is a vector $(c_t, d_t, k_t, s_t, R_t, w_t)_{t \geq 0}$ starting at k_0 and satisfying the conditions (22)–(26) and (28).

Proposition 4 (Decentralization with mandatory retirement age)

For any optimal allocation with positive quantities $(c_t^*, d_t^*, \lambda_t^*, k_t^*)_{t \geq 0}$ starting at k_0 , there exists a transfer system $(\tau_t, p_t)_{t \geq 0}$ satisfying (27) such that this trajectory is an inter-temporal equilibrium with perfect foresight and mandatory retirement age λ_t^* . The sequences $(\tau_t, p_t)_{t \geq 0}$ satisfy:

$$p_t = (n_t + \lambda_t^*)(f_l(k_t^*, n_t + \lambda_t^*) - k_{t+1}^* - c_t^*) \quad (29)$$

$$\tau_t = \frac{f_l(k_t^*, n_t + \lambda_t^*) - k_{t+1}^* - c_t^*}{f_l(k_t^*, n_t + \lambda_t^*)} \quad (30)$$

Proof: See appendix F for a proof. As the retirement age is mandatory, labor taxes are lump-sum taxes, and can be used to set the level of savings so as to obtain the optimal stock of capital. Since the individual allocation rule of consumption over the life cycle is the same as the optimal one, there is no need for another instrument to allocate consumption optimally.

5.2 Decentralization with free retirement age

When the retirement age is chosen by the households, their maximization program becomes

$$\max_{c_t, d_{t+1}, \lambda_{t+1}} u(c_t) + \beta u(d_{t+1}) - v(\lambda_{t+1})$$

and there is an additional first-order condition:

$$\beta u_c(d_{t+1}) w_{t+1} (1 - \tau_{t+1}) = v_l(\lambda_{t+1}). \quad (31)$$

As mentioned above, the decentralization requires another instrument. The government now runs a public debt. Its budget constraint is:

$$R_t b_t + p_t = \tau_t w_t (n_t + \lambda_t) + n_t b_{t+1} \quad (32)$$

where b_t is the debt per old agent B_t/N_{t-1} . The debt is held by households. It thus diverts part of private savings from productive capital:

$$b_{t+1} + k_{t+1} = s(w_t(1 - \tau_t), p_{t+1} + w_{t+1} \lambda_{t+1}(1 - \tau_{t+1}), R_{t+1}) \quad (33)$$

Proposition 5 (Decentralization with free retirement age)

For any optimal allocation with positive quantities $(c_t^, d_t^*, \lambda_t^*, k_t^*)_{t \geq 0}$ starting at k_0 there exists a sequence of proportional wage taxes $(\tau_t)_{t \geq 0}$, transfers $(p_t)_{t \geq 0}$ and public debt $(b_{t+1})_{t \geq 0}$ such that this trajectory is an inter-temporal equilibrium with perfect foresight. The sequence $(\tau_t)_{t \geq 0}$, $(p_t)_{t \geq 0}$ and $(b_{t+1})_{t \geq 0}$ satisfy:*

$$\tau_t = 0 \quad (34)$$

$$b_{t+1} = \frac{d_t^* - f_l(k_t^*, n_t + \lambda_t^*) \lambda_t^* - f_k(k_t^*, n_t + \lambda_t^*) k_t^*}{n_t} \quad (35)$$

$$p_t = n_t b_{t+1} - f_k(k_t^*, n_t + \lambda_t^*) b_t \quad (36)$$

b_0 being given.

Proof: See appendix G for a proof. As the planner's optimality condition related to labor supply is the same as the individual one when the Golden Rule holds, any non-zero labor tax would distort the individual retirement choice. Debt⁸ can be used to obtain the optimal stock of capital and lump-sum transfers to the old balance the budget or vice-versa. Note that, at steady state, the optimal transfer is zero since the golden rule $f_k = n$ holds (See equation (36)).

5.3 Optimal policies

We now consider the effect of a drop in fertility on the optimal policy. Qualitative changes in policy instruments are represented in table 1 for our two examples. When the retirement age is mandatory, transfers and taxes decrease in case 1 and increase in case 2. Although

⁸A negative debt is not excluded and corresponds to a situation where the government detains a part of the productive capital.

Table 1: Optimal policies in the face of a demographic shock

System	instrument	Case 1 ($\sigma = 2$)		Case 2 ($\sigma = .1$)	
		$t = 2$	$t = \infty$	$t = 2$	$t = \infty$
Mandatory retirement	p_t	—	—	+	+
	τ_t	—	—	+	+
Free retirement	p_t	—	0	+	0
	b_t	—	—	+	—

the new optimal capital stock is higher, it does not necessarily require a drop in old age transfer. The rise in capital may be ensured by the “capital-tickening” effect provided that the optimal policies do not depress savings too sharply.

Optimal policies differ because they depend on the preferences of individuals. In case 1, the preferences of the agents make it optimal to substitute young-age consumption for old-age consumption and leisure. This is implemented by reducing the labor taxes and decreasing the transfer level. In case 2, preferences of agents are such that it is now optimal to increase both periods’ consumption and leisure. To increase second period consumption, the planner raises transfers and taxes, but to a moderate extent in order not to depress savings too much.

When the retirement age results from an individual decision, the planner can no longer intervene in the labor market without distorting the optimal individual leisure choice. To induce the optimal level of capital stock, the planner reduces the public debt in both cases. In case 1, the adjustment is rapid and is realized essentially by a sharp decrease of national debt during period 3, implying a sharp reduction of the transfer in period 2. This reflects the burden inflicted upon the old of period 2. In case 2, national debt must also decrease in the long-run. However, debt rises strongly in period 2 in order to increase the consumption of the old in this period. Transfers are higher than in the previous steady-state during the transition, but to an extent that diminishes with the increase of capital stock.

From these simulations it appears that the reduction of the national debt should be implemented in both cases⁹. No clear action on transfers can be recommended if one does not know the magnitude of σ .

6 Modelling the increase in longevity

Another element that affects the demographic structure of most of the OECD countries is the increased longevity. This component of the demographic change has not been treated above, but can easily be introduced in our model. An increase in the longevity enlarges the maximum span which old agents are physically able to work. This corresponds in our settings to a rise in x .

A lengthening of the period of physical ability reduces the disutility of working as well

⁹This prediction holds in the huge majority of the simulations we have carried out. However, for σ extremely low (0.05) optimal debt should increase.

as the marginal disutility to work. This is modelled by assuming that the disutility of work satisfies the following properties:¹⁰

$$v_x < 0 \quad \text{and} \quad v_{\lambda x} < 0$$

Using the same method as in section 3.1, it is straightforward to compute the long-run adjustment of the optimal capital stock and the retirement age to a rise in x :

$$\begin{aligned} \frac{dk/k}{dx/x} &= \frac{\epsilon\mu}{\delta} \frac{n c \sigma(c) + d \sigma(d)}{f - nk + \nu(n c \sigma(c) + d \sigma(d))^{\frac{n+\lambda}{\lambda}}} > 0, \\ \frac{d\lambda/\lambda}{dx/x} &= \frac{-\mu(n c \sigma(c) + d \sigma(d))}{\frac{\lambda}{n+\lambda}(f - nk) + \nu(n c \sigma(c) + d \sigma(d))} > 0, \\ \frac{dq/q}{dx/x} &= \frac{\mu(f - n k)}{f - nk + \nu(n c \sigma(c) + d \sigma(d))^{\frac{n+\lambda}{\lambda}}} < 0 \end{aligned}$$

where

$$\mu \equiv \frac{v_{\lambda x} x}{v_{\lambda}} < 0$$

is the elasticity of the marginal disutility to work with respect to the span of physical ability, and

$$\delta \equiv \frac{f_{kk}k}{f_k} \tag{37}$$

is the elasticity of the marginal productivity of capital with respect to capital.

The first two expressions are unambiguously positive. When the longevity increases, it is optimal to increase the retirement age since the marginal disutility of working is reduced. This fosters the marginal productivity of capital at given capital stock. Capital should thus rise to lower its marginal productivity to its new optimal level. An increase in the longevity constitutes therefore also a positive shock. This allows the planner to increase consumption of every generation: $dq/dx < 0$.

7 Conclusion

In this paper we focus on the consequences of a decline in fertility for optimal retirement policies. Four options to reform current pension systems are generally considered: lowering pension benefits; increasing contribution rates; increasing national savings through an accelerated reduction of public debt; and increasing the (effective and/or legal) retirement age. Most of the literature concerned with the aging of the population adopts a numerical approach and studies the effect of various policy reforms. Our objective is to switch the debate from the design of parametric reforms to the optimal allocation of resources (consumption, leisure and capital). Endowed with the well-defined criterion of Ramsey (1928), we theoretically assess the impact of a fertility drop on the optimal allocation. In contrast with the theoretical contributions in this field, our model includes endogenous retirement decisions.

¹⁰Note that the functional form chosen in the example satisfies these properties.

Our analysis of the long-run optimal allocation allows to conclude the following. First, the optimal stock of capital increases after a drop in population growth except when parameters take extreme values. This enhances the production possibilities of the economy. A drop in fertility constitutes therefore a positive shock permitting to increase per-capita consumption. Second, the optimal response of the retirement age crucially depends on the parameters, and in particular on the concavity of the utility function. When consumption and leisure are highly substitutable, the optimal retirement age should increase. On the contrary, when they are low substitutes, the optimal retirement age should decrease. The policy recommendation of postponing retirement is thus not robust to a wide class of preferences and technologies.

The study of the optimal dynamics highlights that the welfare effects of the fertility drop on the initial generations also depends on the concavity of the utility function. When consumption and leisure are highly substitutable, the transition is rapid but induces a loss for the two initial generations. When they are poor substitutes, every generation is better off.

In order to translate these effects in terms of optimal retirement policies, we characterize the decentralization of the optimum through the control over some instruments. We do so both when the retirement age is compulsory and when the retirement decision is private. It appears that the reduction of the national debt should be implemented in all the cases considered. No clear action on transfers can be recommended if one does not know the magnitude of the agents' elasticity of substitution.

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A Proof of proposition 1

A.1 Existence

The steady state is characterized by the following system of equations

$$f_k(k, n + \lambda) = n \quad (38)$$

$$u_c(c) = q \quad (39)$$

$$u_c(d) = \frac{q}{\beta n} \quad (40)$$

$$v_\lambda(\lambda) = \frac{q}{n} f_l(k, n + \lambda) \quad (41)$$

$$f(k, n + \lambda) = nk + nc + d \quad (42)$$

On the one hand, the golden rule (38) determines a unique relationship in the space $\{k, \lambda\}$ between steady state capital and λ for given n : when taking the total derivative of (38) we get $dk/d\lambda = -f_{kl}/f_{kk} > 0$. On the other hand, equations (39)-(42) establish another relationship between k and λ that should be verified at the steady state. We shall prove that both equations in k and λ intersect at least once, in which case we can conclude that there exists at least one steady state.

The golden rule (38) implies that $\forall \lambda \in (0, 1)$, there exists one level of k which satisfies this rule. Now define

$$\underline{k} \in \mathbb{R}_+ : f_k(\underline{k}, n + \lambda) = n \quad \text{for } \lambda = 0 \quad (43)$$

$$\bar{k} \in \mathbb{R}_+ : f_k(\bar{k}, n + \lambda) = n \quad \text{for } \lambda = x. \quad (44)$$

with $\underline{k} < \bar{k}$. This is, $\underline{k}$ and $\bar{k}$ are the levels of capital satisfying (38) for $\lambda = 0$ and $\lambda = x$ respectively.

In the next step we identify two pairs (k, λ) belonging to the curve defined by equations (39)-(42); one of which lies above and the other below the upwards sloping $(\underline{k}, \bar{k})$ -curve determined by (38). This will allow us to conclude that both curves cross at least once. We define $\tilde{k}$ as the steady state capital which satisfies (39)-(42) for $\lambda = x$. Since $v_\lambda(x) = \infty$, according to (41) q also tends to infinity (f_l is strictly positive and finite by assumption). Given (39) and (40) this implies that consumption in both periods tends to zero (because of the limit condition (1)). Hence the resource constraint (42) becomes $f(\tilde{k}, n + x) = n\tilde{k}$; or

$$\frac{f(\tilde{k}, n + x)}{\tilde{k}} = n > f_k(\tilde{k}, n + x). \quad (45)$$

The inequality derives from the Euler theorem implying that $f(k)/k > f_k$. Given (44) and because of decreasing marginal productivity of capital, we can deduce that $\tilde{k} > \bar{k}$, and thus that $(\tilde{k}, n + 1)$ lies above the $(\underline{k}, \bar{k})$ -curve determined by the golden rule.

It is further straightforward to see that when $k = \underline{k}$, (39)-(42) imply that $\lambda \in (0, x)$: on one hand we have that $f_l(\underline{k}, \cdot)$ is strictly positive and finite. On the other one $f(\underline{k}, \cdot)$ is strictly positive and finite and greater than $n\underline{k}$. Indeed, we know that $f(\tilde{k}, n + 1) = n\tilde{k}$ and $\underline{k} < \tilde{k}$; decreasing returns hence imply that $f(\underline{k}, \cdot) > n\underline{k}$, allowing for strictly positive

and finite consumption c and d . Consequently, q is also strictly positive and finite, given (39)-(40). Further, according to (41), $v_\lambda \in (0, \infty)$, which implies that $\lambda \in (0, x)$ for $k = \underline{k}$. We can conclude that $\forall \lambda \in (0, 1)$, $(\underline{k}, \lambda)$ lies below the $(\underline{k}, \bar{k})$ -curve determined by (38) since the latter is upwards sloping and starts in $(\underline{k}, 0)$. We can thus conclude that given continuity of (38)-(42), there is at least one intersection in the (k, λ) -plane.

A.2 Uniqueness

The dynamics of the economy can be reduced to a system of two equations:

$$k_{t+1} = \Phi(k_t, q_t) \quad (46)$$

$$q_{t+1} = \Psi(k_t, q_t); \quad (47)$$

In appendix E we show that the slope dk/dq of the curves (46) and (47) evaluated at the steady state are negative and positive respectively. In section A we showed that there exists at least one steady state. Hence both curves cross at least once. Suppose there is a second steady state implying that (46) and (47) cross again with a positive and negative slope respectively. By continuity of (46) and (47) this would imply the existence of at least a third steady state, again determined by the intersection of the two curves, but where at least one of the curves has its slope reversed. This is in contradiction with the characterization of the steady state itself. Hence, the steady state is unique.

A.3 Stability

To study the characteristics of the dynamics, we take a first order Taylor expansion of the system around its unique steady state (k, q) in order to study the local dynamics. This leads to

$$\begin{bmatrix} k_{t+1} - k \\ q_{t+1} - q \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} k_t - k \\ q_t - q \end{bmatrix}$$

with the partial derivatives taken at the steady state (k, q) :

$$a_1 = \frac{\partial \Phi}{\partial k_t} = \frac{f_k}{n} + \frac{f_l}{n} \frac{d\lambda}{dk} + \frac{C_z}{n} \frac{q f_{kk}}{(f_k)^2} + \frac{C_z}{n} \frac{q f_{kl}}{(f_k)^2} \frac{d\lambda}{dk}$$

$$a_2 = \frac{\partial \Phi}{\partial q_t} = \frac{1}{n} \left(f_l \frac{d\lambda}{dq} - \frac{C_z}{f_k} + C_z \frac{q f_{kl}}{(f_k)^2} \frac{d\lambda}{dq} \right)$$

$$b_1 = \frac{\partial \Psi}{\partial k_t} = -n \frac{q}{(f_k)^2} \left(f_{kk} + f_{kl} \frac{d\lambda}{dk} \right)$$

$$b_2 = \frac{\partial \Psi}{\partial q_t} = \frac{n}{f_k} \left(1 - \frac{q f_{kl}}{f_k} \frac{d\lambda}{dq} \right)$$

where C_z is the derivative of C with respect to its argument. The derivatives $d\lambda/dk$ and $d\lambda/dq$ are given by

$$\begin{aligned} \frac{d\lambda}{dk} &= \frac{q f_{fl} - v_\lambda f_{kl}}{v_{\lambda\lambda} f_k + v_\lambda f_{kl} - q f_{ll}} > 0 \\ \frac{d\lambda}{dq} &= \frac{f_l}{v_{\lambda\lambda} f_k + v_\lambda f_{kl} - q f_{ll}} > 0. \end{aligned}$$

The characteristic polynomial of the linear approximation is given by

$$p(\mu) = \mu^2 - (a_1 + b_2)\mu + a_1b_2 - a_2b_1. \quad (48)$$

When substituting for $d\lambda/dk$ and $d\lambda/dq$, and rearranging terms we have that

$$a_1 + b_2 = 2 + \frac{C_z q f_k f_{kk} v_{\lambda\lambda} - f_l(f_k)^2 f_{kk} v_\lambda}{n(f_k)^2 (v_{\lambda\lambda} f_k + v_\lambda f_{kl} - q f_{lu})} \quad (49)$$

$$a_1 b_2 - a_2 b_1 = 1. \quad (50)$$

From (48)-(50) we can see that $p(1) > 0$ and $p(-1) < 0$, which corresponds to a steady state stable in the saddle-point sense. Q.E.D.

B Proof of proposition 2

Expression (19) can be rewritten as

$$\frac{-\sigma(c) + \frac{\eta}{\epsilon}(n c \sigma(c) + d \sigma(d))}{N} + \frac{c + k - \frac{n}{n + \lambda} \left(c + \frac{d}{n}\right)}{N} + \frac{n}{n + \lambda} - \frac{1}{\epsilon}, \quad (51)$$

where $N = c + d/n + \nu(n + \lambda)/(n\lambda)(n c \sigma(c) + d \sigma(d)) > 0$. The first term in (51) is unambiguously negative. The second, third, and fourth term can be grouped as follows:

$$\frac{c + k - \frac{n}{n + \lambda} \left(c + \frac{d}{n}\right) + \left(\frac{n}{n + \lambda} - \frac{1}{\epsilon}\right) N}{N}. \quad (52)$$

When developing the numerator and after simplifying, (52) becomes

$$\frac{\nu \left(1 - \frac{n + \lambda}{n\epsilon}\right) \frac{n c \sigma(c) + d \sigma(d)}{\lambda}}{N} + \frac{c + k - \frac{1}{\epsilon} \left(c + \frac{d}{n}\right)}{N}. \quad (53)$$

In the case of a CES production function we can rewrite $1 - (n + \lambda)/(n\epsilon)$ as $1/(nf_l)(nf_l - f/(1 + \rho))$; it is straightforward to see that the first term in (53) is negative for $\rho < 0$. The numerator of the second term is equivalent to $c + k - f/(1 + \rho)$, which is also clearly negative for $\rho < 0$. Q.E.D.

C Proof of corollary 1

When σ tends to zero, (19) becomes

$$-\frac{1}{\epsilon} + \frac{c + k}{c + \frac{d}{n}}.$$

or, over a common denominator:

$$\frac{c + k - \frac{1}{\epsilon} \left(c + \frac{d}{n} \right)}{c + \frac{d}{n}}.$$

In the case of a CES-production function, $c + k - 1/\epsilon(c + d/n)$ can be rewritten as $c + k - f/(1 + \rho)$. This term will be positive for large ρ . Q.E.D.

D Proof of corollary 2

When substituting (19) into (21), the effect of a change in n on λ can be rewritten as

$$\frac{d\lambda}{dn} = \frac{c + k - f_l - c \sigma(c) + (n c \sigma(c) + d \sigma(d)) \frac{\eta}{\epsilon}}{f_l + \frac{\nu(n c \sigma(c) + d \sigma(d))}{\lambda}}$$

From above expression it is straightforward to see that $d\lambda/dn < 0$ for large σ .

E The phase diagram

E.1 The slope of the phase-line $\Phi(k_t, q_t) = k_t$

The slope of this curve is obtained by totally differentiating this expression at the steady state:

$$\frac{dq}{dk} = -\frac{\partial\Phi/\partial k}{\partial\Phi/\partial q} = -\frac{\Phi_k + \Phi_\lambda \Omega_k - 1}{\Phi_q + \Phi_\lambda \Omega_q} \quad (54)$$

with

$$\begin{aligned} \Phi_k &= 1 + \frac{q}{n^3} f_{kk} C_z > 0 \\ \Phi_q &= -\frac{C_z}{n^2} > 0 \\ \Phi_\lambda &= \frac{f_l}{n} + \frac{q}{n^3} f_{kl} C_z \\ \Omega_q &= -\frac{h_q}{h_\lambda} = \frac{f_l}{v_{\lambda\lambda} f_k + v_{\lambda} f_{kl} - q f_{ll}} = \frac{f_l f_k}{v_{\lambda\lambda} f_k^2 + q f_l f_{kl} - q f_k f_{ll}} > 0 \\ \Omega_k &= -\frac{h_k}{h_\lambda} = \frac{-v_{\lambda} f_{kk} + q f_{kl}}{v_{\lambda\lambda} f_k + v_{\lambda} f_{kl} - q f_{ll}} = \frac{-q f_l f_{kk} + q f_k f_{kl}}{v_{\lambda\lambda} f_k^2 + q f_{kl} f_l - q f_k f_{ll}} > 0 \end{aligned}$$

where we substituted $v_\lambda = \frac{q f_l}{f_k}$. Let us show that the numerator of (54) is positive:

$$\begin{aligned} \frac{\partial\Phi}{\partial k} = \Phi_k + \Phi_\lambda \Omega_k - 1 &= \frac{f_l}{n} \Omega_k + \frac{q}{n^3} f_{kk} C_z + \frac{q}{n^3} f_{kl} C_z \Omega_k \\ &= \frac{f_l}{n} \Omega_k + \frac{q C_z}{n^3} (f_{kk} + f_{kl} \Omega_k) \end{aligned}$$

The first term is positive. In order to determine the sign of the second term (with $C_z < 0$) we substitute for Ω_k and we obtain the following expression:

$$(f_{kk} + f_{kl}\Omega_k) = \frac{v_{\lambda\lambda}f_k^2 f_{kk}}{v_{\lambda\lambda}f_k^2 + qf_l f_{kl} - qf_k f_{ll}} < 0$$

Since this expression is negative, the second term is also positive and the numerator is positive.

The denominator of (54)

$$\frac{\partial\Phi}{\partial q} = \Phi_q + \Phi_\lambda\Omega_q = \frac{f_l\Omega_q}{n} + \frac{C_z}{n^2} \left(-1 + \frac{q}{n}f_{kl}\Omega_q \right)$$

can also proven to be positive. The first term is positive. After substituting for Ω_q , the second term becomes:

$$\frac{C_z}{n^2} [-v_{\lambda\lambda}n^2 + qnf_{ll}] \frac{1}{v_{\lambda\lambda}f_k^2 + qf_l f_{kl} - qf_k f_{ll}} > 0.$$

The denominator is therefore positive. We can thus conclude that (54) is negative which implies that the phase-line $\Phi(k_t, q_t) = k_t$ is downwards sloping.

E.2 The slope of the phase-line $\Psi(k_t, q_t) = q_t$

The slope of this phase-line is obtained by totally differentiating the equation at steady state:

$$\frac{dq}{dk} = -\frac{\partial\Psi/\partial k}{\partial\Psi/\partial q} = -\frac{\Psi_k + \Psi_\lambda\Omega_k}{\Psi_q + \Psi_\lambda\Omega_q - 1}$$

with

$$\Psi_k = -\frac{qf_{kk}}{n} > 0$$

$$\Psi_q = 1$$

$$\Psi_\lambda = -\frac{qf_{kl}}{n} < 0$$

The sign of the denominator is negative:

$$\frac{\partial\Psi}{\partial q} = \Psi_\lambda\Omega_q < 0,$$

since $\Omega_q > 0$ and $\Psi_\lambda < 0$.

The sign of the numerator

$$\frac{\partial\Psi}{\partial k} = \Psi_k + \Psi_\lambda\Omega_k > 0.$$

can be proven to be positive. Indeed, this expression can be rewritten as:

$$\Psi_k + \Psi_\lambda\Omega_k = -\frac{q}{n}(f_{kk} + f_{kl}\Omega_k).$$

Since we have proved that $(f_{kk} + f_{kl}\Omega_k) < 0$, $\Psi_k + \Psi_\lambda\Omega_k$ is positive. Consequently, the phase-line $\Psi(k_t, q_t) = q_t$ is upward sloping.

F Proof of proposition 4

We follow the proof in de la Croix and Michel (2000). The government budget constraint and equation (23) allow us to express the transfers in the following way, for all $t \geq 0$:

$$p_t = \frac{n_t + \lambda_t^*}{n_t} (d_t^* - f_k(k_t^*, n_t + \lambda_t^*)k_t^* - f_l(k_t^*, n_t + \lambda_t^*)\lambda_t^*)$$

which are the levels that allow the old agents to consume d_t^* at equilibrium. Using the resource constraint, this implies

$$p_t = \frac{n_t + \lambda_t^*}{n_t} (f(k_t^*, n_t + \lambda_t^*) - n_t k_{t+1}^* - n_t c_t^* - f_k(k_t^*, n_t + \lambda_t^*)k_t^* - f_l(k_t^*, n_t + \lambda_t^*)\lambda_t^*)$$

and, using the Euler theorem,

$$p_t = (n_t + \lambda_t^*)(f_l(k_t^*, n_t + \lambda_t^*) - k_{t+1}^* - c_t^*) \quad (55)$$

The corresponding tax rate is given by

$$\tau_t = \frac{f_l(k_t^*, n_t + \lambda_t^*) - k_{t+1}^* - c_t^*}{f_l(k_t^*, n_t + \lambda_t^*)} \quad (56)$$

Consider any date $t \geq 0$. At the given capital stock k_t^* , and assuming perfect forecasts, the optimal choices c_t, d_{t+1} and s_t of the agents for period t are characterized by

$$\begin{aligned} u_c(c_t) &= f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)\beta u_c(d_{t+1}) \\ c_t &= f_l(k_t^*, n_t + \lambda_t^*)(1 - \tau_t) - s_t \\ d_{t+1} &= f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)s_t + p_{t+1} + f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)\lambda_{t+1}^*(1 - \tau_{t+1}) \end{aligned}$$

After substituting the transfers, the last two equations become

$$\begin{aligned} c_t &= k_{t+1}^* + c_t^* - s_t \\ d_{t+1} &= f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)s_t + (n_{t+1} + \lambda_{t+1}^*)f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*) - n_{t+1}(k_{t+2}^* + c_{t+1}^*) \\ &= f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)(s_t - k_{t+1}^*) + f(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*) - n_{t+1}(k_{t+2}^* + c_{t+1}^*) \\ &= f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)(s_t - k_{t+1}^*) + d_{t+1}^* \end{aligned}$$

$s_t = k_{t+1}^*, c_t = c_t^*$ and $d_{t+1} = d_{t+1}^*$ is the unique solution of the above system. Q.E.D.

G Proof of proposition 5

Using the additional first order condition of the households (31), we can express the tax rate as follows:

$$(1 - \tau_{t+1}) = \frac{v_l(\lambda_{t+1}^*)}{\beta u_c(d_{t+1}^*)f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)} \quad (57)$$

Using (5) and (6) characterizing the optimal solution, we deduce that:

$$\tau_{t+1} = \tau_t = 0$$

The level of public debt that allows the old agents to consume d_t^* at equilibrium can be defined as follows, using (23), the government budget constraint (32) and the capital market equilibrium condition (33):

$$d_t^* = f_k(k_t^*, n_t + \lambda_t^*)k_t^* + n_t b_{t+1} + f_l(k_t^*, n_t + \lambda_t^*)\lambda_t^* \quad (58)$$

or equivalently as

$$b_{t+1} = \frac{d_t^* - f_l(k_t^*, n_t + \lambda_t^*)\lambda_t^* - f_k(k_t^*, n_t + \lambda_t^*)k_t^*}{n_t} \quad (59)$$

From the government budget constraint, the corresponding pension level is:

$$p_t = n_t b_{t+1} - f_k(k_t^*, n_t + \lambda_t^*)b_t \quad (60)$$

Consider any date $t \geq 0$. At the given capital stock k_t^* , and assuming perfect forecasts, the optimal choices c_t, d_{t+1} and s_t of the agents for period t are characterized by

$$u_c(c_t) = f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)\beta u_c(d_{t+1}) \quad (61)$$

$$v_l(\lambda_{t+1}) = \beta u_c(d_{t+1})f_l(k_t^*, n_t + \lambda_t^*)(1 - \tau_{t+1}) \quad (62)$$

$$c_t = f_l(k_t^*, n_t + \lambda_t^*)(1 - \tau_t) - s_t \quad (63)$$

$$d_{t+1} = f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)s_t + p_{t+1} + f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)\lambda_{t+1}^*(1 - \tau_{t+1}) \quad (64)$$

From equations (12), (13), we know that the optimal retirement age is characterized by:

$$\beta u_c(d_{t+1}) = \frac{v_\lambda(\lambda_{t+1}^*)}{f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)} \quad (65)$$

Substituting this expression and the transfer in (62), we obtain:

$$\frac{v_\lambda(\lambda_{t+1}^*)}{f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)} = \frac{v_\lambda(\lambda_{t+1})}{f_l(k_{t+1}, n_{t+1} + \lambda_{t+1})} \quad (66)$$

which implies that $\lambda_{t+1} = \lambda_{t+1}^*$. After substituting the transfers and (33), equation (63) becomes:

$$\begin{aligned} c_t &= f_l(k_t^*, n_t + \lambda_t^*) - k_{t+1}^* - \frac{d_t^*}{n_t} + \frac{f_l(k_t^*, n_t + \lambda_t^*)\lambda_t^*}{n_t} + \frac{f_k(k_t^*, n_t + \lambda_t^*)k_t^*}{n_t} \\ &= \frac{1}{n_t}(f(k_t^*, n_t + \lambda_t^*) - d_t^* - n_t k_{t+1}^*) = c_t^* \end{aligned}$$

Substituting the transfers in equation (64), we get:

$$\begin{aligned} d_{t+1} &= f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)(s_t - b_{t+1}) + f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)\lambda_{t+1}^* + n_{t+1}b_{t+2} \\ &= f_k(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)k_{t+1}^* + f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)\lambda_{t+1}^* + n_{t+1}b_{t+2} \\ &= f(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*) - f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*)n_{t+1} + n_{t+1}(s_{t+1} - k_{t+2}^*) \\ &= f(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*) - n_{t+1}(f_l(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*) - s_{t+1}) - n_{t+1}k_{t+2}^* \\ &= f(k_{t+1}^*, n_{t+1} + \lambda_{t+1}^*) - c_{t+1}^*n_{t+1} - n_{t+1}k_{t+2}^* \\ &= n_{t+1}k_{t+2}^* + d_{t+1}^* - n_{t+1}k_{t+2}^* = d_{t+1}^* \end{aligned}$$

Therefore, $b_{t+1} + s_t = k_{t+1}^*$, $c_t = c_t^*$, $d_{t+1} = d_{t+1}^*$, $\lambda_{t+1} = \lambda_{t+1}^*$ is the unique solution of the above system. Q.E.D.