Finite-horizon covariance control of state-affine nonlinear systems with application to proton beamline calibration

Zheming Wang, Raphaël M. Jungers, Quentin Flandroy, Baptiste Herregods, and Cedric Hernalsteens

Abstract— **In this paper, we address the problem of finite-horizon distribution steering of state-affine nonlinear systems using open-loop control. The terminal distribution is constrained by an isotropy constraint which enforces uniformity in all directions. With the nonlinear dynamics and the isotropy constraint, a constrained nonlinear optimization problem is formulated. To solve this nonlinear problem, this paper proposes a sequential linearized algorithm that generates feasible iterates. The main advantage of the proposed algorithm is that it requires no a priori knowledge of the global Lipschtiz property of the gradients of the cost and constraint functions. The performance of the proposed algorithm is demonstrated by a beamline calibration problem in proton therapy.**

I. Introduction

This paper considers the problem of steering dynamical systems from an initial distribution to some desired terminal distribution while minimizing a given cost function. Such a problem has received considerable attention in recent years and it can be found in a wide range of engineering applications, such as swarm robots [1], [2], nuclear magnetic resonance (NMR) spectroscopy and imaging (MRI) [3], proton radiotherapy [4]–[6], and self-assembly of nanoparticles [7]. It is can also be viewed as an optimal mass transport problem [8] when the terminal distribution is fixed.

One typical distribution steering problem is the covariance steering problem. This problem is especially important for Gaussian distributions as they are fully defined by the first two moments. The covariance steering problem, also known as the covariance assignment problem, was first introduced in [9], [10], where linear feedback controllers were designed for continuous stochastic linear systems such that the state covariance converges to a specified value. This work was then followed by other infinite-horizon optimal solutions [11]–[14] for both continuous and discrete stochastic linear systems. Recently, the finite-horizon covariance control problem has also been extensively studied in [15]–[19], where the state covariance is steered to the targeted value in a finite horizon. In particular, the authors in [15], [16] discussed the sufficient conditions for the existence of the state-feedback gains to achieve finite-horizon optimal steering. These references focus on state-feedback controllers and assume that the state of the system is observable. However, in some circumstances, the state is not always observable and state-feedback controllers cannot be implemented. For instance, in proton radiotherapy, a group of protons will be driven to a desired distribution by external electric fields in the absence of state measurements. For this reason, this paper will focus on open-loop distribution steering and the state is assumed to be unobservable during the course of control. Note that the state covariance of linear systems is always uncontrollable using open-loop control. This paper studies a family of nonlinear systems where the open-loop control input affects the state covariance.

In this paper, we also aim to achieve an isotropy terminal distribution that is uniform in all directions. An exact isotropy distribution will restrict its covariance matrix to be a multiple of identity. However, it is unrealistic to pursuit an exact isotropy distribution in practical applications. Consequently, an inequality of the covariance matrix will be used to achieve an approximate isotropy distribution. The issue of isotropy often arises in proton radiotherapy, where the protons are required to be distributed uniformly in all directions for better treatment. With this consideration, the terminal distribution is not fixed but rather constrained in a feasible set, from which this paper differs from many works in the literature [15]–[19]. In the presence of the nonlinear dynamics and the isotropy constraint, a constrained nonlinear optimization problem has to be solved to achieve open-loop distribution steering. To ensure the feasibility of the isotropy constraint, this paper will use sequential convex approximation methods [20], [21] which generate feasible iterates. Unfortunately, the method in [20] is based on the global Lipschitz property of the gradients of the cost and constraint functions, which is difficult to obtain and may lead to poor numerical performance for general nonlinear problems. Attempts to improve the numerical performance have been made in [21] where the global Lipschitz property of the gradient of the cost function is not needed. While being a great improvement, this method still requires the global Lipschitz property of the gradient of a general nonconvex constraint to construct a feasible approximation set of the original constraint set. In order to circumvent the inconvenience of the global Lipschitz property, this paper will propose an iterative algorithm where the approximation set is parameterized and updated until its feasibility is reached.

The rest of the paper is organized as follows. This section ends with the notations needed, followed by the next section on the formulation of the control problem. Section III discusses the reachability of the covariance matrix of

Zheming Wang (zheming.wang@uclouvain.be) and Raphaël M. Jungers (raphael.jungers@uclouvain.be) are with the ICTEAM Institute, UCLouvain, Louvain-la-Neuve,1348, Belgium. Quentin Flandroy (Quentin.Flandroy@iba-group.com), Baptiste Herregods (Baptiste.Herregods@iba-group.com), and Cedric Hernalsteens (Cedric.Hernalsteens@iba-group.com) are with IBA, Louvain-la-Neuve, 1348, Belgium

Raphaël M. Jungers is a FNRS Research Associate. He is supported by the French Community of Belgium, the Walloon Region and the Innoviris Foundation.

the dynamical system. In Section IV, the solution of the distribution control problem is discussed and a proximal linearized algorithm is proposed with a feasible initialization procedure. Section V shows the application of the proposed algorithm on the beamline calibration in proton therapy. The last section concludes the work. The proofs of some Lemmas and Theorems are not provided in this paper due to page limitation.

The notation used in this paper is as follows. I_n is the $n \times n$ identity matrix and 1_n is the column vector of all ones (subscript omitted when the dimension is clear). For a square matrix Q, $Q \succ (\succeq) 0$ means Q is positive definite (semi-definite). The p-norm of $x \in \mathbb{R}^n$ is $\|x\|_p$ and $\|x\|$ denotes 2-norm. For $Q \succ 0$, $\|x\|_Q^2$ denotes $x^T Q x$. Given a set S and matrices A, B, $ASB = \{Axb : x \in S\}$. For a square matrix $A \in \mathbb{R}^{n\times n}$, $\rho(A)$ denotes the spectral radius of A. For a square matrix $Q \succeq 0$, $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$ denote the maximal and minimal eigenvalues, and $\kappa(Q) := \lambda_{\max}(Q)/\lambda_{\min}(Q)$ denotes its condition number. Additional notation is introduced as required in the text.

II. Problem Formulation

A. System Description

We consider the following nonlinear state-affine system

$$\begin{aligned} x_{t+1} &= f_t(u_t)x_t + g_t(u_t), \quad 0 \le t \le N-1 \\ y_t &= Cx_t, \quad 0 \le t \le N, \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m, y_t \in \mathbb{R}^r$ are the state, input and output vectors, $f_t : \mathbb{R}^m \to \mathbb{R}^{n\times n}$ and $g_t : \mathbb{R}^m \to \mathbb{R}^n$ are proper functions, $C \in \mathbb{R}^{r\times n}$, N is the given time horizon, and x_0 is distributed according to a known Gaussian distribution $\mathcal{N}(\mu_0, \Sigma_0)$. The state x_t is not measurable for all $t \ge 0$. The control input is subject to the constraint

$$u_t \in U \subseteq \mathbb{R}^m, \quad 0 \le t \le N-1. \tag{2}$$

The following assumptions are needed in the sequel.

Assumption 1: $U \subseteq \mathbb{R}^m$ is a convex and compact set.

Assumption 2: For any t, $f_t(u)$ and $g_t(u)$ are continuously differentiable functions with Lipschitz gradients:

$$\|\mathcal{D}_{u^T} f_t(u') - \mathcal{D}_{u^T} f_t(u)\| \le L_f \|u' - u\|, \tag{3}$$

$$\|\mathcal{D}_{u^T} g_t(u') - \mathcal{D}_{u^T} g_t(u)\| \le L_g \|u' - u\|, \ \forall u', u \in U, \tag{4}$$

where $L_f > 0$ and $L_g > 0$ are the Lipschitz constants, and $\mathcal{D}_{u^T}$ denotes the derivative of a matrix-value function with respect to u [22].

From Lemma 6.9.1 in [23], the following lemma is an immediate consequence of Assumption 2.

Lemma 1: Suppose Assumption (2) holds and let $f_t^{ij}(u)$ and $g_t^i(u)$ denote the (i,j)-entry of $f_t(u)$ and i^{th} element of $g_t(u)$ respectively, for all $0 \le i,j \le n$ and $t \ge 0$. Then, $\forall u', u \in U$,

$$|f_t^{ij}(u') - f_t^{ij}(u) - (\nabla f_t^{ij}(u))^T(u' - u)| \le \frac{L_f}{2}\|u' - u\|^2 \tag{5}$$

$$|g_t^{i}(u') - g_t^{i}(u) - (\nabla g_t^{i}(u))^T(u' - u)| \le \frac{L_g}{2}\|u' - u\|^2 \tag{6}$$

The control objective is to steer the output distribution to some desired distribution while minimizing a given cost function. Since the state information is not available, this paper only considers open-loop control strategies and the control signal u_t in (1) serves as a universal input for any realization of the initial state from $\mathcal{N}(\mu_0, \Sigma_0)$.

Let μ_t and Σ_t denote the mean and the covariance matrix at time t. As u_t is independent of x_t for all t, it is easy to verify that μ_t and Σ_t evolve as follows

$$\mu_{t+1} = f_t(u_t)\mu_t + g_t(u_t), \ \Sigma_{t+1} = f_t(u_t)\Sigma_t f_t^T(u_t). \tag{7}$$

The output distribution can be given by

$$\mu_{y_t} = C\mu_t, \ \Sigma_{y_t} = C\Sigma_t C^T, \quad 0 \le t \le N. \tag{8}$$

B. The isotropy constraint

In this paper, we aim to achieve the uniformity of the output distribution in all directions, motivated by the beam specifications in proton therapy [4], [6]. The uniformity can be represented as the isotropy of the distribution. For the terminal output distribution $\mathcal{N}(\mu_{y_N}, \Sigma_{y_N})$, such a property can be quantified by the relative difference between the largest and smallest eigenvalues of the covariance matrix, as shown below,

$$\frac{\lambda_{\max}(\Sigma_{y_N}) - \lambda_{\min}(\Sigma_{y_N})}{\lambda_{\min}(\Sigma_{y_N})} \tag{9}$$

Equivalently, it can be described as the condition number $\kappa(\Sigma_{y_N})$. The following constraint is imposed at the terminal time instant to achieve an isotropy output distribution

$$\kappa(\Sigma_{y_N}) := \frac{\lambda_{\max}(\Sigma_{y_N})}{\lambda_{\min}(\Sigma_{y_N})} \le \bar{\kappa} \tag{10}$$

where $\bar{\kappa} \ge 1$ is some given parameter.

C. The distribution control problem

The control problem is to determine the control inputs $\boldsymbol{u}_N$ such that (10) is satisfied. The objective function is chosen to be

$$J(\boldsymbol{u}_{N-1}) := \mathbb{E}\{\sum_{t=0}^{N-1}(\|y_t\|_Q^2 + \|u_t\|_R^2) + \|y_N\|_P^2\} \tag{11}$$

for some weighting matrices $Q, R \succ 0$. Using this objective function, the control problem can be cast as

$$\min_{\boldsymbol{u}_{N-1}} \mathbb{E}\{\sum_{t=0}^{N-1}(\|y_t\|_Q^2 + \|u_t\|_R^2) + \|y_N\|_P^2\} \tag{12a}$$

$$\text{s.t.} \quad x_{t+1} = f_t(u_t)x_t + g_t(u_t), u_t \in U, t \le N-1 \tag{12b}$$

$$y_t = Cx_t, \quad 0 \le t \le N \tag{12c}$$

$$\kappa(\Sigma_{y_N}) \le \bar{\kappa}, \quad x_0 \sim \mathcal{N}(\mu_0, \Sigma_0) \tag{12d}$$

III. The Reachability of the Covariance Matrix and Its Condition Number

This section discusses the reachability of the covariance matrix of the output distribution and its condition number in system (1).

Let $\Gamma_t(\boldsymbol{u}_t) = f_t(u_t)\dots f_1(u_1)f_0(u_0)$ for all t. For a given initial distribution $\mathcal{N}(\mu_0, \Sigma_0)$, the reachable set of the covariance matrices at time t is defined as $\Omega(t, \Sigma_0) :=$

$\{\Gamma_{t-1}(\boldsymbol{u}_{t-1})\Sigma_0\Gamma_{t-1}^T(\boldsymbol{u}_{t-1}) : \boldsymbol{u}_{t-1} \in \boldsymbol{U}_t\}, t \geq 1$, where $\boldsymbol{U}_t := U \times U \times \cdots \times U \subset \mathbb{R}^{tm}$. Via vectorization, (7) becomes

$$\text{Vec}(\Sigma_{t+1}) = (f_t(u_t) \otimes f_t(u_t))\,\text{Vec}(\Sigma_t). \tag{13}$$

Note that Σ_t is symmetric, $\text{Vec}(\Sigma_t)$ can be represented by the $\frac{n^2+n}{2}$ variables in upper triangle. Let z_t denote the $\frac{n^2+n}{2}$ variables and $\text{Vec}(\Sigma_t)$ can be represented by

$$\text{Vec}(\Sigma_t) = M z_t, \quad \forall t \geq 0, \tag{14}$$

where $z_t \in \mathbb{R}^{\frac{n^2+n}{2}}$ and $M \in \{0,1\}^{n^2 \times \frac{n^2+n}{2}}$ is the selection matrix. Since M is full column rank, (13) can be rewritten as

$$z_{t+1} = M^+ (f_t(u_t) \otimes f_t(u_t))\, M z_t. \tag{15}$$

where M^+ denotes the pseudoinverse of M. Let $\tilde{f}_t(u_t) := M^+ (f_t(u_t) \otimes f_t(u_t))\, M$ and $\tilde{\Gamma}_t(\boldsymbol{u}_t) = \tilde{f}_t(u_t) \ldots \tilde{f}_1(u_1)\tilde{f}_0(u_0)$ for all t. The reachability problem of the covariance matrix boils down to the reachability of the nonlinear system (15). With $z_0 = M^+\text{Vec}(\Sigma_0)$, an equivalent reachable set can be defined $\Omega_z(t, z_0) := \{\tilde{\Gamma}_{t-1}(\boldsymbol{u}_{t-1})z_0 : \boldsymbol{u}_{t-1} \in \boldsymbol{U}_t\}, t \geq 1$. Several algorithms, see, e.g., [24], [25], are available in the literature to compute bounds on $\Omega_z(N, z_0)$. From (8), the reachable set of the output covariance matrices can be given by $C\Omega(t, \Sigma_0)C^T$ for all $t \geq 1$. Considering the vectorization form (15), it can also be expressed by $C \otimes CM\Omega_z(t, z_0)$ for all $t \geq 1$.

As this paper is focused on the isotropy of the output distribution, it is more important to investigate the condition number reachable set. Similarly, we define the following reachable set $\Omega_\kappa(t, \Sigma_0) := \{\kappa\left(C\Gamma_{t-1}(\boldsymbol{u}_{t-1})\Sigma_0\Gamma_{t-1}^T(\boldsymbol{u}_{t-1})C^T\right) : \boldsymbol{u}_{t-1} \in \boldsymbol{U}_t\}$. Let κ_{min} denote the lower bound of $\Omega_\kappa(N, \Sigma_0)$. The feasibility of problem (12) relies on the condition that $\bar{\kappa} \geq \kappa_{min}$. To verify this condition, we do not have to compute the whole set $\Omega_\kappa(N, \Sigma_0)$, but rather, we restrict our attention to κ_{min}. For notational convenience, we drop the subscript of $\boldsymbol{u}_{N-1}$ and let

$$\tilde{G}(\boldsymbol{u}) := C\Gamma_{N-1}(\boldsymbol{u}_{N-1})\Sigma_0\Gamma_{N-1}^T(\boldsymbol{u}_{N-1})C^T. \tag{16}$$

To compute κ_{min}, we need in principle to solve the following problem

$$\min_{\boldsymbol{u} \in \boldsymbol{U}} \kappa(\tilde{G}(\boldsymbol{u})), \tag{17}$$

where $\boldsymbol{U} := U \times U \times \cdots \times U \subseteq \mathbb{R}^{Nm}$. However, it is difficult to obtain the exact κ_{min} for general nonlinear systems. Hence, we will aim to compute an upper bound of κ_{min} as discussed in the next section.

IV. The Distribution Control and Optimization: A Proposed Algorithm

In this section, we will present a proximal linearized algorithm to solve problem (12) with some given $\bar{\kappa}$.

A. *Reformulation*

Before we present the proposed algorithm, we first need to reformulate problem (12). After several manipulations, $J(\boldsymbol{u})$ can be rewritten as

$$J(\boldsymbol{u}) = \sum_{t=0}^{N-1} (trace(QC\Sigma_t C^T) + \|C\mu_t\|_Q^2 + \|u_t\|_R^2) + trace(PC\Sigma_N C^T) + \|C\mu_N\|_P^2 \tag{18}$$

As shown in Chapter 3 of [26], the isotropy constraint (10) is satisfied if and only if there exists a $\gamma > 0$ such that

$$\gamma I \preceq C\Sigma_N C^T \preceq \gamma\bar{\kappa} I \tag{19}$$

Taking (18) and (19) into consideration, the problem (12) can be reformulated as

$$\min_{\boldsymbol{u}, \gamma} J(\boldsymbol{u}) \tag{20a}$$
$$\text{s.t. } \mu_{t+1} = f_t(u_t)\mu_t + g_t(u_t) \tag{20b}$$
$$\Sigma_{t+1} = f_t(u_t)\Sigma_t f_t^T(u_t), u_t \in U, 0 \leq t \leq N-1 \tag{20c}$$
$$\gamma I \preceq C\Sigma_N C^T \leq \gamma\bar{\kappa} I, \gamma \geq 0 \tag{20d}$$

Using the notation in (16), a compact formulation is given

$$\min_{\boldsymbol{u} \in \boldsymbol{U}, \gamma \geq 0} J(\boldsymbol{u}) \tag{21a}$$
$$\text{s.t.} \quad \gamma I \preceq \tilde{G}(\boldsymbol{u}) \preceq \gamma\bar{\kappa} I. \tag{21b}$$

For notational convenience, let

$$H_J(\boldsymbol{u}', \boldsymbol{u}) := J(\boldsymbol{u}) + (\nabla J(\boldsymbol{u}))^T(\boldsymbol{u}' - \boldsymbol{u}) \tag{22}$$
$$H_{\tilde{G}}(\boldsymbol{u}', \boldsymbol{u}) := \tilde{G}(\boldsymbol{u}) + \mathcal{D}_{\boldsymbol{u}^T}\tilde{G}(\boldsymbol{u})((\boldsymbol{u}' - \boldsymbol{u}) \otimes I) \tag{23}$$

for all $\boldsymbol{u}', \boldsymbol{u} \in \boldsymbol{U}$. The following lemma is needed for the proposed algorithm.

Lemma 2: Suppose Assumptions 1 and 2 hold, there exist $L_{\tilde{G}} > 0$ and $L_J > 0$ such that the following inequalities hold

$$J(\boldsymbol{u}') - H_J(\boldsymbol{u}', \boldsymbol{u}) \leq L_J\|\boldsymbol{u}' - \boldsymbol{u}\|^2, \tag{24}$$
$$-L_{\tilde{G}}\|\boldsymbol{u}' - \boldsymbol{u}\|^2 I \preceq \tilde{G}(\boldsymbol{u}') - H_{\tilde{G}}(\boldsymbol{u}', \boldsymbol{u}) \preceq L_{\tilde{G}}\|\boldsymbol{u}' - \boldsymbol{u}\|^2 I, \tag{25}$$

for all $\boldsymbol{u}', \boldsymbol{u} \in \boldsymbol{U}$.

B. *The Proposed Proximal Linearized Algorithm*

Similar to [20], [21], we propose an algorithm that solves (12) via sequential linearization. As will be shown, the proposed algorithm ensures the feasibility of the solution as long as the initialization is feasible.

We use the epigraph form of (21):

$$\min_{\boldsymbol{u}, \gamma, \beta} \beta \tag{26a}$$
$$\text{s.t.} \quad \boldsymbol{u} \in \boldsymbol{U} \tag{26b}$$
$$\gamma I \preceq \tilde{G}(\boldsymbol{u}) \preceq \gamma\bar{\kappa} I, \gamma \geq 0 \tag{26c}$$
$$J(\boldsymbol{u}) \leq \beta. \tag{26d}$$

To convert the problem above into a convex problem, we will linearize the constraints (26c) and (26d) based on Lemma 2. Suppose there is a feasible solution $(\tilde{\boldsymbol{u}}, \tilde{\gamma}, \tilde{\beta})$ that satisfies (26b), (26c) and (26d), the following linearized set is defined based on Lemma 2: $\Phi(\tilde{\boldsymbol{u}}; L_a', L_b') := \{(\boldsymbol{u}, \gamma, \beta) : H_J(\boldsymbol{u}, \tilde{\boldsymbol{u}}) \leq$

$\beta - L_a'\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|^2, (\gamma + L_b'\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|^2)I \preceq H_{\tilde{G}}(\boldsymbol{u},\tilde{\boldsymbol{u}}) \preceq (\gamma\bar{\kappa} - L_b'\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|^2)I, \boldsymbol{u} \in \boldsymbol{U}, \gamma \geq 0\}$ for any $L_a', L_b' \geq 0$. This is a convex set and can be represented as $\Phi(\tilde{\boldsymbol{u}}; L_a', L_b') := \{(\boldsymbol{u},\gamma,\beta) : H_J(\boldsymbol{u},\tilde{\boldsymbol{u}}) \leq \beta - L_a'\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|^2, \lambda_{\max}(H_{\tilde{G}}(\boldsymbol{u},\tilde{\boldsymbol{u}})) \leq \gamma\bar{\kappa} - L_b'\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|^2 \gamma + L_b'\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|^2 \leq \lambda_{\min}(H_{\tilde{G}}(\boldsymbol{u},\tilde{\boldsymbol{u}})), \boldsymbol{u} \in \boldsymbol{U}, \gamma \geq 0\}$. The size of this set largely depends on the parameters L_a' and L_b'. The properties of the linearized set are stated in the following lemma.

Lemma 3: Given a $\tilde{\boldsymbol{u}} \in \boldsymbol{U}$, suppose there is $(\tilde{\gamma}, \tilde{\beta})$ such that (26c) and (26d) are satisfied. Then, the following results hold.

(i) $\Phi(\tilde{\boldsymbol{u}}; L_a', L_b')$ is non-empty and $(\tilde{\boldsymbol{u}}, \tilde{\gamma}, \tilde{\beta}) \in \Phi(\tilde{\boldsymbol{u}}; L_a', L_b')$ for any $L_a', L_b' \geq 0$.

(ii) Suppose $L_a' \geq L_J$ and $L_b' \geq L_{\tilde{G}}$, for any $(\boldsymbol{u},\gamma,\beta) \in \Phi(\tilde{\boldsymbol{u}}; L_a', L_b')$, $(\boldsymbol{u},\gamma,\beta)$ is feasible to problem (26).

With the linearized set, problem (26) can be linearized and the following proximal linearized problem is invoked at every iteration

$$(\boldsymbol{u}^{k+1}, \gamma^{k+1}, \beta^{k+1}) \leftarrow \arg\min_{\boldsymbol{u},\gamma,\beta} \beta + \tau\|\boldsymbol{u}-\boldsymbol{u}^k\|^2 \tag{27a}$$

$$\text{s.t.} \quad (\boldsymbol{u},\gamma,\beta) \in \Phi(\boldsymbol{u}^k; L_J, L_{\tilde{G}}); \tag{27b}$$

where $\tau > 0$. This is a convex constrained optimization problem with a quadratic cost function and can be easily solved by general convex solvers, such as CVX [27] and MOSEK [28]. The convergence properties of the solution of problem (27) are stated in the following theorem.

Theorem 1: Suppose $(\boldsymbol{u}^0, \gamma^0, \beta^0)$ is a feasible solution to (26) and let $\{\boldsymbol{u}^k, \gamma^k, \beta^k\}$ be obtained from solving problem (27) with $\tau > 0$. Then, the following results hold.

(i) $(\boldsymbol{u}^k, \gamma^k, \beta^k)$ is feasible for all $k \geq 0$.

(ii) $\beta^{k+1} \leq \beta^k - \tau\|\boldsymbol{u}^{k+1} - \boldsymbol{u}^k\|^2$ for all $k \geq 0$.

(iii) The sequence $\{\boldsymbol{u}^k, \gamma^k, \beta^k\}$ converges.

Proof of Theorem 1: (i) This statement can be easily proved by induction. Suppose $(\boldsymbol{u}^k, \gamma^k, \beta^k)$ is feasible to (26), from property (ii) of Lemma 3, $(\boldsymbol{u}^{k+1}, \gamma^{k+1}, \beta^{k+1})$ is also a feasible solution.

(ii) From property (i) of Lemma 3, $(\boldsymbol{u}^k, \gamma^k, \beta^k)$ is always a feasible solution to (27). As $(\boldsymbol{u}^{k+1}, \gamma^{k+1}, \beta^{k+1})$ is the optimal solution, we can know from the optimality that $\beta^{k+1} + \tau\|\boldsymbol{u}^{k+1} - \boldsymbol{u}^k\|^2 + \tau\|\gamma^{k+1} - \gamma^k\| \leq \beta^k + \tau\|\boldsymbol{u}^k - \boldsymbol{u}^k\|^2 + \tau\|\gamma^k - \gamma^k\| = \beta^k$.

(iii) From property (ii), we can see that $\{\beta^k\}$ is a non-increasing sequence that is bounded from below by 0. This means that $\{\beta^k\}$ is convergent. Finally, this implies the convergence of $\{\boldsymbol{u}^k\}$ and $\{\gamma^k\}$. □

C. The Proximal Linearized Algorithm without a priori Lipschitz constants

The main limitation of problem (27) is that it requires the knowledge of L_J and $L_{\tilde{G}}$. In practice, it is difficult to obtain tight bounds on L_J and $L_{\tilde{G}}$ and conservative bounds may restrict the size of the set $\Phi(\boldsymbol{u}^k; L_J, L_{\tilde{G}})$ in (27). To circumvent this problem, we propose an iterative algorithm where the lower bounds on L_J and $L_{\tilde{G}}$ is refined during the convergence of the algorithm. At the k^{th} iteration, an inner loop is introduced to update the parameters L_J and $L_{\tilde{G}}$ in $\Phi(\boldsymbol{u}^k; L_J, L_{\tilde{G}})$. Let $\Phi(\boldsymbol{u}^k; L_a^{k,\ell}, L_b^{k,\ell})$ denote the linearized set at the ℓ^{th} inner loop iteration with $L_a^{k,0} = 0$ and $L_b^{k,0} = 0$. With this set, problem (27) is solved and the solution is denoted as $(\boldsymbol{u}^{k+1,\ell}, \gamma^{k+1,\ell}, \beta^{k+1,\ell})$. The parameters $L_a^{k,\ell}$ and $L_b^{k,\ell}$ are updated according to

$$L_a^{k,\ell+1} \leftarrow L_a^{k,\ell} + \frac{\theta^\ell \omega_a^{k,\ell}}{\max\{\|\boldsymbol{u}^{k+1,\ell} - \boldsymbol{u}^k\|^2, 1\}} \tag{28}$$

$$L_b^{k,\ell+1} \leftarrow L_b^{k,\ell} + \frac{\theta^\ell \omega_b^{k,\ell}}{\max\{\|\boldsymbol{u}^{k+1,\ell} - \boldsymbol{u}^k\|^2, 1\}} \tag{29}$$

where

$$\omega_a^{k,\ell} = \max\{J(\boldsymbol{u}^{k+1,\ell}) - \beta^{k+1,\ell}, 0\} \tag{30}$$

$$\omega_b^{k,\ell} = \max\{\lambda_{\max}(\tilde{G}(\boldsymbol{u}^{k+1,\ell})) - \gamma^{k+1,\ell}\bar{\kappa}, \gamma^{k+1,\ell} - \lambda_{\min}(\tilde{G}(\boldsymbol{u}^{k+1,\ell})), 0\}, \tag{31}$$

and $\theta^\ell > 0$ is the stepsize. The inner loop will be terminated when the following conditions are satisfied:

$$J(\boldsymbol{u}^{k+1,\ell}) \leq \beta^{k+1,\ell} \tag{32}$$

$$\gamma^{k+1,\ell} I \preceq \tilde{G}(\boldsymbol{u}^{k+1,\ell}) \preceq \gamma^{k+1,\ell}\bar{\kappa} I. \tag{33}$$

Suppose the inner loop terminates at $\bar{\ell}_k$, let $\boldsymbol{u}^{k+1} \leftarrow \boldsymbol{u}^{k+1,\bar{\ell}_k}$, $\gamma^{k+1} \leftarrow \gamma^{k+1,\bar{\ell}_k}$ and $\beta^{k+1} \leftarrow \beta^{k+1,\bar{\ell}_k}$. The proposed algorithm is summarized in Algorithm 1.

Algorithm 1 The iterative proximal linearized algorithm without a priori Lipschitz constants

1: *Initialization*: set $k = 0$ and obtain $(\boldsymbol{u}^0, \gamma^0, \beta^0)$;
2: *Inner loop Initialization*: Set $\ell = 0$ and $L_a^{k,0} = L_b^{k,0} = 0$;
3: Obtain $(\boldsymbol{u}^{k+1,\ell}, \gamma^{k+1,\ell}, \beta^{k+1,\ell})$ by solving

$$(\boldsymbol{u}^{k+1,\ell}, \gamma^{k+1,\ell}, \beta^{k+1,\ell}) \leftarrow \arg\min_{\boldsymbol{u},\gamma,\beta} \beta + \tau\|\boldsymbol{u}-\boldsymbol{u}^k\|^2 \tag{34a}$$

$$\text{s.t.} \quad (\boldsymbol{u},\gamma,\beta) \in \Phi(\boldsymbol{u}^k; L_a^{k,\ell}, L_b^{k,\ell}); \tag{34b}$$

4: Obtain $\omega_a^{k,\ell}$ and $\omega_b^{k,\ell}$ from (30) and (31);
5: Update $L_a^{k,\ell+1}$ and $L_b^{k,\ell+1}$ according to (28) and (29);
6: If (32) and (33) are satisfied, terminate and let $\bar{\ell}_k = \ell$; otherwise, let $\ell \leftarrow \ell + 1$ and go to Step 3;
7: Let $\boldsymbol{u}^{k+1} \leftarrow \boldsymbol{u}^{k+1,\bar{\ell}_k}$, $\gamma^{k+1} \leftarrow \gamma^{k+1,\bar{\ell}_k}$ and $\beta^{k+1} \leftarrow \beta^{k+1,\bar{\ell}_k}$;
8: $k \leftarrow k + 1$ and go to Step 2.

The convergence properties in Theorem 1 still hold for Algorithm 1 if the inner loop terminate successfully. Hence, we only need to discuss the convergence of the inner loop, which is stated in the following lemma. This lemma also gives the motivation of the inner loop stopping criterion.

Proposition 1: For any $k \geq 0$, suppose $(\boldsymbol{u}^k, \gamma^k, \beta^k)$ is feasible to problem (26), let $\{\boldsymbol{u}^{k+1,\ell}, \gamma^{k+1,\ell}, \beta^{k+1,\ell}\}$ be generated from (34) and $\{L_a^{k,\ell}, L_b^{k,\ell}\}$ be generated from (28) and (29) with $L_a^{k,0} = 0$ and $L_b^{k,0} = 0$, and $\theta^\ell \geq c$ for some $c > 0$. Then, the following results hold.

(i) If $L_a^{k,\ell} \geq L_J$ for some $\ell > 0$, the solution $(\boldsymbol{u}^{k+1,\ell}, \gamma^{k+1,\ell}, \beta^{k+1,\ell})$ satisfies (32);if $L_b^{k,\ell} \geq L_{\tilde{G}}$ for some $\ell > 0$, the solution $(\boldsymbol{u}^{k+1,\ell}, \gamma^{k+1,\ell}, \beta^{k+1,\ell})$ satisfies (33).

(ii) There exists some finite ℓ such that (32) and (33) will be satisfied.

D. Feasible Initialization

As we have mentioned in the beginning of this section, the proposed algorithm requires a feasible initial solution. One way to find a feasible solution is to compute κ_{min} by solving (17). The optimal solution of (17) will be a feasible solution to (21) if $\kappa_{min} \leq \bar{\kappa}$. However, it is impractical to obtain κ_{min} due to the nonlinearity in (17). Usually, we can only get an upper bound of κ_{min} and use the upper bound to evaluate the feasibility of the solution.

A reformulation of problem (17) is given by

$$\min_{\boldsymbol{u}\in\boldsymbol{U},\alpha\geq 0,\eta\geq 0} \alpha \tag{35a}$$

$$\text{s.t.} \quad \eta I \preceq \tilde{G}(\boldsymbol{u}) \preceq \alpha\eta I. \tag{35b}$$

The additional variables α and η are coupled in (35b). To handle this issue, an alternating algorithm will be used. Suppose $(\tilde{\boldsymbol{u}}, \tilde{\alpha}, \tilde{\eta})$ is feasible to problem (35), the constraint (35b) can be linearized around $(\tilde{\boldsymbol{u}}, \tilde{\alpha}, \tilde{\eta})$ and the following linearized set is defined based on Lemma 2: $\bar{\Phi}(\tilde{\boldsymbol{u}}, \tilde{\eta}; L') := \{(\boldsymbol{u}, \alpha) : \boldsymbol{u} \in \boldsymbol{U}, \alpha \geq 0, (\tilde{\eta} + L'\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^2)I \preceq H_{\tilde{G}}(\boldsymbol{u}, \tilde{\boldsymbol{u}}) \preceq (\alpha\tilde{\eta} - L'\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^2)I\}$, for any $L' \geq 0$. It is easy to verify that this is a convex set. Based on this set, a proximal alternating algorithm is proposed in Algorithm 2 to solve problem (35). The convergence of Algorithm 2 can be guaranteed, although not mentioned due to page limitation.

Algorithm 2 The iterative proximal linearized alternating algorithm without an a priori Lipschitz constant

---

1: *Initialization*: set $k = 0$ and obtain $(\boldsymbol{u}^0, \alpha^0, \eta^0)$;
2: *Inner loop initialization*: set $\ell = 0$ and $L^{k,0} = 0$;
3: Obtain $(\boldsymbol{u}^{k+1,\ell}, \alpha^{k+1,\ell})$ from

$$(\boldsymbol{u}^{k+1,\ell}, \alpha^{k+1,\ell}) \leftarrow \arg\min_{(\boldsymbol{u},\alpha)} \alpha + \tau\|\boldsymbol{u} - \boldsymbol{u}^k\|^2 \tag{36a}$$

$$\text{s.t.} \quad (\boldsymbol{u}, \alpha) \in \bar{\Phi}(\boldsymbol{u}^k, \eta^k; L^{k,\ell}) \tag{36b}$$

4: Obtain $\omega^{k,\ell}$ and $L^{k,\ell+1}$ from

$$\omega^{k,\ell} \leftarrow \max\{\lambda_{\max}(\tilde{G}(\boldsymbol{u}^{k+1,\ell})) - \alpha^{k+1,\ell}\eta^k, \eta^k - \lambda_{\min}(\tilde{G}(\boldsymbol{u}^{k+1,\ell})), 0\} \tag{37}$$

$$L^{k,\ell+1} \leftarrow L^{k,\ell} + \frac{\theta^\ell \omega^{k,\ell}}{\max\{\|\boldsymbol{u}^{k+1,\ell} - \boldsymbol{u}^k\|^2, 1\}} \tag{38}$$

5: If $\eta^k I \preceq \tilde{G}(\boldsymbol{u}^{k+1,\ell}) \preceq \alpha^{k+1,\ell}\eta^k I$, terminate and let $\bar{\ell}_k = \ell$; otherwise, set $\ell \leftarrow \ell + 1$ and go to Step 3;
6: Let $\boldsymbol{u}^{k+1} \leftarrow \boldsymbol{u}^{k+1,\bar{\ell}_k}$ and $\alpha^{k+1} \leftarrow \alpha^{k+1,\bar{\ell}_k}$, and update η^{k+1} from

$$\eta^{k+1} \leftarrow \frac{\lambda_{\max}(\tilde{G}(\boldsymbol{u}^{k+1}))}{2\alpha^{k+1}} + \frac{\lambda_{\min}(\tilde{G}(\boldsymbol{u}^{k+1}))}{2} \tag{39}$$

7: Set $k \leftarrow k + 1$ and go to Step 2.

V. Application to the Beamline Calibration in Proton Therapy

Proton therapy uses a beam of protons to irradiate the diseased tissues. It is an attractive cancer treatment modality for its precision and low overall toxicity as opposed to the conventional X-ray radiotherapy. Recently, a compact proton therapy technology, called Proteus®ONE, has been developed by IBA, a world leader in proton therapy. Figure 1 shows the configuration of Proteus®ONE, which consists of the extraction beam line and the compact gantry. The gantry is equipped with energy selection and beam transport elements.

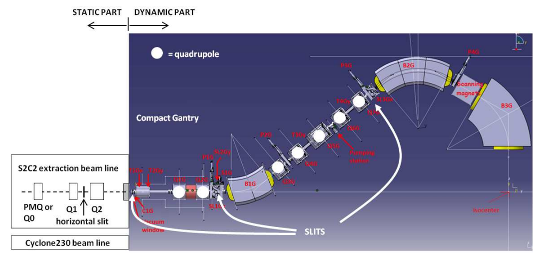


Fig. 1: Proteus®ONE Configuration

A therapeutic proton beam is typically specified by a number of characteristic quantities at the isocenter, where the tumor is positioned for treatment. The quantities include the size and the shape and they are determined by the transit of the beam through a number of beamline magnets, whose magnetic properties are set by the bias currents feeding the magnets. An important beam feature is the uniformity in the horizontal and vertical directions and it is mainly affected by the quadrupoles in the beamline as they are used to focus (or defocus) the beam. Since the whole beamline calibration is complicated, we restrict the problem to an easier one where only 5 quadruples(between B1G and B2G in Figure 1) are involved. We want to achieve an isotropy beam at exit of the last quadrupole. For the entire beamline, there are more elements and constraints. We will handle those using a similar idea. This is a work in progress.

The behavior of the protons can be described by dynamical equations, see [6] for details,

$$x_{t+1} = M_t(u_t)x_t, \quad 0 \leq t \leq 4 \tag{40}$$

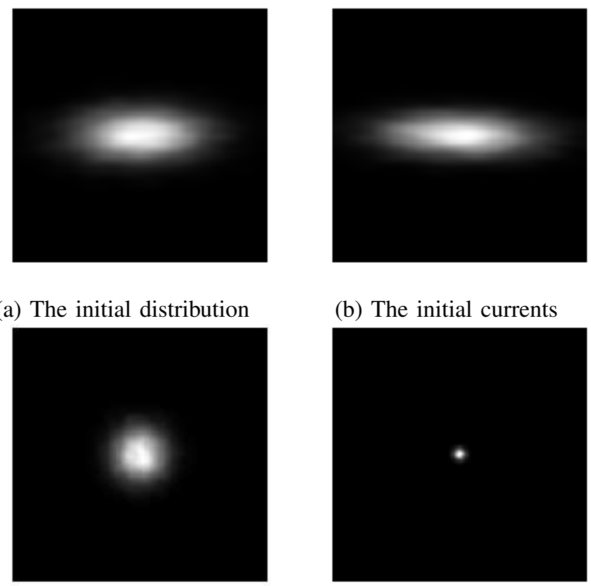
where x_t is the state of the proton, u_t is the current, and M_t denotes the transfer matrix of the t^{th} element.

In the simulation, the energy of the proton beam is set to be 200.23MeV and the maximal current is 100A. The time horizon in system (1) is set to be $N = 5$. The constraint is given by

$$\frac{\lambda_{max}(\Sigma_{y_5}) - \lambda_{min}(\Sigma_{y_5})}{\lambda_{min}(\Sigma_{y_5})} \leq 9\%, \text{ i.e., } \kappa(\Sigma_{y_5}) \leq 1.09$$

Let $Q = 0$, $R = 0$ and $P = I$ in (18). Before we solve (21), we first need to get a feasible solution using Algorithm 2. Let $\tau = 10^{-4}$ and $\theta^\ell = 0.1 + 0.01e^{0.3\ell}$. In the initialization step, $\boldsymbol{u}^0$ is set to be $\boldsymbol{u}_{init} = (4.265,\ 63.52,\ 28.19,\ 53.86,\ 69.52,\ 12.7609)$, η^0 is set to be $\lambda_{\min}(\tilde{G}(\boldsymbol{u}^0))$ and α^0 is set to be $\kappa(\tilde{G}(\boldsymbol{u}^0))$. The stopping criterion is $\|\boldsymbol{u}^{k+1} - \boldsymbol{u}^k\|/\|\boldsymbol{u}^k\| \leq 10^{-3}$. At the termination, a feasible solution $\boldsymbol{u}_f = (23.87,\ 77.53,\ 35.87,\ 48.25,\ 71.55)$ is obtained and the condition number is reduced from

12.7609 to 1. With the feasible initialization, we can use Algorithm 1 to solve (21). We also let $\tau = 10^{-4}$ and $\theta^{\ell} = 0.1 + 0.01e^{0.3\ell}$ in Algorithm 1. In the initialization step, let $\boldsymbol{u}^0$ be the feasible solution $\boldsymbol{u}_f$ from Algorithm 2, $\gamma^0 = \lambda_{\min}(\tilde{G}(\boldsymbol{u}^0))$ and $\beta^0 = J(\boldsymbol{u}^0)$. With the stopping criterion $\|\boldsymbol{u}^{k+1} - \boldsymbol{u}^k\|/\|\boldsymbol{u}^k\| \leq 10^{-5}$, the optimal solution $\boldsymbol{u}_{opt} = (0,\ 58.57,\ 44.05,\ 0,\ 100)$ is obtained. The cost function is reduced from 88.2353 to 4.3892. At the optimal solution, the condition number is 1.09. Hence, the constraint is satisfied. Note that the optimal solution in this section refers to the solution obtained from Algorithm 1 with the stopping criterion $\|\boldsymbol{u}^{k+1} - \boldsymbol{u}^k\|/\|\boldsymbol{u}^k\| \leq 10^{-5}$. For different sets of currents, the proton distributions along horizontal and vertical directions are shown in Figure 2.



(a) The initial distribution (b) The initial currents

(c) The feasible currents (d) The optimal currents

Fig. 2: The distributions of the proton beam for different sets of currents

VI. Conclusions

We studied the distribution steering of state-affine nonlinear systems using open-loop control. An isotropy constraint is imposed on the terminal distribution to achieve the uniformity in all directions. The nonlinear dynamics and the isotropy constraint result in a constrained nonlinear optimization problem. In order to ensure the feasibility of the solution, a sequential linearized algorithm is proposed without any a priori knowledge of the global Lipschitz property. In this paper, we also discuss the application of the proposed algorithm to the beamline calibration in proton therapy. Numerical simulation of a set of quadrupoles is made to demonstrate the performance of the proposed algorithm.

References

[1] L. C. A. Pimenta, N. Michael, R. C. Mesquita, G. A. S. Pereira, and V. Kumar. Control of swarms based on hydrodynamic models. In *IEEE International Conference on Robotics and Automation*, pages 1948–1953, 2008.

[2] S. Bandyopadhyay, S. J. Chung, and F. Y. Hadaegh. Probabilistic and distributed control of a large-scale swarm of autonomous agents. *IEEE Transactions on Robotics*, 33(5):1103–1123, 2017.

[3] J. S. Li and N. Khaneja. Ensemble control of bloch equations. *IEEE Transactions on Automatic Control*, 54(3):528–536, 2009.

[4] K. L. Brown, F. Rothacker, D. C. Carey, and F. C. Iselin. Transport: a computer program for designing charged-particle beam-transport systems. Technical report, SLAC-R-91-REV-3, 1980.

[5] W. P. Levin, H. Kooy, J. S. Loeffler, and T. F. DeLaney. Proton beam therapy. *British journal of Cancer*, 93(8):849, 2005.

[6] H. Wiedemann. *Particle accelerator physics*. Springer, 2015.

[7] A. Komaee and P. I. Barton. Potential canals for control of nonlinear stochastic systems in the absence of state measurements. *IEEE Transactions on Control Systems Technology*, 25(1):161–174, 2017.

[8] C. Villani. *Optimal transport: old and new*, volume 338. Springer Science & Business Media, 2008.

[9] A. Hotz and R. E. Skelton. A covariance control theory. In *Proceedings of the 24th IEEE Conference on Decision and Control*, page 552–557, 1985.

[10] A. Hotz and R. E. Skelton. Covariance control theory. *International Journal of Control*, 46(1):13–32, 1987.

[11] E. Collins and R. Skelton. A theory of state covariance assignment for discrete systems. *IEEE Transactions on Automatic Control*, 32(1):35–41, 1987.

[12] J. H. Xu and R. E. Skelton. An improved covariance assignment theory for discrete systems. *IEEE transactions on Automatic Control*, 37(10):1588–1591, 1992.

[13] K. Yasuda, R. E. Skelton, and K. M. Grigoriadis. Covariance controllers: A new parametrization of the class of all stabilizing controllers. *Automatica*, 29(3):785–788, 1993.

[14] K. M. Grigoriadis and R. E. Skelton. Minimum-energy covariance controllers. *Automatica*, 33(4):569–578, 1997.

[15] Y. Chen, T. T. Georgiou, and M. Pavon. Optimal steering of a linear stochastic system to a final probability distribution, part i. *IEEE TRANSACTIONS ON AUTOMATIC CONTROL*, 61(5), 2016.

[16] Y. Chen, T. T. Georgiou, and M. Pavon. Optimal steering of a linear stochastic system to a final probability distribution, part ii. *IEEE Transactions on Automatic Control*, 61(5):1170–1180, 2016.

[17] E. Bakolas. Optimal covariance control for discrete-time stochastic linear systems subject to constraints. In *Proceedings of 55th Conference on Decision and Control*, pages 1153–1158, 2016.

[18] Maxim Goldshtein and Panagiotis Tsiotras. Finite-horizon covariance control of linear time-varying systems. In *Proceedings of the 56th IEEE Conference on Decision and Control*, pages 3606–3611. IEEE, 2017.

[19] E. Bakolas. Finite-horizon covariance control for discrete-time stochastic linear systems subject to input constraints. *Automatica*, 91:61–68, 2018.

[20] A. Auslender, R. Shefi, and M. Teboulle. A moving balls approximation method for a class of smooth constrained minimization problems. *SIAM Journal on Optimization*, 20(6):3232–3259, 2010.

[21] F. Facchinei, L. Lampariello, and G. Scutari. Feasible methods for nonconvex nonsmooth problems with applications in green communications. *Mathematical Programming*, 164(1-2):55–90, 2017.

[22] W. J. Vetter. Matrix calculus operations and taylor expansions. *SIAM review*, 15(2):352–369, 1973.

[23] D. P. Bertsekas. *Convex optimization theory*. Athena Scientific, 2009.

[24] E. Asarin, T. Dang, and A. Girard. Reachability analysis of nonlinear systems using conservative approximation. In *International Workshop on Hybrid Systems: Computation and Control*, pages 20–35. Springer, 2003.

[25] J. K. Scott and P. I. Barton. Bounds on the reachable sets of nonlinear control systems. *Automatica*, 49(1):93–100, 2013.

[26] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics (SIAM), 1994.

[27] Michael Grant and Stephen Boyd. CVX: Matlab software for disciplined convex programming, version 2.1. http://cvxr.com/cvx, 2014.

[28] E. D. Andersen, C. Roos, and T. Terlaky. On implementing a primal-dual interior-point method for conic quadratic optimization. *Mathematical Programming*, 95(2):249–277, 2003.