PIP-space valued reproducing pairs of measurable functions

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Abstract

We analyze the notion of reproducing pair of weakly measurable functions, a generalization of continuous frame. The aim is to represent elements of an abstract space Y as superpositions of weakly measurable functions belonging to a space $V = V(X, \mu)$, where (X, μ) is a measure space. Three cases are envisaged, with increasing generality: (i) Y and V are both Hilbert spaces; (ii) Y is a Hilbert space, but V is a PIP-space; (iii) Y and V are both PIP-spaces. It is shown, in particular, that the requirement that a pair of measurable functions be reproducing strongly constraints the structure of the initial space Y. Examples are presented for each case.

1 Introduction

Representing functions in terms of simple ones, with a small number of them, if possible, is a standard problem in analysis, in particular in signal and image processing, where transmission imposes severe constraints. Signals are usually taken as square integrable functions on some manifold, hence they constitute a Hilbert space.

More precisely, given a separable Hilbert space $\mathcal{H}$, one wishes to represent an arbitrary element $f \in \mathcal{H}$ by a superposition of simpler, basic elements $\Psi = (\psi_k)$, $k \in \Gamma$, with Γ a countable index set:

$$f = \sum_{k \in \Gamma} c_k \psi_k, . \tag{1.1}$$

One usually requires that the sum converges adequately (e.g. in norm or unconditionally) and that the coefficients c_k are (preferably) unique and easy to compute. There are many possibilities for obtaining that result, the simplest ones being that Ψ be an orthonormal basis or a Riesz basis.

These two notions indeed solve the problem, but they are very rigid and lead usually to slowly converging infinite expansions. Thus frames were introduced for ensuring a better flexibility, originally in 1952 by Duffin and Schaeffer [17] in the context of nonharmonic analysis. The notion was revived by Daubechies, Grossmann and Meyer [15] in the context of wavelet theory and then became a very popular topic, in particular in Gabor and wavelet analysis [13, 14, 16, 20]. The reason is that a good frame in a Hilbert space is almost as good as an orthonormal basis for expanding arbitrary elements (albeit non-uniquely) and is often easier to construct. Let us first recall that a sequence $\Psi = (\psi_k)$ is a discrete frame for a Hilbert space $\mathcal{H}$ if there exist constants $0 < \mathsf{m} \leq \mathsf{M} < \infty$ (the frame bounds) such that

$$\mathsf{m} \, \|f\|^2 \leq \sum_{k \in \Gamma} |\langle f|\psi_k \rangle|^2 \leq \mathsf{M} \, \|f\|^2 , \forall f \in \mathcal{H}. \tag{1.2}$$

As a matter of fact, most frames considered in applications are discrete, for instance in signal or image processing [14]. Yet continuous frames offer interesting mathematical problems. They have been introduced originally by Ali, Gazeau and one of us [1, 2] and also, independently, by Kaiser [24]. Since then, many papers dealt with various aspects of the concept, see for instance [10, 18, 19] or [26].

However, there are situations where it is impossible to satisfy both frame bounds at the same time. Therefore, several generalizations of frames have been introduced. Semi-frames [4, 5], for example, are obtained when functions only satisfy one of the two frame bounds. Thus one speaks of upper or lower semi-frames. It turns out that a large portion of frame theory can be extended to this larger framework.

More recently, a new generalization of frames was introduced by Balazs and Speckbacher [28], called reproducing pairs. Here, given a measure space (X, μ), one considers a couple of weakly measurable functions (ψ, ϕ), instead of a single mapping, and one studies the correlation between the two (a precise definition is given below). This definition also includes the original definition of a continuous frame [1, 2] to which it reduces when $\psi = \phi$. The choice of the mappings ψ and ϕ gives more freedom, but it leads to the problem of characterizing the range of the analysis operators, which in general need no more be contained in $L^2(X, \mathrm{d}\mu)$, as in the frame case. Therefore, it is natural to extend the theory to the case where the weakly measurable functions take their values in a partial inner product space (PIP-space) [3]. Actually we will go further and consider a more general construction. Namely we want to represent elements of an abstract space $f \in Y$ by weakly measurable functions $(C_\psi f)(x) := \langle f|\psi_x\rangle$, belonging to a space $V(X, \mu)$, in such a way that the (formal) inner product

$$\langle C_\psi f|C_\phi g\rangle = \int_X (C_\psi f)(x)\overline{(C_\phi g)(x)}\,\mathrm{d}\mu(x) \tag{1.3}$$

is well defined. Nevertheless, since the techniques needed in this general case are very similar to those used in the previous ones, we have sketched them in these simpler cases first, in order to make the paper self-contained. Further information may be found in the original papers or in the review paper [9] that we follow closely.

The paper is organized as follows. In Section 2, we review the notions of frames, semi-frames and reproducing pairs and we recall their salient properties. In Section 3, we consider the case where Y and V are both Hilbert spaces, which corresponds to genuine frames. In particular we construct and analyze the so-called coefficient spaces $W_\psi(X, \mu), W_\phi(X, \mu)$, which are Hilbert spaces in conjugate duality when (ψ, ϕ) are a reproducing pair. In Section 4, motivated by the relation (1.3), we take for target space a PIP-space $V = \{V_p(X, \mu)\}$, in particular a lattice of Hilbert or Banach spaces (LHS/LBS). In Section 5, we go one step further and take for the initial space another PIP-space $Y = \{Y_u\}$. In particular, we see how the requirement of having a reproducing pair affects the structure of the initial space Y. Finally, we conclude by an Appendix, in which we recall the main definitions about PIP-spaces (LHS/LBS) and operators on them.

2 Preliminaries

Before proceeding, we list our definitions and conventions. We work in a (separable) Hilbert space $\mathcal{H}$, with the inner product $\langle\cdot|\cdot\rangle$ linear in the first factor. If A is an operator on $\mathcal{H}$, we denote its domain by $D(A)$, its range by $\mathsf{Ran}\,(A)$ and its kernel by $\mathsf{Ker}\,(A)$. The set of all invertible bounded operators on $\mathcal{H}$ with bounded inverse is denoted $GL(\mathcal{H})$ We will consider throughout weakly measurable functions $\psi : X \to \mathcal{H}$, where (X, μ) is a locally compact, σ-compact space with a Radon measure μ, i.e., $\langle f|\psi_x\rangle$ is μ-measurable for every $f \in \mathcal{H}$.

The weakly measurable function ψ is called *continuous frame* if there exist constants $\mathsf{m} > 0$ and $\mathsf{M} < \infty$ (the frame bounds) such that

$$\mathsf{m}\,\|f\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2\,\mathrm{d}\mu(x) \leq \mathsf{M}\,\|f\|^2\,, \forall\, f \in \mathcal{H}. \tag{2.1}$$

Remark 2.1 The geometry of the Hilbert space imposes a number of constraints that severely limit the existence of continuous families acting as bases or frames. Apart from the case of a separable Hilbert space where orthonormal or Riesz bases cannot be uncountable, there are more general situations where, for instance, Riesz bases cannot be continuous, but they are necessarily *discrete*, in a certain sense [21, 22, 29]. Recently a similar result has also been shown for frames [12]. Nevertheless, this result is essentially of theoretical nature and does not affect the interest for continuous frames in applications. On the other hand, continuous frame do really exist in more general frameworks (rigged Hilbert space, PIP-space) as shown in [8, 11, 30] and in the present paper. This fact constitutes a further motivation for *going beyond Hilbert spaces.*

Given the continuous frame ψ, we define the *analysis* operator $C_\psi : \mathcal{H} \to L^2(X, \mathrm{d}\mu)$ [1] as

$$(C_\psi f)(x) = \langle f|\psi_x\rangle, \; f \in \mathcal{H}, \tag{2.2}$$

and the corresponding *synthesis operator* $C_\psi^* : L^2(X, \mathrm{d}\mu) \to \mathcal{H}$ as

$$C_\psi^* \xi = \int_X \xi(x)\, \psi_x \; \mathrm{d}\mu(x), \; \text{for } \; \xi \in L^2(X, \mathrm{d}\mu). \tag{2.3}$$

(the integral being understood in the weak sense, as usual). We set $S_\psi := C_\psi^* C_\psi$, *i.e.,*

$$\langle f|S_\psi f\rangle = \int_X |\langle f|\psi_x\rangle|^2 \; \mathrm{d}\mu(x). \tag{2.4}$$

Thus the so-called *frame or resolution operator* S_ψ is self-adjoint, invertible, bounded with bounded inverse S_ψ^{-1}, that is, $S_\psi \in GL(\mathcal{H})$.

In particular, if X is a discrete set with μ being a counting measure, we recover the standard definition (1.2) of a (discrete) frame [13, 14, 17].

The weakly measurable function ψ is said to be μ*-total* if $\langle \psi_x|g\rangle = 0$, μ-a.e., implies $g = 0$, that is, $\mathsf{Ker}\, C_\phi = \{0\}$. Following [4, 5], we will say that a family $\Psi = \{\psi_x\}$ is an *upper (resp. lower) semi-frame*, if it is μ-total in $\mathcal{H}$ and satisfies the upper (resp. lower) frame inequality. For the sake of completeness, we recall the definitions. A weakly measurable function ψ is an *upper semi-frame* if there exists $\mathsf{M} < \infty$ such that

$$0 < \int_X |\langle f|\psi_x\rangle|^2 \; \mathrm{d}\mu(x) \leq \mathsf{M} \, \|f\|^2 \,, \; \forall\, f \in \mathcal{H}, \; f \neq 0. \tag{2.5}$$

Note that an upper semi-frame is also called a (total) Bessel mapping [19].

On the other hand, a function ϕ_x is called a *lower semi-frame* if it satisfies the lower frame condition, that is, there exists a constant $\mathsf{m} > 0$ such that

$$\mathsf{m} \, \|f\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2 \; \mathrm{d}\mu(x), \;\; \forall\, f \in \mathcal{H}. \tag{2.6}$$

Clearly, (2.6) implies that the family $\Phi = \{\phi_x\}$ is μ-total in $\mathcal{H}$.

If ψ is a measurable function, the operator C_ψ, formally defined as for frames, takes values in the space of measurable functions on (X, μ) and the adjoint C_ψ^* is, in general, meaningless. C_ψ has a natural domain in Hilbert space, namely,

$$D(C_\psi) = \{f \in \mathcal{H} : \langle f|\psi_\cdot\rangle \in L^2(X, \mathrm{d}\mu)\}$$

but this domain, in general, is not dense and nothing guarantees that it does not reduce to $\{0\}$.

If the measurable function ψ is Bessel or an upper semi-frame , the definition implies that $\mathsf{Ran}\,(C_\psi)$ is contained in $L^2(X, \mathrm{d}\mu)$. On the contrary, if a measurable function ϕ is a lower semi-frame, $\mathsf{Ran}\,(C_\psi)$ contains $L^2(X, \mathrm{d}\mu)$.

[1] As usual, we identify a function f with its residue class in $L^2(X, \mathrm{d}\mu)$.

In the lower case, the definition of S_ψ must be changed, since C_ψ need not be densely defined, so that C_ψ^* may not exist. Instead, following [4, Sec.2] one defines the synthesis operator as

$$D_\psi F = \int_X F(x)\, \psi_x \;\, \mathrm{d}\mu(x), \quad F \in L^2(X, \mathrm{d}\mu), \tag{2.7}$$

on the domain of all elements F for which the integral in (2.7) converges weakly in $\mathcal{H}$, and then $S_\psi := D_\psi C_\psi$. With this definition, it is shown in [4, Sec.2] that S_ψ is unbounded and S_ψ^{-1} is bounded.

All these objects are studied in detail in our previous papers [4, 5]. In particular, it is shown there that a natural notion of duality exists, namely, two weakly measurable functions ψ, ϕ are dual to each other (the relation is symmetric) if one has

$$\langle f|g\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \;\, \mathrm{d}\mu(x), \; \forall\, f, g \in \mathcal{H}.$$

A new generalization of frames was introduced recently by Balazs and Speckbacher [28], namely, reproducing pairs. Given a measure space (X, μ), one considers a couple of weakly measurable functions (ψ, ϕ), instead of a single mapping. The advantage is that no further conditions are imposed on these functions, which results in an increased flexibility.

More precisely, the couple of weakly measurable functions (ψ, ϕ) is called a *reproducing pair* [7, 28] if (i) The sesquilinear form

$$\Omega_{\psi,\phi}(f, g) = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\, \mathrm{d}\mu(x) \tag{2.8}$$

is well-defined and bounded on $\mathcal{H} \times \mathcal{H}$, that is, $|\Omega_{\psi,\phi}(f, g)| \leq c\, \|f\|\, \|g\|$, for some $c > 0$ and every $f, g \in \mathcal{H}$; and (ii) The corresponding bounded (resolution) operator $S_{\psi,\phi}$ belongs to $GL(\mathcal{H})$.

Under these hypotheses, one has

$$S_{\psi,\phi} f = \int_X \langle f|\psi_x\rangle \phi_x\, \mathrm{d}\mu(x), \; \forall\, f \in \mathcal{H}, \tag{2.9}$$

the integral on the r.h.s. being defined in weak sense. If $\psi = \phi$, we recover the notion of continuous frame, as introduced in [1, 2], so that we have indeed a genuine generalization of the latter.

Notice that, if $\psi \neq \phi$, the frame operator $S_{\psi,\phi}$ is in general neither positive, nor self-adjoint, since $S_{\psi,\phi}^* = S_{\phi,\psi}$. However, if (ψ, ϕ) is a reproducing pair, then $(\psi, S_{\psi,\phi}^{-1}\phi)$ is also a reproducing pair, for which the corresponding frame operator is the identity, that is, ψ and ϕ are in duality. Thus, there is no restriction of generality to assume that $S_{\phi,\psi} = I$ [28]. The only thing that can happen is to replace some norms by equivalent ones.

3 The case $Y = \mathcal{H}$ and $V = L^2(X, \mu)$, both Hilbert spaces

As mentioned in Section 1, we begin with the case where the initial space $Y = \mathcal{H}$ and the target space $V = L^2(X, \mu)$ are both Hilbert spaces, which is characteristic of a reproducing pair as originally defined [28].

Given a weakly measurable function ϕ, let us denote by $K_\phi(X, \mu)$ the space of all measurable functions $\xi : X \to \mathbb{C}$ such that the integral $\int_X \xi(x)\langle\phi_x|g\rangle\, \mathrm{d}\mu(x)$ exists for every $g \in \mathcal{H}$ and defines a bounded conjugate linear functional on $\mathcal{H}$, i.e., $\exists\, c > 0$ such that

$$\left| \int_X \xi(x)\langle\phi_x|g\rangle\, \mathrm{d}\mu(x) \right| \leq c\, \|g\|\, , \; \forall\, g \in \mathcal{H}. \tag{3.1}$$

Clearly, if (ψ, ϕ) is a reproducing pair, all functions $\xi(x) = \langle f|\psi_x\rangle = (C_\psi f)(x)$ belong to $K_\phi(X, \mu)$.

By the Riesz lemma, we can define a linear map $T_\phi : K_\phi(X,\mu) \to \mathcal{H}$, which we call the *synthesis operator*, by the weak relation

$$\langle T_\phi \xi | g\rangle = \int_X \xi(x)\langle \phi_x | g\rangle \, \mathrm{d}\mu(x), \ \forall\, \xi \in K_\phi(X,\mu), g \in \mathcal{H}. \tag{3.2}$$

Next, we define the vector space

$$W_\phi(X,\mu) = K_\phi(X,\mu)/\mathsf{Ker}\ T_\phi \tag{3.3}$$

and equip it with the norm

$$\|[\xi]_\phi\|_\phi := \sup_{\|g\|\leq 1} \left| \int_X \xi(x)\langle \phi_x | g\rangle \, \mathrm{d}\mu(x) \right| = \sup_{\|g\|\leq 1} |\langle T_\phi \xi | g\rangle| \,, \tag{3.4}$$

where we have put $[\xi]_\phi = \xi + \mathsf{Ker}\ T_\phi$ for $\xi \in K_\phi(X,\mu)$. Clearly, $W_\phi(X,\mu)$ is a normed space, called the *coefficient space* of ϕ. However, the norm $\|\cdot\|_\phi$ derives from an inner product. First, the map $\widehat{T}_\phi : W_\phi(X,\mu) \to \mathcal{H}$, $\widehat{T}_\phi[\xi]_\phi := T_\phi \xi$ is an isometry of $W_\phi(X,\mu)$ into $\mathcal{H}$. Next, one defines on $W_\phi(X,\mu)$ an inner product by setting

$$\langle [\xi]_\phi | [\eta]_\phi \rangle_{(\phi)} := \langle \widehat{T}_\phi [\xi]_\phi | \widehat{T}_\phi [\eta]_\phi \rangle, \ [\xi]_\phi, [\eta]_\phi \in W_\phi(X,\mu).$$

Finally, the norm defined by $\langle \cdot|\cdot\rangle_{(\phi)}$ coincides with the norm $\|\cdot\|_\phi$ defined in (3.4), since one has indeed

$$\|[\xi]_\phi\|_{(\phi)} = \left\| \widehat{T}_\phi [\xi]_\phi \right\| = \|T_\phi \xi\| = \sup_{\|g\|\leq 1} |\langle T_\phi \xi | g\rangle| = \|[\xi]_\phi\|_\phi \,.$$

Thus $W_\phi(X,\mu)$ is a pre-Hilbert space.

We denote by $W_\phi(X,\mu)^*$ the Hilbert dual space of $W_\phi(X,\mu)$, that is, the set of continuous linear functionals on $W_\phi(X,\mu)$. The (dual) norm $\|\cdot\|_{\phi^*}$ of $W_\phi(X,\mu)^*$ is defined, as usual, by

$$\|F\|_{\phi^*} = \sup_{\|[\xi]_\phi\|_\phi \leq 1} |F([\xi]_\phi)|.$$

Now we define a conjugate linear map $\mathsf{C}_\phi : \mathcal{H} \to W_\phi(X,\mu)^*$ by

$$(\mathsf{C}_\phi f)([\xi]_\phi) := \int_X \xi(x)\langle \phi_x | f\rangle \, \mathrm{d}\mu(x), \ f \in \mathcal{H}. \tag{3.5}$$

Of course, (3.5) means that $(\mathsf{C}_\phi f)([\xi]_\phi) = \langle T_\phi \xi | f\rangle = \langle \widehat{T}_\phi [\xi]_\phi | f\rangle$, for every $f \in \mathcal{H}$. Thus $\mathsf{C}_\phi = \widehat{T}_\phi^*$, the adjoint map of $\widehat{T}_\phi$. By (3.1) it follows that C_ϕ is continuous . This implies that

$$\mathcal{H} = [\mathsf{Ran}\, \widehat{T}_\phi]^{\sim} \oplus \mathsf{Ker}\, \mathsf{C}_\phi, \tag{3.6}$$

where the first summand denotes the closure of $\mathsf{Ran}\, \widehat{T}_\phi$. Hence $\mathsf{C}_\phi^* = \widehat{T}_\phi^{**} = \widehat{T}_\phi$, if $W_\phi(X,\mu)$ is complete.

By modifying in an obvious way the definition given in Section 2, we say that ϕ is *μ-total* if $\mathsf{Ker}\, \mathsf{C}_\phi = \{0\}$.

A first preliminary result reads as follows.

Proposition 3.1 *If (ψ,ϕ) is a reproducing pair, then $\mathsf{Ran}\, \widehat{T}_\phi = \mathcal{H}$.*

For the proof, notice that, since $S_{\psi,\phi} \in GL(\mathcal{H})$, for every $h \in \mathcal{H}$, there exists a unique $f \in \mathcal{H}$ such that $S_{\psi,\phi} f = h$. By (2.9), we can conclude that

$$\langle h | g\rangle = \int_X \langle f | \psi_x\rangle \langle \phi_x | g\rangle \, \mathrm{d}\mu(x), \ \forall\, g \in \mathcal{H},$$

that is, $h = \widehat{T}_\phi [C_\psi f]_\phi \in \mathsf{Ran}\, \widehat{T}_\phi$. Notice that, if (ψ,ϕ) is a reproducing pair, both functions are necessarily μ-total, which already implies that $\mathsf{Ran}\, \widehat{T}_\phi$ is dense in $\mathcal{H}$, by (3.6).

The crucial step for characterizing the elements of $W_\phi(X,\mu)^*$ is the following result.

Theorem 3.2 *If (ψ, ϕ) is a reproducing pair, then every bounded linear functional F on $W_\phi(X, \mu)$, i.e., $F \in W_\phi(X, \mu)^*$, can be represented as*

$$F([\xi]_\phi) = \int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x),\ \forall\,[\xi]_\phi \in W_\phi(X, \mu), \tag{3.7}$$

with $\eta \in K_\psi(X, \mu)$. The residue class $[\eta]_\psi \in W_\psi(X, \mu)$ is uniquely determined.

Proof. We proceed in several steps. First, if F is a continuous linear functional on $W_\phi(X, \mu)$, then there exists a unique $g \in \mathcal{H}$, such that

$$F([\xi]_\phi) = \int_X \xi(x)\langle \phi_x | g\rangle\,\mathrm{d}\mu(x),\ \forall\,\xi \in K_\phi(X, \mu). \tag{3.8}$$

Indeed, given $F \in W_\phi(X, \mu)^*$, there exists $c > 0$ such that

$$|F([\xi]_\phi)| \leq c\,\|[\xi]_\phi\|_\phi = c\,\|T_\phi \xi\|\,,\ \forall\,\xi \in K_\phi(X, \mu).$$

Let $\widetilde{F}$ be the continuous functional on $\mathcal{H}$ defined by

$$\widetilde{F}(T_\phi \xi) := F([\xi]_\phi),\ \xi \in K_\phi(X, \mu).$$

The functional $\widetilde{F}$ is well-defined. Indeed, if $T_\phi\xi = T_\phi\xi'$, then $\xi - \xi' \in \mathsf{Ker}\ T_\phi$. Hence, $[\xi]_\phi = [\xi']_\phi$ and $F([\xi]_\phi) = F([\xi']_\phi)$. Thus there exists a unique $g \in \mathcal{H}$ such that

$$\widetilde{F}(T_\phi \xi) = \langle \widehat{T}_\phi[\xi]_\phi | g\rangle = \int_X \xi(x)\langle \phi_x | g\rangle\,\mathrm{d}\mu(x)$$

and $\|g\| = \|\widetilde{F}\|$. In conclusion, (3.8) holds true and $\|F\|_{\phi^*} = \|g\|$.

Conversely, every $g \in \mathcal{H}$ obviously defines a continuous linear functional F by (3.8) since $|F([\xi]_\phi)| \leq \|g\|\,\|[\xi]_\phi\|_\phi$ for every $[\xi]_\phi \in W_\phi(X, \mu)$.

Finally, since $\eta(x) := \langle g|\phi_x\rangle \in K_\psi(X, \mu)$, we have the representation (3.7). Uniqueness is immediate. $\square$

Remark 3.3 Several of the previous statements hold true without the assumption that (ψ, ϕ) is a reproducing pair, see [7, Sec.3] for details.

The lesson of Theorem 3.2 is that the map

$$\Lambda : F \in W_\phi(X, \mu)^* \mapsto [\eta]_\psi \in W_\psi(X, \mu) \tag{3.9}$$

is well-defined and conjugate linear. On the other hand, $\Lambda(F) = \Lambda(F')$ implies easily $F = F'$. Therefore $W_\phi(X, \mu)^*$ can be identified with a closed subspace of $\overline{W_\psi}(X, \mu) := \{\overline{[\xi]_\psi} : \xi \in K_\psi(X, \mu)\}$, where the overbar denotes the complex conjugate.

Actually, there is more: if (ψ, ϕ) is a reproducing pair, $W_\phi(X, \mu)^*$ and $\overline{W_\psi}(X, \mu)$ can be identified. The proof relies on two technical lemmas which can be found in [7, Lemmas 3.10 and 3.11]. These imply that the map Λ defined in (3.9) is surjective. Hence, $W_\phi(X, \mu)^* \simeq \overline{W_\psi}(X, \mu)$, where $\simeq$ denotes a bounded isomorphism and the norm $\|\cdot\|_\psi$ is the dual norm of $\|\cdot\|_\phi$. Finally, for every $\eta \in W_\psi(X, \mu)$, there exists $g \in \mathcal{H}$ such that $\eta(x) = \langle \phi_x | g\rangle$. In conclusion, one obtains the main result of [7], namely:

Theorem 3.4 *If (ψ, ϕ) is a reproducing pair, the spaces $W_\phi(X, \mu)$ and $W_\psi(X, \mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form*

$$\langle \xi | \eta \rangle := \int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x), \tag{3.10}$$

which coincides with the inner product of $L^2(X, \mu)$ whenever the latter makes sense.

This is true, in particular, for $\phi = \psi$, since then ψ is a continuous frame and $W_\psi(X, \mu)$ is a closed subspace of $L^2(X, \mu)$.

In addition, the converse of Theorem 3.4 holds if ϕ and ψ are both μ-total: in that case, (ψ, ϕ) is a reproducing pair if and only if $W_\phi(X, \mu)$ and $W_\psi(X, \mu)$ are Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form (3.10).

4 The case Y= Hilbert space and $V = \{V_p(X, \mu)\}$ = PIP-space

Clearly, the inner product (3.10) need not always be defined, it is a *partial* inner product on the space of measurable functions on (X, μ). This obviously suggests to take for $V(X, \mu)$ a PIP-space around $L^2(X, \mu)$, as we have proposed in [8], that we mostly follow in this section. As for the relevant notions about PIP-spaces, we have collected them in the Appendix.

4.1 The case where V is a rigged Hilbert space

The simplest example of a PIP-space is a rigged Hilbert space (RHS). Let indeed $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a RHS with $\mathcal{D}[t]$ reflexive (hence t and $t^\times$ coincide with the respective Mackey topologies). Given the measure space (X, μ), we denote by $\langle \cdot, \cdot \rangle$ the sesquilinear form expressing the duality between $\mathcal{D}$ and $\mathcal{D}^\times$. This form replaces the inner product $\langle \cdot | \cdot \rangle$ used so far. As usual, we suppose that this sesquilinear form extends the inner product of $\mathcal{D}$ (and $\mathcal{H}$). This allows to build the triplet above.

Let $x \in X \mapsto \psi_x$, $x \in X \mapsto \phi_x$ be weakly measurable functions from X into $\mathcal{D}^\times$. Instead of (2.8), we consider the sesquilinear form

$$\Omega^{\mathcal{D}}_{\psi,\phi}(f, g) = \int_X \langle f, \psi_x \rangle \langle \phi_x, g \rangle \, \mathrm{d}\mu(x), \ f, g \in \mathcal{D}, \tag{4.1}$$

and we assume that it is jointly continuous on $\mathcal{D} \times \mathcal{D}$. Writing

$$\langle S_{\psi,\phi} f, g \rangle := \int_X \langle f, \psi_x \rangle \langle \phi_x, g \rangle \, \mathrm{d}\mu(x), \ \forall \, f, g \in \mathcal{D}, \tag{4.2}$$

we see that the operator $S_{\psi,\phi}$ belongs to $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$, the space of all continuous linear maps from $\mathcal{D}$ into $\mathcal{D}^\times$.

At this point, we have a choice. A first possibility is to require that the sesquilinear form $\Omega^{\mathcal{D}}$ be well-defined and bounded on $\mathcal{D} \times \mathcal{D}$ in the topology of $\mathcal{H}$. Then $\Omega^{\mathcal{D}}_{\psi,\phi}$ extends to a bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$ and the discussion of Section 3 may be essentially repeated verbatim. Thus this choice gives almost nothing new.

Another possibility is to assume that the form $\Omega^{\mathcal{D}}$ is jointly continuous on $\mathcal{D} \times \mathcal{D}$, with no other regularity requirement. In that case, the vector space $K_\phi(X, \mu)$ must be defined differently, taking into account the (locally convex) topology of $\mathcal{D}$ and $\mathcal{D}^\times$. Instead of (3.1), we require that the integral $\int_X \xi(x) \langle \phi_x, g \rangle \, \mathrm{d}\mu(x)$ exists for every $g \in \mathcal{D}$ and defines a continuous conjugate linear functional on $\mathcal{D}$. As before, we define (weakly) a linear map $T_\phi : K_\phi(X, \mu) \to \mathcal{D}^\times$ by the following relation

$$\langle T_\phi \xi, g \rangle = \int_X \xi(x) \langle \phi_x, g \rangle \, \mathrm{d}\mu(x), \ \forall \, \xi \in K_\phi(X, \mu), g \in \mathcal{D}. \tag{4.3}$$

Next, we define again the vector space $W_\phi(X, \mu) = K_\phi(X, \mu)/\mathsf{Ker}\, T_\phi$. In order to introduce a topology on $W_\phi(X, \mu)$, we consider a bounded subset $\mathcal{M}$ of $\mathcal{D}[t]$ and we define the seminorm

$$\widehat{\mathsf{p}}_{\mathcal{M}}([\xi]_\phi) := \sup_{g \in \mathcal{M}} |\langle T_\phi \xi, g \rangle| \, . \tag{4.4}$$

This means, we are defining the topology of $W_\phi(X, \mu)$ by means of the strong dual topology $t^\times$ of $\mathcal{D}^\times$ which we recall is defined by the seminorms

$$\|F\|_{\mathcal{M}} = \sup_{g \in \mathcal{M}} |\langle F | g \rangle|, \quad F \in \mathcal{D}^\times,$$

where $\mathcal{M}$ runs over the family of bounded subsets of $\mathcal{D}[t]$. As said above, the reflexivity of $\mathcal{D}$ entails that $t^\times$ is equal to the Mackey topology $\tau(\mathcal{D}^\times, \mathcal{D})$. Thus $W_\phi(X, \mu)$ is a locally convex space.

Next we consider the dual $W_\phi(X,\mu)^*$ of the space $W_\phi(X,\mu)$, that is, the set of continuous linear functionals on $W_\phi(X,\mu)$. As topology on $W_\phi(X,\mu)^*$, we take the strong dual topology. This being done, almost all the results of Section 3 may be deduced. In particular, Theorem 3.2 remains true, under the form

Theorem 4.1 *Under the condition* (4.1), *every continuous linear functional F on $W_\phi(X,\mu)$, i.e., $F \in W_\phi(X,\mu)^*$, can be represented as*

$$F([\xi]_\phi) = \int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x),\ \forall\,[\xi]_\phi \in W_\phi(X,\mu), \tag{4.5}$$

with $\eta \in K_\psi(X,\mu)$. The residue class $[\eta]_\psi \in W_\psi(X,\mu)$ is uniquely determined.

As in the previous case, one may prove again that $W_\phi(X,\mu)^*$ can be identified with a closed subspace of $\overline{W_\psi}(X,\mu) := \{\overline{[\xi]_\psi} : \xi \in K_\psi(X,\mu)\}$, but we cannot go further. Indeed the previous results rely heavily on Hilbert space methods, which are not available here.

4.2 The case where V is a LHS/LBS

The next choice is to take for V a genuine PIP-space. For simplicity, we restrict ourselves to a lattice of Banach spaces (LBS) or a lattice of Hilbert spaces (LHS) [3], which is amply sufficient for applications. For instance, $\{L^p[0,1], 1 \leq p \leq \infty\}$, the lattice generated by $\{L^p(\mathbb{R}), 1 \leq p \leq \infty\}$ or a lattice of weighted L^2 spaces.

Let $V_J = \{V_p, p \in J\}$ be a LBS or a LHS of μ-measurable functions with the property

$$\xi \in V_p,\ \eta \in V_{\overline{p}} \implies \xi\overline{\eta} \in L^1(X,\mu) \qquad \text{and} \qquad \left|\int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x)\right| \leq \|\xi\|_p\, \|\eta\|_{\overline{p}}. \tag{4.6}$$

Thus the central Hilbert space is $\mathcal{H} := V_o = L^2(X,\mu)$ and the spaces $V_p, V_{\overline{p}}$ are dual of each other with respect to the L^2 inner product. The partial inner product, which extends the inner product of $L^2(X,\mu)$, is denoted again by $\langle\cdot|\cdot\rangle$. As usual we put $V = \sum_{p\in J} V_p$ and $V^\# = \bigcap_{p\in J} V_p$. According to the general theory of PIP-spaces [3], V is the algebraic inductive limit of the V_p's (see the Appendix). Thus $\psi \in V$ means that $\psi \in V_p$ for some $p \in J$.

Let ψ, ϕ be weakly measurable functions from X into $\mathcal{H}$. In the case of Section 3, if (ψ,ϕ) is a reproducing pair, the frame operator may be written as

$$\langle S_{\psi,\phi} f|g\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x) = \int_X C_\psi f(x)\,\overline{C_\phi g(x)}\,\mathrm{d}\mu(x), \tag{4.7}$$

which is well defined for all $f, g \in \mathcal{H}$. But now the inner product is only partial, and this motivates the next assumption. In view of (4.6), (4.7), and the definition of V, we assume that the following condition holds:

(p) $\exists\, p \in J$ such that $C_\psi f = \langle f|\psi.\rangle \in V_p$ and $C_\phi g = \langle g|\phi.\rangle \in V_{\overline{p}}, \forall\, f, g \in \mathcal{H}$.

We recall that $V_{\overline{p}}$ is the conjugate dual of V_p. In this case, then

$$\Omega_{\psi,\phi}(f,g) := \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x),\ f, g \in \mathcal{H},$$

defines a sesquilinear form on $\mathcal{H}\times\mathcal{H}$ and one has

$$|\Omega_{\psi,\phi}(f,g)| \leq \|C_\psi f\|_p\, \|C_\phi g\|_{\overline{p}},\ \forall\, f, g \in \mathcal{H}. \tag{4.8}$$

If $\Omega_{\psi,\phi}$ is bounded as a form on $\mathcal{H}\times\mathcal{H}$ (this is not automatic), there exists a bounded operator $S_{\psi,\phi}$ in $\mathcal{H}$ such that

$$\langle S_{\psi,\phi} f|g\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x),\ \forall\, f, g \in \mathcal{H}. \tag{4.9}$$

Then (ψ, ϕ) is a *reproducing pair* if $S_{\psi,\phi} \in GL(\mathcal{H})$.

In the sequel, we will often meet the statement $C_\psi : \mathcal{H} \to V_r$. This implies that $C_\psi : \mathcal{H} \to V_r$ is continuous, provided the space V_r satisfies an additional condition, called condition (k). If we don't know whether the condition holds, we will have to assume explicitly that $C_\psi : \mathcal{H} \to V_r$ is continuous. The condition (k) will be discussed in Section 5.4.

If $C_\psi : \mathcal{H} \to V_r$ is continuous, then $C_\psi^* : V_{\overline{r}} \to \mathcal{H}$ exists and it is continuous too. By definition, if $\xi \in V_{\overline{r}}$,

$$\langle C_\psi f | \xi \rangle = \int_X \langle f | \psi_x \rangle \overline{\xi(x)} \, \mathrm{d}\mu(x), \ \forall f \in \mathcal{H}. \tag{4.10}$$

Thus,

$$C_\psi^* \xi = \int_X \psi_x \xi(x) \, \mathrm{d}\mu(x).$$

Of course, what we have said about C_ψ holds in the very same way for C_ϕ. Assume now that for some $p \in J, C_\psi : \mathcal{H} \to V_p$ and $C_\phi : \mathcal{H} \to V_{\overline{p}}$ continuously. Then, $C_\phi^* : V_p \to \mathcal{H}$ so that $C_\phi^* C_\psi$ is a well-defined bounded operator in $\mathcal{H}$, given by

$$C_\phi^* C_\psi f = \int_X \langle f | \psi_x \rangle \phi_x \, \mathrm{d}\mu(x) = S_{\psi,\phi} f, \quad \forall f \in \mathcal{H},$$

the last equality following also from (4.9). Of course, this does not yet imply that $S_{\psi,\phi} \in GL(\mathcal{H})$, thus we don't know whether (ψ, ϕ) is a reproducing pair.

Let us now return to the pre-Hilbert space $K_\phi(X, \mu)$. First, the defining relation (3.1) must be written as

$$\left| \int_X \xi(x) \overline{(C_\phi g)(x)} \, \mathrm{d}\mu(x) \right| \leq c \, \|g\| \, , \ \forall \xi \in K_\phi(X, \mu), g \in \mathcal{H}. \tag{4.11}$$

Since $C_\phi : \mathcal{H} \to V_{\overline{p}}$, the integral is well defined for all $\xi \in V_p$. This means, the inner product on the l.h.s. is again the partial inner product of V. Hence, we may rewrite (4.11) as

$$|\langle \xi | C_\phi g \rangle| \leq c \, \|g\| \, , \forall \, g \in \mathcal{H}, \ \xi \in V_p.$$

Next, by (4.6), one has, for $\xi \in V_p, g \in \mathcal{H}$,

$$|\langle \xi | C_\phi g \rangle| \leq \|\xi\|_p \ \|C_\phi g\|_{\overline{p}} \leq c \, \|\xi\|_p \, \|g\| \, ,$$

where the last inequality follows from the assumption of continuity of C_ϕ. Hence indeed $\xi \in K_\phi(X, \mu)$, so that $V_p \subset K_\phi(X, \mu)$.

As for the adjoint operator, we have $C_\phi^* : V_p \to \mathcal{H}$. Then we may write, for $\xi \in V_p, g \in \mathcal{H}$, $\langle \xi | C_\phi g \rangle = \langle T_\phi \xi | g \rangle$, thus C_ϕ^* is the restriction from $K_\phi(X, \mu)$ to V_p of the operator $T_\phi : K_\phi \to \mathcal{H}$ given in (3.2), which reads now as

$$\langle T_\phi \xi | g \rangle = \int_X \xi(x) \langle \phi_x | g \rangle \, \mathrm{d}\mu(x), \ \forall \, \xi \in V_p, g \in \mathcal{H}. \tag{4.12}$$

Thus $C_\phi^* \subset T_\phi$.

Next, the construction proceeds as in Section 3. The space $W_\phi(X, \mu) = K_\phi(X, \mu)/\mathsf{Ker}\, T_\phi$, with the norm $\|[\xi]_\phi\|_\phi = \|T_\phi \xi\|$, is a pre-Hilbert space. Then Theorem 3.4 remains true, namely,

Theorem 4.2 *If (ψ, ϕ) is a reproducing pair, the spaces $W_\phi(X, \mu)$ and $W_\psi(X, \mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form* (3.10)*, namely,*

$$\langle \xi | \eta \rangle_\mu := \int_X \xi(x) \overline{\eta(x)} \, \mathrm{d}\mu(x),$$

which coincides with the inner product of $L^2(X, \mu)$ whenever the latter makes sense.

More precisely, we may state

Proposition 4.3 *Let (ψ, ϕ) be a reproducing pair. Then, if $C_\psi : \mathcal{H} \to V_p$ and $C_\phi : \mathcal{H} \to V_{\overline{p}}$ continuously, one has*

$$V_{\overline{p}}/\mathsf{Ker}\ T_\psi = W_\psi(X,\mu) \simeq \overline{W_\phi}(X,\mu)^* \ \ \text{and}\ \ V_p/\mathsf{Ker}\ T_\phi = W_\phi(X,\mu) \simeq \overline{W_\psi}(X,\mu)^*. \tag{4.13}$$

In these relations, the equality sign means an isomorphism of vector spaces, whereas $\simeq$ denotes an isomorphism of Hilbert spaces.

Since $W_\psi(X,\mu)$ and $W_\phi(X,\mu)$ are both Hilbert spaces, the relations (4.13) suggest to take for $V_p, V_{\overline{p}}$ Hilbert spaces as well, that is, take for V a LHS. The simplest case is then a Hilbert chain, for instance, the scale (A.2) $\{\mathcal{H}_k, k \in \mathbb{Z}\}$ built on the powers of a self-adjoint operator $A > I$.

Choose a reproducing pair (ψ, ϕ). Without loss of generality, we suppose that ψ, ϕ are dual to each other, that is, $S_{\psi,\phi} = I$. If ψ and ϕ are both frames, there is nothing to say, since then $C_\psi(\mathcal{H}), C_\phi(\mathcal{H}) \subset L^2(X,\mu) = \mathcal{H}_o$. Thus we assume that ψ is an upper semi-frame and ϕ is a lower semi-frame, dual to each other. It follows that $C_\psi(\mathcal{H}) \subset L^2(X,\mu)$. Hence Condition (p) becomes: There is an index $k \geq 1$ such that $C_\psi : \mathcal{H} \to \mathcal{H}_k$ and $C_\phi : \mathcal{H} \to \mathcal{H}_{\overline{k}}$ continuously, thus $V_p \equiv \mathcal{H}_k$ and $V_{\overline{p}} \equiv \mathcal{H}_{\overline{k}}$. This means we are working in the Hilbert triplet

$$V_p \equiv \mathcal{H}_k \subset \mathcal{H}_o = L^2(X,\mu) \subset \mathcal{H}_{\overline{k}} \equiv V_{\overline{p}}\,. \tag{4.14}$$

Then, according to Proposition 4.3, we have $W_\psi(X,\mu) = \mathcal{H}_{\overline{k}}/\mathsf{Ker}\ T_\psi$ and $W_\phi(X,\mu) = \mathcal{H}_k/\mathsf{Ker}\ T_\phi$, as vector spaces. Concrete examples will be given in Section 6.

5 The case where Y and V are both PIP-spaces

In the previous sections, we have taken for the initial space Y a single abstract Hilbert space $\mathcal{H}$. Now we will go one step further and take instead an abstract LBS or a LHS $Y = \{Y_u, u \in U\}$, with $Y_o = \mathcal{H}$, while keeping $V = \{V_p(X,\mu), p \in J\}$, a LBS or a LHS of complex measurable functions over the usual measure space (X,μ), with $V_o(X) = L^2(X,\mu)$.

Let ψ, ϕ be two weakly measurable functions on X, with values in Y. This means that there exists $u \in U$ such that $\psi_x \in Y_u, \forall\, x \in X$, and $v \in U$ such that $\phi_x \in Y_v, \forall\, x \in X$, and the complex functions $x \in X \mapsto \langle f|\psi_x\rangle, x \in X \mapsto \langle g|\phi_x\rangle$ are μ-measurable for every $f \in Y_{\overline{u}}$, $g \in Y_{\overline{v}}$.

Define two *analysis operators* $C_\psi, C_\phi \in \mathrm{Op}(Y)$ as follows :

$$(C_\psi f)(x) = \langle f|\psi_x\rangle,\ f \in Y_{\overline{u}},\ \psi_x \in Y_u, \quad (C_\phi g)(x) = \langle g|\phi_x\rangle,\ g \in Y_{\overline{v}},\ \phi_x \in Y_v,$$

where the inner product is the partial inner product of Y.

Definition 5.1 Given (u, v) such that $\psi. \in Y_u$ and $\phi. \in Y_v$, assume there exists $p \in J$ such that[2]

$$C_\psi f \in V_p(X),\ \forall\, f \in Y_{\overline{u}} \quad \text{and} \quad C_\phi g \in V_{\overline{p}}(X),\ \forall\, g \in V_{\overline{v}},$$

with $C_\psi : Y_{\overline{u}} \to V_p(X)$ and $C_\phi : Y_{\overline{v}} \to V_{\overline{p}}(X)$ continuous, , so that $C_\psi^* : V_{\overline{p}}(X) \to Y_u$ and $C_\phi^* : V_p(X) \to Y_v$ are continuous also. In that case, we say that (u, v) is an *admissible couple* for (ψ, ϕ).

Given an admissible couple (u, v) for (ψ, ϕ), the sesquilinear form

$$\Omega_{\psi,\phi}(f,g) = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu = \int_X (C_\psi f)(x)\overline{(C_\phi g)(x)}\,\mathrm{d}\mu \tag{5.1}$$

is well-defined on $Y_{\overline{u}} \times Y_{\overline{v}}$ and we have

$$|\Omega_{\psi,\phi}(f,g)| \leq \|C_\psi f\|_p \|C_\phi g\|_{\overline{p}} \leq c'\|f\|_{\overline{u}}\|g\|_{\overline{v}}, \quad \forall\, f \in Y_{\overline{u}}, g \in Y_{\overline{v}}. \tag{5.2}$$

[2]For simplicity, we write $V_p(X))$ instead of $V_p(X,\mu)$.

Then there exists a continuous operator $S_{\psi,\phi} : Y_{\overline{u}} \to Y_v$ such that

$$\Omega_{\psi,\phi}(f,g) = \langle S_{\psi,\phi} f | g\rangle, \quad \forall\, f \in Y_{\overline{u}}, g \in Y_{\overline{v}}.$$

More precisely, in the PIP-space operator language (see the Appendix), $S_{\psi,\phi}$ has a (necessarily continuous) representative $[S_{\psi,\phi}]_{v\,\overline{u}} : Y_{\overline{u}} \to Y_v$. It is easily checked that $S_{\psi,\phi} = C_\phi^* C_\psi$. The operator $S_{\psi,\phi}$ is called the *frame operator* associated to the pair (ψ, ϕ).

The map $S_{\psi,\phi}$ has an adjoint $S^\dagger_{\psi,\phi} : Y_{\overline{v}} \to Y_u$ defined by

$$\langle S^\dagger_{\psi,\phi} g | f\rangle = \overline{\langle S_{\psi,\phi} f | g\rangle}, \quad f \in Y_{\overline{u}}, g \in Y_{\overline{v}}.$$

An easy computation shows that

$$\langle S^\dagger_{\psi,\phi} g | f\rangle = \int_X \langle g|\phi_x\rangle\langle\psi_x|f\rangle \,\mathrm{d}\mu, \quad f \in Y_{\overline{u}}, g \in Y_{\overline{v}}.$$

Hence, $S^\dagger_{\psi,\phi} = S_{\phi,\psi}$.

5.1 Construction of coefficient spaces

Fix an admissible couple (u, v) for (ψ, ϕ). Denote by K_ϕ the space of all measurable functions $\xi : X \to \mathbb{C}$ such that the integral

$$F_\phi(g) := \int_X \xi(x)\langle\phi_x|g\rangle \,\mathrm{d}\mu$$

exists for every $g \in Y_{\overline{v}}$ and defines a continuous conjugate linear functional on $Y_{\overline{v}}$. That is, there exists $c > 0$ such that

$$\left| \int_X \xi(x)\langle\phi_x|g\rangle \,\mathrm{d}\mu(x)\right| \le c\, \|g\|_{\overline{v}}\,, \ \forall\, g \in Y_{\overline{v}}. \tag{5.3}$$

As usual we do not distinguish functions ξ which differ on μ-null subsets of X. As in the previous cases, we refer to K_ϕ as the *coefficient space* of ϕ. The space K_ψ is defined in a similar way.

Since the sesquilinear form $\Omega_{\psi,\phi}$ is bounded, by (5.2), it is clear that all functions $\xi(x) = \langle f|\psi_x\rangle$ belong to K_ϕ since, by assumption,

$$\int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \,\mathrm{d}\mu(x)$$

exists and is bounded.

Then, following the familiar pattern, we can define a linear map $T_\phi : K_\phi \to Y_v$, which we call again the *synthesis operator* by the following relation

$$\langle T_\phi \xi | g\rangle = \int_X \xi(x)\langle\phi_x|g\rangle \,\mathrm{d}\mu, \quad \xi \in K_\phi, g \in Y_{\overline{v}}. \tag{5.4}$$

Set $W_\phi = K_\phi / \mathsf{Ker}\, T_\phi$ and $[\xi]_\phi := \xi + \mathsf{Ker}\, T_\phi$. By definition, ϕ is called μ-independent whenever $\mathsf{Ker}\, T_\phi = \{0\}$; in that case, of course, $W_\phi = K_\phi$.

The space W_ϕ is a normed space under the norm

$$\|[\xi]_\phi\|_\phi = \|T_\phi \xi\|_v\,, \tag{5.5}$$

which implies that the operator $\widehat{T}_\phi : W_\phi \to Y_v$ defined by $\widehat{T}_\phi[\xi]_\phi := T_\phi \xi$ is continuous, injective and an isometry. Note that, if Y is a LHS, the norm $\|\cdot\|_\phi$ is Hilbertian, so that $K_\phi[\|\cdot\|_\phi]$ is a pre-Hilbert space (same argument as in Section 3).

The same result applies to K_ψ, which is also a pre-Hilbert space in the LHS case, i.e., $T_\psi : K_\psi \to Y_u$ is an isometry.

It is shown in [8, Theorem 3.4], that a linear functional F on $W_\phi[\| \cdot \|_\phi]$ is continuous if, and only if, there exists $g \in Y_{\overline{v}}$ such that

$$F([\xi]_\phi) = \int_X \xi(x) \langle \phi_x | g \rangle \, \mathrm{d}\mu, \quad \forall \, \xi \in K_\phi \, ,$$

Then, taking into account the relations (5.1)-(5.2) and the fact that $\langle \phi_x | g \rangle \in K_\psi$, we see that every $F \in W_\phi(X, \mu)^*$ can be represented as

$$F([\xi]_\phi) = \int_X \xi(x) \overline{\eta(x)} \, \mathrm{d}\mu(x), \; \forall \, [\xi]_\phi \in W_\phi(X, \mu), \tag{5.6}$$

with $\eta \in K_\psi(X, \mu)$. Hence, the dual space W_ϕ^* of W_ϕ, with respect to the sesquilinear form given by the L^2 inner product, can be identified with a space E_ϕ of measurable functions containing all functions $\{\langle g | \phi_x \rangle, g \in Y_{\overline{v}}\}$.

5.2 Compatible pairs

Fix an admissible couple (u, v) for (ψ, ϕ). As seen above, the sesquilinear form

$$\Omega_{\psi,\phi}(f, g) = \int_X \langle f | \psi_x \rangle \langle \phi_x | g \rangle \, \mathrm{d}\mu = \int_X (C_\psi f)(x) \overline{(C_\phi g)(x)} \, \mathrm{d}\mu$$

is well defined and bounded on $Y_{\overline{u}} \times Y_{\overline{v}}$. With a proof similar to that of [8, Theorem 3.6] or that of Section 4, one shows that the dual W_ϕ^* can be identified with a closed subspace of $\overline{W_\psi}$, the space of conjugates of elements of W_ψ. On the other hand, it was proved in [7] that, for a *reproducing pair* of $\mathcal{H}$-valued weakly measurable functions (ψ, ϕ), the spaces W_ϕ and W_ψ are both Hilbert spaces in conjugate duality to each other (see Theorem 3.4).

In the same vein as in Section 3, in particular Theorem 3.4, we define a new concept, generalizing to the present situation what has been done in [30].

Definition 5.2 Let (u, v) be an admissible couple for (ψ, ϕ). We say that ψ, ϕ are *compatible* if W_ϕ^* is topologically isomorphic to $\overline{W_\psi}$ and $W_\psi^\times$, the conjugate dual of W_ψ, is topologically isomorphic to W_ϕ. Thus we identify W_ϕ^* with $\overline{W_\psi}$ and $W_\psi^\times$ with W_ϕ.

This definition implies that the spaces W_ϕ and W_ψ are reflexive spaces enjoying the duality properties mentioned above, which we write shortly as $W_\phi^* \approx \overline{W}_\psi$ and $W_\psi^\times \approx W_\phi$. Then we have

$$W_\psi^* \approx \overline{W_\psi^\times} \approx \overline{W_\phi} \;\text{ and }\; W_\phi^\times \approx \overline{W_\psi^*} \approx W_\psi .$$

This proves that (ϕ, ψ) is a compatible pair if and only if (ψ, ϕ) is a compatible pair.

In what follows, for simplicity, we will consider pairs (ψ, ϕ) with ψ, ϕ both μ-independent. In this case $W_\psi = K_\psi$, $W_\phi = K_\phi$. Hence we will use the notation K_ψ, K_ϕ for these spaces. The general case may be treated in a similar way.

As a matter of fact, compatible pairs and reproducing pairs are closely related, as shown in the following result.

Theorem 5.3 *Let (u, v) be an admissible couple for (ψ, ϕ), with ψ, ϕ both μ-independent and μ-total. Consider the following statements :*

(i) (ψ, ϕ) is a compatible pair;

(ii) The operator $S_{\psi,\phi} : Y_{\overline{u}} \to Y_v$ is bounded with bounded inverse, that is, (ψ, ϕ) is a reproducing pair.

Then, if K_ϕ is complete, (i) implies (ii). On the other hand, (ii) always implies (i). Thus, if K_ϕ is complete, the two statements are equivalent.

Proof. Since ψ, ϕ are μ-independent, $W_\phi = K_\phi$, $W_\psi = K_\psi$, $\widehat{T}_\phi = T_\phi$.

(i) $\Rightarrow$ (ii): If (u, v) is an admissible couple for (ψ, ϕ) the operator $S_{\psi,\phi}$ is continuous from $Y_{\overline{u}}$ into Y_v. It is injective, since ϕ is μ-independent and ψ is μ-total. Next we show that it is also surjective. We first provethat $\mathsf{Ran}\, T_\phi$ is dense in Y_v. Suppose that $g \in Y_{\overline{v}}$ is such that $\langle T_\phi \xi | g\rangle = 0$ for every $\xi \in K_\phi$. Thus,

$$\langle T_\phi \xi | g\rangle = \int_X \xi(x)\langle \phi_x | g\rangle \, \mathrm{d}\mu = 0, \quad \forall\, \xi \in K_\phi.$$

In particular this is true if we take $\xi(x) = \langle f|\psi_x\rangle$, with $f \in Y_{\overline{u}}$. The μ-independence of ψ implies that $\langle \phi_x|g\rangle = 0$ for almost every $x \in X$. Hence $g = 0$, because ϕ is μ-total.

The definition of $\|\cdot\|_\phi$ and the completeness of K_ϕ imply that $\mathsf{Ran}\, T_\phi$ is closed in Y_v. Hence, $\mathsf{Ran}\, T_\phi = Y_v$. Therefore, for every $\Phi \in Y_v$ there exists a unique $\xi \in K_\phi$ such that $T_\phi \xi = \Phi$. This implies that:

$$\langle \Phi | g\rangle = \int_X \xi(x)\langle \phi_x|g\rangle \, \mathrm{d}\mu, \quad \forall\, g \in Y_{\overline{v}}.$$

By the definition of compatible pairs, the conjugate linear functional F_ξ on K_ψ defined by:

$$F_\xi(\eta) = \int_X \xi(x)\overline{\eta(x)} \, \mathrm{d}\mu, \quad \eta \in K_\psi.$$

is continuous. Then, as discussed at the end of Section 5.1, there exists a unique $f \in Y_{\overline{u}}$ such that

$$F_\xi(\eta) = \int_X \langle f|\psi_x\rangle\overline{\eta(x)} \, \mathrm{d}\mu, \quad \eta \in K_\psi.$$

Hence, in particular

$$F_\xi(\langle g|\phi_x\rangle) = \int_X \langle f|\psi_x\rangle\langle \phi_x|g\rangle \, \mathrm{d}\mu, \quad \forall\, g \in Y_{\overline{v}}.$$

In conclusion,

$$\langle \Phi | g\rangle = \int_X \langle f|\psi_x\rangle\langle \phi_x|g\rangle \, \mathrm{d}\mu, \quad \forall\, g \in Y_{\overline{v}}.$$

This implies that $\Phi = S_{\psi,\phi} f$. Hence $S_{\psi,\phi}^{-1}$ is bounded, by the inverse mapping theorem.

(ii) $\Rightarrow$ (i): Let $H \in K_\phi^*$, the dual of K_ϕ; then, as discussed at the end of Section 5.1 there exists $g \in Y_{\overline{v}}$ such that:

$$H(\xi) = H_g(\xi) = \int_X \xi(x)\langle \phi_x|g\rangle \, \mathrm{d}\mu, \quad \forall\, \xi \in K_\phi.$$

We show that g is unique. Suppose that there exists another g' satisfying the same condition. Then it follows that:

$$\int_X \xi(x)\langle \phi_x|g - g'\rangle \, \mathrm{d}\mu = 0, \quad \forall\, \xi \in K_\phi.$$

In particular, this is true for $\xi(x) = \langle f|\psi_x\rangle$, for every $f \in Y_{\overline{u}}$. Thus:

$$\int_X \langle f|\psi_x\rangle\langle \phi_x|g - g'\rangle \, \mathrm{d}\mu = 0, \quad \forall\, \xi \in K_\phi.$$

Hence, $\langle S_{\psi,\phi} f|g - g'\rangle = 0$, for every $f \in Y_{\overline{u}}$. But, since $S_{\psi,\phi} Y_{\overline{u}} = Y_v$, we conclude that $\langle G|g - g'\rangle = 0$, for every $G \in Y_v$ and this, in turn, implies that $g = g'$. Therefore, we can define a map $\Xi : H \in K_\phi^* \mapsto g \in Y_{\overline{v}}$, where g is the unique element of $Y_{\overline{v}}$ such that $H = H_g$. This map is an isomorphism of vector spaces. Indeed, it is clearly injective. On the other hand, if $g \in Y_{\overline{v}}$, the functional

$$H(\xi) = H_g(\xi) = \int_X \xi(x)\langle \phi_x|g\rangle \, \mathrm{d}\mu, \quad \xi \in K_\phi,$$

is in K_ϕ^* and satisfies $\Xi(H_g) = g$. It is clear that the function $\eta(x) = \langle g|\phi_x\rangle$ is an element of K_ψ. Assume that there exists another function $\eta' \in K_\psi$ such that

$$H(\xi) = \int_X \xi(x)\overline{\eta'(x)}\,\mathrm{d}\mu, \quad \xi \in K_\phi.$$

This implies that:

$$\int_X \xi(x)\overline{(\eta(x) - \eta'(x))}\,\mathrm{d}\mu = 0, \quad \forall\, \xi \in K_\phi.$$

Take $\xi(x) = \langle f|\psi_x\rangle$, $f \in Y_{\overline{u}}$, then $\eta - \eta' \in \mathsf{Ker}\, T_\psi$. Hence $\eta = \eta'$. Then we can define a linear map:

$$\Theta : H \in K_\phi^* \mapsto \eta \in K_\psi, \tag{5.7}$$

where K_ϕ^* denote the dual space of K_ϕ. The map Θ is clearly injective. As we have seen $T_\phi^\times g = \phi_g \in K_\psi$, for every $g \in Y_{\overline{v}}$, where $\phi_g(x) = \langle g|\phi_x\rangle$. Indeed, $T_\phi : K_\phi \to Y_v$. Hence $T_\phi^* : Y_{\overline{v}} \to K_\phi^*$, thus $T_\phi^\times : Y_{\overline{v}} \to K_\phi^\times \approx K_\psi$. Now we put $C_{\phi,\psi} g = T_\phi^\times g$. Hence $C_{\phi,\psi}$ is a conjugate linear map of $Y_{\overline{v}}$ into K_ψ. We want to prove that $\mathsf{Ran}\, C_{\phi,\psi}$ coincides with K_ψ.

First we have

$$\begin{aligned}\|g\|_{\overline{v}} = \|S_{\phi,\psi}^{-1} S_{\phi,\psi} g\|_{\overline{v}} \leq c\,\|S_{\phi,\psi} g\|_u &= c \sup_{\|h\|_{\overline{u}} \leq 1} |\langle S_{\phi,\psi} g|h\rangle| \\ = c \sup_{\|h\|_{\overline{u}} \leq 1} \left| \int_X \langle g|\phi_x\rangle\langle\psi_x|h\rangle\,\mathrm{d}\mu \right| &= c\,\|T_\psi \phi_g\|_u = c\|\phi_g\|_\psi,\end{aligned}$$

This inequality implies that $\mathsf{Ran}\, C_{\phi,\psi}$ is closed in K_ψ.

Next, we prove that $\mathsf{Ran}\, C_{\phi,\psi}$ is also dense in K_ψ. If it was not so, there would be a nonzero continuous linear functional F on K_ψ such that $F(\langle f|\phi_x\rangle) = 0$ for every $f \in \mathcal{D}$. Hence, there would exist $g \neq 0$ such that:

$$F(\xi) = \int_X \xi(x)\langle\psi_x|g\rangle\,\mathrm{d}\mu, \quad \forall\, \xi \in K_\psi,$$

and therefore

$$F(\phi_f) = \int_X \langle f|\phi_x\rangle\langle\psi_x|g\rangle\,\mathrm{d}\mu, \quad \forall\, f \in Y_{\overline{v}}.$$

This implies that $\langle S_{\phi,\psi} f|g\rangle = 0$ for all $f \in Y_{\overline{v}}$. But, since $S_{\phi,\psi} Y_{\overline{v}} = Y_u$, this yields a contradiction. Let us now consider the map Θ defined in (5.7). An immediate consequence of the equality $\mathsf{Ran}\, C_{\phi,\psi} = K_\psi$ is that Θ is surjective. This implies that the conjugate dual of K_ϕ can be identified with $\overline{K_\psi}$, the space of conjugates of elements of K_ψ. □

5.3 The general case

Take again an admissible couple (u, v) for (ψ, ϕ), assuming both ϕ, ψ to be μ-independent and μ-total, for simplicity. We know from Section 5.2 that the dual K_ϕ^* can be identified with a closed subspace of $\overline{K_\psi}$, the space of conjugates of elements of K_ϕ. Also K_ϕ^* is complete as a dual.

Assume first that (ψ, ϕ) is a compatible pair. Then we have $K_\phi^* = \overline{K_\psi}$ and $K_\psi^* = \overline{K_\phi}$. Notice that both spaces K_ψ and K_ϕ are Banach spaces in general, but they are Hilbert spaces in the case of a LHS since then the norm (5.5) is Hilbertian [7]. Now $T_\psi : K_\psi \to Y_u$ and $T_\phi : K_\phi \to Y_v$ are continuous and bijective, hence the inverse maps $T_\psi^{-1} : Y_u \to K_\psi$ and $T_\phi^{-1} : Y_v \to K_\phi$ are also continuous by the inverse mapping theorem. Thus T_ψ and T_ϕ are isomorphisms. In addition, they are isometries, by (5.5). Thus K_ψ et Y_u are isometric and so are the respective duals K_ψ^* and $Y_{\overline{v}}$ and the same for $K_\phi \sim Y_v, K_\phi^* \sim Y_{\overline{v}}$. Since $K_\phi^* = \overline{K_\psi}$ (as spaces of measurable functions), we can say that $Y_{\overline{v}}$ and $\overline{Y_u}$ have equal norms, hence we have

$\overline{v} = u$. Thus this is a *necessary* condition for (ψ, ϕ) to be a compatible pair for the admissible couple (u, v).

If the condition $\overline{v} = u$ is satisfied, it follows that the frame operator $S_{\psi,\phi}$ map Y_v onto itself continuously and has a bounded inverse, since the representative $[S_{\psi,\phi}]_{vv}$ is invertible, which means that $S_{\psi,\phi}$ is an invertible PIP-space operator [6].

More generally, given an admissible couple (u, v) for (ψ, ϕ), we may ask under which conditions it is a compatible pair, which is the same thing as a reproducing pair, according to Theorem 5.3. First, we need $\overline{v} = u$, that is, there must exist $p \in J$ such that $C_\psi : Y_u \to V_p(X)$ and $C_\phi : Y_{\overline{u}} \to V_{\overline{p}}(X)$ continuously. In that case, since $[S_{\psi,\phi}]_{vv}$ exists, so are the representatives $[S_{\psi,\phi}]_{v"v'} : Y_{v'} \to Y_{v"}$, where $v' \le v, v" \ge v$. The required condition is then that one of these representatives $[S_{\psi,\phi}]_{v"v'}$ be invertible. This replaces the condition $S_{\psi,\phi} \in GL(\mathcal{H})$ used for reproducing pairs [8, 7].

5.4 Comparison with the case $Y = \mathcal{H}$

As we said in Section 3, the case treated in [8, Sec.4] amounts to take for the initial space Y a single Hilbert space $\mathcal{H}$. We will adapt here the results of that paper to the new situation, described in Definition 5.1.

Let us suppose that the spaces V_r have the following property:

(k) If $\xi_n \to \xi$ in V_r, then, for every compact subset $K \subset X$, there exists a subsequence $\{\xi_n^K\}$ of $\{\xi_n\}$ which converges to ξ almost everywhere in K.

The property (k) is satisfied by L^p-spaces [27, p.73, Ex.18], but presumably not in general. Notice that it concerns only the target space V, it is independent of the initial space Y.

Let (u, v) be an admissible couple for (ψ, ϕ). Instead of $C_\psi : \mathcal{H} \to V$, we have that $\psi_x \in Y_u, \forall\, x \in X$, and $C_\psi : Y_{\overline{u}} \to V_p(X)$. Let V_r be an arbitrary element of V (an assaying subspace [3], which is a Banach space or a Hilbert space). Following [8, Sec.4], we define

$$D_{r,\overline{u}}(C_\psi) = \{f \in Y_{\overline{u}} : \, C_\psi f \in V_r\}, \; r \in J.$$

In particular, $D_{r,\overline{u}}(C_\psi) = Y_{\overline{u}}$ means $C_\psi(Y_{\overline{u}}) \subset V_r$. According to Definition 5.1, this condition is satisfied for $r = p$, but not necessarily otherwise.

Then we have the following results, corresponding to Proposition 4.2 and Corollaries 4.3 and 4.4 of [8].

Proposition 5.4 *Assume that* (k) *holds for V_r. Then :*

(i) $C_\psi : D_{r,\overline{u}}(C_\psi) \to V_r$ *is a closed linear map.*

(ii) If for some $r \in J$, $C_\psi(Y_{\overline{u}}) \subset V_r$, *then* $C_\psi : Y_{\overline{u}} \to V_r$ *is continuous.*

(iii) If $C_\psi(Y_{\overline{u}}) \subset V_p$ *and* $C_\phi(Y_{\overline{v}}) \subset V_{\overline{p}}$, *the form* Ω *is bounded on* $Y_{\overline{u}} \times Y_{\overline{v}}$, *that is,* $|\Omega_{\psi,\phi}(f, g)| \le c\, \|f\|_{\overline{u}}\, \|g\|_{\overline{v}}$.

Proof. The proof of (i) follows *verbatim* that of [8, Prop. 4.2], simply replacing $\mathcal{H}$ by $Y_{\overline{u}}$. The same is true for (ii) and (iii) with the two corollaries. □

By definition, $C_\psi : Y_{\overline{u}} \to V_p$ is continuous. If $r \ne p$ and condition (k) holds, $C_\psi(Y_{\overline{u}}) \subset V_r$ implies that $C_\psi : Y_{\overline{u}} \to V_r$ is continuous. If $r \ne p$ and we don't know whether the condition holds, we will have to assume explicitly that $C_\psi : Y_{\overline{u}} \to V_r$ is continuous.

Suppose now that for some $r \in J$, the maps $C_\psi : Y_{\overline{u}} \to V_r$ and $C_\phi : Y_{\overline{v}} \to V_{\overline{r}}$ are continuous (automatic for $r = p$).. Then, $C_\phi^* : V_r \to Y_{\overline{u}}$, so that $S_{\psi,\phi} := C_\phi^* C_\psi$ is a well-defined bounded operator from $Y_{\overline{u}}$ to Y_v.. Of course, this does not yet imply that $[S_{\psi,\phi}]_{v\,\overline{u}}$, or any other representative, has a bounded inverse. Hence we don't know whether (ψ, ϕ) is a reproducing pair, even if we impose the necessary condition $\overline{v} = u$.

6 Examples

In this section, we present concrete examples that illustrate all three types of reproducing pairs described in the previous sections.

6.1 A purely hilbertian reproducing pair

Several discrete examples may be found in [7, Sec.6.1]. Here, however, we restrict ourselves to continuous examples, which are more spectacular.

If $\phi \in L^2(\mathbb{R}, \mathrm{d}x)$, consider the continuous wavelet system $\phi_{b,a}(x) = a^{-1/2}\phi(a^{-1}(x-b))$, $b \in \mathbb{R}, a > 0$. Let $\psi, \phi \in L^2(\mathbb{R}, \mathrm{d}x)$. Then the well-known orthogonality relations of the corresponding wavelet transforms [20, Theorem 10.1] imply that

$$c_{\psi,\phi} := \int_{\mathbb{R}} |\widehat{\psi}(\omega)\widehat{\phi}(\omega)| \frac{\mathrm{d}\omega}{|\omega|} < \infty, \tag{6.1}$$

Thus, (ψ, ϕ) is a reproducing pair for $L^2(\mathbb{R}, dx)$ with $S_{\psi,\phi} = c_{\psi,\phi} I$. For $\psi = \phi$, the cross-admissibility condition (6.1) reduces to the classical admissibility condition

$$c_{\phi} := \int_{\mathbb{R}} |\widehat{\phi}(\omega)|^2 \frac{\mathrm{d}\omega}{|\omega|} < \infty. \tag{6.2}$$

Considering the obvious inequalities

$$|c_{\psi,\phi}| \leq \int_{\mathbb{R}} |\widehat{\psi}(\omega)\widehat{\phi}(\omega)| \frac{\mathrm{d}\omega}{|\omega|} \leq c_{\phi}^{1/2} c_{\psi}^{1/2},$$

we see that condition (6.1) is automatically satisfied whenever ϕ and ψ are both admissible, so that indeed (ψ, ϕ) is a reproducing pair.

However, nonadmissible wavelets may also generate reproducing pairs, as shown by the following trivial example. Take the Gaussian window $\phi(x) = e^{-\pi x^2}$, then $c_\phi = \infty$ which implies that ϕ is not a continuous wavelet frame. However, if one defines $\psi \in L^2(\mathbb{R}, dx)$ in the Fourier domain via $\widehat{\psi}(\omega) = |\omega|\widehat{\phi}(\omega)$, it follows that $0 < c_{\psi,\phi} = \|\phi\|_2^2 < \infty$. Thus (ψ, ϕ) is a reproducing pair. This example clearly shows the increasing flexibility obtained when replacing continuous frames by reproducing pairs.

6.2 A reproducing pair with a PIP-space target

As discussed after Proposition 4.3, we take for V a LHS, more precisely the scale (A.2) built on the powers of the self-adjoint operator $A > 1$. Thus, according to (4.14), there is an index $k \geq 1$ such that $C_\psi : \mathcal{H} \to \mathcal{H}_k$ and $C_\phi : \mathcal{H} \to \mathcal{H}_{\overline{k}}$ continuously, where we have

$$\mathcal{H}_k \subset \mathcal{H}_o = L^2(X, \mu) \subset \mathcal{H}_{\overline{k}}. \tag{6.3}$$

Actually one may give an explicit example, using a Sobolev-type scale [8, Sec.5]. Let $\mathcal{H}_K$ be a reproducing kernel Hilbert space (RKHS) of (nice) functions on the measure space (X, μ_K), with kernel function $k_x, x \in X$, that is, $f(x) = \langle f|k_x\rangle_K$, $\forall f \in \mathcal{H}_K$ (we could also take another measure space (Z, μ_K)). The corresponding reproducing kernel is $K(x, y) = k_y(x) = \langle k_y|k_x\rangle_K$. Choose a weight function $m(x) > 1$, the analog of the weight $(1+|x|^2)$ considered in the Sobolev case (see Appendix A.1). Define the Hilbert scale $\mathcal{H}_r$, $r \in \mathbb{Z}$, determined by the multiplication operator $Af(x) = m(x)f(x)$, $\forall x \in X$. Then, for some $n \geq 1$, define the measurable functions $\psi_x = k_x m^{-n}(x), \phi_x = k_x m^n(x)$, so that $C_\psi : \mathcal{H}_K \to \mathcal{H}_n$, $C_\phi : \mathcal{H}_K \to \mathcal{H}_{\overline{n}}$ continuously, and they are dual of each other. One has indeed $\langle \phi_x|g\rangle_K = \langle k_x m^n(x)|g\rangle_K = \langle k_x|g\, m^n(x)\rangle_K = \overline{g(x)}\, m^n(x) \in \mathcal{H}_{\overline{n}}$ and $\langle \psi_x|g\rangle_K = \overline{g(x)}\, m^{-n}(x) \in \mathcal{H}_n$, which implies duality. Thus (ψ, ϕ) is a reproducing pair with $S_{\psi,\phi} = I$, where ψ is an upper semi-frame and ϕ a lower semi-frame.

In this case, one can compute the operators T_ψ, T_ϕ explicitly. For all $\xi \in \mathcal{H}_n, g \in \mathcal{H}_K$, the definition (4.12) reads as

$$\langle T_\phi \xi | g \rangle_K = \int_X \xi(x) \langle \phi_x | g \rangle_K \,\mathrm{d}\mu(x), = \int_X \xi(x)\, \overline{g(x)}\, m^n(x)\,\mathrm{d}\mu,$$

that is, $(T_\phi \xi)(x) = \xi(x)\, m^n(x)$ or $T_\phi \xi = \xi\, m^n$. However, since the weight $m(x) > 1$ is invertible, $\overline{g}\, m^n$ runs over the whole of $\mathcal{H}_{\overline{n}}$ whenever g runs over H_K. Hence $\xi \in \mathsf{Ker}\ T_\phi \subset \mathcal{H}_n$ means that $\langle T_\phi \xi | g \rangle_K = 0$, $\forall g \in H_K$, which implies $\xi = 0$, since the duality between $\mathcal{H}_n$ and $\mathcal{H}_{\overline{n}}$ is separating. The same reasoning yields $\mathsf{Ker}\ T_\psi = \{0\}$. Therefore $W_\phi(X, \mu) = K_\phi(X, \mu) = \mathcal{H}_n$ and $W_\psi(X, \mu) = K_\psi(X, \mu) = \mathcal{H}_{\overline{n}}$.

Similar examples concerning spaces of sequences may be found in [8, Se. 5.2], possibly more general, in the sense that the space V is no longer a Hllbert scale, but a genuine LHS.

6.3 A reproducing pair with two PIP-spaces

An easy example may be derived from the previous one. For the target space V, we keep the same Hilbert scale (A.2) built on the powers of the self-adjoint operator of multiplication by $m(x) > 1$. For the initial space Y, we take a similar Hilbert scale (A.2), but around $\mathcal{H}_K$, instead of $L^2(X, \mu)$. Thus for each $l \geq 1$, we have

$$\mathcal{H}_l \subset \mathcal{H}_0 \equiv \mathcal{H}_K \subset \mathcal{H}_{\overline{l}}\,.$$

Hence, $f \in \mathcal{H}_l$ means that $\int_X |f(x)|^2 m^{2l}(x)\ \mathrm{d}\mu_K(x) < \infty$.

Given two positive integers l, n, we take for analyzing functions $\psi_x = k_x m^{-(l+n)}(x), \phi_x = k_x m^{(l+n)}(x)$, where k_x is again the reproducing function of $\mathcal{H}_K$. Then we have:

- $\psi_x \in \mathcal{H}_l$ and $\phi_x \in \mathcal{H}_{\overline{l}}$; for instance,

$$\begin{aligned}\|\psi_x\|_l^2 &= \int_X |\psi_x|^2\, m^{2l}(x)\,\mathrm{d}\mu_K(x) = \int_X |k_x|^2\, m^{-2n}(x)\,\mathrm{d}\mu_K(x)\\ &< \int_X |k_x|^2\ \mathrm{d}\mu_K(x) < \infty,\ \text{since}\ m(x) > 1.\end{aligned}$$

- $\forall f \in \mathcal{H}_{\overline{l}}$, $C_\psi f = \langle f | \psi_x \rangle_K = f\, m^{-(l+n)} \in \mathcal{H}_n$.
- $\forall g \in \mathcal{H}_l$, $C_\phi g = \langle g | \phi_x \rangle_K = g\, m^{(l+n)} \in \mathcal{H}_{\overline{n}}$.

In conclusion, as for the previous example, (ψ, ϕ) is a reproducing pair with $S_{\psi,\phi} = I$, where ψ is an upper semi-frame and ϕ a lower semi-frame.

Here too, one can compute the operators T_ψ, T_ϕ explicitly. For all $\xi \in \mathcal{H}_n, g \in \mathcal{H}_{\overline{l}}$, the definition (4.12) reads as

$$\langle T_\phi \xi | g \rangle_K = \int_X \xi(x) \langle \phi_x | g \rangle_K \,\mathrm{d}\mu_K(x) = \int_X \xi(x)\, \overline{g(x)}\, m^{(l+n)}\,\mathrm{d}\mu_K(x),$$

that is, $(T_\phi \xi)(x) = \xi(x)\, m^{(l+n)}(x)$ or $T_\phi \xi = \xi\, m(l+n)$.

As compared with the general scheme of Definition 5.1, we have taken $Y_u \equiv \mathcal{H}_l, Y_v \equiv \mathcal{H}_{\overline{l}}, V_p \equiv \mathcal{H}_n, V_{\overline{p}} \equiv \mathcal{H}_{\overline{n}}$. This means, in particular, that the condition $v = \overline{u}$ is satisfied, as expected for a reproducing pair

A Lattices of Banach or Hilbert spaces and operators on them

A.1 Lattices of Banach or Hilbert spaces

For the convenience of the reader, we summarize in this Appendix the basic facts concerning PIP-spaces and operators on them. However, we will restrict the discussion to the simpler case of a lattice of Banach (LBS) or Hilbert spaces (LHS). Further information may be found in our monograph [3] or the review paper [9].

Let thus $\mathcal{I} = \{V_p,\, p \in I\}$ be a family of Hilbert spaces or reflexive Banach spaces, partially ordered by inclusion. Then $\mathcal{I}$ generates an involutive lattice $\mathcal{J}$, indexed by J, through the operations ($p, q, r \in I$):

- . involution: $V_r \;\leftrightarrow\; V_{\overline{r}} = V_r^{\times}$, the conjugate dual of V_r
- . infimum: $V_{p\wedge q} \;:=\; V_p \wedge V_q = V_p \cap V_q$
- . supremum: $V_{p\vee q} \;:=\; V_p \vee V_q = V_p + V_q$.

It turns out that both $V_{p\wedge q}$ and $V_{p\vee q}$ are Hilbert spaces, resp. reflexive Banach spaces, under appropriate norms (the so-called projective, resp. inductive norms). Assume that the following conditions are satisfied:

(i) $\mathcal{I}$ contains a unique self-dual, Hilbert subspace $V_o = V_{\overline{o}}$.

(ii) for every $V_r \in \mathcal{I}$, the norm $\|\cdot\|_{\overline{r}}$ on $V_{\overline{r}} = V_r^{\times}$ is the conjugate of the norm $\|\cdot\|_r$ on V_r.

In addition to the family $\mathcal{J} = \{V_r,\, r \in J\}$, it is convenient to consider the two spaces $V^{\#}$ and V defined as

$$V = \sum_{q\in I} V_q, \quad V^{\#} = \bigcap_{q\in I} V_q. \tag{A.1}$$

These two spaces themselves usually do *not* belong to $\mathcal{I}$. According to the general theory of PIP-spaces [3], V is the algebraic inductive limit of the V_p's and $V^{\#}$ is the projective limit of the V_p's .

We say that two vectors $f, g \in V$ are *compatible*[3] if there exists $r \in J$ such that $f \in V_r, g \in V_{\overline{r}}$. Then a *partial inner product* on V is a Hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors. In particular, the partial inner product $\langle\cdot|\cdot\rangle$ coincides with the inner product of V_o on the latter. A *partial inner product space* (PIP-space) is a vector space V equipped with a partial inner product. Clearly LBSs and LHSs are particular cases of PIP-spaces.

From now on, we will assume that our PIP-space $(V, \langle\cdot|\cdot\rangle)$ is *nondegenerate*, that is, $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$. As a consequence, $(V^{\#}, V)$ and every couple $(V_r, V_{\overline{r}})$, $r \in J$, are a dual pair in the sense of topological vector spaces [25]. In particular, the original norm topology on V_r coincides with its Mackey topology $\tau(V_r, V_{\overline{r}})$, so that indeed its conjugate dual is $(V_r)^{\times} = V_{\overline{r}}$, $\forall\, r \in J$. Then, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r. In the sequel, we also assume the partial inner product to be positive definite, $\langle f|f\rangle > 0$ whenever $f \neq 0$.

A standard, albeit trivial, example is that of a Rigged Hilbert space (RHS) $\Phi \subset \mathcal{H} \subset \Phi^{\#}$ (it is trivial because the lattice $\mathcal{I}$ contains only three elements).

Familiar concrete examples of PIP-spaces are sequence spaces, with $V = \omega$ the space of *all* complex sequences $x = (x_n)$, and spaces of locally integrable functions with $V = L^1_{\rm loc}(\mathbb{R}, {\rm d}x)$, the space of Lebesgue measurable functions, integrable over compact subsets.

Among LBSs, the simplest example is that of a chain of reflexive Banach spaces. The prototype is the chain $\mathcal{I} = \{L^p := L^p([0,1]; dx),\ 1 < p < \infty\}$ of Lebesgue spaces over the interval [0, 1].

As for a LHS, the simplest example is the Hilbert scale generated by a self-adjoint operator $A > I$ in a Hilbert space $\mathcal{H}_o$. Let $\mathcal{H}_n$ be $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{\overline{n}} := \mathcal{H}_{-n} = \mathcal{H}_n^{\times}$ (conjugate dual):

$$\mathcal{D}^{\infty}(A) := \bigcap_n \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 \ \subset \mathcal{H}_{\overline{1}} \subset \mathcal{H}_{\overline{2}} \ldots \subset \mathcal{D}_{\overline{\infty}}(A) := \bigcup_n \mathcal{H}_n. \tag{A.2}$$

Note that here the index n may be integer or real, the link between the two cases being established by the spectral theorem for self-adjoint operators. Here again the inner product of $\mathcal{H}_0$ extends to each pair $\mathcal{H}_n, \mathcal{H}_{-n}$, but on $\mathcal{D}_{\overline{\infty}}(A)$ it yields only a *partial* inner product. For standard examples, take $\mathcal{H}_0 = L^2(\mathbb{R}, dx)$ and the following operators:

[3]This notion is the basis of the theory of PIP-spaces [3], it should not be confused with that of Definition 5.2 !

- $(A_{\rm p}f)(x) = (1+x^2)^{1/2}f(x)$ yields the Fourier transform of the Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$.
- $(A_{\rm m}f)(x) = (1 - \frac{d^2}{dx^2})^{1/2}f(x)$ yields the Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$.
- $(A_{\rm osc}f)(x) = (1 + x^2 - \frac{d^2}{dx^2})f(x)$ yields the harmonic oscillator representation of the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions of fast decay and its conjugate dual $\mathcal{S}^\times(\mathbb{R})$, the space of of tempered distributions.

A.2 Operators on LBSs and LHSs

Let V_J be a LHS or a LBS. Then an *operator* on V_J is a map from a subset $\mathcal{D}(A) \subset V$ into V, such that

(i) $\mathcal{D}(A) = \bigcup_{q\in \mathsf{d}(A)} V_q$, where $\mathsf{d}(A)$ is a nonempty subset of J;

(ii) For every $q \in \mathsf{d}(A)$, there exists $p \in J$ such that the restriction of A to V_q is a continuous linear map into V_p (we denote this restriction by A_{pq});

(iii) A has no proper extension satisfying (i) and (ii).

We denote by $\mathrm{Op}(V_J,)$ the set of all operators on V_J . The continuous linear operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. The properties of A are conveniently described by the set $\mathsf{j}(A)$ of all pairs $(q,p) \in J \times J$ such that A maps V_q continuously into V_p Thus the operator A may be identified with the collection of its representatives,

$$A \simeq \{A_{pq} : V_q \to V_p : (q,p) \in \mathsf{j}(A)\}. \tag{A.3}$$

It is important to notice that an operator is uniquely determined by *any* of its representatives, in virtue of Property (iii): there are no extensions for PIP-space operators.

If A_{pq} is a representative of the operator A, then so are $A_{pq'} = A_{pq}E_{qq'}$, for any $q' < q$, where $E_{qq'}$ is a representative of the unit operator, and also $A_{p'q} = E_{p'p}A_{pq}$, for any $p' > p$. Operators between a LHS/LBS Y_K into another one V_J are defined in the same way. They are denoted as $\mathrm{Op}(Y_K, V_J)$.

Although an operator may be identified with a separately continous sesquilinear form on $V^\# \times V^\#$, or a conjugate linear continuous map $V^\#$ into V, it is more useful to keep also the *algebraic operations* on operators, namely:

(i) *Adjoint:* every $A \in \mathrm{Op}(V_J)$ has a unique adjoint $A^\times \in \mathrm{Op}V_J)$, defined by

$$\langle A^\times y|x\rangle = \langle y|Ax\rangle, \text{ for } x \in V_q,\ y \in V_{\overline{p}} \text{ and } A_{pq} \text{ exists}. \tag{A.4}$$

that is, $(A^\times)_{\overline{q}\overline{p}} = (A_{pq})^*$, where $(A_{pq})^* : V_{\overline{p}} \to V_{\overline{q}}$ is the adjoint map of A_{pq}. Furthermore, one has $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_J)$: no extension is allowed, by the maximality condition (iii) of the definition.

(ii) *Partial multiplication:* Let $A, B \in \mathrm{Op}(V_J)$. The product BA is defined if and only if there is a continuous factorization through some V_r:

$$V_q \xrightarrow{A} V_r \xrightarrow{B} V_p, \quad \text{i.e.,} \quad (BA)_{pq} = B_{pr}A_{rq}. \tag{A.5}$$

Several classes of operators are of particular interest, for instance:

(i) *Symmetric* operators, defined as those operators satisfying the relation $A^\times = A$, since these are the ones that may generate self-adjoint operators in the central Hilbert space [3, Section 3.3].

(ii) *Invertible* operators: A is invertible if it has at least one invertible representative; in that case, A has a unique inverse operator $A^{-1} \in \mathrm{Op}(V_J)$ [6].

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