

The hard-to-soft edge transition: exponential moments, central limit theorems and rigidity

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Abstract

The local eigenvalue statistics of large random matrices near a hard edge transitioning into a soft edge are described by the Bessel process associated with a large parameter α . For this point process, we obtain 1) exponential moment asymptotics, up to and including the constant term, 2) asymptotics for the expectation and variance of the counting function, 3) several central limit theorems and 4) a global rigidity upper bound.

AMS SUBJECT CLASSIFICATION (2020): 41A60, 60B20, 35Q15.

KEYWORDS: Exponential moments, Bessel point process, Random matrix theory, Asymptotic analysis, Rigidity, Riemann–Hilbert problems, Airy point process.

1 Introduction and statement of results

The hard-to-soft edge transition is a universal phenomenon which arises in a wide class of unitary invariant random matrix ensembles [5], but to make the explanation more concrete we are going to focus on Wishart matrices. Let X denote an $n \times (n + \alpha)$ matrix whose elements are independent standard complex Gaussian random variables, and consider the matrix XX^* , where X^* is the conjugate transpose of X . Such matrices were first studied in 1928 by Wishart [51], and since then have found applications in various areas, such as numerical analysis [26], information theory [48] and finance [2]. Of particular interest is the local statistics of the smallest eigenvalues of XX^* in the large dimensional limit. There are two distinct regimes that have been observed [39]. In the limit $n \rightarrow +\infty$ while α is kept fixed, the smallest eigenvalues accumulate near the “hard edge” 0, and after rescaling, a limiting point process arises [28, 29], known as the Bessel point process (which depends on α). On the other hand, if α/n tends to a positive constant as $n \rightarrow +\infty$, then the smallest eigenvalues are pushed away from 0 and instead are located near a “soft edge”. In this case, after rescaling, they give rise to the Airy point process [28, 29] (which is independent of α). The hard-to-soft edge transition corresponds to the regime where $\alpha/n \rightarrow 0$ but $\alpha \rightarrow +\infty$, and is described by the Bessel process associated with a large parameter α .

The above example of Wishart matrices only makes sense for $\alpha \in \mathbb{N}_{\geq 0} := \{0, 1, 2, \dots\}$, but in fact the Bessel process can be defined for all values $\alpha \in (-1, +\infty)$ [28, 29]. The Bessel point process is universal in random matrix theory: it appears near hard edges of a wide class of random matrix models [50].

It is known, see [5, Theorem 1] or (1.16) below, that the gap probabilities of the Bessel process, when properly rescaled, converge as $\alpha \rightarrow +\infty$ to the gap probabilities of the Airy process. In this paper, we establish various other properties of the large α Bessel process.

The Bessel point process is a determinantal point process on $[0, +\infty)$ whose kernel is given by

$$K_{\alpha}^{\text{Be}}(x, y) = \frac{J_{\alpha}(\sqrt{x})\sqrt{y}J'_{\alpha}(\sqrt{y}) - \sqrt{x}J'_{\alpha}(\sqrt{x})J_{\alpha}(\sqrt{y})}{2(x - y)}, \quad \alpha > -1, \quad (1.1)$$

where J_{α} is the Bessel function of the first kind of order α . As α increases, the Bessel process increasingly favors configurations whose points are further away from 0. We refer to [47, 4, 35] for several surveys on determinantal point processes.

We emphasize that the Bessel process studied in this paper arises near hard edges of unitary invariant random matrix ensembles. The generalization of this process to the so-called β -ensembles has been introduced in [43] and is not determinantal. Its transition to the β soft edge has been studied in [43, 44, 25]. We also mention that other types of hard-to-soft edge transitions than the one considered here have been studied in e.g. [19, 36, 52].

Let us introduce the parameters

$$m \in \mathbb{N}_{>0}, \quad \vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m \quad \text{and} \quad \vec{x} = (x_1, \dots, x_m) \in \mathbb{R}_{\text{ord}}^{+,m}, \quad (1.2)$$

where $\mathbb{R}_{\text{ord}}^{+,m} := \{\vec{x} = (x_1, \dots, x_m) : 0 < x_1 < x_2 < \dots < x_m < +\infty\}$. Our main interest in this work lies in the generating function of the Bessel process, which can be written as the following exponential moments

$$E_\alpha(\vec{x}, \vec{u}) := \mathbb{E} \left[\prod_{j=1}^m e^{u_j N_\alpha(x_j)} \right], \quad (1.3)$$

where $N_\alpha(x)$ denotes the random variable that counts the number of points $\leq x$ in the Bessel process. It is known that $E_\alpha(\vec{x}, \vec{u})$ can be naturally expressed in terms of the solution to a system of m coupled Painlevé V equations [49, 17]. The goal of this paper is to obtain precise *exponential moment asymptotics* for $E_\alpha(r\vec{x}, \vec{u})$ as $r \rightarrow +\infty$ in the regimes outlined below which involve letting $\alpha \rightarrow +\infty$. Exponential moment asymptotics have attracted considerable attention recently, partly due to their relevance for the global rigidity of the associated point process, see Section 1.3. Such asymptotics have been obtained for various point processes, see [3, 9, 6, 13] for the sine process, [8, 12] for the Bessel process with α fixed (or bounded), [7, 15] for the Airy process, [16] for the Meijer- G and Wright's generalized Bessel process, and [20] for the Pearcey process. In Theorem 1.1 below, we obtain asymptotics for $E_\alpha(r\vec{x}, \vec{u})$ as $r \rightarrow +\infty$ in three regimes for the varying parameters α and $\vec{x}$; this allows us to discuss the transition to the Airy exponential moment asymptotics, and to obtain a precise matching, up to and including the constant term, with the known exponential moment asymptotics from [8, 12] for the Bessel process with α bounded. More precisely, our main result is stated in Theorem 1.1 and describes the asymptotics of $E_\alpha(r\vec{x}, \vec{u})$ in the following three regimes:

1. $r \rightarrow +\infty$, $\alpha = a\sqrt{r} \rightarrow +\infty$ with $a, x_1, \dots, x_m > 0$ fixed (or mildly varying),
2. $r \rightarrow +\infty$, $\alpha = a\sqrt{r} \rightarrow +\infty$, $a \rightarrow 0$ with $x_1, \dots, x_m > 0$ fixed (or mildly varying),
3. $r \rightarrow +\infty$, $\alpha = a\sqrt{r} \rightarrow +\infty$, a fixed, and the points x_j converging sufficiently slowly to a^2 .

The second regime, in which $a \rightarrow 0$, requires a different (and more delicate) proof than the other two, and allows for a matching with the results of [8, 12] for the Bessel process with α bounded, see Corollary 1.2. The third regime is needed to discuss the transition to the Airy exponential moment asymptotics obtained in [7, 15], see Corollary 1.3.

Theorem 1.1 encodes significant information about the hard-to-soft edge transition: we can deduce from it large r asymptotics for the expectation and the variance of $N_\alpha(rx)$, see Corollary 1.4, several central limit theorems (CLTs), see Corollaries 1.7 and 1.8, and an upper bound for the global rigidity of the process, see Theorem 1.11.

As mentioned, as $\alpha = a\sqrt{r}$ gets large, the Bessel process increasingly favors point configurations with fewer points near 0. The quantity $\alpha^2 = a^2 r$, which appears at various places within the statement of our results, is the lower end-point of the support of the points in the large α Bessel process (see also the discussion at the beginning of Subsection 3.1).

Theorem 1.1 (Exponential moment asymptotics for the large α Bessel process).

1. Let $m \in \mathbb{N}_{>0}$, and let

$$a \in (0, +\infty), \quad \vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m \quad \text{and} \quad \vec{x} = (x_1, \dots, x_m) \in \mathbb{R}_{\text{ord}}^{+,m}, \quad (1.4)$$

be such that

$$0 < x_1 < \dots < x_{n-1} < a^2 < x_n < \dots < x_m,$$

with $n \in \{1, \dots, m+1\}$, $x_0 := 0$, $x_{m+1} := +\infty$, and for $r > 0$, define $\alpha = a\sqrt{r}$. As $r \rightarrow +\infty$, we have

$$E_\alpha(r\vec{x}, \vec{u}) = \exp \left(\sum_{j=n}^m u_j \mu_\alpha(rx_j) + \sum_{j=n}^m \frac{u_j^2}{2} \sigma_\alpha^2(rx_j) + \sum_{n \leq j < k \leq m} u_j u_k \Sigma_a(x_k, x_j) \right. \\ \left. + \sum_{j=n}^m \log \left(G\left(1 + \frac{u_j}{2\pi i}\right) G\left(1 - \frac{u_j}{2\pi i}\right) \right) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) \right), \quad (1.5)$$

where G is Barnes' G -function, and μ_α , σ_α^2 and Σ_a are given by

$$\mu_\alpha(rx) = \frac{1}{\pi} \int_{\alpha^2}^{rx} \frac{\sqrt{u - \alpha^2}}{2u} du = \frac{\sqrt{r}}{\pi} \int_{a^2}^x \frac{\sqrt{u - a^2}}{2u} du, \quad (1.6)$$

$$\sigma_\alpha^2(rx) = \frac{\log(4(rx - \alpha^2)^{3/2}(rx)^{-1})}{2\pi^2} = \frac{\log r}{4\pi^2} + \frac{\log(4(x - a^2)^{3/2}x^{-1})}{2\pi^2}, \quad (1.7)$$

$$\Sigma_a(x_k, x_j) = \frac{1}{2\pi^2} \log \left(\frac{\sqrt{x_j - a^2} + \sqrt{x_k - a^2}}{|\sqrt{x_j - a^2} - \sqrt{x_k - a^2}|} \right). \quad (1.8)$$

Furthermore, the asymptotics (1.5) are uniform for $(x_1, \dots, x_{n-1}, a^2, x_n, \dots, x_m)$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+, m+1}$, and uniform for $u_1, \dots, u_m$ in compact subsets of $\mathbb{R}$.

2. Let $m \in \mathbb{N}_{>0}$, $\vec{u} \in \mathbb{R}^m$ and $\vec{x} \in \mathbb{R}_{\text{ord}}^{+, m}$, and for $r > 0$ and $a \in (0, \sqrt{x_1})$, define $\alpha = a\sqrt{r}$. The asymptotics of $E_\alpha(r\vec{x}, \vec{u})$ as $r \rightarrow +\infty$ and simultaneously $a \rightarrow 0$, $a\sqrt{r} \rightarrow +\infty$ are also given by the right-hand side of (1.5) (necessarily with $n = 1$). Furthermore, these asymptotics are uniform for $u_1, \dots, u_m$ in compact subsets of $\mathbb{R}$, and uniform for $\vec{x}$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+, m}$.
3. Let $m \in \mathbb{N}_{>0}$ and $a \in (0, +\infty)$ be fixed, let $\vec{u} \in \mathbb{R}^m$ and $\vec{x} \in \mathbb{R}_{\text{ord}}^{+, m}$, and for $r > 0$, define $\alpha = a\sqrt{r}$. Assume that $0 < a^2 < x_1 < \dots < x_m$ and that the points $\{x_j\}_1^m$ tend to a^2 at the same sufficiently slow rate in the sense that the following hold:

$$|x_m - a^2| \rightarrow 0 \text{ as } r \rightarrow +\infty, \quad (1.9a)$$

$$\frac{x_j - a^2}{x_m - a^2} \text{ stays in a bounded subset of } (0, +\infty) \text{ for each } j = 1, \dots, m-1, \quad (1.9b)$$

$$\frac{\log r}{|x_m - a^2|^4 \sqrt{r}} \rightarrow 0 \text{ as } r \rightarrow +\infty. \quad (1.9c)$$

In this regime, the asymptotics of $E_\alpha(r\vec{x}, \vec{u})$ as $r \rightarrow +\infty$ are given by the right-hand side of (1.5) but with the error term $\mathcal{O}(\log r / \sqrt{r})$ replaced by

$$\mathcal{O}\left(\frac{\log r}{|x_m - a^2|^4 \sqrt{r}}\right). \quad (1.10)$$

Furthermore, these asymptotics are also uniform for $u_1, \dots, u_m$ in compact subsets of $\mathbb{R}$.

The asymptotic formulas in each of the above three regimes can be differentiated any number of times with respect to $u_1, \dots, u_m$ at the cost of increasing slightly the error term as follows. Let $\tilde{E}_\alpha(r\vec{x}, \vec{u})$ be the right-hand side of (1.5) without the error term and let $\mathcal{E} = \log E_\alpha(r\vec{x}, \vec{u}) - \log \tilde{E}_\alpha(r\vec{x}, \vec{u})$ be the error term. If $k_1, \dots, k_m \in \mathbb{N}_{\geq 0}$, $k = k_1 + \dots + k_m \geq 1$ and $\partial_u^k = \partial_{u_1}^{k_1} \dots \partial_{u_m}^{k_m}$, then

$$\partial_u^k \mathcal{E} = \mathcal{O}\left(\frac{(\log r)^k}{\sqrt{r}}\right) \quad \text{for regimes 1 and 2,} \quad (1.11)$$

$$\partial_u^k \mathcal{E} = \mathcal{O}\left(\frac{(\log r)^k}{|x_m - a^2|^4 \sqrt{r}}\right) \quad \text{for regime 3.} \quad (1.12)$$

Remark 1. If $a^2 > x_m$, i.e. $n = m+1$, then each sum in (1.5) should be interpreted as 0.

Remark 2. For the regimes 1 and 2 of Theorem 1.1, the points $x_1, \dots, x_m$ remain bounded from each other, while for regime 3, the points $x_1, \dots, x_m$ are merging. We believe the convergence rate (1.9c) and the error estimate (1.10) are not optimal, and that they could be improved with some extra work.

As stated in part 2 of Theorem 1.1, the error term in (1.5) does not get worse as $a \rightarrow 0$. It is however important in our proof that $\alpha = a\sqrt{r} \rightarrow +\infty$. The asymptotics of $E_\alpha(r\vec{x}, \vec{u})$ as $r \rightarrow +\infty$ with α bounded are not covered by Theorem 1.1, but they have been obtained in [8, eq (1.35)] for $m = 1$ and in [12, Theorem 1.1] for $m \geq 2$. These asymptotics are given by the right-hand side of (1.5) with $n = 1$ and with μ_α , σ_α^2 and Σ_a replaced by

$$\tilde{\mu}_\alpha(rx) = \frac{\sqrt{rx}}{\pi} - \frac{\alpha}{2}, \quad \tilde{\sigma}_\alpha^2(rx) = \frac{\log(4\sqrt{rx})}{2\pi^2}, \quad \tilde{\Sigma}(x_k, x_j) = \frac{1}{2\pi^2} \log \left(\frac{\sqrt{x_j} + \sqrt{x_k}}{|\sqrt{x_j} - \sqrt{x_k}|} \right), \quad (1.13)$$

respectively. Corollary 1.2 below shows that Theorem 1.1 matches explicitly with [12, Theorem 1.1] in the regime $a \rightarrow 0$, $\alpha = a\sqrt{r} \rightarrow +\infty$, up to an error $\mathcal{O}\left(\frac{\log r}{\sqrt{r}} + a^2\sqrt{r}\right)$. In other words, Theorem 1.1 and [12, Theorem 1.1] taken together describe the asymptotics of $E_{a\sqrt{r}}(r\vec{x}, \vec{u})$ as $r \rightarrow +\infty$, up to and including the constant, uniformly for $a \in [\frac{\tilde{\alpha}}{\sqrt{r}}, a_0]$ for any fixed $a_0 \in (0, x_1)$ and fixed $\tilde{\alpha} \in (-1, +\infty)$.

Corollary 1.2 (Matching with the Bessel exponential moment asymptotics with α bounded). *Let $m \in \mathbb{N}_{>0}$, $\vec{u} \in \mathbb{R}^m$ and $\vec{x} \in \mathbb{R}_{\text{ord}}^{+,m}$, and for $r > 0$ and $a > 0$, define $\alpha = a\sqrt{r}$. The asymptotics of $E_\alpha(r\vec{x}, \vec{u})$ as $r \rightarrow +\infty$ and simultaneously $a \rightarrow 0$, $a\sqrt{r} \rightarrow +\infty$ can be written as*

$$E_\alpha(r\vec{x}, \vec{u}) = \exp \left(\sum_{j=1}^m u_j \tilde{\mu}_\alpha(rx_j) + \sum_{j=1}^m \frac{u_j^2}{2} \tilde{\sigma}_\alpha^2(rx_j) + \sum_{1 \leq j < k \leq m} u_j u_k \tilde{\Sigma}(x_k, x_j) \right. \\ \left. + \sum_{j=1}^m \log G(1 + \frac{u_j}{2\pi i}) G(1 - \frac{u_j}{2\pi i}) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) + \mathcal{O}(a^2\sqrt{r}) \right), \quad (1.14)$$

where $\tilde{\mu}_\alpha$, $\tilde{\sigma}_\alpha^2$, $\tilde{\Sigma}$ are defined in (1.13). Furthermore, these asymptotics are uniform for $u_1, \dots, u_m$ in compact subsets of $\mathbb{R}$, and uniform for $\vec{x}$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m}$.

Proof. Since the error term in Theorem 1.1 is uniform as $a \rightarrow 0$, the claim follows after letting $a \rightarrow 0$ in (1.6)–(1.8) with $n = 1$, and noting that

$$\mu_\alpha(rx) = \tilde{\mu}_\alpha(rx) + \mathcal{O}(a^2\sqrt{r}), \quad \sigma_\alpha^2(rx) = \tilde{\sigma}_\alpha^2(rx) + \mathcal{O}(a^2), \quad \Sigma_a(x_k, x_j) = \tilde{\Sigma}(x_k, x_j) + \mathcal{O}(a^2) \quad (1.15)$$

uniformly for $\vec{x}$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m}$ and for $r \geq 1$. $\square$

The Airy process is a determinantal point process on $\mathbb{R}$ whose kernel is given by

$$K_{\text{Ai}}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}, \quad x, y \in \mathbb{R},$$

where Ai denotes the Airy function. It has been proved by Borodin and Forrester in [5, Theorem 1] that

$$\mathbb{P}_{\text{Bessel}} \left[\text{there is a gap on } [0, \alpha^2 + 2^{\frac{2}{3}}\alpha^{\frac{4}{3}}y + \mathcal{O}(\alpha)] \right] \rightarrow \mathbb{P}_{\text{Airy}} \left[\text{there is a gap on } [-y, +\infty] \right] \quad (1.16)$$

as $\alpha \rightarrow \infty$, for any $y \in \mathbb{R}$. Here “a gap” on an interval $A \subset \mathbb{R}$ means that no points in the process fall in A . It is important to note that y is fixed in (1.16). In Corollary 1.3 below, we show an analogue of (1.16) at the level of the exponential moment asymptotics, in the double scaling limit where both $\alpha \rightarrow +\infty$ and simultaneously $y \rightarrow +\infty$ at a slow speed. Let $N_{\text{Ai}}(y)$ denote the random variable that counts the number of points in $[-y, +\infty)$ in the Airy process, $y \in \mathbb{R}$, and consider the exponential moment

$$E_{\text{Ai}}(\vec{y}, \vec{u}) := \mathbb{E} \left[\prod_{j=1}^m e^{u_j N_{\text{Ai}}(y_j)} \right], \quad \vec{y} = (y_1, \dots, y_m) \in \mathbb{R}^m, \quad \vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m.$$

The asymptotics for $E_{\text{Ai}}(r_{\text{Ai}}\vec{y}, \vec{u})$ as $r_{\text{Ai}} \rightarrow +\infty$ have been obtained in [7] for $m = 1$ and in [15, Theorem 1.1] for $m \geq 2$, and are as follows:

$$E_{\text{Ai}}(r_{\text{Ai}}\vec{y}, \vec{u}) = \exp \left(\sum_{j=1}^m u_j \mu_{\text{Ai}}(r_{\text{Ai}}y_j) + \sum_{j=1}^m \frac{u_j^2}{2} \sigma_{\text{Ai}}^2(r_{\text{Ai}}y_j) + \sum_{1 \leq j < k \leq m} u_j u_k \Sigma_{\text{Ai}}(y_k, y_j) \right. \\ \left. + \sum_{j=1}^m \log G(1 + \frac{u_j}{2\pi i}) G(1 - \frac{u_j}{2\pi i}) + \mathcal{O}\left(\frac{\log r_{\text{Ai}}}{r_{\text{Ai}}^{3/2}}\right) \right), \quad (1.17)$$

where

$$\mu_{\text{Ai}}(y) = \frac{2}{3\pi} y^{3/2}, \quad \sigma_{\text{Ai}}^2(y) = \frac{3}{4\pi^2} \log(4y), \quad \Sigma_{\text{Ai}}(y_k, y_j) = \frac{1}{2\pi^2} \log \left(\frac{\sqrt{y_j} + \sqrt{y_k}}{|\sqrt{y_j} - \sqrt{y_k}|} \right). \quad (1.18)$$

Furthermore, the error term in (1.17) is uniform for $u_1, \dots, u_m$ in compact subsets of $\mathbb{R}$. The result (1.16) suggests that the asymptotics of $E_\alpha(r\vec{x}, \vec{u})$ as $r \rightarrow +\infty$, $\alpha \rightarrow +\infty$ and simultaneously

$$rx_j = \alpha^2 + 2^{\frac{2}{3}} \alpha^{\frac{4}{3}} r_{\text{Ai}} y_j, \quad j = 1, \dots, m, \quad 0 < y_1 < \dots < y_m,$$

with $r_{\text{Ai}} \rightarrow +\infty$ at a sufficiently slow speed should somehow be related to the asymptotics of $E_{\text{Ai}}(r_{\text{Ai}}\vec{y}, \vec{u})$. Corollary 1.3 below confirms this expectation.

Corollary 1.3 (Matching with the Airy exponential moment asymptotics). *Let $m \in \mathbb{N}_{>0}$, $a \in (0, +\infty)$ and $0 < y_1 < \dots < y_m$ be fixed. For $r > 0$, define $\alpha = a\sqrt{r}$, and let $\vec{u} \in \mathbb{R}^m$ and $\vec{x} \in \mathbb{R}_{\text{ord}}^{+,m}$ be such that*

$$x_j = \frac{\alpha^2}{r} + 2^{\frac{2}{3}} \frac{\alpha^{\frac{4}{3}} r_{\text{Ai}}}{r} y_j = a^2 + 2^{\frac{2}{3}} \frac{a^{\frac{4}{3}} r_{\text{Ai}}}{r^{\frac{1}{3}}} y_j, \quad j = 1, \dots, m, \quad r_{\text{Ai}} > 0. \quad (1.19)$$

As $r \rightarrow +\infty$ and simultaneously

$$M_r r^{\frac{5}{24}} (\log r)^{\frac{1}{4}} \leq r_{\text{Ai}} \leq \frac{1}{M_r} r^{\frac{1}{3}} \quad (1.20)$$

where $M_r > 0$ tends to infinity at an arbitrarily slow rate as $r \rightarrow +\infty$ (i.e. $M_r^{-1} = o(1)$), we have

$$\begin{aligned} E_\alpha(r\vec{x}, \vec{u}) = \exp \left(\sum_{j=1}^m u_j \mu_{\text{Ai}}(r_{\text{Ai}} y_j) \left(1 + \mathcal{O}\left(\frac{r_{\text{Ai}}}{r^{\frac{1}{3}}}\right) \right) + \sum_{j=1}^m \frac{u_j^2}{2} \left(\sigma_{\text{Ai}}^2(r_{\text{Ai}} y_j) + \mathcal{O}\left(\frac{r_{\text{Ai}}}{r^{\frac{1}{3}}}\right) \right) \right. \\ \left. + \sum_{1 \leq j < k \leq m} u_j u_k \Sigma_{\text{Ai}}(y_k, y_j) + \sum_{j=1}^m \log G\left(1 + \frac{u_j}{2\pi i}\right) G\left(1 - \frac{u_j}{2\pi i}\right) + \mathcal{O}\left(\frac{r^{\frac{5}{6}} \log r}{r_{\text{Ai}}^4}\right) \right). \end{aligned} \quad (1.21)$$

Furthermore, the error term is uniform for $u_1, \dots, u_m$ in compact subsets of $\mathbb{R}$.

Proof. Substituting $rx = \alpha^2 + 2^{\frac{2}{3}} \alpha^{\frac{4}{3}} r_{\text{Ai}} y$ in (1.6)–(1.8) and letting $\frac{r_{\text{Ai}}}{\alpha^{\frac{2}{3}}} \rightarrow 0$, we get

$$\mu_\alpha(rx) = \mu_{\text{Ai}}(r_{\text{Ai}} y) \left(1 + \mathcal{O}\left(\frac{r_{\text{Ai}}}{\alpha^{\frac{2}{3}}}\right) \right), \quad \sigma_\alpha^2(rx) = \sigma_{\text{Ai}}^2(r_{\text{Ai}} y) + \mathcal{O}\left(\frac{r_{\text{Ai}}}{\alpha^{\frac{2}{3}}}\right), \quad \Sigma_\alpha(x_k, x_j) = \Sigma_{\text{Ai}}(y_k, y_j). \quad (1.22)$$

The claim now follows directly from part 3 of Theorem 1.1. $\square$

1.1 Large r asymptotics of $\mathbb{E}[N_\alpha(rx_1)]$, $\text{Var}[N_\alpha(rx_1)]$ and $\text{Cov}[N_\alpha(rx_1), N_\alpha(rx_2)]$

It is directly seen from (1.3) with $m = 1$ and $m = 2$ that

$$\partial_u \log E_\alpha(x_1, u)|_{u=0} = \mathbb{E}[N_\alpha(x_1)], \quad \partial_u^2 \log E_\alpha(x_1, u)|_{u=0} = \text{Var}[N_\alpha(x_1)], \quad (1.23)$$

$$\partial_u^2 \log \left(\frac{E_\alpha((x_1, x_2), (u, u))}{E_\alpha(x_1, u) E_\alpha(x_2, u)} \right) \Big|_{u=0} = 2 \text{Cov}(N_\alpha(x_1), N_\alpha(x_2)). \quad (1.24)$$

Recall that the asymptotics of $E_\alpha(rx_1, u)$ and $E_\alpha((rx_1, rx_2), (u, u))$ as $r \rightarrow +\infty, \alpha \rightarrow +\infty$, which are given by Theorem 1.1 with $m = 1$ and $m = 2$, can be differentiated at the cost of increasing the error term as in (1.11)–(1.12). Hence, from (1.23)–(1.24), we obtain the following asymptotics for $\mathbb{E}[N_\alpha(rx_1)]$, $\text{Var}[N_\alpha(rx_1)]$ and $\text{Cov}[N_\alpha(rx_1), N_\alpha(rx_2)]$ as $r \rightarrow +\infty, \alpha \rightarrow +\infty$.

Corollary 1.4. *Let $0 < x_1 < x_2$ be fixed, let $a \in (0, \sqrt{x_1})$, and for $r > 0$, define $\alpha = a\sqrt{r}$. As $r \rightarrow +\infty$, we have*

$$\mathbb{E}[N_\alpha(rx_1)] = \mu_\alpha(rx_1) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) = \frac{\sqrt{r}}{\pi} \int_{a^2}^{x_1} \frac{\sqrt{u - a^2}}{2u} du + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \quad (1.25)$$

$$\begin{aligned} \text{Var}[N_\alpha(rx_1)] &= \sigma_\alpha^2(rx_1) + \frac{1 + \gamma_E}{2\pi^2} + \mathcal{O}\left(\frac{\log^2 r}{\sqrt{r}}\right) \\ &= \frac{\log r}{4\pi^2} + \frac{1 + \log(4(x_1 - a^2)^{3/2} x_1^{-1}) + \gamma_E}{2\pi^2} + \mathcal{O}\left(\frac{\log^2 r}{\sqrt{r}}\right), \end{aligned} \quad (1.26)$$

$$\text{Cov}[N_\alpha(rx_1), N_\alpha(rx_2)] = \Sigma_\alpha(x_2, x_1) + \mathcal{O}\left(\frac{\log^2 r}{\sqrt{r}}\right) = \frac{1}{2\pi^2} \log \left(\frac{\sqrt{x_2 - a^2} + \sqrt{x_1 - a^2}}{\sqrt{x_2 - a^2} - \sqrt{x_1 - a^2}} \right) + \mathcal{O}\left(\frac{\log^2 r}{\sqrt{r}}\right),$$

where $\gamma_E \approx 0.5772$ is Euler's gamma constant. Furthermore, these asymptotics are uniform for a in compact subsets of $(0, \sqrt{x_1})$.

Remark 3. The asymptotics (1.26), without the error term, have been obtained independently in the recent work [45]. The main result of [45] is a general asymptotic formula for the variance of the counting function of an arbitrary *non-interacting fermion process*. Non-interacting fermion processes are determinantal point processes whose kernel can be written in the form

$$K_\mu(x, y) = \sum_{k=1}^n \overline{\psi_k(x)} \psi_k(y), \quad n = \#\{k : \epsilon_k \leq \mu\}, \quad \mu \in \mathbb{R},$$

where $\epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \dots$ denote the eigenvalues of a Schrödinger operator $\mathcal{H} = -\partial_x^2 + V$ associated to a certain potential $V : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$, and ψ_k denotes the eigenfunction associated to ϵ_k . It has been observed, see [38, Section 6.2], that the Bessel kernel, after a proper rescaling, arises as the limit $\mu \rightarrow +\infty$ of the correlation kernel of the non-interacting fermion process associated to the potential

$$V(x) = \begin{cases} \frac{\alpha^2 - 1/4}{2x^2}, & \text{if } x > 0, \\ +\infty, & \text{if } x \leq 0. \end{cases} \quad (1.27)$$

The general formula of [45] specialized to the potential (1.27) agrees with (1.26) up to and including the constant; this consistency has been verified in detail in [45, Section V (ii)].

The large r asymptotics of $\mathbb{E}[N_\alpha(rx_1)]$, $\text{Var}[N_\alpha(rx_1)]$ and $\text{Cov}[N_\alpha(rx_1), N_\alpha(rx_2)]$ with α bounded have been obtained in [46] (for the leading terms) and in [12, Corollary 1.2] (for the subleading terms). Using (1.15), it is straightforward to verify that if $a \rightarrow 0$ sufficiently fast, the asymptotics of Corollary 1.4 agree with [12, Corollary 1.2], up to and including the constant term. More precisely, we have the following.

Corollary 1.5 (Matching with the Bessel asymptotics with α bounded). *Let $0 < x_1 < x_2$ be fixed, and for $a > 0$, $r > 0$, define $\alpha = a\sqrt{r}$. As $r \rightarrow +\infty$, $\alpha \rightarrow +\infty$ and $a^2\sqrt{r} \rightarrow 0$, we have*

$$\begin{aligned} \mathbb{E}[N_\alpha(rx_1)] &= \tilde{\mu}_\alpha(rx_1) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}} + a^2\sqrt{r}\right), \\ \text{Var}[N_\alpha(rx_1)] &= \tilde{\sigma}_\alpha^2(rx_1) + \frac{1 + \gamma_E}{2\pi^2} + \mathcal{O}\left(\frac{\log^2 r}{\sqrt{r}} + a^2\right), \\ \text{Cov}[N_\alpha(rx_1), N_\alpha(rx_2)] &= \tilde{\Sigma}(x_2, x_1) + \mathcal{O}\left(\frac{\log^2 r}{\sqrt{r}} + a^2\right). \end{aligned}$$

The following asymptotics related to the Airy process have been obtained in [46] (for the leading terms) and in [15] (for the subleading terms), and are valid as $r_{\text{Ai}} \rightarrow +\infty$ with fixed $0 < y_1 < y_2$:

$$\begin{aligned} \mathbb{E}[N_{\text{Ai}}(r_{\text{Ai}}y_1)] &= \mu_{\text{Ai}}(r_{\text{Ai}}y_1) + o(1), \\ \text{Var}[N_{\text{Ai}}(r_{\text{Ai}}y_1)] &= \sigma_{\text{Ai}}^2(r_{\text{Ai}}y_1) + \frac{1 + \gamma_E}{2\pi^2} + o(1), \\ \text{Cov}[N_{\text{Ai}}(r_{\text{Ai}}y_1), N_{\text{Ai}}(r_{\text{Ai}}y_2)] &= \Sigma_{\text{Ai}}(y_1, y_2) + o(1). \end{aligned}$$

Using the expansions given in the proof of Corollary 1.3, we establish the following asymptotics (the result follows in the same way as Corollary 1.4).

Corollary 1.6 (Matching with the Airy asymptotics). *Let $0 < y_1 < y_2$ and $a \in (0, +\infty)$ be fixed. For $r > 0$, define $\alpha = a\sqrt{r}$, and let x_1, x_2 be such that*

$$x_j = \frac{\alpha^2}{r} + 2^{\frac{2}{3}} \frac{\alpha^{\frac{4}{3}} r_{\text{Ai}}}{r} y_j = a^2 + 2^{\frac{2}{3}} \frac{a^{\frac{4}{3}} r_{\text{Ai}}}{r^{\frac{1}{3}}} y_j, \quad j = 1, 2.$$

As $r_{\text{Ai}} \rightarrow +\infty$ and $r \rightarrow +\infty$ such that (1.20) holds, we have

$$\begin{aligned} \mathbb{E}[N_\alpha(rx_1)] &= \mu_{\text{Ai}}(r_{\text{Ai}}y_1) \left(1 + \mathcal{O}\left(\frac{r_{\text{Ai}}}{r^{\frac{1}{3}}}\right)\right) + \mathcal{O}\left(\frac{r^{\frac{5}{6}} \log r}{r_{\text{Ai}}^4}\right), \\ \text{Var}[N_\alpha(rx_1)] &= \sigma_{\text{Ai}}^2(r_{\text{Ai}}y_1) + \frac{1 + \gamma_E}{2\pi^2} + \mathcal{O}\left(\frac{r_{\text{Ai}}}{r^{\frac{1}{3}}} + \frac{r^{\frac{5}{6}} \log^2 r}{r_{\text{Ai}}^4}\right), \\ \text{Cov}[N_\alpha(rx_1), N_\alpha(rx_2)] &= \Sigma_{\text{Ai}}(y_1, y_2) + \mathcal{O}\left(\frac{r^{\frac{5}{6}} \log^2 r}{r_{\text{Ai}}^4}\right). \end{aligned}$$

Remark 4. We believe the error estimates in the statement of Corollary 1.6 are not optimal.

1.2 Central limit theorems

We first obtain a CLT for the counting function of the large α Bessel process.

Corollary 1.7. *Let $\alpha = a\sqrt{r}$, $a > 0$, $r > 0$, and let $m \in \mathbb{N}_{>0}$ and $\vec{x} \in \mathbb{R}_{\text{ord}}^{+,m}$ be such that $0 < a^2 < x_1 < \dots < x_m$. Consider the random variables $\mathcal{N}_j^{(r)}$ defined by*

$$\mathcal{N}_j^{(r)} = \frac{N_\alpha(rx_j) - \mu_\alpha(rx_j)}{\sqrt{\sigma_\alpha^2(rx_j)}}, \quad j = 1, \dots, m.$$

1. As $r \rightarrow +\infty$ and $\alpha \rightarrow +\infty$, we have

$$(\mathcal{N}_1^{(r)}, \mathcal{N}_2^{(r)}, \dots, \mathcal{N}_m^{(r)}) \xrightarrow{d} \mathcal{N}(\vec{0}, I_m), \quad (1.28)$$

uniformly for $(a^2, x_1, \dots, x_m)$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m+1}$, where “ $\xrightarrow{d}$ ” means convergence in distribution, I_m is the $m \times m$ identity matrix, and $\mathcal{N}(\vec{0}, I_m)$ is a multivariate normal random variable of mean $\vec{0} = (0, \dots, 0)$ and covariance matrix I_m .

2. The convergence in distribution (1.28) still holds in the regime where $r \rightarrow +\infty$ and $\alpha = a\sqrt{r} \rightarrow +\infty$ such that $a \rightarrow 0$ and $\vec{x}$ lies in a compact subset of $\mathbb{R}_{\text{ord}}^{+,m}$.

3. The convergence in distribution (1.28) still holds in the regime where $r \rightarrow +\infty$, $\alpha = a\sqrt{r} \rightarrow +\infty$, $a \in (0, +\infty)$ is fixed and $\vec{x} \in \mathbb{R}_{\text{ord}}^{+,m}$ satisfies $a^2 < x_1 < \dots < x_m$ and (1.9).

Proof. Recall that the asymptotics of part 1 of Theorem 1.1 are valid uniformly for $u_1, \dots, u_m$ in compact subsets of $\mathbb{R}$, and uniformly for $\vec{x}$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m}$. Therefore, it follows from (1.5) with $n = 1$ and $u_j = t_j / \sqrt{\sigma_\alpha^2(rx_j)}$ that

$$\mathbb{E}_\alpha \left[\exp \left(\sum_{j=1}^m t_j \mathcal{N}_j^{(r)} \right) \right] = \exp \left(\sum_{j=1}^m \frac{t_j^2}{2} + o(1) \right), \quad \text{as } r \rightarrow +\infty, \alpha \rightarrow +\infty,$$

uniformly for $\vec{x}$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m}$, which implies the convergence in distribution (1.28) stated in part 1 of the claim. The regimes considered in parts 2 and 3 of the claim follow similarly from parts 2 and 3 of Theorem 1.1. $\square$

The Bessel process is locally finite, has almost surely all points distinct and possesses almost surely a smallest point. Let $0 < \xi_{\alpha,1} < \xi_{\alpha,2} < \dots$ be the points in the Bessel point process. Our next result is a CLT for the fluctuations of the points around their classical locations. We will use the notation $[x] = \lfloor x + \frac{1}{2} \rfloor$, i.e. $[x]$ is the closest integer to x . Also, since $\mu_\alpha : [\alpha^2, +\infty) \rightarrow \mathbb{R}$, defined by (1.6), is strictly increasing, it possesses an inverse which we denote by μ_α^{-1} .

Corollary 1.8. *Let $\alpha = a\sqrt{r}$, $a > 0$, $r > 0$, let $m \in \mathbb{N}_{>0}$ and $\vec{x} \in \mathbb{R}_{\text{ord}}^{+,m}$ be such that $0 < a^2 < x_1 < \dots < x_m$, and let $k_j = [\mu_\alpha(rx_j)]$, $j = 1, \dots, m$. Consider the random variables $Y_j^{(r)}$ defined by*

$$Y_j^{(r)} = \frac{\mu_\alpha(\xi_{\alpha,k_j}) - k_j}{\sqrt{\sigma_\alpha^2 \circ \mu_\alpha^{-1}(k_j)}}, \quad j = 1, \dots, m.$$

1. As $r \rightarrow +\infty$, $\alpha \rightarrow +\infty$, such that $(a^2, x_1, \dots, x_m)$ lies in a compact subset of $\mathbb{R}_{\text{ord}}^{+,m+1}$, we have

$$(Y_1^{(r)}, Y_2^{(r)}, \dots, Y_m^{(r)}) \xrightarrow{d} \mathcal{N}(\vec{0}, I_m). \quad (1.29)$$

2. The convergence in distribution (1.29) still holds in the regime where $r \rightarrow +\infty$ and $\alpha = a\sqrt{r} \rightarrow +\infty$ such that $a \rightarrow 0$ and $\vec{x}$ lies in a compact subset of $\mathbb{R}_{\text{ord}}^{+,m}$.

3. The convergence in distribution (1.29) still holds in the regime where $r \rightarrow +\infty$, $\alpha = a\sqrt{r} \rightarrow +\infty$, $a \in (0, +\infty)$ is fixed and $\vec{x}$ satisfies (1.9).

Proof. The proof is inspired by (but different from) the proof of [31, Theorem 1.2]. Given $y_1, \dots, y_m \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{P}[Y_j^{(r)} \leq y_j \text{ for all } j = 1, \dots, m] &= \mathbb{P}\left[\xi_{\alpha, k_j} \leq \mu_\alpha^{-1}\left(k_j + y_j \sqrt{\sigma_\alpha^2 \circ \mu_\alpha^{-1}(k_j)}\right) \text{ for all } j = 1, \dots, m\right], \\ &= \mathbb{P}\left[N_\alpha\left(\mu_\alpha^{-1}\left(k_j + y_j \sqrt{\sigma_\alpha^2 \circ \mu_\alpha^{-1}(k_j)}\right)\right) \geq k_j \text{ for all } j = 1, \dots, m\right]. \end{aligned} \quad (1.30)$$

Let us define

$$\tilde{x}_j := \frac{1}{r} \mu_\alpha^{-1}\left(k_j + y_j \sqrt{\sigma_\alpha^2 \circ \mu_\alpha^{-1}(k_j)}\right), \quad j = 1, \dots, m.$$

As $r \rightarrow +\infty$, $\alpha \rightarrow +\infty$ such that $(a, x_1, \dots, x_m)$ lies in a compact subset of $\mathbb{R}_{\text{ord}}^{+, m+1}$, we verify from (1.6) and (1.7) that

$$k_j = [\mu_\alpha(r x_j)] = \mathcal{O}(\sqrt{r}), \quad \tilde{x}_j = x_j \left(1 + \mathcal{O}\left(\sqrt{\frac{\log r}{r}}\right)\right),$$

and in particular $(a^2, \tilde{x}_1, \dots, \tilde{x}_m)$ lies also in a compact subset of $\mathbb{R}_{\text{ord}}^{+, m+1}$ for all sufficiently large r . Now, we rewrite (1.30) as

$$\begin{aligned} &\mathbb{P}[Y_j^{(r)} \leq y_j \text{ for all } j = 1, \dots, m] \\ &= \mathbb{P}\left[\frac{N_\alpha(r \tilde{x}_j) - \mu_\alpha(r \tilde{x}_j)}{\sqrt{\sigma_\alpha^2(r \tilde{x}_j)}} \geq \frac{k_j - \mu_\alpha(r \tilde{x}_j)}{\sqrt{\sigma_\alpha^2(r \tilde{x}_j)}} \text{ for all } j = 1, \dots, m\right] \\ &= \mathbb{P}\left[\frac{N_\alpha(r \tilde{x}_j) - \mu_\alpha(r \tilde{x}_j)}{\sqrt{\sigma_\alpha^2(r \tilde{x}_j)}} \geq -y_j \frac{\sqrt{\sigma_\alpha^2 \circ \mu_\alpha^{-1}(k_j)}}{\sqrt{\sigma_\alpha^2(r \tilde{x}_j)}} \text{ for all } j = 1, \dots, m\right] \\ &= \mathbb{P}\left[\frac{\mu_\alpha(r \tilde{x}_j) - N_\alpha(r \tilde{x}_j)}{\sqrt{\sigma_\alpha^2(r \tilde{x}_j)}} \leq y_j(1 + o(1)) \text{ for all } j = 1, \dots, m\right], \end{aligned} \quad (1.31)$$

where $o(1)$ in the last equality is understood as $r \rightarrow +\infty$, $\alpha \rightarrow +\infty$ such that $(a^2, \tilde{x}_1, \dots, \tilde{x}_m)$ lies in a compact subset of $\mathbb{R}_{\text{ord}}^{+, m+1}$. Part 1 of the claim now follows from (1.31) and part 1 of Corollary 1.7, because if $\mathcal{N}$ is a multivariate normal random variable of mean $\vec{0}$ and covariance matrix I_m , then so is $-\mathcal{N}$. Part 2 follows similarly from (1.31) and part 2 of Corollary 1.7. For part 3 of the claim, we note that

$$k_j = \mathcal{O}(\sqrt{r}|x_m - a^2|^{3/2}), \quad \tilde{x}_j = x_j \left(1 + \mathcal{O}\left(\frac{\sqrt{\log(r|x_m - a^2|^{3/2})}}{\sqrt{r}|x_m - a^2|^{3/2}}\right)\right).$$

Since $\frac{\sqrt{\log(r|x_m - a^2|^{3/2})}}{\sqrt{r}|x_m - a^2|^{3/2}} = \mathcal{O}\left(\frac{\log r}{|\tilde{x}_m - a^2|^4 \sqrt{r}}\right)$ as $r \rightarrow +\infty$, $\alpha \rightarrow +\infty$ with $a \in (0, +\infty)$ fixed and $\vec{x}$ satisfying (1.9), the claim follows from (1.31) and part 3 of Corollary 1.7. $\square$

1.3 Rigidity

There exist several notions of *rigidity* in the literature. For example, the Bessel process is said to be *rigid* in [10, 11, 40] because $N_\alpha(x)$ is almost surely determined by the points on $(x, +\infty)$. In this work, the *rigidity* of a point process refers to the study of the maximal deviation of the points with respect to their classical locations. This notion of rigidity has been widely studied in recent years, see e.g. [27, 1] for important early works, [32] for the sine process, [16] for the Airy and Bessel point processes, and [14] for the Pearcey process. A global rigidity upper bound for the Bessel process with α bounded has been established in [16]. In this work, we contribute in this direction by providing a global rigidity upper bound for the large α Bessel process. Our techniques are inspired by previous works and rely on the first exponential moment asymptotics, which are given by (1.5) with $m = 1$. There already exist several general rigidity theorems that are available in the literature, see [18, 16], but none of them can be directly applied to our case. The reason is that here the Bessel process varies as r increases (recall that $\alpha \rightarrow +\infty$ as $r \rightarrow +\infty$).

In fact, we will go slightly further and prove a general rigidity result which can be applied to varying point processes that possess a smallest point almost surely and for which the first exponential moment asymptotics are known. This result, which is stated in Theorem 1.10 below, generalizes [16, Theorem 1.2]

and allows us to obtain a global rigidity upper bound for the large α Bessel process. It can also be applied to other varying point processes, such as the conditional Airy and Bessel point processes considered in [15, Theorem 1.2] and [12, Theorem 1.4], but we do not pursue this matter here.

Assume that we have a family of point processes $\{X_r\}_{r \geq 0}$, each of them with an almost surely smallest point. Let $N_r(x)$ be the random variable that counts the number of points of X_r that are $\leq x$. Our general rigidity result will be valid under the following assumptions on $\{X_r\}_{r \geq 0}$.

Assumptions 1.9. There exist constants $\mathfrak{a} > 0$, $M > \sqrt{2/\mathfrak{a}}$, $C > 0$, $r_0 > 0$, $\{\eta_{r,2}, \eta_{r,1}, \delta_r\}_{r \geq r_0}$ with $0 < \eta_{r,1} < \eta_{r,2} < +\infty$, $\delta_r \in (0, \frac{\eta_{r,2} - \eta_{r,1}}{5})$ and functions $\{\mu_r, \sigma_r : [\eta_{r,1}r, \eta_{r,2}r] \rightarrow [0, +\infty)\}_{r \geq r_0}$ such that the following statements hold:

(1) We have

$$\mathbb{E}[e^{\gamma N_r(xr)}] \leq C e^{\gamma \mu_r(xr) + \frac{\gamma^2}{2} \sigma_r^2(xr)}, \quad (1.32)$$

for all $\gamma \in [-M, M]$, all $x \in (\eta_{r,1} + \frac{\delta_r}{2}, \eta_{r,2} - \frac{\delta_r}{2})$, and all $r > r_0$.

(2) For each $r > r_0$, the functions $x \mapsto \mu_r(x)$ and $x \mapsto \sigma_r(x)$ are strictly increasing and differentiable, and satisfy

$$\lim_{r \rightarrow +\infty} \mu_r((\eta_{r,1} + \delta_r)r) = +\infty \quad \text{and} \quad \lim_{r \rightarrow +\infty} \sigma_r((\eta_{r,1} + \delta_r)r) = +\infty.$$

Moreover, $x \mapsto x \mu'_r(x)$ is non-decreasing and

$$\lim_{r \rightarrow +\infty} \inf_{x, y \in (\eta_{r,1} + \frac{\delta_r}{2}, \eta_{r,2} - \frac{\delta_r}{2})} \frac{\delta_r r \mu'_r(xr)}{\sigma_r^2(yr)} = +\infty. \quad (1.33)$$

(3) For each $r > r_0$, the function $\sigma_r^2 \circ \mu_r^{-1} : [\mu_r(\eta_{r,1}r), \mu_r(\eta_{r,2}r)] \rightarrow [0, +\infty)$ is strictly concave, strictly increasing, and

$$(\sigma_r^2 \circ \mu_r^{-1})(k) = \mathfrak{a} \log k + \mathcal{O}(1) \quad \text{as } r \rightarrow +\infty, \quad (1.34)$$

uniformly for $k \in [\mu_r((\eta_{r,1} + \frac{\delta_r}{2})r), \mu_r((\eta_{r,2} - \frac{\delta_r}{2})r)]$.

(4) The quantities $\eta_{r,1} + \delta_r$ and $\eta_{r,2} - \eta_{r,1} - 4\delta_r$ remain bounded away from 0 as $r \rightarrow +\infty$, and $\eta_{r,2}$ remains bounded away from $+\infty$ as $r \rightarrow +\infty$.

Theorem 1.10 (Rigidity). Suppose that $\{X_r\}_{r \geq 0}$ is a family of locally finite point processes on the real line with almost surely a smallest point and such that Assumptions 1.9 hold. Let $\mathfrak{a}, \eta_{r,1}, \eta_{r,2}, \delta_r$ be the constants appearing in Assumptions 1.9, and let $\xi_{r,k}$ denote the k -th smallest point of X_r , $k \geq 1$, $r \geq 0$. Let $\lambda > 1$. There exist constants $c > 0$ and $r_0 > 0$ such that for any small enough $\epsilon > 0$ and for all $r \geq r_0$,

$$\mathbb{P} \left(\max_{k \in I_r \cap \mathbb{N}_{\geq 0}} \frac{|\mu_r(\xi_{r,k}) - k|}{(\sigma_r^2 \circ \mu_r^{-1})(k)} \leq \sqrt{\frac{2}{\mathfrak{a}}(1 + \epsilon)} \right) \geq 1 - \frac{c \mu_r((\eta_{r,1} + \delta_r)r)^{-\frac{\epsilon}{\lambda}}}{\epsilon}, \quad (1.35)$$

where $I_r = (\mu_r((\eta_{r,1} + 2\delta_r)r), \mu_r((\eta_{r,2} - 2\delta_r)r))$. In particular, for any $\epsilon > 0$,

$$\lim_{r \rightarrow +\infty} \mathbb{P} \left(\max_{k \in I_r \cap \mathbb{N}_{\geq 0}} \frac{|\mu_r(\xi_{r,k}) - k|}{(\sigma_r^2 \circ \mu_r^{-1})(k)} \leq \sqrt{\frac{2}{\mathfrak{a}}} + \epsilon \right) = 1. \quad (1.36)$$

Remark 5. We expect (1.35) to hold also for $\lambda = 1$, but this sharper case requires a more delicate analysis and we do not pursue this here.

Remark 6. Assumptions 1.9 imply that the Lebesgue measure of I_r tends to $+\infty$ as $r \rightarrow +\infty$. Indeed, using parts (2) and (4) of Assumptions 1.9, one has

$$|I_r| \geq (\eta_{r,2} - \eta_{r,1} - 4\delta_r) \inf_{x \in (\eta_{r,1} + 2\delta_r)r, \eta_{r,2} - 2\delta_r} r \mu'_r(xr) \geq \frac{(\eta_{r,2} - \eta_{r,1} - 4\delta_r)(\eta_{r,1} + 2\delta_r)}{\eta_{r,2} - 2\delta_r} r \mu'_r((\eta_{r,1} + 2\delta_r)r) \geq c_1 r \mu'_r((\eta_{r,1} + 2\delta_r)r),$$

where the infimum is taken over $x \in (\eta_{r,1} + 2\delta_r)r, \eta_{r,2} - 2\delta_r$ and $c_1 > 0$ is independent of r . It follows from part (2) of Assumptions 1.9 that the right-hand side tends to $+\infty$ as $r \rightarrow +\infty$. This means that the maximum in (1.36) is taken over a large number of values of k as r gets large.

The parameters $\eta_{r,2}$ and $\eta_{r,1}$ influence the size of $|I_r|$. Since the bound (1.36) gives more information about $\{X_r\}_{r \geq 0}$ if $|I_r|$ is larger, it is important, in concrete examples, to choose $\eta_{r,2}$ as large as possible, and $\eta_{r,1}$ and δ_r as small as possible, such that (1.32) holds.

In the next subsection, we use Theorem 1.10 to obtain a global rigidity upper bound for the large α Bessel process.

1.4 Rigidity of the large α Bessel point process

Assume that $\alpha = \alpha(r) > 0$ is an increasing function of r such that $\{a := \alpha^2/r\}_{r \geq 1}$ is bounded and $\alpha \rightarrow +\infty$ as $r \rightarrow +\infty$. Let X_r be the Bessel process associated with α . It follows from parts 1 and 2 of Theorem 1.1 that the sequence $\{X_r\}_{r \geq 1}$ satisfies Assumptions 1.9 with $\eta_{r,1} = \alpha^2/r$, $\eta_{r,2} = K$, r_0 sufficiently large,

$$\mathfrak{a} = \frac{1}{2\pi^2}, \quad \mu_r(\xi) = \mu_\alpha(\xi), \quad \sigma_r^2(\xi) = \sigma_\alpha^2(\xi), \quad \delta_r = \delta > 0, \quad \text{and} \quad M > \sqrt{2/\mathfrak{a}},$$

$C = 2 \sup_{u \in [-M, M]} G(1 + \frac{u}{2\pi i}) G(1 - \frac{u}{2\pi i})$, where μ_α and σ_α^2 are defined in (1.6)–(1.7), where $M > \sqrt{2/\mathfrak{a}}$ and $K > \sup_{r \geq 1} \alpha^2/r$ can be chosen arbitrarily large but finite and where $\delta > 0$ can be chosen arbitrarily small but fixed. Indeed, all statements in Assumptions 1.9 follow from straightforward computations except (1.34). To verify (1.34), suppose $k \in [\mu_r((\eta_{r,1} + \frac{\delta}{2})r), \mu_r((\eta_{r,2} - \frac{\delta}{2})r)]$. Since μ_α is increasing, it follows that $\mu_\alpha^{-1}(k) \in [(\eta_{r,1} + \frac{\delta}{2})r, (\eta_{r,2} - \frac{\delta}{2})r]$ and hence $\alpha^2 + cr \leq \mu_\alpha^{-1}(k) \leq Cr$ for some constants $c, C > 0$. Since, by (1.7),

$$\sigma_\alpha^2(\xi) = \frac{\log \xi}{4\pi^2} + \mathcal{O}(1), \quad \text{as } \alpha \rightarrow +\infty \text{ uniformly for } \xi \in [\alpha^2 + cr, Cr],$$

we conclude that

$$(\sigma_r^2 \circ \mu_r^{-1})(k) = (\sigma_\alpha^2 \circ \mu_\alpha^{-1})(k) = \frac{\log(\mu_\alpha^{-1}(k))}{4\pi^2} + \mathcal{O}(1) = \frac{\log r}{4\pi^2} + \mathcal{O}(1), \quad r \rightarrow +\infty. \quad (1.37)$$

Moreover, by (1.6), there exist constants $c_1, C_1 > 0$ such that $c_1\sqrt{\xi} \leq \mu_\alpha(\xi) \leq C_1\sqrt{\xi}$ whenever $\xi \in [\alpha^2 + cr, Cr]$; thus $c_2\sqrt{r} \leq k \leq C_2\sqrt{r}$ for some constants $c_2, C_2 > 0$. This implies that $\log r = \log(k^2) + \mathcal{O}(1)$ and (1.34) then follows from (1.37). Therefore, by specializing Theorem 1.10 to the large α Bessel process, we obtain the following.

Theorem 1.11 (Rigidity of the large α Bessel point process). *Assume that α is an increasing function of r such that $\{a := \alpha^2/r\}_{r \geq 1}$ is bounded and $\alpha \rightarrow +\infty$ as $r \rightarrow +\infty$, and let $\xi_{\alpha,1} < \xi_{\alpha,2} < \dots$ be the points in the Bessel point process X_r . Fix $\lambda > 1$. For any $K > \sup_{r \geq 1} \alpha^2/r$, any $\delta \in (0, K)$ and any small enough $\epsilon > 0$, there exist $r_0 > 0, c > 0$ such that*

$$\mathbb{P} \left(\max_{k \in (\delta\sqrt{r}, K\sqrt{r}) \cap \mathbb{N}_{\geq 0}} \frac{|\mu_\alpha(\xi_{\alpha,k}) - k|}{\log k} \leq \frac{\sqrt{1+\epsilon}}{\pi} \right) \geq 1 - \frac{c}{\epsilon} \sqrt{r}^{-\frac{\epsilon}{\lambda}}, \quad (1.38)$$

for all $r \geq r_0$. In particular, for any $\epsilon > 0$,

$$\lim_{r \rightarrow +\infty} \mathbb{P} \left(\max_{k \in (\delta\sqrt{r}, K\sqrt{r}) \cap \mathbb{N}_{\geq 0}} \frac{|\frac{1}{\pi} \int_{\alpha^2}^{\xi_{\alpha,k}} \frac{\sqrt{u-\alpha^2}}{2u} du - k|}{\log k} \leq \frac{1}{\pi} + \epsilon \right) = 1. \quad (1.39)$$

Proof. Applying Theorem 1.10 to the large α Bessel process, we infer that there exist c', ϵ'_0 and r'_0 such that

$$\begin{aligned} \mathbb{P} \left(\max_{k \in I_r \cap \mathbb{N}_{\geq 0}} \frac{|\mu_\alpha(\xi_{\alpha,k}) - k|}{(\sigma_\alpha^2 \circ \mu_\alpha^{-1})(k)} \leq 2\pi\sqrt{1+\epsilon'} \right) &\geq 1 - \frac{c' \mu_\alpha(\alpha^2 + \delta r)^{-\frac{\epsilon'}{\lambda}}}{\epsilon'} \\ &= 1 - \frac{c'}{\epsilon'} \left(\frac{\sqrt{r}}{\pi} \int_{a^2}^{a^2+\delta} \frac{\sqrt{u-a^2}}{2u} du \right)^{-\frac{\epsilon'}{\lambda}} \geq 1 - \frac{\hat{c}}{\epsilon'} \sqrt{r}^{-\frac{\epsilon'}{\lambda}}, \end{aligned}$$

for all $r \geq r'_0$ and $\epsilon' \in (0, \epsilon'_0]$, where $\hat{c} := c' \sup_{r \geq r'_0, \epsilon' \in (0, \epsilon'_0]} \left(\frac{1}{\pi} \int_{a^2}^{a^2+\delta} \frac{\sqrt{u-a^2}}{2u} du \right)^{-\frac{\epsilon'}{\lambda}} < +\infty$ and

$$I_r = (\mu_\alpha(\alpha^2 + 2\delta r), \mu_\alpha(Kr - 2\delta r)) = \left(\frac{\sqrt{r}}{\pi} \int_{a^2}^{a^2+2\delta} \frac{\sqrt{u-a^2}}{2u} du, \frac{\sqrt{r}}{\pi} \int_{a^2}^{K-2\delta} \frac{\sqrt{u-a^2}}{2u} du \right).$$

Note that

$$I_r \supseteq (c_\delta \sqrt{r}, c_K \sqrt{r}), \quad \text{where } c_\delta := \sup_{r \geq r_0} \frac{1}{\pi} \int_{a^2}^{a^2+2\delta} \frac{\sqrt{u-a^2}}{2u} du, \quad c_K := \inf_{r \geq r_0} \frac{1}{\pi} \int_{a^2}^{K-2\delta} \frac{\sqrt{u-a^2}}{2u} du.$$

Let δ be sufficiently small and/or let K be sufficiently large such that $c_\delta < c_K$. Then,

$$\mathbb{P} \left(\max_{k \in (c_\delta \sqrt{r}, c_K \sqrt{r}) \cap \mathbb{N}_{\geq 0}} \frac{|\mu_\alpha(\xi_{\alpha,k}) - k|}{(\sigma_\alpha^2 \circ \mu_\alpha^{-1})(k)} \leq 2\pi\sqrt{1+\epsilon'} \right) \geq 1 - \frac{\hat{c}}{\epsilon'} \sqrt{r}^{-\frac{\epsilon'}{\lambda}},$$

for all $r \geq r'_0$ and $\epsilon' \in (0, \epsilon'_0]$. Using that (cf. (1.37))

$$\max_{k \in (c_\delta \sqrt{r}, c_K \sqrt{r}) \cap \mathbb{N}_{\geq 0}} \left| (\sigma_\alpha^2 \circ \mu_\alpha^{-1})(k) \left(\frac{\log k}{2\pi^2} \right)^{-1} - 1 \right| = \mathcal{O} \left(\frac{1}{\log r} \right) \quad \text{as } r \rightarrow +\infty,$$

we conclude that there exists a $c_0 > 0$ such that

$$\mathbb{P} \left(\max_{k \in (c_\delta \sqrt{r}, c_K \sqrt{r}) \cap \mathbb{N}_{\geq 0}} \frac{|\mu_\alpha(\xi_{\alpha,k}) - k|}{\log k} \leq \frac{\sqrt{1+\epsilon'}}{\pi(1 - \frac{c_0}{\log r})} \right) \geq 1 - \frac{\hat{c}}{\epsilon'} \sqrt{r}^{-\frac{\epsilon'}{\lambda}},$$

for all sufficiently large r and all $\epsilon' \in (0, \epsilon'_0]$. Hence for each $\epsilon' \in (0, \epsilon'_0]$,

$$\mathbb{P} \left(\max_{k \in (c_\delta \sqrt{r}, c_K \sqrt{r}) \cap \mathbb{N}_{\geq 0}} \frac{|\mu_\alpha(\xi_{\alpha,k}) - k|}{\log k} \leq \frac{\sqrt{1+\epsilon}}{\pi} \right) \geq 1 - \frac{2\hat{c}}{\epsilon} \sqrt{r}^{-\frac{\epsilon'}{\lambda}} = 1 - \frac{2\hat{c}e^{\frac{c_1}{2\lambda}}}{\epsilon} \sqrt{r}^{-\frac{\epsilon}{\lambda}},$$

for all sufficiently large r , where $\epsilon = \epsilon' + \frac{c_1}{\log r}$ and c_1 is sufficiently large but fixed. Since c_δ can be made arbitrarily small and c_K can be made arbitrarily large by choosing δ sufficiently small and K sufficiently large, the claim follows. $\square$

If we let $\alpha \rightarrow +\infty$ at a slow speed so that $\alpha^2/\sqrt{r} \rightarrow 0$ as $r \rightarrow +\infty$ in Theorem 1.11, we obtain the following corollary, which is consistent with the known rigidity result [16, Theorem 1.6] for the Bessel process with α fixed.

Corollary 1.12 (Matching with the rigidity of the Bessel process with α bounded). *Assume that α is an increasing function of r such that $\alpha^2/\sqrt{r} \rightarrow 0$ and $\alpha \rightarrow +\infty$ as $r \rightarrow +\infty$, and let $\xi_{\alpha,1} < \xi_{\alpha,2} < \dots$ be the points in the Bessel point process X_r . For any $K > \delta > 0$ and any $\epsilon > 0$, we have*

$$\lim_{r \rightarrow +\infty} \mathbb{P} \left(\max_{k \in I_r \cap \mathbb{N}_{\geq 0}} \frac{|\frac{\sqrt{\xi_{\alpha,k}}}{\pi} - \frac{\alpha}{2} - k|}{\log k} \leq \frac{1}{\pi} + \epsilon \right) = 1, \quad \text{where } I_r = (\delta\sqrt{r}, K\sqrt{r}).$$

Proof. The claim follows easily from Theorem 1.11 and the first expansion in (1.15). $\square$

Outline. The proof of Theorem 1.1 is divided into several steps that are carried out in Sections 2-5. In Section 2, we present a differential identity from [12] that relates E_α to a model Riemann-Hilbert (RH) problem whose solution is denoted by Φ . In Section 3, we obtain asymptotics for Φ in the three regimes of Theorem 1.1 via the Deift-Zhou [24] steepest descent method. As mentioned in the introduction, the regime $\alpha/\sqrt{r} \rightarrow 0$ (part 2 of Theorem 1.1) requires a different analysis from the regimes 1 and 3 where $\alpha/\sqrt{r}$ remains bounded away from 0. In Section 5, we substitute the asymptotics of Φ into the differential identity to obtain large r asymptotics for $E_\alpha(r\vec{x}, \vec{u})$. The proof of Theorem 1.10 is independent from the other proofs and is given in Section 6. The appendix collects three model RH problems that are important for the construction of local parametrices in the steepest descent analysis. Of particular interest is that we describe and employ the Bessel model problem in the regime where the order α of the Bessel functions tends to infinity. This appears to be the first time that the Bessel model problem is used in this way.

Acknowledgements

We are grateful to Grégory Schehr for suggesting this problem to us, for interesting discussions and for pointing out a mistake in an early draft of this manuscript. The authors acknowledge support from the European Research Council, Grant Agreement No. 682537, the Swedish Research Council, Grant No. 2015-05430, and the Ruth and Nils-Erik Stenbäck Foundation.

2 Background from [17, 12]

We will prove Theorem 1.1 via a differential identity that was established in [12]. This identity relates E_α with the solution Φ to a model RH problem. We first recall the properties of Φ in Subsection 2.1, and then present the differential identity in Subsection 2.2.

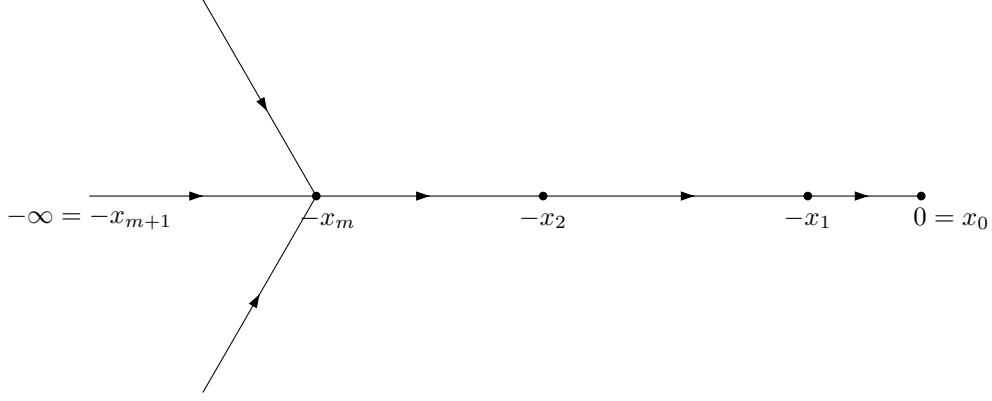


Figure 1: The contour Σ_Φ with $m = 3$.

2.1 Model RH problem Φ from [17]

The model RH problem for Φ depends on parameters α , $\vec{x} = (x_1, \dots, x_m)$, $\vec{s} = (s_1, \dots, s_m)$ satisfying

$$\alpha > -1, \quad 0 < x_1 < \dots < x_m < +\infty, \quad s_1, \dots, s_m \in [0, +\infty), \quad (2.1)$$

and is as follows.

RH problem for $\Phi(\cdot) = \Phi(\cdot; \vec{x}, \vec{s}, \alpha)$

- (a) $\Phi : \mathbb{C} \setminus \Sigma_\Phi \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where the contour is given by

$$\Sigma_\Phi = (-\infty, 0] \cup (-x_m + [0, e^{\frac{2\pi i}{3}} \infty)) \cup (-x_m + [0, e^{-\frac{2\pi i}{3}} \infty)),$$

and is oriented as shown in Figure 1.

- (b) Let Φ_+ (resp. Φ_-) denote the left (resp. right) boundary values of Φ . Here, “left” and “right” are to be understood with respect to the orientation of Σ_Φ . Φ_+ and Φ_- are continuous on $\Sigma_\Phi \setminus \{0, -x_1, \dots, -x_m\}$ and are related by:

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \quad z \in -x_m + (0, e^{\frac{2\pi i}{3}} \infty), \quad (2.2)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, -x_m), \quad (2.3)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \quad z \in -x_m + (0, e^{-\frac{2\pi i}{3}} \infty), \quad (2.4)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} e^{\pi i \alpha} & s_j \\ 0 & e^{-\pi i \alpha} \end{pmatrix}, \quad z \in (-x_j, -x_{j-1}), \quad (2.5)$$

where $j = 1, \dots, m$.

- (c) As $z \rightarrow \infty$, we have

$$\Phi(z) = \left(I + \Phi_1 z^{-1} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N e^{\sqrt{z} \sigma_3}, \quad (2.6)$$

where the principal branch is chosen for each root, the matrix $\Phi_1 = \Phi_1(\vec{x}, \vec{s}, \alpha)$ is independent of z and traceless, and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

As z tends to $-x_j$, $j \in \{1, \dots, m\}$, we have

$$\Phi(z) = G_j(z) \begin{pmatrix} 1 & \frac{s_j + 1 - s_j}{2\pi i} \log(z + x_j) \\ 0 & 1 \end{pmatrix} V_j(z) e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} H_{-x_m}(z), \quad (2.7)$$

where $G_j(z) = G_j(z; \vec{x}, \vec{s}, \alpha)$ is analytic in a neighborhood of $-x_j$ and satisfies $\det G_j \equiv 1$, and θ , V_j , H_{-x_m} are piecewise constant and given by

$$\theta(z) = \begin{cases} +1, & \text{Im } z > 0, \\ -1, & \text{Im } z < 0, \end{cases} \quad V_j(z) = \begin{cases} I, & \text{Im } z > 0, \\ \begin{pmatrix} 1 & -s_j \\ 0 & 1 \end{pmatrix}, & \text{Im } z < 0, \end{cases} \quad (2.8)$$

and

$$H_{-x_m}(z) = \begin{cases} I, & \text{for } -\frac{2\pi}{3} < \arg(z + x_m) < \frac{2\pi}{3}, \\ \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg(z + x_m) < \pi, \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \text{for } -\pi < \arg(z + x_m) < -\frac{2\pi}{3}. \end{cases} \quad (2.9)$$

As z tends to 0, we have

$$\Phi(z) = G_0(z) z^{\frac{\alpha}{2} \sigma_3} \begin{pmatrix} 1 & s_1 h(z) \\ 0 & 1 \end{pmatrix}, \quad (2.10)$$

where $\det G_0 \equiv 1$, $G_0(z) = G_0(z; \vec{x}, \vec{s}, \alpha)$ is analytic in a neighborhood of 0, and

$$h(z) = \begin{cases} \frac{1}{2i \sin(\pi \alpha)}, & \alpha > -1, \alpha \notin \mathbb{N}_{\geq 0}, \\ \frac{(-1)^\alpha}{2\pi i} \log z, & \alpha \in \mathbb{N}_{\geq 0}. \end{cases} \quad (2.11)$$

The solution Φ to the above RH problem exists and is unique, and satisfies $\det \Phi \equiv 1$, see [17].

2.2 Differential identity from [12]

Recall that $E_\alpha(\vec{x}, \vec{u})$ is the exponential moment (1.3), where $\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$. To relate $E_\alpha(\vec{x}, \vec{u})$ with the solution Φ of the model RH problem of Subsection 2.1, we define $\vec{s} = (s_1, \dots, s_m)$ in terms of $\vec{u}$ as follows:

$$s_j = e^{u_j + u_{j+1} + \dots + u_m}, \quad j = 1, \dots, m. \quad (2.12)$$

Since $u_1, \dots, u_m \in \mathbb{R}$, we have $s_1, \dots, s_m \in (0, +\infty)$, and in particular the s_j 's meet the requirement in (2.1). For convenience, we define

$$F_\alpha(\vec{x}, \vec{s}) = E_\alpha(\vec{x}, \vec{u}). \quad (2.13)$$

Based on the facts that

- $F_\alpha(\vec{x}, \vec{s})$ can be written as a Fredholm determinant with m discontinuities and is an entire function of $s_1, \dots, s_m$ (see [47, Theorem 2]),
- the kernel of this Fredholm determinant is *integrable* in the sense of [33],
- Φ admits a Lax pair (studied in [17]),

the following differential identity was obtained in [12, eqs (2.34)–(2.37)] for $k = 1, \dots, m$:

$$\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) = K_\infty + \sum_{j=1}^m K_{-x_j} + K_0, \quad \alpha \neq 0, \quad (2.14)$$

where

$$K_\infty = -\frac{i}{2} \partial_{s_k} \Phi_{1,12}(r\vec{x}, \vec{s}, \alpha), \quad (2.15)$$

$$K_{-x_j} = \frac{s_{j+1} - s_j}{2\pi i} \left(G_{j,11} \partial_{s_k} G_{j,21} - G_{j,21} \partial_{s_k} G_{j,11} \right) (-rx_j; r\vec{x}, \vec{s}, \alpha), \quad (2.16)$$

$$K_0 = \alpha \left(G_{0,21} \partial_{s_k} G_{0,12} - G_{0,11} \partial_{s_k} G_{0,22} \right) (0; r\vec{x}, \vec{s}, \alpha). \quad (2.17)$$

We mention that the above differential identity is not valid for $\alpha = 0$, but since our analysis deals with the case $\alpha \rightarrow +\infty$, this fact is not important for us.

3 Large r asymptotics for Φ with $\alpha \rightarrow +\infty$: regimes 1 and 3

In this section we perform an asymptotic analysis of $\Phi(rz; r\vec{x}, \vec{s}, \alpha)$ in the two regimes of the parameters that are relevant to prove parts 1 and 3 of Theorem 1.1. In Subsections 3.1–3.5, we deal with the regime which is relevant for part 1; in Subsection 3.6, we indicate how to adapt this analysis to the regime in part 3.

Let $m \in \mathbb{N}_{>0}$, $a \in (0, +\infty)$, $\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ and $\vec{x} = (x_1, \dots, x_m) \in \mathbb{R}_{\text{ord}}^{+,m}$, be such that

$$0 < x_1 < \dots < x_{n-1} < a^2 < x_n < \dots < x_m,$$

with $n \in \{1, \dots, m+1\}$, $x_0 := 0$, $x_{m+1} := +\infty$, and define $\vec{s} = (s_1, \dots, s_m)$ in terms of $\vec{u}$ as in (2.12). For $r > 0$, define $\alpha = a\sqrt{r}$. In this section, we first obtain asymptotics for $\Phi(rz; r\vec{x}, \vec{s}, \alpha)$ as $r \rightarrow +\infty$ uniformly for $(x_1, \dots, x_{n-1}, a^2, x_n, \dots, x_m)$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m+1}$, uniformly for $u_1, \dots, u_m$ in compact subsets of $\mathbb{R}$, and uniformly for $z \in \mathbb{C}$. This analysis is based on the Deift–Zhou steepest descent method [24] and is carried out in Subsections 3.1–3.5.

In what follows, we assume without loss of generality that $a^2 < x_m$, or in other words that $n \in \{1, 2, \dots, m\}$. Indeed, if $a^2 > x_m$, then it suffices to increase m by 1 and to define $x_m = a^2 + 1$ and $s_m = 1$.

3.1 First transformation: $\Phi \rightarrow T$

As mentioned in the introduction, as $\alpha = a\sqrt{r}$ gets large, the Bessel process increasingly favors point configurations with fewer points near 0. It turns out that a typical point configuration has its smallest point located near the soft edge a^2 . Therefore, at least at a heuristic level, one expects the interval $(-\infty, -a^2)$ to play a special role in the large r analysis of $\Phi(rz; r\vec{x}, \vec{s}, a\sqrt{r})$.¹ The first transformation has multiple purposes: it normalizes the RH problem at ∞ and transforms the jumps that are not on $(-\infty, -a^2)$ into exponentially decaying jumps. The g -function is the main ingredient of this transformation and is defined by

$$g(z) = f(z) + \theta(z) \frac{\pi i a}{2}, \quad f(z) = \int_{-a^2}^z \frac{\sqrt{s+a^2}}{2s} ds, \quad (3.1)$$

where the path does not intersect $(-\infty, -a^2) \cup (0, +\infty)$, and θ is defined in (2.8). We summarize the properties of the g -function in the following lemma.

Lemma 3.1. *The g -function is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and satisfies the following properties:*

$$g_+(z) + g_-(z) = 0, \quad \text{for } z \in (-\infty, -a^2), \quad (3.2)$$

$$g_+(z) - g_-(z) = \pi i a + f_+(z) - f_-(z), \quad \text{for } z \in (-\infty, -a^2), \quad (3.3)$$

$$g_+(z) - g_-(z) = \pi i a, \quad \text{for } z \in (-a^2, 0), \quad (3.4)$$

$$g(z) = \sqrt{z} - \frac{a^2}{2\sqrt{z}} + \mathcal{O}(z^{-3/2}), \quad \text{as } z \rightarrow \infty, \quad (3.5)$$

$$g(z) = \frac{a}{2} \log z + g_0 + \mathcal{O}(z), \quad \text{as } z \rightarrow 0, \quad (3.6)$$

$$g(z) = \overline{g(\bar{z})}, \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0], \quad (3.7)$$

where $g_0 \in \mathbb{R}$. Furthermore, there exists $\epsilon > 0$ and an open set $\mathcal{V} \subset \mathbb{C}$ such that

$$\{z \in \mathbb{C} : |z| \geq \frac{1}{\epsilon}\} \cup \{z \in \mathbb{C} : \operatorname{Re} z < -a^2 \text{ and } |\operatorname{Im} z| \leq \epsilon\} \subset \mathcal{V},$$

and such that

$$\operatorname{Re} f(z) \geq 0, \quad \text{for all } z \in \mathcal{V}, \quad (3.8)$$

$$\operatorname{Re} f(z) < 0, \quad \text{for all } z \in (-a^2, 0), \quad (3.9)$$

with equality in (3.8) if and only if $z \in (-\infty, -a^2)$.

¹The model RH problem for Φ is directly related to the reverse Bessel process for convenience. One needs to identify $(-\infty, -a^2)$ in the reverse Bessel process with $(a^2, +\infty)$ in the classical Bessel process.

Proof. The properties (3.2)–(3.7) and (3.9) follow directly from (3.1) and some straightforward computations. Let us prove (3.8). Since $\operatorname{Re} f(z)/\operatorname{Re} \sqrt{z} = 1 + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$, there exists $\epsilon_1 > 0$ such that $\operatorname{Re} f(z) \geq 0$ for all $|z| \geq \frac{1}{\epsilon_1}$, with equality if and only if $z \in (-\infty, \frac{1}{\epsilon_1}]$. On the other hand, as $z \rightarrow -a^2$, we have $\operatorname{Re} f(z) \sim c \operatorname{Re}(z + a^2)^{3/2}$ for a certain $c < 0$. Hence, there exists $\epsilon_2 > 0$ such that $\operatorname{Re} f(z) \geq 0$ for all $z \in \{z : \operatorname{Re} z \in [-a^2 - \epsilon_2, -a^2] \text{ and } |\operatorname{Im} z| \leq \epsilon_2\}$ with equality if and only if $z \in [-a^2 - \epsilon_2, -a^2]$. Also, we note from (3.1) that $f(z) = \overline{f(\bar{z})}$ and that $\operatorname{Im} f_+(x)$ is decreasing as $x \in (-\infty, -a^2)$ increases. Therefore, by Cauchy-Riemann, there exists $\epsilon_3 > 0$ such that $\operatorname{Re} f(z) > 0$ for all z satisfying $\operatorname{Re} z \in [-\frac{1}{\epsilon_1}, -a^2 - \epsilon_2]$ and $|\operatorname{Im} z| \in (0, \epsilon_3]$. Indeed, suppose such an ϵ_3 does not exist. Then there is a sequence $\{z_n\}_1^\infty$ of zeros of $\operatorname{Re} f$ such that $\operatorname{Re} z_n \in [-\frac{1}{\epsilon_1}, -a^2 - \epsilon_2]$ and $\operatorname{Im} z_n > 0$ for each n , and such that $\operatorname{Im} z_1 > \operatorname{Im} z_2 > \dots$ and $\lim_{n \rightarrow \infty} \operatorname{Im} z_n = 0$. For each n , there exists by the mean value theorem a point ξ_n with the same real part as z_n such that $0 < \operatorname{Im} \xi_n < \operatorname{Im} z_n$ and $\partial_y \operatorname{Re} f(x + iy) = 0$ for $x + iy = \xi_n$. Moreover, since the interval $[-\frac{1}{\epsilon_1}, -a^2 - \epsilon_2]$ is compact, there is a subsequence z_{n_k} such that $z_{n_k} \rightarrow x_*$ as $k \rightarrow \infty$ where $x_* \in [-\frac{1}{\epsilon_1}, -a^2 - \epsilon_2]$. Since the restriction of f to $[-\frac{1}{\epsilon_1}, -a^2 - \epsilon_2] \times [0, 1]$ is C^1 and $\xi_{n_k} \rightarrow x_*$, we infer that $\partial_y \operatorname{Re} f(x_* + iy)|_{y=0^+} = 0$, which is a contradiction. This proves (3.8) with $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. $\square$

The first transformation is defined by

$$T(z) = \begin{pmatrix} 1 & 0 \\ -\frac{ia^2}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} \Phi(rz; r\vec{x}, \vec{s}, \alpha) e^{-\sqrt{r}g(z)\sigma_3}. \quad (3.10)$$

Using Lemma 3.1 together with the properties of Φ listed in Subsection 2.1, it can be verified that T satisfies the following RH problem.

RH problem for T

(a) $T : \mathbb{C} \setminus \Sigma_\Phi \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps for T are given by

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & 0 \\ e^{-2\sqrt{r}f(z)} & 1 \end{pmatrix}, \quad z \in -x_m + (0, e^{\pm \frac{2\pi i}{3}} \infty), \quad (3.11)$$

$$T_+(z) = T_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, -x_m), \quad (3.12)$$

$$T_+(z) = T_-(z) \begin{pmatrix} e^{-\sqrt{r}(f_+(z) - f_-(z))} & s_j \\ 0 & e^{\sqrt{r}(f_+(z) - f_-(z))} \end{pmatrix}, \quad z \in \mathcal{B}_j, \quad j \in \{n, \dots, m\}, \quad (3.13)$$

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & s_j e^{2\sqrt{r}f(z)} \\ 0 & 1 \end{pmatrix}, \quad z \in \mathcal{C}_j, \quad j \in \{1, \dots, n\}, \quad (3.14)$$

where

$$\mathcal{B}_j := \begin{cases} (-x_j, -x_{j-1}), & \text{if } j > n, \\ (-x_n, -a^2), & \text{if } j = n, \end{cases} \quad \mathcal{C}_j := \begin{cases} (-x_j, -x_{j-1}), & \text{if } j < n, \\ (-a^2, -x_{n-1}), & \text{if } j = n, \end{cases}$$

with $x_0 := 0$ and $x_{m+1} = +\infty$.

(c) As $z \rightarrow \infty$, we have

$$T(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N, \quad (T_1)_{1,2} = -\frac{ia^2\sqrt{r}}{2} + \frac{\Phi_{1,12}(r\vec{x}, \vec{s}, a\sqrt{r})}{\sqrt{r}} \quad (3.15)$$

where the principal branch is chosen for the root, and the matrix $T_1 = T_1(r\vec{x}, \vec{s}, a\sqrt{r})$ is independent of z and traceless.

As $z \rightarrow -x_j$, $j \in \{1, \dots, m\}$, we have

$$T(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z + x_j)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z + x_j)) \end{pmatrix}. \quad (3.16)$$

As $z \rightarrow -a^2$, we have $T(z) = \mathcal{O}(1)$.

As $z \rightarrow 0$, $T(z) = \mathcal{O}(1)$. More precisely, we have

$$T(z) = \begin{pmatrix} 1 & 0 \\ -\frac{ia^2}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_0(rz; r\vec{x}, \vec{s}, \alpha)(rz)^{\frac{a\sqrt{r}}{2}\sigma_3} \begin{pmatrix} 1 & s_1 h(rz) \\ 0 & 1 \end{pmatrix} z^{-\frac{a\sqrt{r}}{2}\sigma_3} e^{-\sqrt{r}g_0\sigma_3} (I + \mathcal{O}(z)).$$

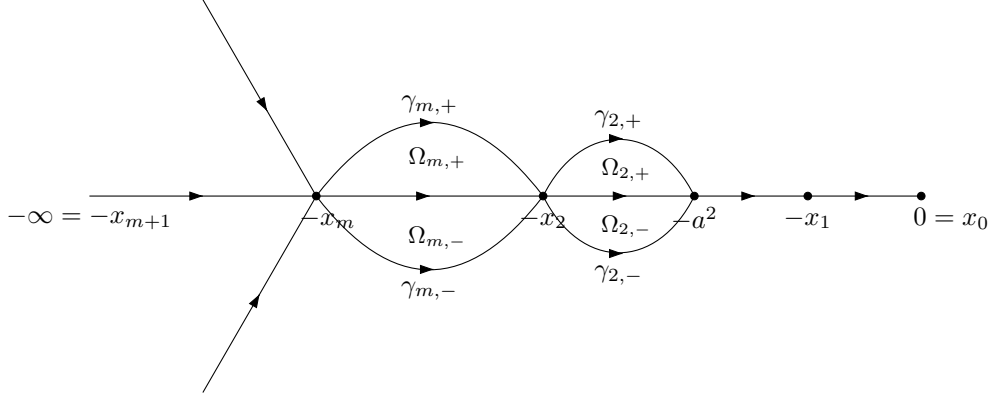


Figure 2: The contour Γ_S with $m = 3$ and $n = 2$.

Using that $f_+(x) + f_-(x) = 0$ for all $x \in (-\infty, -a^2)$, the jumps in (3.13) can be factorized as follows

$$\begin{pmatrix} e^{-2\sqrt{r}f_+(z)} & s_j \\ 0 & e^{-2\sqrt{r}f_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_j^{-1}e^{-2\sqrt{r}f_-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_j^{-1}e^{-2\sqrt{r}f_+(z)} & 1 \end{pmatrix}. \quad (3.17)$$

3.2 Second transformation: $T \mapsto S$

Here we proceed with the opening of the lenses, which is a standard step of the steepest descent method [21]. Let $\Omega_{j,+}$ and $\Omega_{j,-}$ be open regions located above and below $\mathcal{B}_j$, respectively. We let $\gamma_{j,+}$ and $\gamma_{j,-}$ denote the parts of $\partial\Omega_{j,+} \cup \partial\Omega_{j,-}$ lying strictly in the upper and lower half-plane, respectively. The contours $\gamma_{j,+}, \gamma_{j,-}$, $j = n, \dots, m$, are represented in Figure 2 in a situation where $m = 3$ and $n = 2$. The second transformation is defined by

$$S(z) = T(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ -s_j^{-1}e^{-2\sqrt{r}f(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,+}, j \in \{n, \dots, m\} \\ \begin{pmatrix} 1 & 0 \\ s_j^{-1}e^{-2\sqrt{r}f(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,-}, j \in \{n, \dots, m\} \\ I, & \text{if } z \in \mathbb{C} \setminus \bigcup_{j=n}^m (\Omega_{j,+} \cup \Omega_{j,-}). \end{cases} \quad (3.18)$$

Since T is analytic in $\mathbb{C} \setminus \Sigma_\Phi$, we conclude that S is analytic in $\mathbb{C} \setminus \Gamma_S$, where

$$\Gamma_S := (-\infty, 0) \cup \gamma_+ \cup \gamma_- \quad \text{with} \quad \gamma_\pm := \bigcup_{j=n}^{m+1} \gamma_{j,\pm}, \quad \gamma_{m+1,\pm} := -x_m + (0, e^{\pm \frac{2\pi i}{3}} \infty).$$

The contour Γ_S is oriented as shown in Figure 2. Using the factorization (3.17) and the jumps (3.11)–(3.14), we verify that S satisfies the following jumps:

$$\begin{aligned} S_+(z) &= S_-(z) \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, & z \in \mathcal{B}_j, j \in \{n, \dots, m+1\} \\ S_+(z) &= S_-(z) \begin{pmatrix} 1 & 0 \\ s_j^{-1}e^{-2\sqrt{r}f(z)} & 1 \end{pmatrix}, & z \in \gamma_{j,\pm}, j \in \{n, \dots, m+1\}, \\ S_+(z) &= S_-(z) \begin{pmatrix} 1 & s_j e^{2\sqrt{r}f(z)} \\ 0 & 1 \end{pmatrix}, & z \in \mathcal{C}_j, j \in \{1, \dots, n\}, \end{aligned}$$

where $x_0 = 0$, $x_{m+1} = +\infty$ and $s_{m+1} := 1$.

Deforming the contour if necessary, we can (and do) assume without loss of generality that $\gamma_\pm \subset \mathcal{V}$, where $\mathcal{V}$ is as described in the statement of Lemma 3.1. We infer from (3.8)–(3.9) that the jumps $S_-(z)^{-1}S_+(z)$ are exponentially close to the identity matrix as $r \rightarrow +\infty$ for $z \in \gamma_+ \cup \gamma_- \cup (-a^2, 0)$. This convergence is uniform for z bounded away from $(-\infty, -a^2)$, but only pointwise for z close to $(-\infty, -a^2)$.

3.3 Global parametrix

The global parametrix is denoted $P^{(\infty)}$ and is defined as the solution to a RH problem whose jumps are obtained by ignoring the (pointwise) exponentially small jumps of S . We will show in Subsection 3.5 that $P^{(\infty)}$ is a good approximation to S outside small neighborhoods of $-a^2, -x_1, \dots, -x_m$.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, -a^2] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps for $P^{(\infty)}$ are given by

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, \quad z \in \mathcal{B}_j, j \in \{n, \dots, m+1\}.$$

(c) As $z \rightarrow \infty$, we have

$$P^{(\infty)}(z) = \left(I + \frac{P_1^{(\infty)}}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N, \quad (3.19)$$

where $P_1^{(\infty)}$ is a matrix independent of z .

(d) As $z \rightarrow -x_j, j \in \{n, \dots, m\}$, we have $P^{(\infty)}(z) = \mathcal{O}(1)$.

As $z \rightarrow -a^2$, we have $P^{(\infty)}(z) = \mathcal{O}((z + a^2)^{-1/4})$.

A slightly more complicated version of this RH problem has actually been solved in [12]. The solution $P^{(\infty)}$ is obtained by setting the parameters α and $-x_1$ of [12, Section 5.3] to 0 and $-a^2$, respectively. The construction is as follows. Define

$$\beta_j := \frac{1}{2\pi i} \log \frac{s_{j+1}}{s_j}, \quad j = 1, \dots, m, \quad \text{with } s_{m+1} := 1, \quad (3.20)$$

and for $\ell \in \{0, 1, 2, \dots\}$, define

$$d_\ell = \frac{2i(-1)^\ell}{2\ell - 1} \sum_{j=n}^m \beta_j (x_j - a^2)^{\ell - \frac{1}{2}}. \quad (3.21)$$

The unique solution to the RH problem for $P^{(\infty)}$ is given by

$$P^{(\infty)}(z) = \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} (z + a^2)^{-\frac{\sigma_3}{4}} N D(z)^{-\sigma_3}, \quad (3.22)$$

where

$$D(z) = \exp \left(-\sqrt{z + a^2} \sum_{j=n}^m i\beta_j \int_{a^2}^{x_j} \frac{du}{\sqrt{u - a^2}(z + u)} \right).$$

Furthermore, $P^{(\infty)}$ satisfies

$$P_{1,12}^{(\infty)} = id_1, \quad (3.23)$$

and, for any $p \in \mathbb{N}_{>0} = \{1, 2, \dots\}$ and any $j \in \{n, \dots, m\}$, we have

$$D(z) = \exp \left(\sum_{\ell=1}^p \frac{d_\ell}{(z + a^2)^{\ell - \frac{1}{2}}} + \mathcal{O}(z^{-p - \frac{1}{2}}) \right), \quad \text{as } z \rightarrow \infty, \quad (3.24)$$

$$D(z) = \sqrt{s_j} (4(x_j - a^2))^{-\beta_j} \left(\prod_{\substack{k=n \\ k \neq j}}^m T_{k,j}^{-\beta_k} \right) (z + x_j)^{\beta_j} (1 + \mathcal{O}(z + x_j)), \quad \text{as } z \rightarrow -x_j, \operatorname{Im} z > 0,$$

$$D(z) = \sqrt{s_n} \left(1 - d_0 \sqrt{z + a^2} + \mathcal{O}(z + a^2) \right), \quad \text{as } z \rightarrow -a^2, \quad (3.25)$$

$$\log D(0) = -a \sum_{j=n}^m i\beta_j \mathcal{I}_j, \quad (3.26)$$

where

$$T_{k,j} = \frac{\sqrt{x_j - a^2} + \sqrt{x_k - a^2}}{|\sqrt{x_j - a^2} - \sqrt{x_k - a^2}|}, \quad \mathcal{I}_j = \int_{a^2}^{x_j} \frac{du}{u\sqrt{u - a^2}}. \quad (3.27)$$

3.4 Local parametrices

We consider small open disks $\mathcal{D}_p$ centered at $p \in \{-a^2, -x_1, \dots, -x_m\}$. The radii of the disks $\mathcal{D}_{-x_j}$, $j = 1, \dots, m$ are all chosen to be equal to $\delta > 0$, where

$$\delta = \frac{1}{M} \min_{0 \leq j < k \leq m} \{x_k - x_j, x_n - a^2, a^2 - x_{n-1}\}, \quad M \geq 3, \quad M \text{ fixed}, \quad (3.28)$$

where we recall that $x_0 := 0$. The condition $M \geq 3$ ensures that the disks do not intersect each other, and that $0 \notin \mathcal{D}_{-x_1} \cup \mathcal{D}_{-a^2}$. Since $(x_1, \dots, x_{n-1}, a^2, x_n, \dots, x_m)$ lies in a compact subset of $\mathbb{R}_{\text{ord}}^{+, m+1}$, δ remains bounded away from 0. In this regime, there is no need to consider a disk around the origin.

In each of the disks, we will build a so-called local parametrix. We will show in Subsection 3.5 that the local parametrices are good approximations to S inside the disks.

The local parametrix $P^{(p)}$ is defined in $\mathcal{D}_p \cup \partial\mathcal{D}_p$ as the solution to a RH problem whose jumps are identical to those of S . On the boundary of the disk, we require $P^{(p)}$ to “match” with $P^{(\infty)}$, in the sense that

$$P^{(p)}(z) = (I + o(1))P^{(\infty)}(z), \quad \text{as } r \rightarrow +\infty, \quad (3.29)$$

uniformly for $z \in \partial\mathcal{D}_p$. We need to distinguish three different types of local parametrices:

- in $\mathcal{D}_{-x_j}$, $j = n, \dots, m$, $P^{(-x_j)}$ is built in terms of hypergeometric functions,
- in $\mathcal{D}_{-x_j}$, $j = 1, \dots, n-1$, $P^{(-x_j)}$ can be solved explicitly using elementary functions,
- in $\mathcal{D}_{-a^2}$, $P^{(-a^2)}$ is built in terms of Airy functions.

We will construct the local parametrices with the help of three model RH problems that have already been studied in the literature and which we recall in the appendix (Section A).

3.4.1 Local parametrices around $-x_j$, $j = n, \dots, m$

The construction of $P^{(-x_j)}$ for $j \in \{n, \dots, m\}$ relies on a model RH problem whose solution Φ_{HG} is built in terms of confluent hypergeometric functions. This model RH problem has been studied in [34, 30] and we recall the properties of Φ_{HG} in Section A.3 for the convenience of the reader. The function

$$f_{-x_j}(z) := -2 \begin{cases} f(z) - f_+(-x_j), & \text{if } \text{Im } z > 0 \\ -(f(z) - f_-(-x_j)), & \text{if } \text{Im } z < 0 \end{cases} = -2i \int_{-x_j}^z \frac{\sqrt{-s - a^2}}{2s} ds \quad (3.30)$$

has the following expansion as $z \rightarrow -x_j$

$$f_{-x_j}(z) = ic_{-x_j}(z + x_j)(1 + \mathcal{O}(z + x_j)), \quad \text{with} \quad c_{-x_j} = \frac{\sqrt{x_j - a^2}}{x_j} > 0. \quad (3.31)$$

This shows that f_{-x_j} is a conformal map in $\mathcal{D}_{-x_j}$, provided that M in (3.28) is chosen sufficiently large. In order to use the model RH problem for Φ_{HG} , we need f_{-x_j} to map the contour $\Gamma_S \cap \mathcal{D}_{-x_j}$ to a subset of the contour Σ_{HG} , where Σ_{HG} is shown in Figure 7. Note that the function f_{-x_j} automatically satisfies $f_{-x_j}(\mathbb{R} \cap \mathcal{D}_{-x_j}) \subset i\mathbb{R}$. In Subsection 3.2, we had some freedom in the choice of the lenses $\gamma_{\pm}$. Now, we use this freedom to ensure that the lenses in a neighborhood of $-x_j$ are such that

$$f_{-x_j}((\gamma_{j+1,+} \cup \gamma_{j,+}) \cap \mathcal{D}_{-x_j}) \subset \Gamma_3 \cup \Gamma_2, \quad f_{-x_j}((\gamma_{j+1,-} \cup \gamma_{j,-}) \cap \mathcal{D}_{-x_j}) \subset \Gamma_5 \cup \Gamma_6, \quad (3.32)$$

where Γ_j , $j = 2, 3, 5, 6$ are the contours displayed in Figure 7. Let us define

$$P^{(-x_j)}(z) = E_{-x_j}(z) \Phi_{\text{HG}}(\sqrt{r} f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{-\sqrt{r} f(z) \sigma_3}, \quad (3.33)$$

where E_{-x_j} is analytic in $\mathcal{D}_{-x_j}$ and given by

$$E_{-x_j}(z) = P^{(\infty)}(z)(s_j s_{j+1})^{\frac{\sigma_3}{4}} \begin{cases} \sqrt{\frac{s_j}{s_{j+1}}}, & \text{Im } z > 0 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{Im } z < 0 \end{cases} e^{\sqrt{r} f_+(-x_j) \sigma_3} (\sqrt{r} f_{-x_j}(z))^{\beta_j \sigma_3}. \quad (3.34)$$

One can verify from the RH problem for Φ_{HG} that $S(z)P^{(-x_j)}(z)^{-1}$ has no jumps in $\mathcal{D}_{-x_j}$ and has a removable singularity at $z = -x_j$. Furthermore, using (A.19), we obtain

$$P^{(-x_j)}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{\sqrt{r}f_{-x_j}(z)}E_{-x_j}(z)\Phi_{\text{HG},1}(\beta_j)E_{-x_j}(z)^{-1} + \mathcal{O}(r^{-1}), \quad (3.35)$$

as $r \rightarrow +\infty$, uniformly for $z \in \partial\mathcal{D}_{-x_j}$. In particular $P^{(-x_j)}$ satisfies (3.29). Finally, using (3.22), (3.24) and (3.31), we get

$$E_{-x_j}(-x_j) = \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3} (x_j - a^2)^{-\frac{\sigma_3}{4}} N\Lambda_j^{\sigma_3}, \quad (3.36)$$

where

$$\Lambda_j = (4(x_j - a^2))^{\beta_j} \left(\prod_{\substack{k=n \\ k \neq j}}^m T_{k,j}^{\beta_k} \right) e^{\sqrt{r}f_+(-x_j)} r^{\frac{\beta_j}{2}} c_{-x_j}^{\beta_j}. \quad (3.37)$$

The quantities (3.36) and (3.37) will be useful in Section 5.

3.4.2 Local parametrices around $-x_j$, $j = 1, \dots, n-1$

We note from (3.16) and (3.18) that S has a logarithmic singularity at $-x_j$, and from (3.22) that $P^{(\infty)}(z)$ remains bounded as $z \rightarrow -x_j$, $j = 1, \dots, n-1$. Therefore, even though the jumps for S are exponentially small as $r \rightarrow +\infty$ uniformly for $z \in \mathcal{D}_{-x_j} \cap \mathbb{R} \setminus \{-x_j\}$, $P^{(\infty)}$ cannot be a good approximation to S in $\mathcal{D}_{-x_j}$, and we need to construct a local parametrix. We define

$$P^{(-x_j)}(z) = P^{(\infty)}(z) \begin{pmatrix} 1 & h_{-x_j}(z) \\ 0 & 1 \end{pmatrix}, \quad (3.38)$$

where

$$h_{-x_j}(z) = \frac{s_{j+1}}{2\pi i} \int_{-x_j-2\delta}^{-x_j} \frac{e^{2\sqrt{r}f(s)}}{s-z} ds + \frac{s_j}{2\pi i} \int_{-x_j}^{-x_j+2\delta} \frac{e^{2\sqrt{r}f(s)}}{s-z} ds.$$

We easily check from the definition of h_{-x_j} that $P^{(-x_j)}$ has the same jumps as S inside $\mathcal{D}_{-x_j}$. Furthermore, we infer from (3.1) and (3.9) that

$$P^{(-x_j)}(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(e^{-c\sqrt{r}}), \quad \text{as } r \rightarrow +\infty, \quad (3.39)$$

uniformly for $z \in \partial\mathcal{D}_{-x_j}$, for a certain $c > 0$. In particular, $P^{(-x_j)}$ satisfies (3.29). Finally, using the expansion

$$h_{-x_j}(z) = \frac{s_{j+1} - s_j}{2\pi i} e^{2\sqrt{r}f(-x_j)} \log(z + x_j) + \mathcal{O}(1), \quad \text{as } z \rightarrow -x_j, \quad (3.40)$$

one verifies that $S(z)P^{(-x_j)}(z)^{-1}$ has a removable singularity at $z = -x_j$.

3.4.3 Local parametrix around $-a^2$

Recall that in this section a remains bounded away from 0 as $r \rightarrow +\infty$. The construction of $P^{(-a^2)}$ is standard and relies on the model RH problem from [23], whose solution is denoted Φ_{Ai} . For the reader's convenience, we recall the properties of Φ_{Ai} in Section A.1. The function

$$f_{-a^2}(z) = \left(-\frac{3}{2}f(z) \right)^{2/3} = \left(-\frac{3}{2} \int_{-a^2}^z \frac{\sqrt{s+a^2}}{2s} ds \right)^{2/3} \quad (3.41)$$

has the following expansion

$$f_{-a^2}(z) = \frac{z+a^2}{(2a^2)^{2/3}} \left(1 + \frac{2}{5a^2}(z+a^2) + \mathcal{O}((z+a^2)^2) \right), \quad \text{as } z \rightarrow -a^2. \quad (3.42)$$

From (3.42), we infer that the map f_{-a^2} is conformal in $\mathcal{D}_{-a^2}$, provided that M in (3.28) is chosen sufficiently large. In view of the jump contour for Φ_{Ai} displayed in Figure 6 (left), we deform the lenses in a neighborhood of $-a^2$ such that

$$f_{-a^2}(\gamma_{n,+}) \subset e^{\frac{2\pi i}{3}} \mathbb{R}^+, \quad f_{-a^2}(\gamma_{n,-}) \subset e^{-\frac{2\pi i}{3}} \mathbb{R}^+. \quad (3.43)$$

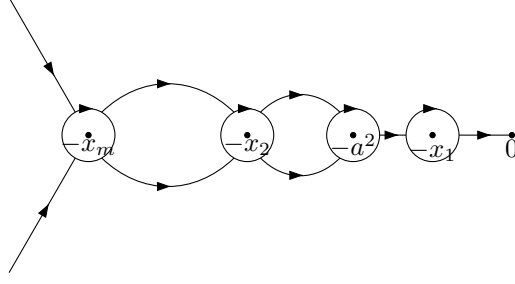


Figure 3: The contour Γ_R with $m = 3$ and $n = 2$.

We define

$$P^{(-a^2)}(z) = E_{-a^2}(z) \Phi_{\text{Ai}}(r^{\frac{1}{3}} f_{-a^2}(z)) s_n^{-\frac{\sigma_3}{2}} e^{-\sqrt{r} f(z) \sigma_3}, \quad (3.44)$$

where E_{-a^2} is analytic in $\mathcal{D}_{-a^2}$ and given by

$$E_{-a^2}(z) = P^{(\infty)}(z) s_n^{\frac{\sigma_3}{2}} N^{-1} f_{-a^2}(z)^{\frac{\sigma_3}{4}} r^{\frac{\sigma_3}{12}}. \quad (3.45)$$

From (3.43), (3.44) and the RH problem for Φ_{Ai} , we verify that $S(z)P^{(-a^2)}(z)^{-1}$ has no jumps in $\mathcal{D}_{-a^2}$ and remains bounded as $z \rightarrow -a^2$. Also, using (A.2), we obtain

$$P^{(-a^2)}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{\sqrt{r} f_{-a^2}(z)^{3/2}} P^{(\infty)}(z) s_n^{\frac{\sigma_3}{2}} \Phi_{\text{Ai},1} s_n^{-\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1} + \mathcal{O}(r^{-1}), \quad (3.46)$$

as $r \rightarrow +\infty$ uniformly for $z \in \partial\mathcal{D}_{-a^2}$. In particular, $P^{(-a^2)}$ satisfies (3.29). Finally, after a long computation using (3.22), (3.25) and (3.42), we obtain

$$E_{-a^2}(-a^2) = \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -id_0 \\ 0 & 1 \end{pmatrix} (r^{\frac{1}{12}} 2^{-\frac{1}{6}} a^{-\frac{1}{3}})^{\sigma_3}. \quad (3.47)$$

The above identity will be used in Section 5.

3.5 Small norm problem

In this section, c and C denote generic positive constants that may change within a computation. We will think of c as being sufficiently small (but fixed), and C as being sufficiently large (but fixed). Define

$$R(z) := \begin{cases} S(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus (\bigcup_{j=1}^m \mathcal{D}_{-x_j} \cup \mathcal{D}_{-a^2}), \\ S(z)P^{(-x_j)}(z)^{-1}, & \text{for } z \in \mathcal{D}_{-x_j}, j \in \{1, \dots, m\}, \\ S(z)P^{(-a^2)}(z)^{-1}, & \text{for } z \in \mathcal{D}_{-a^2}. \end{cases} \quad (3.48)$$

From the analysis of Subsections 3.4.1–3.4.3, R has no jumps inside the $m+1$ disks, and the singularities of R at $-x_1, \dots, -x_m, -a^2$ are removable. Therefore, R is analytic inside the $m+1$ disks and R has jumps only on the contour

$$\Gamma_R = \left(\partial\mathcal{D}_{-a^2} \cup \bigcup_{j=1}^m \partial\mathcal{D}_{-x_j} \cup \gamma_+ \cup \gamma_- \cup (-a^2, 0) \right) \setminus \left(\mathcal{D}_{-a^2} \cup \bigcup_{j=1}^m \mathcal{D}_{-x_j} \right),$$

see also Figure 3, and for convenience we orient the boundaries of the disks in the clockwise direction. The jumps $J_R(z) := R_-(z)^{-1}R_+(z)$ are given by

$$J_R(z) = \begin{cases} P^{(\infty)}(z)J_S(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in (\gamma_+ \cup \gamma_- \cup (-a^2, 0)) \setminus (\mathcal{D}_{-a^2} \cup \bigcup_{j=1}^m \mathcal{D}_{-x_j}), \\ P^{(-x_*)}(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \partial\mathcal{D}_{-x_*}, x_* \in \{x_1, \dots, x_m, a^2\}. \end{cases}$$

where $J_S(z) := S_-^{-1}(z)S_+(z)$. Using Lemma 3.1 and (3.22), (3.35), (3.39), (3.46), we infer that there exists $c > 0$ such that, as $r \rightarrow +\infty$ we have

$$J_R(z) - I = \begin{cases} \mathcal{O}(e^{-c\sqrt{r}\sqrt{z}}), & \text{unif. for } z \in (\gamma_+ \cup \gamma_-) \cap \Gamma_R, \\ \mathcal{O}(e^{-c\sqrt{r}}), & \text{unif. for } z \in \bigcup_{j=1}^{n-1} \partial\mathcal{D}_{-x_j} \cup (-a^2, 0) \setminus (\mathcal{D}_{-a^2} \cup \bigcup_{j=1}^{n-1} \mathcal{D}_{-x_j}), \\ \frac{J_R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}(\frac{1}{r}), & \text{unif. for } z \in \partial\mathcal{D}_{-a^2} \cup \bigcup_{j=n}^m \partial\mathcal{D}_{-x_j}, \end{cases} \quad (3.49)$$

where $J_R^{(1)}$ is given by

$$J_R^{(1)}(z) := \begin{cases} \frac{1}{f_{-x_j}(z)} E_{-x_j}(z) \Phi_{\text{HG},1}(\beta_j) E_{-x_j}(z)^{-1}, & z \in \partial\mathcal{D}_{-x_j}, j = n, \dots, m, \\ \frac{1}{f_{-a^2}(z)^{3/2}} P^{(\infty)}(z) s_n^{\frac{\sigma_3}{2}} \Phi_{\text{Ai},1} s_n^{-\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1}, & z \in \partial\mathcal{D}_{-a^2}. \end{cases} \quad (3.50)$$

In particular, for each $1 \leq p \leq \infty$ and any choice of $k_1, \dots, k_m \in \mathbb{N}_{\geq 0}$, there exist constants $c > 0$ and $C > 0$ such that for all $r \geq 2$ we have

$$\|\partial_\beta^k(J_R - I)\|_{L^p(\Gamma_R \setminus (\partial\mathcal{D}_{-a^2} \cup \bigcup_{l=n}^m \partial\mathcal{D}_{-x_l}))} \leq e^{-c\sqrt{r}}, \quad (3.51a)$$

$$\|\partial_\beta^k(J_R - I - \frac{J_R^{(1)}(z)}{\sqrt{r}})\|_{L^p(\partial\mathcal{D}_{-a^2} \cup \bigcup_{l=n}^m \partial\mathcal{D}_{-x_l})} \leq C \frac{\log^k r}{r}, \quad (3.51b)$$

$$\|\partial_\beta^k(J_R - I)\|_{L^p(\Gamma_R)} \leq C \frac{\log^k r}{\sqrt{r}}, \quad (3.51c)$$

where $k = k_1 + \dots + k_m$ and $\partial_\beta^k = \partial_{\beta_1}^{k_1} \dots \partial_{\beta_m}^{k_m}$. The factors $\log^k r$ in (3.51) is due to the factors $r^{\pm\beta_j}$ appearing in the entries of J_R , see (3.33)–(3.35). It is also easy to see from (3.22), (3.35), (3.46), (3.38) that the estimates (3.49) and (3.51) hold uniformly for $s_1, \dots, s_m$ in compact subsets of $(0, +\infty)$, and uniformly for $(x_1, \dots, x_{n-1}, a^2, x_n, \dots, x_m)$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m+1}$. Finally, we also note that

- since $S(z)$ and $P^{(\infty)}(z)$ are $\mathcal{O}(1)$ as $z \rightarrow 0$, $R(z)$ remains bounded for z near the endpoint 0,
- (3.22), (3.33), (3.44) and (3.38) imply that $R(z)$ remains bounded as z approaches an intersection point of Γ_R ,
- (3.15), (3.19) and (3.48) imply that $R(z) = I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.

Therefore, R satisfies a so-called small norm RH problem, and one can prove existence of R for sufficiently large r and compute its asymptotics following the method of Deift and Zhou [24]. Let $\mathcal{C} : L^2(\Gamma_R) \rightarrow L^2(\Gamma_R)$ be the operator defined by

$$\mathcal{C}f(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(s)}{s - z} dz, \quad f \in L^2(\Gamma_R),$$

and let $\mathcal{C}_+f$ and $\mathcal{C}_-f$ denote the left and right non-tangential limits of $\mathcal{C}f$. Since $J_R - I \in L^2(\Gamma_R) \cap L^\infty(\Gamma_R)$, we can define the Cauchy operator

$$\mathcal{C}_{J_R} : L^2(\Gamma_R) + L^\infty(\Gamma_R) \rightarrow L^2(\Gamma_R), \quad \mathcal{C}_{J_R}f = \mathcal{C}_-((J_R - I)f), \quad f \in L^2(\Gamma_R) + L^\infty(\Gamma_R).$$

Since

$$\|\mathcal{C}_{J_R}\|_{L^2(\Gamma_R) \rightarrow L^2(\Gamma_R)} \leq C \|J_R - I\|_{L^\infty(\Gamma_R)}, \quad (3.52)$$

the estimate (3.51c) with $p = +\infty$ and $k_1 = \dots = k_m = 0$ implies that for all sufficiently large r the operator $I - \mathcal{C}_{J_R}$ can be inverted as a Neumann series:

$$(I - \mathcal{C}_{J_R})^{-1} = \sum_{\ell=0}^{+\infty} \mathcal{C}_{J_R}^\ell. \quad (3.53)$$

Hence, for all sufficiently large r , we have

$$R = I + \mathcal{C}(\mu_R(J_R - I)), \quad \text{where} \quad \mu_R := I + (I - \mathcal{C}_{J_R})^{-1} \mathcal{C}_{J_R}(I). \quad (3.54)$$

Furthermore, using (3.51c), one deduces that, for $k_1, \dots, k_m \in \mathbb{N}_{\geq 0}$,

$$\|\partial_\beta^k(\mu_R - I)\|_{L^2(\Gamma_R)} = \|\partial_\beta^k((I - \mathcal{C}_{J_R})^{-1} \mathcal{C}_{J_R}(I))\|_{L^2(\Gamma_R)} \leq C \frac{\log^k r}{\sqrt{r}}, \quad (3.55)$$

where as before $k = k_1 + \dots + k_m$ and $\partial_\beta^k = \partial_{\beta_1}^{k_1} \dots \partial_{\beta_m}^{k_m}$. Using (3.51), (3.55), and the representation

$$R = I + \mathcal{C}\left(\frac{J_R^{(1)}(z)}{\sqrt{r}}\right) + \mathcal{C}\left(J_R - I - \frac{J_R^{(1)}(z)}{\sqrt{r}}\right) + \mathcal{C}((\mu_R - I)(J_R - I)),$$

we get

$$\partial_\beta^k R(z) = \partial_\beta^k \left(I + \frac{R^{(1)}(z)}{\sqrt{r}} \right) + \mathcal{O}\left(\frac{\log^k r}{r}\right), \quad \partial_\beta^k R^{(1)}(z) = \mathcal{O}(\log^k r), \quad \text{as } r \rightarrow +\infty, \quad (3.56)$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_R$, for $(x_1, \dots, x_{n-1}, a^2, x_n, \dots, x_m)$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m+1}$ and for $\beta_1, \dots, \beta_m$ in compact subsets of $i\mathbb{R}$, where

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}_{-a^2}} \frac{J_R^{(1)}(s)}{s-z} ds + \sum_{j=n}^m \frac{1}{2\pi i} \int_{\partial \mathcal{D}_{-x_j}} \frac{J_R^{(1)}(s)}{s-z} ds. \quad (3.57)$$

The fact that (3.56) holds uniformly for z close to Γ_R can be seen, for example, by deforming the contour Γ_R slightly. From (3.50), we infer that $J_R^{(1)}$ can be analytically continued to

$$\left(\mathcal{D}_{-a^2} \cup \bigcup_{j=n}^m \mathcal{D}_{-x_j} \right) \setminus \{-a^2, -x_n, \dots, -x_m\},$$

and has a double pole at $-a^2$ and simple poles at $-x_n, \dots, -x_m$. Hence,

$$\begin{aligned} R^{(1)}(z) &= \frac{1}{z+a^2} \text{Res}(J_R^{(1)}(s), s=-a^2) + \frac{1}{(z+a^2)^2} \text{Res}((s+a^2)J_R^{(1)}(s), s=-a^2) \\ &\quad + \sum_{j=n}^m \frac{1}{z+x_j} \text{Res}(J_R^{(1)}(s), s=-x_j), \quad \text{for } z \in \mathbb{C} \setminus \left(\mathcal{D}_{-a^2} \cup \bigcup_{j=n}^m \mathcal{D}_{-x_j} \right). \end{aligned} \quad (3.58)$$

A long but straightforward computation using (3.22), (3.25), (3.42) and (3.50) gives

$$\text{Res}((s+a^2)J_R^{(1)}(s), s=-a^2) = \frac{5a^2 d_1}{24} \begin{pmatrix} 1 & id_1^{-1} \\ id_1 & -1 \end{pmatrix}, \quad (3.59a)$$

$$\text{Res}(J_R^{(1)}(s), s=-a^2) = \begin{pmatrix} \frac{1}{8}(-d_1 + 4a^2 d_0(1+d_0 d_1)) & \frac{i}{8}(-1 + 4a^2 d_0^2) \\ \frac{i}{24}(-3d_1^2 + a^2(7 + 24d_0 d_1 + 12d_0^2 d_1^2)) & \frac{1}{8}(d_1 - 4a^2 d_0(1+d_0 d_1)) \end{pmatrix}. \quad (3.59b)$$

Also, using (3.31), (3.36), (3.37) and (3.50), for $j \in \{n, \dots, m\}$, we obtain

$$\begin{aligned} \text{Res}(J_R^{(1)}(s), s=-x_j) &= \frac{\beta_j^2}{ic_{-x_j}} \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3(x_j - a^2) - \frac{\sigma_3}{4}} N \begin{pmatrix} -1 & \tilde{\Lambda}_{j,1} \\ -\tilde{\Lambda}_{j,2} & 1 \end{pmatrix} \\ &\quad \times N^{-1}(x_j - a^2)^{\frac{\sigma_3}{4}} e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & 0 \\ -id_1 & 1 \end{pmatrix}, \end{aligned} \quad (3.60)$$

where

$$\tilde{\Lambda}_{j,1} = \tau(\beta_j)\Lambda_j^2 \quad \text{and} \quad \tilde{\Lambda}_{j,2} = \tau(-\beta_j)\Lambda_j^{-2}. \quad (3.61)$$

3.6 Large r asymptotics for Φ with $\alpha \rightarrow +\infty$: regime 3

In this subsection we consider the regime where $r \rightarrow +\infty$, $a \in (0, +\infty)$ is fixed and where $\vec{x} \in \mathbb{R}_{\text{ord}}^{+,m}$ satisfies $0 < a^2 < x_1 < \dots < x_m$ and (1.9). In particular, this means that $n = 1$. The asymptotic analysis of Φ in this regime is essentially the same as the one carried out in Subsections 3.1–3.5 (with $n = 1$), but some extra care is needed since the x_j 's converge to a^2 . In particular, the radius δ of the disks in (3.28) must now depend on r and decrease as $r \rightarrow +\infty$, which means that the disks shrink as $r \rightarrow +\infty$. We next discuss the consequences this has for the construction of the local parametrices and of the solution R to the small norm RH problem. Note that (1.9b) implies that δ is of the same order as $|x_m - a^2|$.

3.6.1 Modification of the error terms in the construction of $P^{(-x_j)}$, $j = 1, \dots, m$

By (3.22) and (3.33), we have

$$\begin{aligned} P^{(-x_j)}(z)P^{(\infty)}(z)^{-1} &= E_{-x_j}(z)\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{-\sqrt{r}f(z)\sigma_3} \\ &\quad \times D(z)^{\sigma_3} N^{-1}(z+a^2)^{\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ -id_1 & 1 \end{pmatrix}. \end{aligned} \quad (3.62)$$

The map f_{-x_j} defined in (3.30) satisfies

$$f_{-x_j}(z) = ic_{-x_j}(z + x_j) \left(1 + \mathcal{O}\left(\frac{z + x_j}{x_j - a^2}\right) \right), \quad c_{-x_j} = \frac{\sqrt{x_j - a^2}}{x_j} > 0, \quad \text{as } z \rightarrow -x_j. \quad (3.63)$$

As can be seen from the above error term, f_{-x_j} is a conformal map in $\mathcal{D}_{-x_j}$ provided that $\delta \leq c|x_j - a^2|$ where $c > 0$ is a sufficiently small constant. Note also that $c_{-x_j} = \mathcal{O}(\sqrt{x_j - a^2}) = \mathcal{O}(\delta^{1/2}) \rightarrow 0$ as $x_j \rightarrow a^2$. Hence, $|f_{-x_j}(z)| \asymp \delta^{3/2}$ uniformly for $z \in \partial\mathcal{D}_{-x_j}$. In particular, the argument $\sqrt{r}f_{-x_j}(z)$ of Φ_{HG} in (3.62) satisfies $|\sqrt{r}f_{-x_j}(z)| \asymp \delta^{3/2}\sqrt{r}$. Moreover, for $z \in \partial\mathcal{D}_{-x_j}$, we have $E_{-x_j}(z) = \mathcal{O}(\delta^{-1/4})$, $D(z) = \mathcal{O}(1)$, and $(z + a^2)^{\sigma_3/4} = \mathcal{O}(\delta^{-1/4})$. Consequently, using the asymptotic formula (A.19) for Φ_{HG} in (3.62), we conclude that

$$P^{(-x_j)}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{\sqrt{r}f_{-x_j}(z)}E_{-x_j}(z)\Phi_{\text{HG},1}(\beta_j)E_{-x_j}(z)^{-1} + \mathcal{O}(\delta^{-7/2}r^{-1}) \quad (3.64a)$$

and

$$\frac{1}{\sqrt{r}f_{-x_j}(z)}E_{-x_j}(z)\Phi_{\text{HG},1}(\beta_j)E_{-x_j}(z)^{-1} = \mathcal{O}(\delta^{-2}r^{-1/2}) \quad (3.64b)$$

uniformly for $z \in \partial\mathcal{D}_{-x_j}$ as $r \rightarrow +\infty$ and $\delta \rightarrow 0$. Equation (3.64a) should be compared to the expansion (3.35) of $P^{(-x_j)}(z)P^{(\infty)}(z)^{-1}$ obtained earlier. It follows from (3.64) that $P^{(-x_j)}$ satisfies (3.29), provided that $\sqrt{r}\delta^2$ tends to infinity as $r \rightarrow +\infty$.

3.6.2 Modification of the error terms in the construction of $P^{(-a^2)}$

By (3.42), we have $|f_{-a^2}(z)| \asymp \delta$, and hence $\sqrt{r}f_{-a^2}(z)^{3/2} \asymp \delta^{3/2}\sqrt{r}$, uniformly for $z \in \partial\mathcal{D}_{-a^2}$. Moreover, $P^{(\infty)}(z) = \mathcal{O}(\delta^{-1/4})$ uniformly for $z \in \partial\mathcal{D}_{-a^2}$. Using these observations, arguments similar to those used to obtain (3.64) show that the matching condition (3.46) becomes

$$P^{(-a^2)}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{\sqrt{r}f_{-a^2}(z)^{3/2}}P^{(\infty)}(z)s_n^{\frac{\sigma_3}{2}}\Phi_{\text{Ai},1}s_n^{-\frac{\sigma_3}{2}}P^{(\infty)}(z)^{-1} + \mathcal{O}(\delta^{-7/2}r^{-1}) \quad (3.65a)$$

and that

$$\frac{1}{\sqrt{r}f_{-a^2}(z)^{3/2}}P^{(\infty)}(z)s_n^{\frac{\sigma_3}{2}}\Phi_{\text{Ai},1}s_n^{-\frac{\sigma_3}{2}}P^{(\infty)}(z)^{-1} = \mathcal{O}(\delta^{-2}r^{-1/2}) \quad (3.65b)$$

uniformly for $z \in \partial\mathcal{D}_{-a^2}$ as $r \rightarrow +\infty$ and $\delta \rightarrow 0$. In particular, $P^{(-a^2)}$ satisfies (3.29), provided that $\sqrt{r}\delta^2$ tends to infinity as $r \rightarrow +\infty$.

3.6.3 Modification of the error terms in the construction of R

Let R be defined as in (3.48) with $n = 1$. Using Lemma 3.1, (3.22), (3.64) and (3.65), we infer that there exists a $c > 0$ such that, as $r \rightarrow +\infty$,

$$J_R(z) - I = \begin{cases} \mathcal{O}(e^{-c\delta^{3/2}\sqrt{r}\sqrt{z}}), & \text{uniformly for } z \in (\gamma_+ \cup \gamma_-) \cap \Gamma_R, \\ \mathcal{O}(e^{-c\delta^{3/2}\sqrt{r}}), & \text{uniformly for } z \in (-a^2, 0) \setminus \mathcal{D}_{-a^2}, \\ \frac{J_R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}\left(\frac{1}{\delta^{7/2}r}\right), & \text{uniformly for } z \in \partial\mathcal{D}_{-a^2} \cup \bigcup_{j=n}^m \partial\mathcal{D}_{-x_j}, \\ \mathcal{O}\left(\frac{1}{\delta^2\sqrt{r}}\right), & \text{uniformly for } z \in \partial\mathcal{D}_{-a^2} \cup \bigcup_{j=n}^m \partial\mathcal{D}_{-x_j}, \end{cases} \quad (3.66)$$

where $J_R^{(1)}$ is given by (3.50). In particular, for each $1 \leq p \leq \infty$ and any $k_1, \dots, k_m \in \mathbb{N}_{\geq 0}$, there exist constants $c > 0$ and $C > 0$ such that for all $r \geq 2$ we have

$$\|\partial_\beta^k(J_R - I)\|_{L^p(\Gamma_R \setminus (\partial\mathcal{D}_{-a^2} \cup \bigcup_{l=1}^m \partial\mathcal{D}_{-x_l}))} \leq e^{-c\delta^{3/2}\sqrt{r}}, \quad (3.67a)$$

$$\|\partial_\beta^k(J_R - I - \frac{J_R^{(1)}(z)}{\sqrt{r}})\|_{L^p(\partial\mathcal{D}_{-a^2} \cup \bigcup_{l=1}^m \partial\mathcal{D}_{-x_l})} \leq C\delta^{1/p} \frac{\log^k r}{\delta^{7/2}r} \quad (3.67b)$$

$$\|\partial_\beta^k(J_R - I)\|_{L^p(\Gamma_R)} \leq C\delta^{1/p} \frac{\log^k r}{\delta^2\sqrt{r}}, \quad (3.67c)$$

where $k = k_1 + \dots + k_m$ and $\partial_\beta^k = \partial_{\beta_1}^{k_1} \dots \partial_{\beta_m}^{k_m}$. By (3.52) and (3.67c), we see that $I - \mathcal{C}_{J_R}$ is invertible and R is given by (3.54) provided that $\delta^2 \sqrt{r}$ is large enough. In a similar way as in (3.56), we infer from (3.67) that R satisfies

$$\partial_\beta^k R(z) = \partial_\beta^k \left(I + \frac{R^{(1)}(z)}{\sqrt{r}} \right) + \mathcal{O} \left(\frac{\log^k r}{\delta^4 r} \right), \quad \partial_\beta^k R^{(1)}(z) = \mathcal{O} \left(\frac{\log^k r}{\delta^2} \right), \quad \text{as } r \rightarrow +\infty, \quad (3.68)$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_R$ and with $\vec{x} \in \mathbb{R}_{\text{ord}}^{+,m}$ satisfying $a^2 < x_1$ and (1.9), where $R^{(1)}$ is given by (3.57).

4 Large r asymptotics for Φ with $\alpha \rightarrow +\infty$: regime 2

In this section we analyze $\Phi(rz; r\vec{x}, \vec{s}, \alpha)$ as $r \rightarrow +\infty$ in the regime where $a \rightarrow 0$, $\alpha = a\sqrt{r} \rightarrow +\infty$ and simultaneously $\vec{x}$ lies in a compact subset of $\mathbb{R}_{\text{ord}}^{+,m}$ and $\vec{s}$ lies in a compact subset of $(0, +\infty)^m$. This regime is relevant for part 2 of Theorem 1.1.

In this regime, if one tries to repeat the same construction of $P^{(-a^2)}$ as in Subsection 3.4.3, one faces several issues. As can be seen from (3.42), for f_{-a^2} to be a conformal map in $\mathcal{D}_{-a^2}$, one needs to choose the radius δ small compared to a^2 . As a consequence, the expansion (3.46) converges more slowly to I if $a \rightarrow 0$ at a slow rate, and the construction completely breaks down if $a \rightarrow 0$ at a fast rate. This is a serious obstacle to the study of the regime $a \rightarrow 0$, which we circumvent by constructing a local parametrix $P^{(0)}$ in a disk $\mathcal{D}_0$ centered at 0.

We take the radii of $\mathcal{D}_{-x_j}$, $j = 1, \dots, m$ as

$$\delta = \frac{1}{M} \min_{0 \leq j < k \leq m} \{x_k - x_j, x_1 - a^2\}, \quad M \geq 3, \quad M \text{ fixed}, \quad (4.1)$$

and we take $\delta_0 = a^2 + \frac{x_1 - a^2}{M}$ for the radius of $\mathcal{D}_0$. Since the x_j 's remain bounded away from each other and from 0, both δ and δ_0 are bounded away from 0. We choose $M \geq 3$ so that the disks do not intersect each other, and by definition of δ_0 we have $-a^2 \in \mathcal{D}_0$.

The local parametrices around $-x_j$, $j = 1, \dots, m$, are identical to those constructed in Section 3 (with $n = 1$), so we do repeat their construction here.

4.1 Local parametrix around 0

Our construction is based on the Bessel model RH problem from [37], whose solution is denoted Φ_{Be} and given by (A.4). To the best of our knowledge, this is the first time that Φ_{Be} is used for large values of α . A main difference compared to the case where α is bounded lies in the rather complicated asymptotics (A.15) for $\Phi_{\text{Be}}(z, \alpha)$ as $z \rightarrow \infty$ and simultaneously $\alpha \rightarrow +\infty$. An important observation is that the function ξ defined in (A.12) is related to the g -function (3.1) by the relation

$$\sqrt{r}g(z) = \sqrt{r}f(z) + \frac{\pi i \alpha}{2} \theta(z) = \sqrt{rz + \alpha^2} + \alpha \log \frac{\sqrt{rz}}{a + \sqrt{a^2 + z}} = \alpha \xi \left(\frac{\sqrt{rz}}{\alpha} \right). \quad (4.2)$$

Here, we will simply use $z \mapsto \frac{rz}{4}$ as the conformal map, and we deform the lenses in a neighborhood of 0 such that

$$(\gamma_{1,+} \cap \mathcal{D}_0) \subset -a^2 + e^{\frac{2\pi i}{3}}(0, +\infty), \quad (\gamma_{1,-} \cap \mathcal{D}_0) \subset -a^2 + e^{-\frac{2\pi i}{3}}(0, +\infty). \quad (4.3)$$

We seek a local parametrix $P^{(0)}$ of the form (see Figure 4)

$$P^{(0)}(z) = E_0(z) \Phi_{\text{Be}} \left(\frac{rz}{4}; \alpha \right) \left\{ \begin{array}{ll} \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & z \in \mathfrak{I}_+ \\ \begin{pmatrix} 1 & 0 \\ -e^{-\pi i \alpha} & 1 \end{pmatrix}, & z \in \mathfrak{I}_- \\ I, & \text{else} \end{array} \right\} s_1^{-\frac{\sigma_3}{2}} e^{-\sqrt{r}f(z)\sigma_3} e^{-\frac{\pi i \alpha}{2} \theta(z)\sigma_3}, \quad (4.4)$$

where $\mathfrak{I}_\pm = \{z : \pm \arg z \in (\frac{2\pi}{3}, \pi) \text{ and } \pm \arg(z + a^2) \in (0, \frac{2\pi}{3})\}$, and E_0 is an analytic matrix-valued function in $\mathcal{D}_0$ that will be determined below. It can be verified from the jumps (A.5) of Φ_{Be} that $P^{(0)}$ has the same jumps as S inside $\mathcal{D}_0$, as desired.

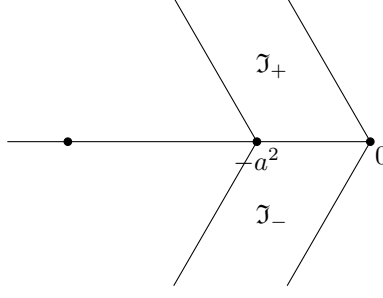


Figure 4: The subsets $\mathfrak{J}_\pm$ of the complex z -plane.

We now choose E_0 such that the matching condition (3.29) holds. Since δ_0 remains bounded away from 0 as $r \rightarrow +\infty$, $\sqrt{rz}\alpha^{-1} \rightarrow \infty$ for all $z \in \partial\mathcal{D}_0$. Therefore, we can use (A.15) with z replaced by $\frac{rz}{4}$, and we obtain

$$P^{(0)}(z)P^{(\infty)}(z)^{-1} = E_0(z)(\sqrt{\pi}(\alpha^2 + rz)^{\frac{1}{4}})^{-\sigma_3} N\left(I + \frac{2\Phi_{\text{Be},1}\left(\frac{rz}{4}; \alpha\right)}{\sqrt{rz}} + \mathcal{O}(r^{-1})\right) \\ \times \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-2\sqrt{r}f(z)} & 1 \end{pmatrix}, & z \in \mathfrak{J}_+ \\ \begin{pmatrix} 1 & 0 \\ -e^{-2\sqrt{r}f(z)} & 1 \end{pmatrix}, & z \in \mathfrak{J}_- \\ I, & \text{else} \end{cases} s_1^{-\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1} \quad \text{as } r \rightarrow +\infty, \quad (4.5)$$

uniformly for $z \in \partial\mathcal{D}_0$. On the other hand, we deduce from (3.1) that

$$f(z) = \sqrt{z} + o(1), \quad \text{as } a \rightarrow 0,$$

uniformly for z in compacts subsets of $\mathbb{C} \setminus \{0\}$, which implies

$$\begin{pmatrix} 1 & 0 \\ \pm e^{-2\sqrt{r}f(z)} & 1 \end{pmatrix} = I + \mathcal{O}(e^{-c\sqrt{r}}), \quad \text{as } r \rightarrow +\infty, a \rightarrow 0, \quad (4.6)$$

uniformly for $z \in \partial\mathcal{D}_0 \cap (\mathfrak{J}_+ \cup \mathfrak{J}_-)$, for a certain $c > 0$. In view of (4.5) and (4.6), we define

$$E_0(z) = P^{(\infty)}(z)s_1^{\frac{\sigma_3}{2}} N^{-1}(\sqrt{\pi}(\alpha^2 + rz)^{\frac{1}{4}})^{\sigma_3}, \quad (4.7)$$

and we verify that E_0 is indeed analytic inside $\mathcal{D}_0$, as desired. Furthermore, we have

$$P^{(0)}(z)P^{(\infty)}(z)^{-1} = I + \frac{2}{\sqrt{rz}} P^{(\infty)}(z)s_1^{\frac{\sigma_3}{2}} \Phi_{\text{Be},1}\left(\frac{rz}{4}; \alpha\right) s_1^{-\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1} + \mathcal{O}(r^{-1}), \quad (4.8)$$

as $r \rightarrow +\infty$ uniformly for $z \in \partial\mathcal{D}_0$. Finally, it is directly seen from (4.7) that

$$E_0(0) = P^{(\infty)}(0)s_1^{\frac{\sigma_3}{2}} N^{-1}(\sqrt{\pi a} r^{\frac{1}{4}})^{\sigma_3}. \quad (4.9)$$

4.2 Small norm problem

The function

$$R(z) := \begin{cases} S(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus (\mathcal{D}_0 \cup \bigcup_{j=1}^m \mathcal{D}_{-x_j}), \\ S(z)P^{(-x_j)}(z)^{-1}, & \text{for } z \in \mathcal{D}_{-x_j}, j \in \{1, \dots, m\}, \\ S(z)P^{(0)}(z)^{-1}, & \text{for } z \in \mathcal{D}_0, \end{cases} \quad (4.10)$$

is analytic in $\mathbb{C} \setminus \Gamma_R$, where

$$\Gamma_R = \left(\partial\mathcal{D}_0 \cup \bigcup_{j=1}^m \partial\mathcal{D}_{-x_j} \cup \gamma_+ \cup \gamma_- \right) \setminus \left(\mathcal{D}_0 \cup \bigcup_{j=1}^m \mathcal{D}_{-x_j} \right),$$

see Figure 5. From (3.35) and (4.8), as $r \rightarrow +\infty$ the jumps $J_R = R_-^{-1} R_+$ satisfy

$$J_R(z) - I = \begin{cases} \mathcal{O}(e^{-c\sqrt{r}\sqrt{z}}), & \text{unif. for } z \in (\gamma_+ \cup \gamma_-) \cap \Gamma_R, \\ \frac{J_R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}(\frac{1}{r}), & \text{unif. for } z \in \partial\mathcal{D}_0 \cup \bigcup_{j=1}^m \partial\mathcal{D}_{-x_j}, \end{cases} \quad (4.11)$$

where $J_R^{(1)}$ is given by

$$J_R^{(1)}(z) := \begin{cases} \frac{1}{f_{-x_j}(z)} E_{-x_j}(z) \Phi_{\text{HG},1}(\beta_j) E_{-x_j}(z)^{-1}, & z \in \partial\mathcal{D}_{-x_j}, j = 1, \dots, m, \\ \frac{2}{\sqrt{z}} P^{(\infty)}(z) s_1^{\frac{\sigma_3}{2}} \Phi_{\text{Be},1}\left(\frac{rz}{4}; \alpha\right) s_1^{-\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1}, & z \in \partial\mathcal{D}_0. \end{cases} \quad (4.12)$$

As in Subsection 3.5, we obtain that, for $k_1, \dots, k_m \in \mathbb{N}_{\geq 0}$,

$$\partial_\beta^k R(z) = \partial_\beta^k \left(I + \frac{R^{(1)}(z)}{\sqrt{r}} \right) + \mathcal{O}\left(\frac{\log^k r}{r}\right), \quad \partial_\beta^k R^{(1)}(z) = \mathcal{O}(\log^k r), \quad \text{as } r \rightarrow +\infty, \quad (4.13)$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_R$, for $a \rightarrow 0$, for $\vec{x}$ in compact subsets of $\mathbb{R}_{\text{ord}}^{+,m}$ and for $\beta_1, \dots, \beta_m$ in compact subsets of $i\mathbb{R}$, where $k = k_1 + \dots + k_m$, $\partial_\beta^k = \partial_{\beta_1}^{k_1} \dots \partial_{\beta_m}^{k_m}$ and

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{D}_0} \frac{J_R^{(1)}(s)}{s-z} ds + \sum_{j=1}^m \frac{1}{2\pi i} \int_{\partial\mathcal{D}_{-x_j}} \frac{J_R^{(1)}(s)}{s-z} ds. \quad (4.14)$$

We deduce from (4.12) and (A.16) that $J_R^{(1)}(z)$ can be analytically continued to

$$\left(\mathcal{D}_0 \cup \bigcup_{j=1}^m \mathcal{D}_{-x_j} \right) \setminus \{-a^2, -x_1, \dots, -x_m\},$$

and has a double pole at $-a^2$ and simple poles at $-x_1, \dots, -x_m$.

In Section 5, we will need the explicit values of $R^{(1)}(z)$ for z outside the disks, and also for $z = 0$. Although the computations involved in the evaluation of the residues at $-a^2$ are quite different from the ones in Subsection 3.5, we find, somewhat remarkably, that the exact same formula (3.58) (with $n = 1$ and the residues given by (3.59) and (3.60)) holds for $R^{(1)}(z)$ for z outside the disks. To obtain an explicit expression for $R(0)$, we evaluate the integrals in (4.14) by means of residue calculations, and we get

$$\begin{aligned} R^{(1)}(0) &= \frac{1}{a^2} \text{Res}(J_R^{(1)}(s), s = -a^2) + \frac{1}{a^4} \text{Res}((s + a^2)J_R^{(1)}(s), s = -a^2) \\ &\quad + \sum_{j=1}^m \frac{1}{x_j} \text{Res}(J_R^{(1)}(s), s = -x_j) - J_R^{(1)}(0), \end{aligned} \quad (4.15)$$

with the residues given by (3.59)–(3.60) and

$$\begin{aligned} J_R^{(1)}(0) &= \sqrt{r} P^{(\infty)}(0) s_1^{\frac{\sigma_3}{2}} \begin{pmatrix} \frac{-3p(0)+p(0)^3}{24\alpha} & \frac{i(-p(0)+p(0)^3)}{4\alpha} \\ \frac{i(-p(0)+p(0)^3)}{4\alpha} & \frac{3p(0)-p(0)^3}{24\alpha} \end{pmatrix} s_1^{-\frac{\sigma_3}{2}} P^{(\infty)}(0)^{-1} \\ &= \sqrt{r} P^{(\infty)}(0) s_1^{\frac{\sigma_3}{2}} \begin{pmatrix} -\frac{1}{12\alpha} & 0 \\ 0 & \frac{1}{12\alpha} \end{pmatrix} s_1^{-\frac{\sigma_3}{2}} P^{(\infty)}(0)^{-1} = \frac{1}{12a^2} \begin{pmatrix} d_1 & i \\ -i(a^2 - d_1^2) & -d_1 \end{pmatrix}, \end{aligned}$$

and $P^{(\infty)}$ is given by (3.22) and the function p is given by (A.12).

5 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 in three steps using the following differential identity stated already in (2.14):

$$\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) = K_\infty + \sum_{j=1}^m K_{-x_j} + K_0. \quad (5.1)$$

First, we obtain large r asymptotics for $K_\infty, K_{-x_1}, \dots, K_{-x_m}, K_0$ using the analysis of Sections 3 and 4. Second, we substitute these asymptotics into the differential identity (5.1) and use the change of

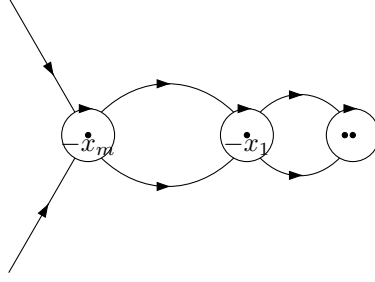


Figure 5: The contour Γ_R with $m = 2$ and $a \rightarrow 0$. The two unlabeled dots are $-a^2$ and 0 .

variables (3.20) to get large r asymptotics for $\partial_{\beta_k} \log F_\alpha(r\vec{x}, \vec{s})$, $k = 1, \dots, m$. Third, we integrate these asymptotics in $\beta_1, \dots, \beta_m$ and obtain large r asymptotics for $\log F_\alpha(r\vec{x}, \vec{s})$.

The three regimes considered in Sections 3 and 4 can all be treated at once, save for the analysis of K_0 . A main difference between these regimes is that the terms that are of order $\mathcal{O}(r^{-1/2})$ and $\mathcal{O}(r^{-1})$ in regimes 1 and 2 become of order $\mathcal{O}(\delta^{-2}r^{-1/2})$ and $\mathcal{O}(\delta^{-4}r^{-1})$ in regime 3. Since $\delta \asymp |x_m - a^2|$ is of order 1 in regimes 1 and 2, we can include all the factors of δ in the error terms, and then simply replace δ by 1 in regimes 1 and 2 at the end. Moreover, we will present all calculations for any value of $n \in \{1, \dots, m+1\}$, but we recall that $n = 1$ in regimes 2 and 3.

Asymptotics for K_∞ . By (3.15), (3.19) and (3.48), we have $T_1 = R_1 + P_1^{(\infty)}$, where R_1 is defined via the expansion $R(z) = I + \frac{R_1}{z} + \mathcal{O}(z^{-2})$ as $z \rightarrow \infty$. Therefore, by (3.56), (3.68) and (3.20), for any $k \in \{1, \dots, m\}$, we have

$$\partial_{s_k} T_1 = \partial_{s_k} P_1^{(\infty)} + \frac{\partial_{s_k} R_1^{(1)}}{\sqrt{r}} + \mathcal{O}\left(\frac{\log r}{\delta^4 r}\right), \quad \text{as } r \rightarrow +\infty, \quad (5.2)$$

where $R_1^{(1)}$ is such that $R^{(1)}(z) = \frac{R_1^{(1)}}{z} + \mathcal{O}(z^{-2})$ as $z \rightarrow \infty$. Here we have used that the expansions in r and z can be computed in any order, which is a consequence of (3.54). The matrix $\partial_{s_k} R_1^{(1)}$ can be calculated explicitly using (3.58). Recalling (2.15), (3.15) and (3.23), a computation yields

$$\begin{aligned} K_\infty &= -\frac{i}{2} \sqrt{r} \partial_{s_k} T_{1,12} = -\frac{i}{2} \left(\partial_{s_k} P_{1,12}^{(\infty)} \sqrt{r} + \partial_{s_k} R_{1,12}^{(1)} + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right) \right) \\ &= \frac{1}{2} \partial_{s_k} d_1 \sqrt{r} - \sum_{j=n}^m \frac{\partial_{s_k} (\beta_j^2 (\tilde{\Lambda}_{j,1} - \tilde{\Lambda}_{j,2} + 2i))}{4ic_{-x_j} \sqrt{x_j - a^2}} + \frac{a^2}{4} \partial_{s_k} (d_0^2) + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right). \end{aligned} \quad (5.3)$$

Asymptotics for K_{-x_j} with $j \in \{1, \dots, n-1\}$. Suppose $j \in \{1, \dots, n-1\}$. For z outside the lenses such that $z \in \mathcal{D}_{-x_j}$ and $\text{Im } z > 0$, using (3.18), (3.48), (3.38), (3.10), (2.7) and (4.2), we find

$$\begin{aligned} T(z) &= R(z) P^{(x_j)}(z) = R(z) P^{(\infty)}(z) \begin{pmatrix} 1 & h_{-x_j}(z) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\frac{ia^2}{2} \sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_j(rz; r\vec{x}, \vec{s}, \alpha) \begin{pmatrix} 1 & \frac{s_{j+1} - s_j}{2\pi i} \log(r(z + x_j)) \\ 0 & 1 \end{pmatrix} e^{-\sqrt{r} f(z) \sigma_3}. \end{aligned}$$

Therefore, using also (3.40),

$$G_j(-rx_j; r\vec{x}, \vec{s}, \alpha) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ \frac{ia^2}{2} \sqrt{r} & 1 \end{pmatrix} R(-x_j) P^{(\infty)}(-x_j) e^{\sqrt{r} f(-x_j) \sigma_3} \begin{pmatrix} 1 & \hat{c} \\ 0 & 1 \end{pmatrix},$$

where the exact expression for $\hat{c} = \hat{c}(r)$ is unimportant for us. Substituting the above expression into (2.16) and using (3.56), we find that

$$K_{-x_j} = \mathcal{O}(e^{-c\sqrt{r}}), \quad \text{as } r \rightarrow +\infty, \quad (5.4)$$

for some constant $c > 0$.

Asymptotics for K_{-x_j} with $j \in \{n, \dots, m\}$. Suppose $j \in \{n, \dots, m\}$. For z outside the lenses such that $z \in \mathcal{D}_{-x_j}$ and $\text{Im } z > 0$, using now (3.18), (3.48) and (3.33), we have

$$T(z) = R(z)E_{-x_j}(z)\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{-\sqrt{r}f(z)\sigma_3}. \quad (5.5)$$

By (3.20), (3.31) and (A.22), as $z \rightarrow -x_j$ from outside the lenses, $\text{Im } z > 0$, we find

$$\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} = \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix} (I + \mathcal{O}(z + x_j)) \begin{pmatrix} 1 & \frac{s_{j+1}-s_j}{2\pi i} \log(r(z+x_j)) \\ 0 & 1 \end{pmatrix}, \quad (5.6)$$

where

$$\Psi_{j,11} = \frac{\Gamma(1-\beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, \quad \Psi_{j,21} = \frac{\Gamma(1+\beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, \quad \Psi_{j,11}\Psi_{j,21} = \beta_j \frac{2\pi i}{s_{j+1} - s_j}. \quad (5.7)$$

The exact values of $\Psi_{j,12}$ and $\Psi_{j,22}$ can also be computed explicitly, but they are unimportant for us. By combining (5.5)–(5.6) with (2.7), (3.10) and (4.2), we find

$$G_j(-rx_j; r\vec{x}, \vec{s}, \alpha) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ i\frac{a^2}{2}\sqrt{r} & 1 \end{pmatrix} R(-x_j)E_{-x_j}(-x_j) \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix}. \quad (5.8)$$

The change of variables (3.20) shows that $\partial_{s_1} = -(2\pi i s_1)^{-1} \partial_{\beta_1}$ and $\partial_{s_k} = (2\pi i s_k)^{-1} (\partial_{\beta_{k-1}} - \partial_{\beta_k})$ for $k = 2, \dots, m$. Hence (3.56) and (3.68) imply that, for each $k = 1, \dots, m$,

$$R(-x_j) = I + \mathcal{O}\left(\frac{1}{\delta^2 \sqrt{r}}\right) \quad \text{and} \quad \partial_{s_k} R(-x_j) = \mathcal{O}\left(\frac{\log r}{\delta^2 \sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \quad (5.9)$$

Substituting (5.8) into (2.16) and using (3.36), (5.9) as well as the identities $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we find after a long computation that

$$\begin{aligned} K_{-x_j} &= \beta_j \partial_{s_k} \log \frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j)} - 2\beta_j \partial_{s_k} \log \Lambda_j \\ &+ \frac{\partial_{s_k} d_1}{2\sqrt{x_j - a^2}} \left(\beta_j^2 (\tilde{\Lambda}_{j,1} + \tilde{\Lambda}_{j,2}) + 2i\beta_j \right) + \mathcal{O}\left(\frac{\log r}{\delta^{5/2} \sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \end{aligned} \quad (5.10)$$

Asymptotics for K_0 in the regimes where a is bounded away from 0. For z near 0, we use (3.18), (3.48), (3.10) and (2.10) to obtain

$$T(z) = R(z)P^{(\infty)}(z) = \begin{pmatrix} 1 & 0 \\ -i\frac{a^2}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_0(rz; r\vec{x}, \vec{s}, \alpha)(rz)^{\frac{a\sqrt{r}}{2}\sigma_3} \begin{pmatrix} 1 & s_1 h(rz) \\ 0 & 1 \end{pmatrix} e^{-\sqrt{r}g(z)\sigma_3},$$

from which we get

$$\begin{aligned} G_0(rz; r\vec{x}, \vec{s}, \alpha) &= r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ i\frac{a^2}{2}\sqrt{r} & 1 \end{pmatrix} R(z)P^{(\infty)}(z) e^{\sqrt{r}g(z)\sigma_3} \begin{pmatrix} 1 & -s_1 h(rz) \\ 0 & 1 \end{pmatrix} (rz)^{-\frac{a\sqrt{r}}{2}\sigma_3} \\ &= r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ i\frac{a^2}{2}\sqrt{r} & 1 \end{pmatrix} R(z)P^{(\infty)}(z) e^{\sqrt{r}g(z)\sigma_3} (rz)^{-\frac{a\sqrt{r}}{2}\sigma_3} \begin{pmatrix} 1 & -s_1 h(rz)(rz)^{a\sqrt{r}} \\ 0 & 1 \end{pmatrix} \\ &= r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ i\frac{a^2}{2}\sqrt{r} & 1 \end{pmatrix} R(z)P^{(\infty)}(z) \tilde{\mathcal{C}}_2^{\sigma_3}(I + o(1)), \quad \text{as } z \rightarrow 0, \end{aligned}$$

for a certain $\hat{c}_2$ that is independent of $s_1, \dots, s_m$ and whose exact expression is unimportant for us. Therefore, we have

$$G_0(0; r\vec{x}, \vec{s}, \alpha) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ i\frac{a^2}{2}\sqrt{r} & 1 \end{pmatrix} R(0)P^{(\infty)}(0) \tilde{\mathcal{C}}_2^{\sigma_3}. \quad (5.11)$$

Substituting this expression into (2.17) and simplifying, we find

$$K_0 = a\sqrt{r} \left(H_{0,21} \partial_{s_k} H_{0,12} - H_{0,11} \partial_{s_k} H_{0,22} - \partial_{s_k} \log D(0) \right),$$

where

$$H_0 := R(0)P^{(\infty)}(0)D(0)^{\sigma_3} = R(0) \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} a^{-\frac{\sigma_3}{2}} N.$$

Recall that $D(0)$ is given by (3.26) and $R(0)$ by (3.56) (see also (3.68)) and (3.58) with $z = 0$. After a direct computation, we obtain

$$\begin{aligned} K_0 &= \left(\frac{\partial_{s_k} d_1}{2} + ia^2 \sum_{j=n}^m \mathcal{I}_j \partial_{s_k} \beta_j \right) \sqrt{r} + \frac{d_0}{2} \partial_{s_k} (d_1 - a^2 d_0) - \sum_{j=n}^m \frac{\beta_j^2 (\tilde{\Lambda}_{j,1} + \tilde{\Lambda}_{j,2}) \partial_{s_k} d_1}{2c_{-x_j} x_j} \\ &+ \sum_{j=n}^m \frac{\partial_{s_k} ([4a^2 - 2x_j - ix_j (\tilde{\Lambda}_{j,1} - \tilde{\Lambda}_{j,2})] \beta_j^2)}{4c_{-x_j} x_j \sqrt{x_j - a^2}} + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \end{aligned} \quad (5.12)$$

Asymptotics for K_0 in the regime where $a \rightarrow 0$ and x_1 remains bounded away from 0 as $r \rightarrow +\infty$. Let us consider regime 2, where $a \rightarrow 0$ while the points x_j stay bounded away from 0 as $r \rightarrow +\infty$. In this regime, the analysis of K_0 involves the local parametrix $P^{(0)}$ rather than $P^{(\infty)}$. For $z \in \mathcal{D}_0$, we use (3.18), (4.10), (3.10) and (2.10) to conclude that

$$T(z) = R(z)P^{(0)}(z) = \begin{pmatrix} 1 & 0 \\ -\frac{ia^2}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_0(rz; r\vec{x}, \vec{s}, \alpha)(rz)^{\frac{a\sqrt{r}}{2}\sigma_3} \begin{pmatrix} 1 & s_1 h(rz) \\ 0 & 1 \end{pmatrix} e^{-\sqrt{r}g(z)\sigma_3},$$

and in the same way as for (5.11), we find

$$G_0(0; r\vec{x}, \vec{s}, \alpha) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ \frac{ia^2}{2}\sqrt{r} & 1 \end{pmatrix} R(0)P^{(0)}(0)\hat{\mathcal{C}}_2^{\sigma_3} \quad (5.13)$$

for a certain $\hat{c}_2$ that is independent of $s_1, \dots, s_m$ and whose exact expression is unimportant for us. We now find an explicit expression for $P^{(0)}(0)$. As $z \rightarrow 0$, $\text{Im } z > 0$, z outside the regions $\mathcal{I}_{\pm}$ defined in (4.4) (for example, $z \rightarrow 0$ with $\arg z = \frac{\pi}{2}$), we use (4.4) and (A.9) to obtain

$$\begin{aligned} P^{(0)}(z) &= E_0(z)\Phi_{\text{Be}}\left(\frac{rz}{4}; \alpha\right) s_1^{-\frac{\sigma_3}{2}} e^{-\sqrt{r}f(z)\sigma_3} e^{-\frac{\pi i \alpha}{2}\theta(z)\sigma_3} \\ &= E_0(z) \begin{pmatrix} \frac{1}{\Gamma(1+\alpha)} & \frac{i\Gamma(\alpha)}{2\pi} \\ \frac{i\pi}{\Gamma(\alpha)} & \frac{\Gamma(1+\alpha)}{2} \end{pmatrix} \left(\frac{rz}{4}\right)^{\frac{\sigma_3}{2}\sigma_3} \begin{pmatrix} 1 & h\left(\frac{rz}{4}\right) \\ 0 & 1 \end{pmatrix} s_1^{-\frac{\sigma_3}{2}} e^{-\sqrt{r}g(z)\sigma_3} \\ &= E_0(0) \begin{pmatrix} \frac{1}{\Gamma(1+\alpha)} & \frac{i\Gamma(\alpha)}{2\pi} \\ \frac{i\pi}{\Gamma(\alpha)} & \frac{\Gamma(1+\alpha)}{2} \end{pmatrix} s_1^{-\frac{\sigma_3}{2}} \hat{\mathcal{C}}_3^{\sigma_3} (I + o(1)), \end{aligned}$$

where here too, $\hat{c}_3$ is a function independent of $s_1, \dots, s_m$ whose exact expression is unimportant for us. Hence, using (4.9), we find

$$\begin{aligned} P^{(0)}(0) &= E_0(0) \begin{pmatrix} \frac{1}{\Gamma(1+\alpha)} & \frac{i\Gamma(\alpha)}{2\pi} \\ \frac{i\pi}{\Gamma(\alpha)} & \frac{\Gamma(1+\alpha)}{2} \end{pmatrix} s_1^{-\frac{\sigma_3}{2}} \hat{\mathcal{C}}_3^{\sigma_3} \\ &= P^{(\infty)}(0) s_1^{\frac{\sigma_3}{2}} N^{-1} (\sqrt{\pi a} r^{\frac{1}{4}})^{\sigma_3} \begin{pmatrix} \frac{1}{\Gamma(1+\alpha)} & \frac{i\Gamma(\alpha)}{2\pi} \\ \frac{i\pi}{\Gamma(\alpha)} & \frac{\Gamma(1+\alpha)}{2} \end{pmatrix} s_1^{-\frac{\sigma_3}{2}} \hat{\mathcal{C}}_3^{\sigma_3}. \end{aligned} \quad (5.14)$$

Defining H_0 by $H_0 := R(0)P^{(\infty)}(0)$, equations (5.13) and (5.14) imply that

$$G_0(0; r\vec{x}, \vec{s}, \alpha) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ \frac{ia^2}{2}\sqrt{r} & 1 \end{pmatrix} H_0 s_1^{\frac{\sigma_3}{2}} N^{-1} (\sqrt{\pi a} r^{\frac{1}{4}})^{\sigma_3} \begin{pmatrix} \frac{1}{\Gamma(1+\alpha)} & \frac{i\Gamma(\alpha)}{2\pi} \\ \frac{i\pi}{\Gamma(\alpha)} & \frac{\Gamma(1+\alpha)}{2} \end{pmatrix} s_1^{-\frac{\sigma_3}{2}} \hat{\mathcal{C}}_3^{\sigma_3} \hat{\mathcal{C}}_2^{\sigma_3}.$$

Substituting this expression for $G_0(0; r\vec{x}, \vec{s}, \alpha)$ into (2.17) and using the identities $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ and $\det H_0 \equiv 1$, we obtain

$$K_0 = a\sqrt{r} \left(H_{0,21} \partial_{s_k} H_{0,12} - H_{0,11} \partial_{s_k} H_{0,22} \right). \quad (5.15)$$

The fact that $P^{(\infty)}(0) = \mathcal{O}(a^{-\frac{1}{2}})$ as $a \rightarrow 0$ could a priori worsen the error in the large r asymptotics of K_0 . However, the following exact formula, which is obtained by substituting the explicit expression (3.22) for $P^{(\infty)}(0)$ into (5.15) with $H_0 = R(0)P^{(\infty)}(0)$, shows that this is not the case:

$$K_0 = \frac{i\sqrt{r}}{2} \left(2ia\partial_{s_k} \log D(0) - i\partial_{s_k} d_1 + (a^2 - d_1^2) \left(R(0)_{22}\partial_{s_k} R(0)_{12} - R(0)_{12}\partial_{s_k} R(0)_{22} \right) \right. \\ \left. + \left(R(0)_{21}\partial_{s_k} R(0)_{11} - R(0)_{11}\partial_{s_k} R(0)_{21} \right) + 2id_1 \left(R(0)_{21}\partial_{s_k} R(0)_{12} - R(0)_{11}\partial_{s_k} R(0)_{22} \right) \right).$$

The right-hand side of the above equation can now be expanded as $r \rightarrow +\infty$ using (4.13) and the expressions (3.26) for $D(0)$ and (4.15) for $R^{(1)}(0)$. A calculation shows that the term $J_R^{(1)}(0)$ in (4.15) leads to a contribution to K_0 of $\mathcal{O}(\log(r)/\sqrt{r})$ and in the end we arrive at the following asymptotic formula which is identical to (5.12):

$$K_0 = \left(\frac{\partial_{s_k} d_1}{2} + ia^2 \sum_{j=n}^m \mathcal{I}_j \partial_{s_k} \beta_j \right) \sqrt{r} + \frac{d_0}{2} \partial_{s_k} (d_1 - a^2 d_0) - \sum_{j=n}^m \frac{\beta_j^2 (\tilde{\Lambda}_{j,1} + \tilde{\Lambda}_{j,2}) \partial_{s_k} d_1}{2c_{-x_j} x_j} \\ + \sum_{j=n}^m \frac{\partial_{s_k} ([4a^2 - 2x_j - ix_j(\tilde{\Lambda}_{j,1} - \tilde{\Lambda}_{j,2})] \beta_j^2)}{4c_{-x_j} x_j \sqrt{x_j - a^2}} + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \quad (5.16)$$

Asymptotics for $\partial_{\beta_k} \log F_\alpha(r\vec{x}, \vec{s})$. Recall that c_{-x_j} is given by (3.31), and d_0 and d_1 by (3.21). Hence, by substituting into (2.14) the large r asymptotics of K_∞ , K_{-x_j} , $j = 1, \dots, m$ and K_0 given by (5.3), (5.4), (5.10) and (5.12), we find (after simplifications), for $k = 1, \dots, m$,

$$\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) = \left(\partial_{s_k} d_1 + ia^2 \sum_{j=n}^m \mathcal{I}_j \partial_{s_k} \beta_j \right) \sqrt{r} - \sum_{j=n}^m \left(2\beta_j \partial_{s_k} \log \Lambda_j + \partial_{s_k} (\beta_j^2) \right) \\ + \sum_{j=n}^m \beta_j \partial_{s_k} \log \frac{\Gamma(1 + \beta_j)}{\Gamma(1 - \beta_j)} + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \quad (5.17)$$

Using (3.37), we also note that

$$- \sum_{j=n}^m 2\beta_j \partial_{s_k} \log \Lambda_j = -2 \sum_{j=n}^m \beta_j \partial_{s_k} (\beta_j) \log(4(x_j - a^2)c_{-x_j} \sqrt{r}) - 2 \sum_{j=n}^m \beta_j \sum_{\substack{\ell=n \\ \ell \neq j}}^m \partial_{s_k} (\beta_\ell) \log(T_{\ell,j}). \quad (5.18)$$

Since $\beta_n, \dots, \beta_m$ are independent of $s_1, \dots, s_{n-1}$, see (3.20), $\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) = \mathcal{O}(\log(r)/(\delta^4 \sqrt{r}))$ as $r \rightarrow +\infty$ for each $k = 1, \dots, n-1$. Let us define $\tilde{F}_\alpha(r\vec{x}, \vec{\beta}) = F_\alpha(r\vec{x}, \vec{s})$, where $\vec{\beta} = (\beta_1, \dots, \beta_m)$. By (3.20), we have $s_k = \exp(-2\pi i \sum_{j=k}^m \beta_j)$ and hence $\partial_{\beta_k} = -2\pi i \sum_{j=1}^k s_j \partial_{s_j}$ for $k = 1, \dots, m$. In particular, for each $k = 1, \dots, n-1$,

$$\partial_{\beta_k} \log \tilde{F}_\alpha(r\vec{x}, \vec{\beta}) = \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \quad (5.19)$$

Substituting (5.18) into (5.17), for $k \in \{n, \dots, m\}$ we obtain

$$\partial_{\beta_k} \log \tilde{F}_\alpha(r\vec{x}, \vec{\beta}) = (\partial_{\beta_k} d_1 + ia^2 \mathcal{I}_k) \sqrt{r} - 2 \sum_{j=n}^m \beta_j \partial_{\beta_k} (\beta_j) \log \left(4(x_j - a^2)^{3/2} x_j^{-1} \sqrt{r} \right) \\ - 2 \sum_{j=n}^m \beta_j \sum_{\substack{\ell=n \\ \ell \neq j}}^m \partial_{\beta_k} (\beta_\ell) \log(T_{\ell,j}) - \sum_{j=n}^m \partial_{\beta_k} (\beta_j^2) + \sum_{j=n}^m \beta_j \partial_{\beta_k} \log \frac{\Gamma(1 + \beta_j)}{\Gamma(1 - \beta_j)} + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right) \quad (5.20)$$

as $r \rightarrow +\infty$. Recalling that d_1 is given by (3.21), $\mathcal{I}_k$ by (3.27), and f by (3.1), we note that

$$\partial_{\beta_k} d_1 + ia^2 \mathcal{I}_k = -2i\sqrt{x_k - a^2} + ia^2 \int_{a^2}^{x_k} \frac{du}{u\sqrt{u - a^2}} = -i \int_{a^2}^{x_k} \frac{\sqrt{u - a^2}}{u} du = 2f_-(-x_k).$$

Hence, for $k \in \{n, \dots, m\}$, the asymptotics (5.20) can be rewritten as

$$\begin{aligned} \partial_{\beta_k} \log \tilde{F}_\alpha(r\vec{x}, \vec{\beta}) &= 2f_-(-x_k)\sqrt{r} - 2\beta_k \log \left(4(x_k - a^2)^{3/2} x_k^{-1} \sqrt{r} \right) \\ &\quad - 2 \sum_{\substack{j=n \\ j \neq k}}^m \beta_j \log(T_{k,j}) - 2\beta_k + \beta_k \partial_{\beta_k} \log \frac{\Gamma(1+\beta_k)}{\Gamma(1-\beta_k)} + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right). \end{aligned} \quad (5.21)$$

Asymptotics for $\log F_\alpha(r\vec{x}, \vec{s})$. Integrating (5.19) in $\beta_1, \dots, \beta_{n-1}$, we get

$$\frac{\log \tilde{F}_\alpha(r\vec{x}, (\beta_1, \dots, \beta_{n-1}, 0, \dots, 0))}{\log \tilde{F}_\alpha(r\vec{x}, (0, \dots, 0))} = \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \quad (5.22)$$

We will now proceed with the successive integrations of (5.21) in $\beta_n, \dots, \beta_m$. We will use the notation $\vec{\beta}_j := (\beta_1, \dots, \beta_j, 0, \dots, 0)$, $j \geq n-1$. First, we set $\beta_{n+1} = \dots = \beta_m = 0$ and use (5.21) with $k = n$. After integration in β_n , we get

$$\begin{aligned} \log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_n)}{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_{n-1})} &= 2\beta_n f_-(-x_n)\sqrt{r} - \beta_n^2 \log \left(4(x_n - a^2)^{3/2} x_n^{-1} \sqrt{r} \right) \\ &\quad + \log \left(G(1+\beta_n)G(1-\beta_n) \right) + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty, \end{aligned}$$

where G is Barnes' G -function, and where we have used that

$$\int_0^\beta x \partial_x \log \frac{\Gamma(1+x)}{\Gamma(1-x)} dx = \beta^2 + \log \left(G(1+\beta)G(1-\beta) \right). \quad (5.23)$$

Now, we set $\beta_{n+2} = \dots = \beta_m = 0$ and $k = n+1$ in (5.21), and integrate in β_{n+1} . This gives

$$\begin{aligned} \log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_{n+1})}{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_n)} &= 2\beta_{n+1} f_-(-x_{n+1})\sqrt{r} - \beta_{n+1}^2 \log \left(4(x_{n+1} - a^2)^{3/2} x_{n+1}^{-1} \sqrt{r} \right) \\ &\quad - 2\beta_n \beta_{n+1} \log(T_{n+1,n}) + \log \left(G(1+\beta_{n+1})G(1-\beta_{n+1}) \right) + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right) \end{aligned} \quad (5.24)$$

as $r \rightarrow +\infty$. The integrations in $\beta_{n+2}, \dots, \beta_m$ are similar. By adding the asymptotics of $\log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_j)}{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_{j-1})}$ for $j = n, \dots, m$ to (5.22), we obtain

$$\begin{aligned} \log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta})}{\tilde{F}_\alpha(r\vec{x}, \vec{0})} &= \sum_{j=n}^m 2\beta_j f_-(-x_j)\sqrt{r} - \sum_{j=n}^m \beta_j^2 \log \left(4(x_j - a^2)^{3/2} x_j^{-1} \sqrt{r} \right) \\ &\quad - 2 \sum_{n \leq j < k \leq m} \beta_j \beta_k \log(T_{k,j}) + \sum_{j=n}^m \log \left(G(1+\beta_j)G(1-\beta_j) \right) + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right). \end{aligned} \quad (5.25)$$

Since $u_j = -2\pi i \beta_j$, $\tilde{F}_\alpha(r\vec{x}, \vec{\beta}) = F_\alpha(r\vec{x}, \vec{s}) = E_\alpha(r\vec{x}, \vec{u})$, and

$$\tilde{F}_\alpha(r\vec{x}, \vec{0}) = E_\alpha(r\vec{x}, \vec{0}) = 1,$$

we can rewrite (5.25) as

$$\begin{aligned} \log E_\alpha(r\vec{x}, \vec{u}) &= \sum_{j=n}^m \left(\frac{\sqrt{r}}{\pi} \int_{a^2}^{x_j} \frac{\sqrt{u-a^2}}{2u} du \right) u_j + \sum_{j=n}^m \frac{u_j^2}{4\pi^2} \log \left(4(x_j - a^2)^{3/2} x_j^{-1} \sqrt{r} \right) \\ &\quad + \sum_{n \leq j < k \leq m} \frac{u_j u_k}{2\pi^2} \log \left(\frac{\sqrt{x_j - a^2} + \sqrt{x_k - a^2}}{|\sqrt{x_j - a^2} - \sqrt{x_k - a^2}|} \right) + \sum_{j=n}^m \log \left(G(1+\beta_j)G(1-\beta_j) \right) + \mathcal{O}\left(\frac{\log r}{\delta^4 \sqrt{r}}\right), \end{aligned}$$

which finishes the proof of Theorem 1.1.

6 Proof of Theorem 1.10

The proof is an adaptation of [16, Theorem 1.2] to handle varying point processes. Let $\{X_r\}_{r \geq 0}$ be a family of point processes satisfying Assumptions 1.9, and let $N_r(x)$ denote the random variable that counts the number of points of X_r that are $\leq x$. In what follows, we let $\mathfrak{a}, \eta_{r,1}, \eta_{r,2}, \delta_r$ be the constants appearing in Assumptions 1.9. We also write $\xi_{r,k}$ for the k -th smallest point of X_r , and let $\kappa_{r,k} := \mu_r^{-1}(k)$. We divide the proof into two lemmas.

Lemma 6.1. *There exist $c > 0$ and $r_0 > 0$ such that for any $\epsilon > 0$ sufficiently small and $r > r_0$,*

$$\mathbb{P} \left(\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \left| \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} \right| > \sqrt{\frac{2}{\mathfrak{a}}(1 + \epsilon)} \right) \leq \frac{c \mu_r((\eta_{r,1} + \delta_r)r)^{-\epsilon}}{2\epsilon}. \quad (6.1)$$

In particular, for any $\epsilon > 0$,

$$\lim_{r \rightarrow +\infty} \mathbb{P} \left(\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \left| \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} \right| \leq \sqrt{\frac{2}{\mathfrak{a}}(1 + \epsilon)} \right) = 1.$$

Proof. For each large enough r , μ_r and σ_r are increasing by part (2) of Assumptions 1.9, and therefore

$$\frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} \leq \frac{N_r(\kappa_{r,k}) - \mu_r(\kappa_{r,k-1})}{\sigma_r^2(\kappa_{r,k-1})} = \frac{N_r(\kappa_{r,k}) - \mu_r(\kappa_{r,k}) + 1}{\sigma_r^2(\kappa_{r,k-1})}, \quad \text{for all } x \in [\kappa_{r,k-1}, \kappa_{r,k}] \quad (6.2)$$

and for all $k \in \mathcal{K}_r := \{k \in \mathbb{N}_{>0} : \kappa_{r,k} > (\eta_{r,1} + \delta_r)r \text{ and } \kappa_{r,k-1} < (\eta_{r,2} - \delta_r)r\}$. The definition of $\mathcal{K}_r$ implies in particular that

$$((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r) \subset \bigcup_{k \in \mathcal{K}_r} [\kappa_{r,k-1}, \kappa_{r,k}].$$

Note that $\mathcal{K}_r$ is finite and $\#\mathcal{K}_r \rightarrow +\infty$ as $r \rightarrow +\infty$ (this follows directly from Remark 6). Using first (6.2), then a union bound and then Markov's inequality, we find

$$\begin{aligned} \mathbb{P} \left(\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} > \gamma \right) &\leq \mathbb{P} \left(\max_{k \in \mathcal{K}_r} \frac{N_r(\kappa_{r,k}) - \mu_r(\kappa_{r,k}) + 1}{\sigma_r^2(\kappa_{r,k-1})} > \gamma \right) \\ &\leq \sum_{k \in \mathcal{K}_r} \mathbb{P} \left(\frac{N_r(\kappa_{r,k}) - \mu_r(\kappa_{r,k}) + 1}{\sigma_r^2(\kappa_{r,k-1})} > \gamma \right) \leq \sum_{k \in \mathcal{K}_r} \mathbb{E} \left[e^{\gamma N_r(\kappa_{r,k})} \right] e^{-\gamma \mu_r(\kappa_{r,k}) + \gamma - \gamma^2 \sigma_r^2(\kappa_{r,k-1})}, \end{aligned} \quad (6.3)$$

for any $\gamma > 0$. Let $k' := \max\{k : k \in \mathcal{K}_r\}$ and $k'' := \min\{k : k \in \mathcal{K}_r\}$ (note that k' and k'' depends on r , even though this is not indicated in the notation). For all k such that $k'' \leq k < k'$, by definition of $\mathcal{K}_r$ we can directly use (1.32) to obtain an upper bound for $\mathbb{E}[e^{\gamma N_r(\kappa_{r,k})}]$. To also obtain an upper bound for $\mathbb{E}[e^{\gamma N_r(\kappa_{r,k'})}]$ using (1.32), we need to show that the inequality $\kappa_{r,k'} < (\eta_{r,2} - \frac{\delta_r}{2})r$ holds for all sufficiently large r . Let us write $\kappa_{r,k'} = \kappa_{r,k'-1} + m$. We have

$$\begin{aligned} k' &= \mu_r(\mu_r^{-1}(k' - 1) + m) \geq k' - 1 + m \inf_{\xi \in [\kappa_{r,k'-1}, \kappa_{r,k'}]} \mu_r'(\xi) \\ &\geq k' - 1 + \frac{m}{\kappa_{r,k'-1} + m} \inf_{\xi \in [\kappa_{r,k'-1}, \kappa_{r,k'}]} \xi \mu_r'(\xi). \end{aligned} \quad (6.4)$$

Note that $\kappa_{r,k'-1} \geq (\eta_{r,1} + \delta_r)r$ for all sufficiently large r , because $\#\mathcal{K}_r \rightarrow +\infty$. Thus, since $\xi \mapsto \xi \mu_r'(\xi)$ is non-decreasing by Assumptions 1.9,

$$\inf_{\xi \in [\kappa_{r,k'-1}, \kappa_{r,k'}]} \xi \mu_r'(\xi) = \kappa_{r,k'-1} \mu_r'(\kappa_{r,k'-1}) \geq (\eta_{r,1} + \delta_r)r \mu_r'((\eta_{r,1} + \delta_r)r) \quad (6.5)$$

for all large enough r . Also, by definition of $\mathcal{K}_r$, $\kappa_{r,k'-1} \leq (\eta_{r,2} - \delta_r)r$. Hence, (6.4)–(6.5) imply

$$\frac{m}{\delta_r r} \leq \frac{1}{\delta_r} \frac{\eta_{r,2} - \delta_r}{(\eta_{r,1} + \delta_r)r \mu_r'((\eta_{r,1} + \delta_r)r) - 1} \leq \frac{2(\eta_{r,2} - \delta_r)}{(\eta_{r,1} + \delta_r)\delta_r r \mu_r'((\eta_{r,1} + \delta_r)r)} \quad (6.6)$$

for all sufficiently large r , and the right-hand side tends to 0 as $r \rightarrow +\infty$ by parts (2) and (4) of Assumptions 1.9. This shows that $\kappa_{r,k'} < (\eta_{r,2} - \frac{\delta_r}{2})r$ for all sufficiently large r . Therefore, using (1.32) in (6.3), we obtain

$$\mathbb{P} \left(\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} > \gamma \right) \leq C e^\gamma \sum_{k \in \mathcal{K}_r} e^{-\frac{\gamma^2}{2} \sigma_r^2(\kappa_{r,k})} e^{\gamma^2 (\sigma_r^2(\kappa_{r,k}) - \sigma_r^2(\kappa_{r,k-1}))}.$$

Using the assumption that $\sigma_r^2 \circ \mu_r^{-1}$ is concave and increasing, we have

$$\sup_{k \in \mathcal{K}_r} e^{\gamma^2 (\sigma_r^2(\kappa_{r,k}) - \sigma_r^2(\kappa_{r,k-1}))} = e^{\gamma^2 (\sigma_r^2 \circ \mu_r^{-1}(k'') - \sigma_r^2 \circ \mu_r^{-1}(k'' - 1))} \leq e^{\gamma^2 (\sigma_r^2 \circ \mu_r^{-1})'(k'' - 1)}. \quad (6.7)$$

On the other hand, for any $k_1, k_2 \in [\mu_r(\eta_{r,1}r), \mu_r(\eta_{r,2}r)]$ with $k_1 \leq k_2$,

$$(\sigma_r^2 \circ \mu_r^{-1})'(k_2) \leq \frac{(\sigma_r^2 \circ \mu_r^{-1})(k_2) - (\sigma_r^2 \circ \mu_r^{-1})(k_1)}{k_2 - k_1} \leq (\sigma_r^2 \circ \mu_r^{-1})'(k_1). \quad (6.8)$$

Moreover, for any $b_1, b_2 \in [\eta_{r,1}, \eta_{r,2}]$ with $b_2 > b_1$,

$$\mu_r(b_2r) - \mu_r(b_1r) \geq (b_2 - b_1)r \inf_{x \in [b_1r, b_2r]} \mu'_r(x) \geq \left(1 - \frac{b_1}{b_2}\right) \inf_{x \in [b_1r, b_2r]} x \mu'_r(x) = \left(1 - \frac{b_1}{b_2}\right) b_1 r \mu'_r(b_1r). \quad (6.9)$$

If $b_1, b_2 \in (\eta_{r,1} + \frac{\delta_r}{2}, \eta_{r,2} - \frac{\delta_r}{2})$ are chosen such that $1 - \frac{b_1}{b_2} \geq c_1 \delta_r$ for a certain $c_1 > 0$ independent of r , then the right-hand side of (6.9) converges to $+\infty$ as $r \rightarrow +\infty$ by part (2) of Assumptions 1.9. The definition of $\mathcal{K}_r$ implies that $k'' - 1 \in (\mu_r((\eta_{r,1} + \delta_r)r) - 1, \mu_r((\eta_{r,1} + \delta_r)r)]$, and therefore (6.9) with $b_1 = (\eta_{r,1} + \frac{3\delta_r}{4})r$ and $b_2 = (\eta_{r,1} + \delta_r)r$ implies $k'' - 1 \in (\mu_r((\eta_{r,1} + \frac{3\delta_r}{4})r), \mu_r((\eta_{r,1} + \delta_r)r)]$ for all sufficiently large r . Hence, applying the first inequality in (6.8) with $k_2 = k'' - 1$ and $k_1 = \mu_r((\eta_{r,1} + \frac{\delta_r}{2})r)$, and using parts (2) and (4) of Assumptions 1.9, we get

$$\begin{aligned} (\sigma_r^2 \circ \mu_r^{-1})'(k'' - 1) &\leq \frac{(\sigma_r^2 \circ \mu_r^{-1})(k'' - 1) - \sigma_r^2((\eta_{r,1} + \frac{\delta_r}{2})r)}{k'' - 1 - \mu_r((\eta_{r,1} + \frac{\delta_r}{2})r)} \leq \frac{\sigma_r^2((\eta_{r,1} + \delta_r)r)}{\mu_r((\eta_{r,1} + \frac{3\delta_r}{4})r) - \mu_r((\eta_{r,1} + \frac{\delta_r}{2})r)} \\ &\leq \frac{4\sigma_r^2((\eta_{r,1} + \delta_r)r)}{\delta_r \inf_{x \in (\eta_{r,1} + \frac{\delta_r}{2}, \eta_{r,1} + \frac{3\delta_r}{4})} r \mu'_r(rx)} \leq \frac{\eta_{r,1} + \frac{3\delta_r}{4}}{\eta_{r,1} + \frac{\delta_r}{2}} \frac{4\sigma_r^2((\eta_{r,1} + \delta_r)r)}{\delta_r r \mu'_r((\eta_{r,1} + \frac{\delta_r}{2})r)} \leq C'', \end{aligned}$$

for all sufficiently large r , where C'' is independent of r . Therefore, the right-hand side of (6.7) is bounded by a constant $C' = C'(M)$ for all r sufficiently large and all $\gamma \in [0, M]$, where $M > \sqrt{2/a}$ is arbitrary but fixed. Using also the fact that σ_r^2 and μ_r are increasing, we obtain

$$\begin{aligned} \mathbb{P} \left(\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} > \gamma \right) &\leq C C' e^\gamma \sum_{k \in \mathcal{K}_r} e^{-\frac{\gamma^2}{2} \sigma_r^2(\mu_r^{-1}(k))} \\ &\leq C C' e^\gamma \left(e^{-\frac{\gamma^2}{2} \sigma_r^2((\eta_{r,1} + \delta_r)r)} + \int_{\mu_r((\eta_{r,1} + \delta_r)r)}^{\mu_r((\eta_{r,2} - \delta_r)r) + 1} e^{-\frac{\gamma^2}{2} (\sigma_r^2 \circ \mu_r^{-1})(k)} dk \right). \end{aligned} \quad (6.10)$$

The proof for the other bound is similar (and simpler): for any $\gamma > 0$,

$$\begin{aligned} \mathbb{P} \left(\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \frac{\mu_r(x) - N_r(x)}{\sigma_r^2(x)} > \gamma \right) &\leq \sum_{k \in \mathcal{K}_r} \mathbb{P} \left(\frac{\mu_r(\kappa_{r,k-1}) - N_r(\kappa_{r,k-1}) + 1}{\sigma_r^2(\kappa_{r,k-1})} > \gamma \right) \\ &\leq \sum_{k: k+1 \in \mathcal{K}_r} \mathbb{E} \left[e^{-\gamma N_r(\kappa_{r,k})} \right] e^{\gamma \mu_r(\kappa_{r,k}) + \gamma - \gamma^2 \sigma_r^2(\kappa_{r,k})} \leq C e^\gamma \sum_{k: k+1 \in \mathcal{K}_r} e^{-\frac{\gamma^2}{2} \sigma_r^2(\mu_r^{-1}(k))} \\ &\leq C e^\gamma \left(2e^{-\frac{\gamma^2}{2} (\sigma_r^2(\mu_r^{-1}(\mu_r((\eta_{r,1} + \delta_r)r) - 1)))} + \int_{\mu_r((\eta_{r,1} + \delta_r)r)}^{\mu_r((\eta_{r,2} - \delta_r)r) + 1} e^{-\frac{\gamma^2}{2} (\sigma_r^2 \circ \mu_r^{-1})(k)} dk \right). \end{aligned} \quad (6.11)$$

The inequalities (6.10) and (6.11) imply that

$$\begin{aligned} \mathbb{P} \left(\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \left| \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} \right| > \gamma \right) \\ \leq C(C' + 2) e^\gamma \left(e^{-\frac{\gamma^2}{2} (\sigma_r^2 \circ \mu_r^{-1})(\mu_r((\eta_{r,1} + \delta_r)r) - 1)} + \int_{\mu_r((\eta_{r,1} + \delta_r)r)}^{\mu_r((\eta_{r,2} - \delta_r)r) + 1} e^{-\frac{\gamma^2}{2} (\sigma_r^2 \circ \mu_r^{-1})(k)} dk \right). \end{aligned}$$

The above inequality is valid for any $\gamma > 0$ but is useful only for $\gamma > \sqrt{2/\mathfrak{a}}$. Indeed, using part 3 of Assumptions 1.9, we find

$$\mathbb{P} \left(\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \left| \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} \right| > \gamma \right) \leq 3C(C' + 2)e^M \frac{\mu_r((\eta_{r,1} + \delta_r)r)^{1 - \frac{\mathfrak{a}\gamma^2}{2}}}{\frac{\mathfrak{a}\gamma^2}{2} - 1}$$

for all sufficiently large r and for all $\gamma \in (\sqrt{2/\mathfrak{a}}, M]$. We obtain the claim with $c = 6C(C' + 2)e^M$ after choosing $\gamma = \sqrt{\frac{2}{\mathfrak{a}}(1 + \epsilon)}$. $\square$

Lemma 6.2. *Let $\epsilon \in (0, 1)$ and $\lambda > 1$. For all sufficiently large r , if the event*

$$\sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \left| \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} \right| \leq \sqrt{\frac{2}{\mathfrak{a}}}(1 + \epsilon) \quad (6.12)$$

holds true, then

$$\max_{k \in (\mu_r((\eta_{r,1} + 2\delta_r)r), \mu_r((\eta_{r,2} - 2\delta_r)r)) \cap \mathbb{N}_{\geq 0}} \frac{|\mu_r(\xi_{r,k}) - k|}{(\sigma_r^2 \circ \mu_r^{-1})(k)} \leq \sqrt{\frac{2}{\mathfrak{a}}}(1 + \lambda\epsilon). \quad (6.13)$$

Proof. First, we observe that if $\xi_{r,k} \leq (\eta_{r,1} + \delta_r)r < (\eta_{r,1} + 2\delta_r)r \leq \kappa_{r,k}$, then

$$\mu_r((\eta_{r,1} + 2\delta_r)r) \leq \mu_r(\kappa_{r,k}) = k = N_r(\xi_{r,k}) \leq N_r((\eta_{r,1} + \delta_r)r),$$

and hence, using Assumptions 1.9,

$$\begin{aligned} \frac{N_r((\eta_{r,1} + \delta_r)r) - \mu_r((\eta_{r,1} + \delta_r)r)}{\sigma_r^2((\eta_{r,1} + \delta_r)r)} &\geq \frac{\mu_r((\eta_{r,1} + 2\delta_r)r) - \mu_r((\eta_{r,1} + \delta_r)r)}{\sigma_r^2((\eta_{r,1} + \delta_r)r)} \\ &\geq \frac{\delta_r \inf_{x \in (\eta_{r,1} + \delta_r)r, \eta_{r,1} + 2\delta_r)r} r\mu'_r(xr)}{\sigma_r^2((\eta_{r,1} + \delta_r)r)} \geq \frac{\eta_{r,1} + \delta_r}{\eta_{r,1} + 2\delta_r} \frac{\delta_r r\mu'_r((\eta_{r,1} + \delta_r)r)}{\sigma_r^2((\eta_{r,1} + \delta_r)r)}, \end{aligned}$$

and since the right-hand side tends to $+\infty$ as $r \rightarrow +\infty$, this contradicts (6.12) if r is large enough. Similarly, if $\xi_{r,k} \geq (\eta_{r,2} - \delta_r)r > (\eta_{r,2} - 2\delta_r)r \geq \kappa_{r,k}$, then

$$\mu_r((\eta_{r,2} - 2\delta_r)r) \geq \mu_r(\kappa_{r,k}) = k = N_r(\xi_{r,k}) \geq N_r((\eta_{r,2} - \delta_r)r),$$

and we find

$$\begin{aligned} \frac{\mu_r((\eta_{r,2} - \delta_r)r) - N_r((\eta_{r,2} - \delta_r)r)}{\sigma_r^2((\eta_{r,2} - \delta_r)r)} &\geq \frac{\mu_r((\eta_{r,2} - \delta_r)r) - \mu_r((\eta_{r,2} - 2\delta_r)r)}{\sigma_r^2((\eta_{r,2} - \delta_r)r)} \\ &\geq \frac{\delta_r \inf_{\xi \in (\eta_{r,2} - 2\delta_r)r, \eta_{r,2} - \delta_r)r} r\mu'_r(r\xi)}{\sigma_r^2((\eta_{r,2} - \delta_r)r)} \geq \frac{\eta_{r,2} - 2\delta_r}{\eta_{r,2} - \delta_r} \frac{\delta_r r\mu'_r((\eta_{r,2} - 2\delta_r)r)}{\sigma_r^2((\eta_{r,2} - \delta_r)r)}. \end{aligned}$$

The right-hand side tends to $+\infty$ as $r \rightarrow +\infty$ by Assumptions 1.9, so this also contradicts (6.12) if r is large enough. Therefore, we have shown that $\xi_{r,k} \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)$ for all $k \in (\mu_r((\eta_{r,1} + 2\delta_r)r), \mu_r((\eta_{r,2} - 2\delta_r)r))$, provided that r is large enough. The rest of the proof follows as in [16, proof of Lemma 2.2, eq (2.9) and below]. In [16], the parameter λ is equal to 2. We consider here the sharper situation $\lambda > 1$, but this improvement can be obtained without modifying the idea of the proof of [16], so we omit the details here. $\square$

Proof of Theorem 1.10. Let $\lambda > 1$. By Lemma 6.1, there exist $c > 0$ and $r_0 > 0$ such that

$$\mathbb{P}(A) \geq 1 - \frac{c\mu_r((\eta_{r,1} + \delta_r)r)^{-\frac{\lambda}{\mathfrak{a}}}}{\epsilon}, \quad \text{for any } r > r_0 \text{ and any small } \epsilon > 0,$$

and Lemma 6.2 implies that $A \subset B$, where A and B are the events

$$\begin{aligned} A &= \left\{ \sup_{x \in ((\eta_{r,1} + \delta_r)r, (\eta_{r,2} - \delta_r)r)} \left| \frac{N_r(x) - \mu_r(x)}{\sigma_r^2(x)} \right| \leq \sqrt{\frac{2}{\mathfrak{a}}}\left(1 + \frac{\epsilon}{\lambda}\right) \right\}, \\ B &= \left\{ \max_{k \in (\mu_r((\eta_{r,1} + 2\delta_r)r), \mu_r((\eta_{r,2} - 2\delta_r)r)) \cap \mathbb{N}_{\geq 0}} \frac{|\mu_r(\xi_{r,k}) - k|}{(\sigma_r^2 \circ \mu_r^{-1})(k)} \leq \sqrt{\frac{2}{\mathfrak{a}}}(1 + \epsilon) \right\}. \end{aligned}$$

Hence $\mathbb{P}(B) \geq \mathbb{P}(A) \geq 1 - \frac{c\mu_r((\eta_{r,1} + \delta_r)r)^{-\frac{\lambda}{\mathfrak{a}}}}{\epsilon}$, which finishes the proof. $\square$

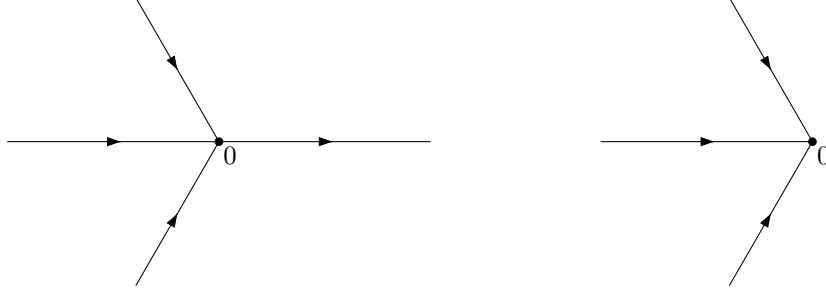


Figure 6: Left: the jump contour Σ_{Ai} for Φ_{Ai} . Right: the jump contour Σ_{Be} for Φ_{Be} .

A Model RH problems

A.1 Airy model RH problem

The following RH problem was introduced in [23], and its unique solution can be explicitly written in terms of Airy functions.

(a) $\Phi_{Ai} : \mathbb{C} \setminus \Sigma_{Ai} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where Σ_{Ai} is shown in Figure 6 (left).

(b) Φ_{Ai} has the jump relations

$$\begin{aligned} \Phi_{Ai,+}(z) &= \Phi_{Ai,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } \mathbb{R}^-, \\ \Phi_{Ai,+}(z) &= \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \text{on } \mathbb{R}^+, \\ \Phi_{Ai,+}(z) &= \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{on } e^{\frac{2\pi i}{3}} \mathbb{R}^+, \\ \Phi_{Ai,+}(z) &= \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{on } e^{-\frac{2\pi i}{3}} \mathbb{R}^+. \end{aligned} \quad (\text{A.1})$$

(c) As $z \rightarrow \infty$, $z \notin \Sigma_{Ai}$, we have

$$\Phi_{Ai}(z) = z^{-\frac{\sigma_3}{4}} N \left(I + \frac{\Phi_{Ai,1}}{z^{3/2}} + \mathcal{O}(z^{-3}) \right) e^{-\frac{2}{3}z^{3/2}\sigma_3}, \quad (\text{A.2})$$

where

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \Phi_{Ai,1} = \frac{1}{8} \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}.$$

As $z \rightarrow 0$, we have

$$\Phi_{Ai}(z) = \mathcal{O}(1). \quad (\text{A.3})$$

A.2 Bessel model RH problem

Given $\alpha > -1$, we define

$$\Phi_{Be}(z; \alpha) = \begin{cases} \begin{pmatrix} I_\alpha(2z^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(2z^{\frac{1}{2}}) \\ 2\pi i z^{\frac{1}{2}} I'_\alpha(2z^{\frac{1}{2}}) & -2z^{\frac{1}{2}} K'_\alpha(2z^{\frac{1}{2}}) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2} H_\alpha^{(1)}(2(-z)^{\frac{1}{2}}) & \frac{1}{2} H_\alpha^{(2)}(2(-z)^{\frac{1}{2}}) \\ \pi z^{\frac{1}{2}} (H_\alpha^{(1)})'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}} (H_\alpha^{(2)})'(2(-z)^{\frac{1}{2}}) \end{pmatrix} e^{\frac{\pi i \alpha}{2} \sigma_3}, & \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \frac{1}{2} H_\alpha^{(2)}(2(-z)^{\frac{1}{2}}) & -\frac{1}{2} H_\alpha^{(1)}(2(-z)^{\frac{1}{2}}) \\ -\pi z^{\frac{1}{2}} (H_\alpha^{(2)})'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}} (H_\alpha^{(1)})'(2(-z)^{\frac{1}{2}}) \end{pmatrix} e^{-\frac{\pi i \alpha}{2} \sigma_3}, & -\pi < \arg z < -\frac{2\pi}{3}, \end{cases} \quad (\text{A.4})$$

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are the Hankel functions of the first and second kind, and I_α and K_α are the modified Bessel functions of the first and second kind.

The matrix-valued function Φ_{Be} was first considered in [22] for $\alpha = 0$ and in [37] for general $\alpha > -1$. For fixed α , it is known that Φ_{Be} enjoys the following properties

(a) $\Phi_{\text{Be}} : \mathbb{C} \setminus \Sigma_{\text{Be}} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where Σ_{Be} is shown in Figure 6 (right).

(b) Φ_{Be} satisfies the jump conditions

$$\begin{aligned} \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathbb{R}^-, \\ \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & z \in e^{\frac{2\pi i}{3}} \mathbb{R}^+, \\ \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & z \in e^{-\frac{2\pi i}{3}} \mathbb{R}^+. \end{aligned} \quad (\text{A.5})$$

(c) As $z \rightarrow \infty$, $z \notin \Sigma_{\text{Be}}$, we have

$$\Phi_{\text{Be}}(z; \alpha) = (2\pi z^{\frac{1}{2}})^{-\frac{\sigma_3}{2}} N \left(I + \sum_{k=1}^{\infty} \tilde{\Phi}_{\text{Be},k}(\alpha) z^{-k/2} \right) e^{2z^{\frac{1}{2}} \sigma_3}, \quad (\text{A.6})$$

for certain matrices $\tilde{\Phi}_{\text{Be},k}(\alpha)$ independent of z .

(d) As z tends to 0,

$$\Phi_{\text{Be}}(z; \alpha) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} z^{\frac{\sigma_3}{2}}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(z^{-\frac{\sigma_3}{2}}) & \mathcal{O}(z^{-\frac{\sigma_3}{2}}) \\ \mathcal{O}(z^{-\frac{\sigma_3}{2}}) & \mathcal{O}(z^{-\frac{\sigma_3}{2}}) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases} \quad \text{if } \alpha > 0. \quad (\text{A.7})$$

Also, in [12, eqs (7.5) and (7.6)] it was shown that, for z in a neighborhood of 0,

$$\Phi_{\text{Be}}(z; \alpha) = \Phi_{\text{Be},0}(z; \alpha) z^{\frac{\sigma_3}{2}} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z), \quad (\text{A.8})$$

where H_0 is given by (2.9), h by (2.11), and $\Phi_{\text{Be},0}$ is analytic in a neighborhood of 0 and satisfies

$$\Phi_{\text{Be},0}(0; \alpha) = \begin{cases} \begin{pmatrix} \frac{1}{\Gamma(1+\alpha)} & \frac{i\Gamma(\alpha)}{2\pi} \\ \frac{i\pi}{\Gamma(\alpha)} & \frac{\Gamma(1+\alpha)}{2} \end{pmatrix}, & \text{if } \alpha \neq 0, \\ \begin{pmatrix} 1 & \frac{\gamma_E}{\pi i} \\ 0 & 1 \end{pmatrix}, & \text{if } \alpha = 0, \end{cases} \quad (\text{A.9})$$

where γ_E is Euler's gamma constant.

We emphasize that the asymptotics (A.6) are valid only as $z \rightarrow \infty$ while α remains in a compact subset of $(-1, +\infty)$. Our task in the rest of this section is to find uniform asymptotics for $\Phi_{\text{Be}}(z, \alpha)$ as $z \rightarrow \infty$ and simultaneously $\alpha \rightarrow +\infty$. The uniform asymptotics of $I_\alpha(\alpha z)$ and $K_\alpha(\alpha z)$ as $z \rightarrow \infty$, $\text{Re } z > 0$, and $\alpha \rightarrow +\infty$ are given in [41, Chapter 10, equations (7.18)–(7.19)]. As $z \rightarrow \infty$ in the right half-plane $\text{Re } z > 0$ and simultaneously $\alpha \rightarrow +\infty$, we have

$$I_\alpha(\alpha z) = \frac{e^{\alpha \xi(z)}}{(2\pi\alpha)^{1/2}(1+z^2)^{1/4}} \left(1 + \frac{3p(z) - 5p(z)^3}{24\alpha} + \mathcal{O}\left(\frac{1}{(\alpha z)^2}\right) \right), \quad (\text{A.10})$$

$$K_\alpha(\alpha z) = \left(\frac{\pi}{2\alpha}\right)^{1/2} \frac{e^{-\alpha \xi(z)}}{(1+z^2)^{1/4}} \left(1 - \frac{3p(z) - 5p(z)^3}{24\alpha} + \mathcal{O}\left(\frac{1}{(\alpha z)^2}\right) \right), \quad (\text{A.11})$$

where the error terms are uniform for $|\arg z| \leq \frac{\pi}{2} - \delta$ for each fixed $\delta > 0$ and

$$p(z) = \frac{1}{\sqrt{1+z^2}}, \quad \xi(z) = \sqrt{1+z^2} + \log \frac{z}{1+\sqrt{1+z^2}}. \quad (\text{A.12})$$

In (A.12), the principal branch is used for the square roots and the logarithm. To evaluate the asymptotics of $I'_\alpha(\alpha z)$ and $K'_\alpha(\alpha z)$, we use (A.10) and (A.11) together with

$$\begin{aligned} I'_\alpha(\alpha z) &= \frac{I_{\alpha+1}(\alpha z) + I_{\alpha-1}(\alpha z)}{2}, & K'_\alpha(\alpha z) &= -K_{\alpha-1}(\alpha z) - \frac{1}{z} K_\alpha(\alpha z), \\ e^{(\alpha+1)\xi(\frac{\alpha z}{\alpha+1})} &= e^{\alpha \xi(z)} \frac{z}{1+\sqrt{1+z^2}} \left(1 - \frac{p(z)}{2\alpha} + \mathcal{O}\left(\frac{1}{(\alpha z)^2}\right) \right), \\ e^{(\alpha-1)\xi(\frac{\alpha z}{\alpha-1})} &= e^{\alpha \xi(z)} \frac{1+\sqrt{1+z^2}}{z} \left(1 - \frac{p(z)}{2\alpha} + \mathcal{O}\left(\frac{1}{(\alpha z)^2}\right) \right). \end{aligned}$$

As $z \rightarrow \infty$, $\operatorname{Re} z > 0$, and simultaneously $\alpha \rightarrow +\infty$, we obtain

$$I'_\alpha(\alpha z) = \frac{e^{\alpha\xi(z)}(1+z^2)^{1/4}}{(2\pi\alpha)^{1/2}z} \left(1 + \frac{-9p(z) + 7p(z)^3}{24\alpha} + \mathcal{O}\left(\frac{1}{(\alpha z)^2}\right) \right), \quad (\text{A.13})$$

$$K'_\alpha(\alpha z) = \left(\frac{\pi}{2\alpha}\right)^{1/2} \frac{e^{-\alpha\xi(z)}(1+z^2)^{1/4}}{-z} \left(1 + \frac{9p(z) - 7p(z)^3}{24\alpha} + \mathcal{O}\left(\frac{1}{(\alpha z)^2}\right) \right). \quad (\text{A.14})$$

Setting $z \mapsto 2\sqrt{z}\alpha^{-1}$ in (A.10)–(A.14), we obtain, as $\alpha \rightarrow +\infty$ and $\sqrt{z}\alpha^{-1} \rightarrow \infty$ uniformly for $|\arg z| < \frac{2\pi}{3}$,

$$\Phi_{\text{Be}}(z; \alpha) = (\sqrt{\pi}(\alpha^2 + 4z)^{\frac{1}{4}})^{-\sigma_3} N \left(I + \Phi_{\text{Be},1}(z; \alpha) z^{-1/2} + \mathcal{O}(z^{-1}) \right) e^{\alpha\xi(\frac{2\sqrt{z}}{\alpha})\sigma_3}, \quad (\text{A.15})$$

where $\Phi_{\text{Be},1}(z; \alpha)$ is uniformly bounded and given by

$$\Phi_{\text{Be},1}(z; \alpha) = \sqrt{z} \begin{pmatrix} \frac{-3p(\frac{2\sqrt{z}}{\alpha}) + p(\frac{2\sqrt{z}}{\alpha})^3}{24\alpha} & \frac{i(-p(\frac{2\sqrt{z}}{\alpha}) + p(\frac{2\sqrt{z}}{\alpha})^3)}{4\alpha} \\ \frac{i(-p(\frac{2\sqrt{z}}{\alpha}) + p(\frac{2\sqrt{z}}{\alpha})^3)}{4\alpha} & \frac{3p(\frac{2\sqrt{z}}{\alpha}) - p(\frac{2\sqrt{z}}{\alpha})^3}{24\alpha} \end{pmatrix}. \quad (\text{A.16})$$

In fact, the asymptotic formula (A.15) is valid as $\alpha \rightarrow +\infty$ and $\sqrt{z}\alpha^{-1} \rightarrow \infty$ uniformly for all values of $\arg z$ (not just in the sector $|\arg z| < \frac{2\pi}{3}$). To see this, we employ the identities (see [42, eq (10.27.8)])

$$\begin{aligned} H_\alpha^{(1)}(z) &= \frac{2}{\pi} e^{-\frac{\pi i}{2}(\alpha+1)} K_\alpha(-iz), & -\frac{\pi}{2} < \arg(z) \leq \pi, \\ H_\alpha^{(2)}(z) &= \frac{2}{\pi} e^{\frac{\pi i}{2}(\alpha+1)} K_\alpha(iz), & -\pi < \arg(z) \leq \frac{\pi}{2}, \end{aligned}$$

to write

$$\Phi_{\text{Be}}(z; \alpha) = \begin{cases} \begin{pmatrix} -\frac{i}{\pi} K_\alpha(-i2(-z)^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(i2(-z)^{\frac{1}{2}}) \\ -2z^{\frac{1}{2}} K'_\alpha(-i2(-z)^{\frac{1}{2}}) & -2z^{\frac{1}{2}} K'_\alpha(i2(-z)^{\frac{1}{2}}) \end{pmatrix}, & \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \frac{i}{\pi} K_\alpha(i2(-z)^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(-i2(-z)^{\frac{1}{2}}) \\ 2z^{\frac{1}{2}} K'_\alpha(i2(-z)^{\frac{1}{2}}) & -2z^{\frac{1}{2}} K'_\alpha(-i2(-z)^{\frac{1}{2}}) \end{pmatrix}, & -\pi < \arg z < -\frac{2\pi}{3}. \end{cases} \quad (\text{A.17})$$

We now observe that the asymptotic formulas (A.11) and (A.14) for $K_\alpha(\alpha z)$ and $K'_\alpha(\alpha z)$ are in fact valid uniformly for all $|\arg z| \leq \pi$ as $z \rightarrow \infty$ and $\alpha \rightarrow +\infty$, provided that the functions $(1+z^2)^{1/4}$, $p(z)$ and $\xi(z)$ are analytically continued in the natural way as z crosses the imaginary axis, see [41, Chapter 10, Section 8.2]. (The formulas (A.10) and (A.13) cannot be analytically continued in the same way.) Substituting the asymptotics (A.11) and (A.14) into (A.17), we conclude after a long calculation that (A.15) holds as $\alpha \rightarrow +\infty$ and $\sqrt{z}\alpha^{-1} \rightarrow \infty$ uniformly also for $\frac{2\pi}{3} < \arg z < \pi$ and $-\pi < \arg z < -\frac{2\pi}{3}$.

A.3 Confluent hypergeometric model RH problem

The following RH problem depends on a parameter $\beta \in i\mathbb{R}$ and was introduced in [34]. Its unique solution can be explicitly written in terms of confluent hypergeometric function.

- (a) $\Phi_{\text{HG}} : \mathbb{C} \setminus \Sigma_{\text{HG}} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where Σ_{HG} is shown in Figure 7.
- (b) For $z \in \Gamma_k$ (see Figure 7), $k = 1, \dots, 6$, Φ_{HG} obeys the jump relations

$$\Phi_{\text{HG},+}(z) = \Phi_{\text{HG},-}(z) J_k, \quad (\text{A.18})$$

where

$$\begin{aligned} J_1 &= \begin{pmatrix} 0 & e^{-i\pi\beta} \\ -e^{i\pi\beta} & 0 \end{pmatrix}, & J_2 &= \begin{pmatrix} 1 & 0 \\ e^{i\pi\beta} & 1 \end{pmatrix}, & J_3 &= \begin{pmatrix} 1 & 0 \\ e^{-i\pi\beta} & 1 \end{pmatrix} \\ J_4 &= \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix}, & J_5 &= \begin{pmatrix} 1 & 0 \\ e^{-i\pi\beta} & 1 \end{pmatrix}, & J_6 &= \begin{pmatrix} 1 & 0 \\ e^{i\pi\beta} & 1 \end{pmatrix}. \end{aligned}$$

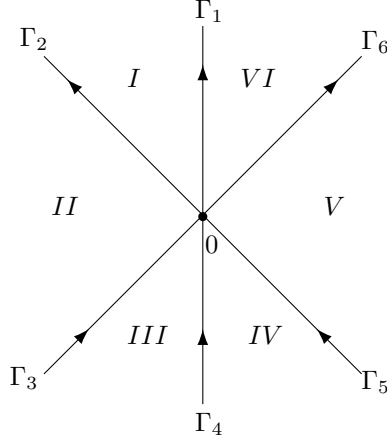


Figure 7: The jump contour Σ_{HG} . For each $k = 1, \dots, 6$, the angle formed by Γ_k and $(0, +\infty)$ is a multiple of $\frac{\pi}{4}$.

(c) As $z \rightarrow \infty$, $z \notin \Sigma_{\text{HG}}$, we have

$$\Phi_{\text{HG}}(z) = \left(I + \frac{\Phi_{\text{HG},1}(\beta)}{z} + \mathcal{O}(z^{-2}) \right) z^{-\beta\sigma_3} e^{-\frac{z}{2}\sigma_3} \begin{cases} e^{i\pi\beta\sigma_3}, & \frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & -\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \end{cases} \quad (\text{A.19})$$

where

$$\Phi_{\text{HG},1}(\beta) = \beta^2 \begin{pmatrix} -1 & \tau(\beta) \\ -\tau(-\beta) & 1 \end{pmatrix}, \quad \tau(\beta) = \frac{-\Gamma(-\beta)}{\Gamma(\beta+1)}. \quad (\text{A.20})$$

In (A.19), the root is defined by $z^\beta = |z|^\beta e^{i\beta \arg z}$ with $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.

As $z \rightarrow 0$, we have

$$\Phi_{\text{HG}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in II \cup V, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in I \cup III \cup IV \cup VI. \end{cases} \quad (\text{A.21})$$

We will need the following more detailed asymptotics: as $z \rightarrow 0$, $z \in II$, we have

$$\Phi_{\text{HG}}(z) = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} (I + \mathcal{O}(z)) \begin{pmatrix} 1 & \frac{\sin(\pi\beta)}{\pi} \log z \\ 0 & 1 \end{pmatrix}, \quad (\text{A.22})$$

where the argument of $\log z = \log |z| + i \arg z$ is such that $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$, and

$$\begin{aligned} \Psi_{11} &= \Gamma(1 - \beta), & \Psi_{12} &= \frac{1}{\Gamma(\beta)} \left(\frac{\Gamma'(1 - \beta)}{\Gamma(1 - \beta)} + 2\gamma_E - i\pi \right), \\ \Psi_{21} &= \Gamma(1 + \beta), & \Psi_{22} &= \frac{-1}{\Gamma(-\beta)} \left(\frac{\Gamma'(-\beta)}{\Gamma(-\beta)} + 2\gamma_E - i\pi \right), \end{aligned}$$

where γ_E is Euler's gamma constant.

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