On the fraction of the water age accounted for by partial ages[1]

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Contents

Abstract. One introduces the Green's function allowing evaluating steady-state water age and partial water ages aimed at assessing ventilation rates in the World Ocean. Elementary physical intuition and Green's function based developments suggest that the ratio of any partial age to the age should scale as the ratio of the volume of the related subdomain to that of the whole domain of interest. Analytical solutions derived in the framework of a water column model fail to fully support this idea. However, potentially interesting properties are established that pertain to the behaviour of the abovementioned time-independent ages in the vicinity of the water-air interface.

Introduction

The past few decades have seen the development of comprehensive theories allowing several diagnostic timescales, including the water age, to be evaluated at any time and position in a variety of domains of interest from the solution of partial differential problems (e.g., Delhez et al. 1999, Holzer and Hall 2000, Delhez et al. 2004, Haine et al. 2025).

The concept of partial age is a generalisation of that of age[2], in which, metaphorically, every water particle[3] is equipped with several clocks rather than only one (e.g., Deleersnijder 2014, Deleersnijder et al. 2014, Mouchet et al. 2016). This approach renders it possible to determine the time spent in any subdomain, i.e., the partial age, and, if necessary, provides

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[1] The present working note borrows ideas and results from Deleersnijder (2014, 2016a, 2016b).

[2] The timescale now referred to as partial age was pioneered by Liu et al. (2012). Moreover, Lin and Liu (2019) extended the partial diagnostic timescale philosophy to the concept of residence time.

[3] A water (or constituent) particle refers to a concept significantly different from that of “water parcel” (e.g., Deleersnijder 2020, Lucas and Deleersnijder 2020).

quantitative connectivity indicators (e.g., Mouchet et al. 2016). Needless to say, the classical age is the sum of all partial ages — provided all of the subdomains cover the whole domain of interest and have no intersection.

In the World Ocean, water age is generally meant to diagnose ventilation, i.e., the "gradual renewal [...] of the deep ocean by water that was once at the sea surface" (England 1995). Accordingly, the age is defined as the time elapsed since leaving the ocean surface (e.g., Thiele and Sarmiento 1990, England 1995).

Hereinafter, we examine whether or not, at a steady state, the fraction of the water age accounted for by the partial age related to a given subdomain approximately scales as the ratio of the volume of this subdomain to the volume of the domain of interest. As is seen below, this property, which seems to be in agreement with elementary physical intuition, cannot be regarded as sufficiently valid. Nonetheless, potentially useful results are established in this working note.

Green's function

Let Ω represent the World Ocean, which is delimited by boundary Γ. The latter is made up of the ocean-atmosphere interface, Γ^s, and impermeable boundaries, Γ^i, i.e. the ocean bottom and "lateral" boundaries. The aforementioned surfaces obviously satisfy the following properties: $\Gamma = \Gamma^s \cup \Gamma^i$ and $\varnothing = \Gamma^s \cap \Gamma^i$ (Figure 1).

The time-independent velocity vector and diffusivity tensor are denoted $\mathbf{v}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$, respectively. The latter is positive-definite (e.g., Deleersnijder 2012), whilst the velocity is divergence-free ($\nabla \bullet \mathbf{v} = 0$, Boussinesq approximation) and satisfies impermeability boundary condition

$$\left[\mathbf{v} \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma} = 0 \tag{1}$$

where $\mathbf{n}$ is the outward unit normal vector to the domain boundary.

The steady-state diagnostic timescales under consideration may be derived from Green's function $G(\mathbf{x} \mid \mathbf{x}_0)$. The latter is the solution of the following partial differential problem

$$\nabla \bullet (\mathbf{K} \bullet \nabla G - \mathbf{v} G) = -\delta(\mathbf{x} - \mathbf{x}_0) \tag{2}$$

$$\left[G(\mathbf{x} \mid \mathbf{x}_0)\right]_{\mathbf{x}\in\Gamma^s} = 0 \tag{3}$$

$$\left[(\mathbf{K} \bullet \nabla G - \mathbf{v} G) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \tag{4a}$$

where δ is the "three-dimensional" Dirac impulse function, whose physical dimension is $(\text{length})^{-3}$. This implies that the physical dimension of the Green's function is $(\text{length})^{-3} \times \text{time}$. Combining (1) and (2) yields

$$\left[(\mathbf{K} \bullet \nabla G) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \tag{4b}$$

Consider an elemental control domain located at $\mathbf{x}_0$. It volume is δV. At point $\mathbf{x}$, the mean age of the water particles that passed through this control domain tends to $G(\mathbf{x} \mid \mathbf{x}_0)\delta V$ in the limit $\delta V \to 0$.

The negative part of the Green's function is

$$G^-(\mathbf{x}\,|\,\mathbf{x}_0) = \frac{G(\mathbf{x}\,|\,\mathbf{x}_0)-\left|G(\mathbf{x}\,|\,\mathbf{x}_0)\right|}{2} = \begin{cases} 0 \text{ if } G(\mathbf{x}\,|\,\mathbf{x}_0)\geq 0 \\ G(\mathbf{x}\,|\,\mathbf{x}_0) \text{ if } G(\mathbf{x}\,|\,\mathbf{x}_0)<0 \end{cases} \tag{5}$$

Accordingly, $G^-(\mathbf{x}\,|\,\mathbf{x}_0)$ is less than or equal to zero. Multiplying (2) by $G^-(\mathbf{x}\,|\,\mathbf{x}_0)$ and integrating this equation over the domain of interest, one obtains after lengthy manipulations (e.g., Lewandowski 1997)

$$\begin{gathered} \overbrace{\int_{\Gamma^s} (G^-\mathbf{K}\bullet\nabla G)\bullet\mathbf{n}\,ds}^{=0,\text{ see (3)}} + \overbrace{\int_{\Gamma^i} (G^-\mathbf{K}\bullet\nabla G)\bullet\mathbf{n}\,ds}^{=0,\text{ see (4b)}} - \overbrace{\int_{\Omega} \nabla G^-\mathbf{K}\bullet\nabla G^-\,d\mathbf{x}}^{\substack{\geq 0,\text{ since } \mathbf{K} \text{ is} \\ \text{positive definite}}} \\ -\frac{1}{2}\underbrace{\int_{\Gamma}\left(G^-\right)^2\mathbf{v}\bullet\mathbf{n}\,ds}_{=0,\text{ see (1)}} = \underbrace{\int_{\Omega}\left(-G^-(\mathbf{x}\,|\,\mathbf{x}_0)\delta(\mathbf{x}-\mathbf{x}_0)\right)d\mathbf{x}}_{=-G^-(\mathbf{x}_0|\mathbf{x}_0)\geq 0,\text{ since } G^-\leq 0} \end{gathered} \tag{6}$$

The left-hand side of (6) is non-positive, whilst the right-hand side is non-negative. As a consequence, $G^-(\mathbf{x}\,|\,\mathbf{x}_0)$ must zero. Therefore, unsurprisingly, the Green's function is non-negative (Deleersnijder 2016a), i.e.,

$$G(\mathbf{x}\,|\,\mathbf{x}_0) \geq 0 \tag{7}$$

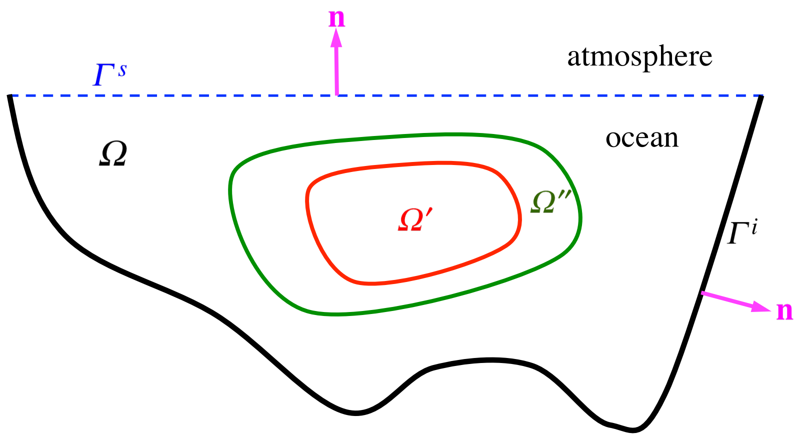

Figure 1. Schematic depiction of domain of interest Ω (i.e., the World Ocean), and its boundaries, the ocean-atmosphere interface (Γ^s) and the remainder of the boundary (Γ^i), which is impermeable. Subdomains Ω' and Ω'', with $\Omega'\subset\Omega''\subset\Omega$, are used in relation with partial age developments. Vector **n** is the outward unit vector ($|\mathbf{n}|=1$) to the domain boundary, $\Gamma=\Gamma^s\cup\Gamma^i$.

Age and partial ages

Steady-state water age $a(\mathbf{x})$ may be evaluated from the Green's function as follows:

$$a(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}\,|\,\mathbf{x}_0)\,d\mathbf{x}_0 \tag{8}$$

By integrating (2)-(4) with respect to $\mathbf{x}_0$ over Ω and using (8), one obtains the partial differential problem obeyed by the age, i.e.,

$$\nabla \bullet (\mathbf{K} \bullet \nabla a - \mathbf{v} a) = -1 \tag{9}$$

$$\left[a(\mathbf{x})\right]_{\mathbf{x} \in \Gamma^s} = 0 \tag{10}$$

$$\left[(\mathbf{K} \bullet \nabla a) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \tag{11}$$

These equations are well known (e.g., Deleersnijder et al. 2001).

The partial age related to subdomain Ω', with $\Omega' \subset \Omega$, is

$$a'(\mathbf{x}) = \int_{\Omega'} G(\mathbf{x} \,|\, \mathbf{x}_0) \, d\mathbf{x}_0 \tag{12}$$

This diagnostic timescale is the time spent in subdomain Ω' after touching the ocean surface for the last time. By integrating (2)-(4) with respect to $\mathbf{x}_0$ over Ω' and using (12), one obtains the partial differential problem obeyed by the aforementioned partial age, i.e.,

$$\nabla \bullet (\mathbf{K} \bullet \nabla a' - \mathbf{v} a') = \begin{cases} -1 \text{ if } \mathbf{x} \in \Omega' \\ 0 \text{ if } \mathbf{x} \notin \Omega' \end{cases} \tag{13}$$

$$\left[a'(\mathbf{x})\right]_{\mathbf{x} \in \Gamma^s} = 0 \tag{14}$$

$$\left[(\mathbf{K} \bullet \nabla a') \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^i} = 0 \tag{15}$$

The time-dependent version of these equations was established by Mouchet et al. (2016).

Diagnostic timescales $a(\mathbf{x})$ and $a'(\mathbf{x})$ satisfy

$$a(\mathbf{x}) = a'(\mathbf{x}) + \underbrace{\int_{\Omega \backslash \Omega'} G(\mathbf{x} \,|\, \mathbf{x}_0) \, d\mathbf{x}_0}_{\substack{\geq 0, \text{ since } G(\mathbf{x}|\mathbf{x}_0) \geq 0 \\ \text{see (7)}}} \geq a'(\mathbf{x}) \geq 0 \tag{16}$$

Obviously, if $\Omega' = \Omega$, then the age and the partial age are equal to eachother, i.e., $a(\mathbf{x}) = a'(\mathbf{x})$. In the same line, consider subdomain Ω'' that is such that $\Omega' \subset \Omega'' \subset \Omega$ (Figure 1). Then, it is readily seen that the partial age related to Ω'', $a''(\mathbf{x})$, satisfies (e.g., Deleersnijder 2016b)

$$a(\mathbf{x}) \geq a''(\mathbf{x}) \geq a'(\mathbf{x}) \geq 0 \tag{17}$$

Scaling of the partial age to the age ratio

Expressions (8) and (12) lead to

$$\frac{a'(\mathbf{x})}{a(\mathbf{x})} = \frac{\int_{\Omega'} G(\mathbf{x} \,|\, \mathbf{x}_0) \, d\mathbf{x}_0}{\int_{\Omega} G(\mathbf{x} \,|\, \mathbf{x}_0) \, d\mathbf{x}_0} \leq \frac{a''(\mathbf{x})}{a(\mathbf{x})} = \frac{\int_{\Omega''} G(\mathbf{x} \,|\, \mathbf{x}_0) \, d\mathbf{x}_0}{\int_{\Omega} G(\mathbf{x} \,|\, \mathbf{x}_0) \, d\mathbf{x}_0} \tag{18}$$

This suggests the following scaling

$$\text{order of magnitude of ratio } \frac{a'(\mathbf{x})}{a(\mathbf{x})} \approx \frac{V'}{V} \tag{19a}$$

or, equivalently,

$$\text{order of magnitude of ratio } \frac{a''(\mathbf{x})}{a(\mathbf{x})} \approx \frac{V''}{V} \tag{19b}$$

with

$$V = \int_{\Omega} d\mathbf{x} \ , \quad V' = \int_{\Omega'} d\mathbf{x} \ , \quad V'' = \int_{\Omega''} d\mathbf{x} \tag{20}$$

Clearly, V, V' and V'' denote the volume of domain of interest Ω, subdomain Ω' and subdomain Ω'', respectively.

If advection is negligible (purely diffusive problem), then the Green's function satisfies Maxwell's reciprocity property (e.g., Deleersnijder 2016a),

$$G(\mathbf{x} \mid \mathbf{x}_0) = G(\mathbf{x}_0 \mid \mathbf{x}) \tag{21}$$

leading to

$$\int_{\Omega'} a(\mathbf{x})\, d\mathbf{x} = \int_{\Omega'} \left(\int_{\Omega} G(\mathbf{x} \mid \mathbf{x}_0)\, d\mathbf{x}_0 \right) d\mathbf{x} = \int_{\Omega} \left(\int_{\Omega'} G(\mathbf{x}_0 \mid \mathbf{x})\, d\mathbf{x} \right) d\mathbf{x}_0 = \int_{\Omega} a'(\mathbf{x}_0)\, d\mathbf{x}_0 \tag{22}$$

This relation does not reinforce the relevance of scaling (19), but does not undermine it either.

Scaling (19) seems to be in line with elementary physical intuition. The clock measuring the age is functioning at any location in the domain of interest, whereas the partial age clock ("attached" to the same water particle) only ticks in the related subdomain. Both clocks are reset to zero whenever the water particle bearing them hits the ocean surface. Assuming that any water particle roughly spends the same time in any similar volume control domain, then (19) is readily obtained.

Below, analytical solutions of a highly idealised problem allow assessing the abovementioned considerations.

Diffusive water column model

Consider a one-dimensional, water column model (Figure 2). The relevant vertical coordinate is denoted z, with $z = -h$ at the lower boundary and $z = 0$ at the upper one. Clearly, the former boundary represents in an idealised manner the ocean bottom (and, to a certain degree, all other impermeable boundaries), whilst the latter is the water-air interface, i.e., the ocean surface, where all ages are prescribed to be zero. Clearly, this model exhibits a simplified version of Γ^s and Γ^i. It takes into account diffusive processes, but ignore the advective ones. In light of the abovementioned considerations, this water column is generally considered as a highly idealised version of the World Ocean — as well as some other marine domains.

Green's function $G(z \mid z_0)$ satisfies

$$\frac{\partial}{\partial z}\left(K \frac{\partial G}{\partial z} \right) = -\,\delta(z - z_0) \tag{23}$$

$$\left[G(z \mid z_0) \right]_{z=0} = 0 \tag{24}$$

$$\left[K\frac{\partial G}{\partial z}\right]_{z=-h} = 0 \tag{25}$$

where $K(z) > 0$ is the eddy diffusivity. In this model, the physical dimension of $G(z\,|\,z_0)$ is $(\text{length})^{-1} \times \text{time}$. The solution to (23)-(25) is (e.g., Deleersnijder 2016a)

$$G(z\,|\,z_0) = \left(\int_{z_0}^{0}\frac{d\eta}{K(\eta)}\right)\Upsilon(z_0 - z) + \left(\int_{z}^{0}\frac{d\eta}{K(\eta)}\right)\Upsilon(z - z_0) \tag{26}$$

where Υ denotes the Heaviside function, i.e., $\Upsilon(\varsigma) = [\operatorname{sign}(\varsigma) + |\operatorname{sign}(\varsigma)|]/2$. Then, in accordance with (8), the water age reads

$$a(z) = G(z\,|\,z_0) = \int_{z}^{0}\left(\int_{z_0}^{0}\frac{d\eta}{K(\eta)}\right)dz_0 + \int_{-h}^{z}\left(\int_{z}^{0}\frac{d\eta}{K(\eta)}\right)dz_0 \tag{27}$$

which simplifies to (Appendix)

$$a(z) = \int_{z}^{0}\frac{\eta + h}{K(\eta)}\,d\eta \tag{28}$$

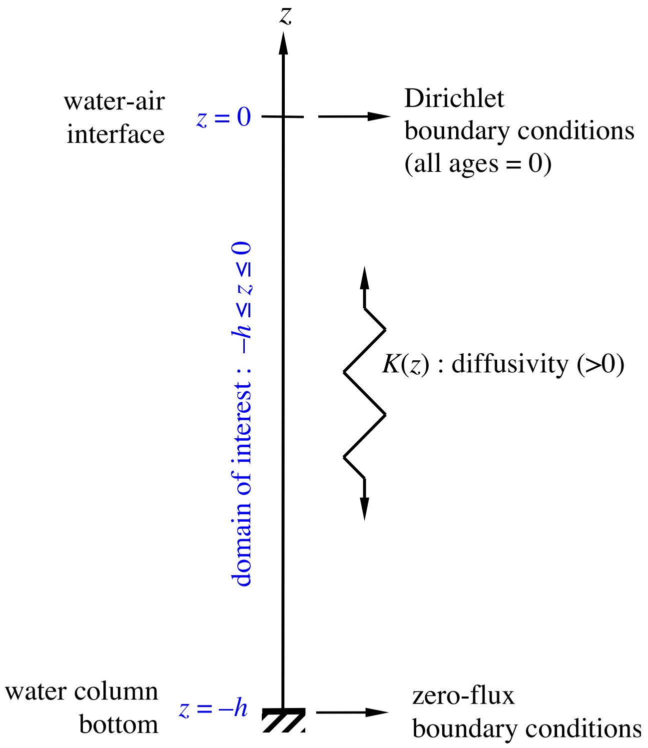

Figure 2. Illustration of the geometry of and key pieces of information about the water column model in which age and partial age developments are carried out herein.

Integrating (23)-(25) with respect to z_0 over the domain of interest $(-h \le z_0 \le 0)$ yields the differential problem obeyed by water age $a(z)$:

$$\frac{d}{dz}\left(K\frac{da}{dz}\right) = -1 \tag{29}$$

$$\left[a(z)\right]_{z=0} = 0 \tag{30}$$

$$\left[K\frac{da}{dz}\right]_{z=-h} = 0 \tag{31}$$

which is a simplified version of the water age equation and boundary conditions that may be found in England (1995), Delhez et al. (1999) or Deleersnijder et al. (2001). The solution to (29)-(31) is (28), indicating that, as expected, the water age may be obtained by means of the relevant Green's function or by directly solving the differential problem governing the water age.

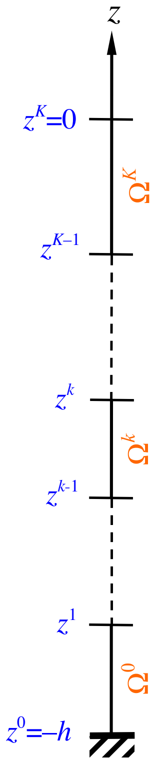

Figure 3. Illustration of the geometry of the K subdomains $(\Omega^0,\ldots\Omega^k,\ldots\Omega^K)$ into which the water column is divided, aiming at evaluating partial ages.

To evaluate partial ages, the water column is divided into K subdomains. They are defined as follows:

$$\Omega^k \ : \ z^{k-1} < z < z^k \ , \quad k = 1,2\ldots\ K \tag{32}$$

with $z^0 = -h$ and $z^K = 0$ (Figure 3). The characteristic function of every subdomain is

$$\omega^k(z) = \begin{cases} 1 & \text{if } z \in \Omega^k \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

with

$$\sum_{k=1}^{K} \omega^k(z) = 1 \tag{34}$$

Then, the partial age associated with the k-th subdomain reads

$$a^k(z) = \int_{z^{k-1}}^{z^k} G(z \mid z_0)\, dz_0 = \int_{-h}^{0} G(z \mid z_0)\, \omega^k(z_0)\, dz_0 \tag{35a}$$

i.e.,

$$\boxed{a^k(z) = \int_{z}^{0} \frac{1}{K(\eta)} \left(\int_{-h}^{\eta} \omega^k(\zeta)\, d\zeta \right) d\eta} \tag{35b}$$

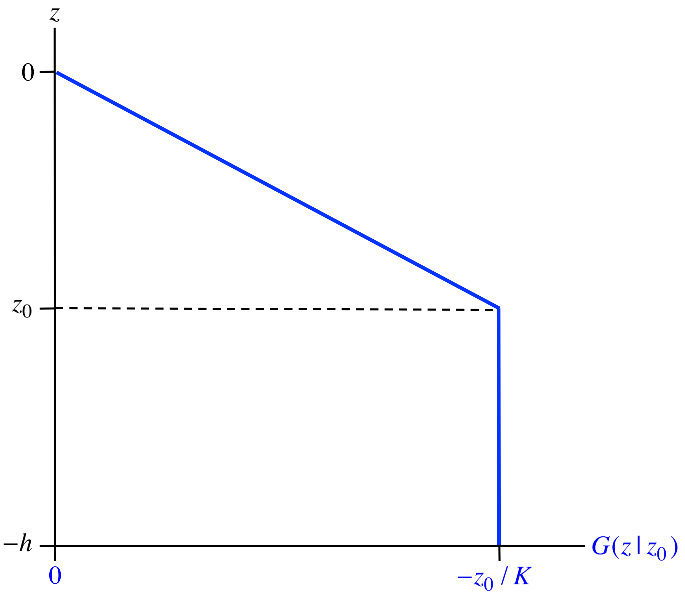

Figure 4. Schematic of Green's function $G(z \mid z_0)$ in accordance with (39). This formulation is derived from (26) following the hypothesis that the eddy diffusivity (K) is constant. Under this simplifying assumption, $G(z \mid z_0)$ is a continuous piecewise linear function of vertical coordinate z.

The differential problem governing the partial ages may be obtained by integrating (23) with respect to z_0 and taking into account boundary conditions (24)-(25). This yields

$$\frac{d}{dz}\left(K \frac{da^k}{dz} \right) = -\omega^k \tag{36}$$

$$a^k(0) = 0 \tag{37}$$

$$\left[K\frac{da^k}{dz}\right]_{z=-h} = 0 \tag{38}$$

Unsurprisingly, the solution of (36)-(38) is (35b).

Assuming that the eddy diffusivity is constant, Green's function (26) simplifies to (Figure 4)

$$G(z\,|\,z_0) = -\frac{z_0}{K}\Upsilon(z_0 - z) - \frac{z}{K}\Upsilon(z - z_0) \tag{39}$$

Then, the water age may be easily evaluated from (28), yielding

$$a(z) = -\frac{z(z+2h)}{2K} \tag{40}$$

In order to examine the behaviour of partial ages, a simple approach is resorted to. The water column is divided into three subdomains of equal heights $(h/3)$, i.e.,

$$\Omega^1: -h \le z < -2h/3 \tag{41a}$$

$$\Omega^2: -2h/3 \le z < -h/3 \tag{41b}$$

$$\Omega^3: -h/3 \le z < 0 \tag{41c}$$

The corresponding partial ages are listed in Table 1 and depicted in Figure 5.

Table 1. Water age and partial ages in the subdomains of equal heights defined by (41). The formulas of this table are based on dimensionless vertical coordinate $\xi = z/h$. Normalised (i.e., dimensionless) ages are listed: all ages are divided by diffusive timescale h^2/K.

subdomain	$\frac{a}{h^2/K}$	$\frac{a^1}{h^2/K}$	$\frac{a^2}{h^2/K}$	$\frac{a^3}{h^2/K}$
$\Omega^3: -\frac{1}{3} < \xi < 0$	$-\xi - \frac{\xi^2}{2}$	$-\frac{\xi}{3}$	$-\frac{\xi}{3}$	$-\frac{\xi}{3} - \frac{\xi^2}{2}$
$\Omega^2: -\frac{2}{3} < \xi < -\frac{1}{3}$	$-\xi - \frac{\xi^2}{2}$	$-\frac{\xi}{3}$	$-\frac{1}{18} - \frac{2\xi}{3} - \frac{\xi^2}{2}$	$\frac{1}{18}$
$\Omega^1: -1 < \xi < -\frac{2}{3}$	$-\xi - \frac{\xi^2}{2}$	$-\frac{2}{9} - \xi - \frac{\xi^2}{2}$	$\frac{1}{6}$	$\frac{1}{18}$

These ages obey the basic property of partial ages (e.g., Mouchet et al. 2016)

$$a(z) = a^1(z) + a^2(z) + a^3(z) \tag{42}$$

and Maxwell's reciprocity property (22), i.e.,

$$\int_{z^{k-1}}^{z^k} a(z)\,dz = \int_{-h}^{0} a^k(z)\,dz \qquad (k = 1, 2, 3) \tag{43}$$

or, specifically,

$$\int_{-h}^{-2h/3} a(z)\,dz = \int_{-h}^{0} a^{1}(z)\,dz = \frac{13}{81}\frac{h^{3}}{K} \tag{44a}$$

$$\int_{-2h/3}^{-h/3} a(z)\,dz = \int_{-h}^{0} a^{2}(z)\,dz = \frac{10}{81}\frac{h^{3}}{K} \tag{44b}$$

$$\int_{-h/3}^{0} a(z)\,dz = \int_{-h}^{0} a^{3}(z)\,dz = \frac{4}{81}\frac{h^{3}}{K} \tag{44c}$$

In addition, the ages satisfy

$$\lim_{z\to 0} \frac{a^{k}(z)}{a(z)} = \frac{h/3}{h} = \frac{1}{3} \tag{45}$$

and

$$a^{k}(z) \sim \frac{a(z)}{3} \sim -\frac{hz}{3K} \ , \quad -\frac{z}{h} << 1 \tag{46}$$

which is in agreement with scaling (19a). Unfortunately, well below the water column surface, the ratios $a^{k}(z)/a(z)$ exhibit values significantly different from 1/3. This is less so for $a^{2}(z)/a(z)$ for reasons that have yet to be unravelled (Figure 6).

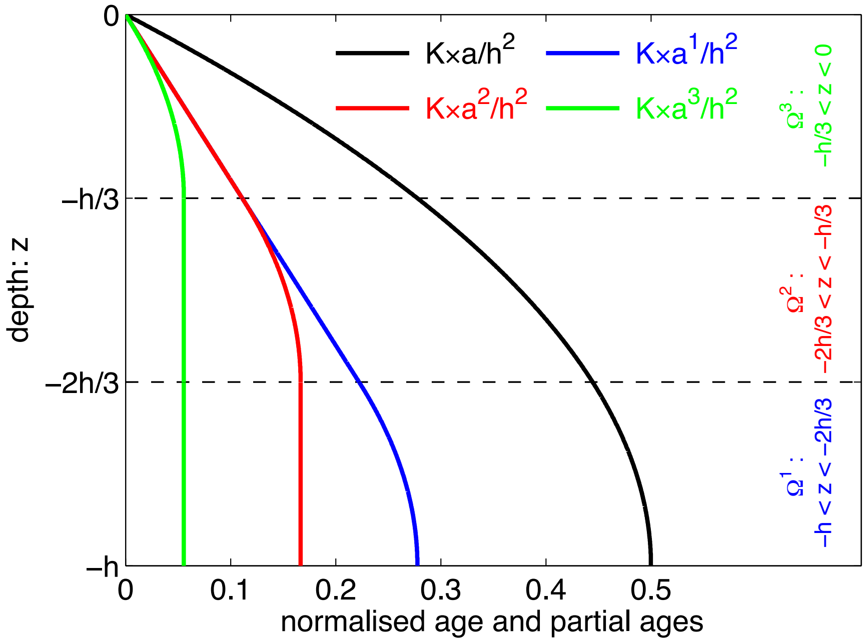

Figure 5. Steady-state age and partial ages as a function of the vertical coordinate. All ages are divided by diffusive timescale h^{2}/K, yielding dimensionless diagnostic timescales. Partial age a^{k} is a quadratic function of vertical coordinate z in the k-th subdomain (Ω^{k}), and a linear function z or a constant outside Ω^{k}.

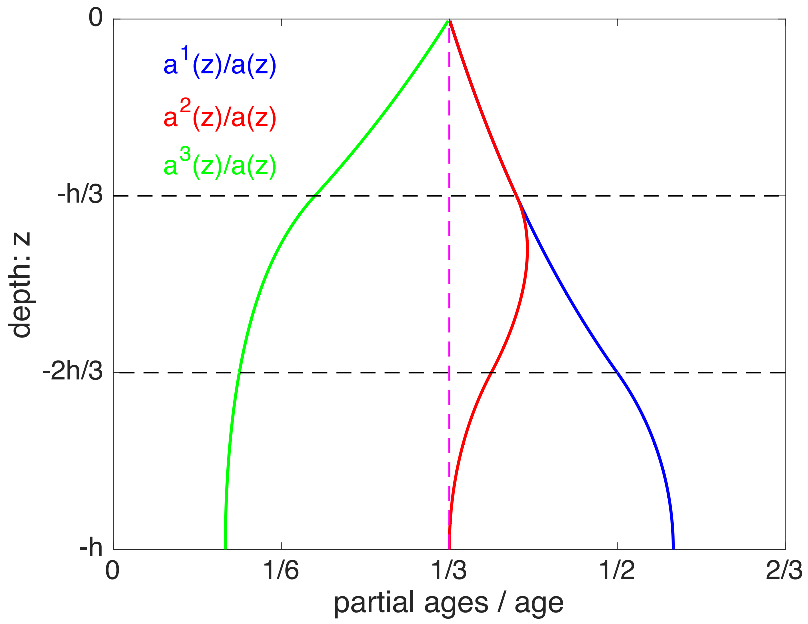

Figure 6. Ratios of the steady-state partial ages to the age, $a^k(z)/a(z)$, as a function of the vertical coordinate. Clearly, scaling (19) is not met, except in the neighbourhood of the water-air interface $(z=0)$.

Surface age and partial age "fluxes"

Integrating (29) over the whole water column and using bottom boundary condition (31) yields

$$\left[K\frac{da}{dz}\right]_{z=0} = -h \tag{47}$$

Then, taking into account Dirichlet boundary condition (30), (47) leads to asymptotic formula

$$a(z) \sim -\frac{hz}{K(0)} \quad , \quad -\frac{z}{h} << 1 \tag{48}$$

Applying similar mathematical operation to (36)-(37), one obtains

$$\left[K\frac{da^k}{dz}\right]_{z=0} = -(z^k - z^{k-1}) \tag{49}$$

and

$$a^k(z) \sim -\frac{(z^k - z^{k-1})z}{K(0)} \quad , \quad -\frac{z}{h} << 1 \tag{50}$$

so that

$$\lim_{z\to 0}\frac{a^k(z)}{a(z)} = \frac{z^k - z^{k-1}}{h} \tag{51}$$

Obviously, (48), (50) and (51) are a generalisation of results derived in the previous Section under the assumption that the eddy diffusivity is constant. In other words, if the eddy diffusivity is a function of the vertical coordinate, scaling (19a) is obeyed in the vicinity of the water-air interface and is likely to be much less relevant in the remainder of the water column.

One must now ask whether these properties remain valid in approaches more sophisticated than a steady-state water column model. One could deal with the leaky funnel model (Mouchet et al., 2008, 2012), a one-dimensional surrogate of the World Ocean that is believed to be more relevant to this domain than a water column model. However, Deleersnijder (2014) hints at little chance of success. Nonetheless, a somewhat interesting property may be obtained from (9)-(11) and (13)-(15).

Integrating (9) over domain of interest Ω, taking into account boundary condition (11) and using the divergence theorem yields

$$\int_{\Gamma^s} (-\mathbf{K} \bullet \nabla a) \bullet \mathbf{n} \, d\Gamma^s = V \tag{52}$$

where V is the value of the volume of Ω, whilst the left-hand side of this relation may be viewed as the “age flux” leaving the ocean through its surface — which is positive definite since the age is prescribed to be zero on Γ^s and is positive inside the ocean. As for the partial age $a'(\mathbf{x})$, (13)-(15) lead to

$$\int_{\Gamma^s} (-\mathbf{K} \bullet \nabla a') \bullet \mathbf{n} \, d\Gamma^s = V' \tag{53}$$

As a consequence, the abovementioned “age fluxes” obey a formula exhibiting a volume ratio similar to (19a), i.e.,

$$\frac{\int_{\Gamma^s} (-\mathbf{K} \bullet \nabla a') \bullet \mathbf{n} \, d\Gamma^s}{\int_{\Gamma^s} (-\mathbf{K} \bullet \nabla a) \bullet \mathbf{n} \, d\Gamma^s} = \frac{V'}{V} \tag{54}$$

Unfortunately, (54) does no imply that age scaling (19a) is valid.

Conclusion

The chief aim of this working was to show that, at a steady state in the World Ocean, the ratio of any partial water age to the water age scales as the ratio of the relevant volumes. The rationale behind this idea was elementary physical intuition and developments using the Green's function of the underlying advection-diffusion problem. Unfortunately, the sought-after demonstration could not be achieved, though potentially interesting formulas pertaining to the behaviour of the ages in the vicinity of the ocean surface have been derived. Whether or not the aforementioned idea should be further investigated is far from clear.

Appendix

According to (27), the water age in the diffusive water column model may be evaluated by means of the Green's function as follows:

$$a(z) = G(z\,|\,z_0) = \int_z^0 \left(\int_{z_0}^0 \frac{d\eta}{K(\eta)} \right) dz_0 + \int_{-h}^z \left(\int_z^0 \frac{d\eta}{K(\eta)} \right) dz_0 \tag{A1}$$

As is seen below, this rather intricate expression may be drastically simplified.

The first term in the right-hand side of (A1) transforms to

$$\begin{aligned} \int_z^0 \left(\int_{z_0}^0 \frac{d\eta}{K(\eta)} \right) dz_0 &= \int_z^0 \left[\frac{d}{dz_0} \left(z_0 \int_{z_0}^0 \frac{d\eta}{K(\eta)} \, dz_0 \right) - z_0 \frac{d}{dz_0} \int_{z_0}^0 \frac{d\eta}{K(\eta)} \right] dz_0 \\ &= \left[z_0 \int_{z_0}^0 \frac{d\eta}{K(\eta)} \right]_{z_0=z}^{z_0=0} + \int_z^0 \frac{z_0}{K(z_0)} \, dz_0 = -z \int_z^0 \frac{d\eta}{K(\eta)} + \int_z^0 \frac{\eta}{K(\eta)} \, d\eta \end{aligned} \tag{A2}$$

The rightmost term of (A1) leads to

$$\begin{aligned} \int_{-h}^z \left(\int_z^0 \frac{d\eta}{K(\eta)} \right) dz_0 &= \int_{-h}^z \left[\frac{d}{dz_0} \left(z_0 \int_z^0 \frac{d\eta}{K(\eta)} \right) - z_0 \frac{d}{dz_0} \int_z^0 \frac{d\eta}{K(\eta)} \right] dz_0 \\ &= \left[z_0 \int_z^0 \frac{d\eta}{K(\eta)} \right]_{z_0=-h}^{z_0=z} - 0 = z \int_z^0 \frac{d\eta}{K(\eta)} + h \int_z^0 \frac{d\eta}{K(\eta)} \end{aligned} \tag{A3}$$

Finally, combining (A1)-(A3) yields

$$a(z) = \int_z^0 \frac{\eta + h}{K(\eta)} \, d\eta \tag{A4}$$

which is equivalent to (28).

Q.E.D.

References

Deleersnijder E., 2012, *Homogenisation of a passive tracer concentration in an isolated domain*, Working note, Université catholique de Louvain, Louvain-la-Neuve, Belgium, 7 pages, available at URL http://hdl.handle.net/2078.1/155297

Deleersnijder E., 2014, *Developing the concept of partial age, a generalisation of the notion of age*, Working note, Université catholique de Louvain, Louvain-la-Neuve, Belgium, 37 pages, available at URL http:// hdl.handle.net/2078.1/155519

Deleersnijder E., 2016a, *On some properties of steady-state partial water ages in the World*

Ocean, Working note, Université catholique de Louvain, Louvain-la-Neuve, Belgium, 16 pages, available at URL http://hdl.handle.net/2078.1/176922

Deleersnijder E., 2016b, *Is the partial age an increasing function of the size of the associated subdomain?*, Working note, Université catholique de Louvain, Louvain-la-Neuve, Belgium, 1 page, available at URL http://hdl.handle.net/2078.1/302285

Deleersnijder E., 2020, *The uneasy collaboration of Leonhard Euler (1707-1783) and Joseph Louis de Lagrange (1736-1813) in environmental fluid mechanics*, Working note, Université catholique de Louvain, Louvain-la-Neuve, Belgium, 5 pages, available at URL http://hdl.handle.net/2078.1/229071

Deleersnijder E., J.-M. Campin and E.J.M. Delhez, 2001, The concept of age in marine modelling: I. Theory and preliminary model results, *Journal of Marine Systems*, 28, 229-267

Deleersnijder E., A. Mouchet, A. Debrauwere, E.J.M. Delhez and E. Hanert, 2014, *The concept of partial age, a generalisation of the notion of age: theory, idealised illustrations and realistic applications*, JONSMOD 2014 Workshop (12-14 May 2014, Brussels, Belgium), 37 pages, slides available at http://hdl.handle.net/2078.1/153807

Delhez E.J.M., J.-M. Campin, A.C. Hirst and E. Deleersnijder, 1999, Toward a general theory of the age in ocean modelling, *Ocean Modelling*, 1, 17-27

Delhez E.J.M., A.W. Heemink and E. Deleersnijder, 2004, Residence time in a semi-enclosed domain from the solution of an adjoint problem, *Estuarine, Coastal and Shelf Science*, 61, 691-702

England M.H., 1995, The age of water and ventilation timescales in a global ocean model, *Journal of Physical Oceanography*, 25, 2756-2777

Haine T.W.N., S.M. Griffies, G. Gebbie and W. Jiang, 2025, A review of Green's function methods for tracer timescales and pathways in ocean models, *Journal of Advances in Modeling Earth Systems*, 17, e2024MS004637, doi: 10.1029/2024MS004637

Holzer M. and T.M. Hall, 2000, Transit-time and tracer-age distributions in geophysical flows, *Journal of the Atmospheric Sciences*, 57, 3539-3558

Lewandowski R., 1997, *Analyse Mathématique et Océanographie*, Masson, Paris, 304 pages

Liu Z., H. Wang, X. Guo, Q. Wang and H. Gao, 2012, The age of Yellow River water in the Bohai Sea, *Journal of Geophysical Research*, 117, C11006, doi: 10.1029/2012JC008263

Lin L. and Z. Liu, 2019, Partial residence times: determining residence time composition in different subregions, *Ocean Dynamics*, 69, 1023-1036

Lucas L.V. and E. Deleersnijder, 2020, Timescale methods for simplifying, understanding and modeling biophysical and water quality processes in coastal aquatic acosystems: a review, *Water*, 12, 2717, doi: 10.3390/w12102717

Mouchet A., F. Cornaton, E. Deleersnijder and E.J.M. Delhez, 2016, Partial ages: diagnosing transport processes by means of multiple clocks, *Ocean Dynamics,* 66, 367-386

Mouchet A. and E. Deleersnijder, 2008, The leaky funnel model, a metaphor of the ventilation of the World Ocean as simulated in an OGCM, *Tellus*, 60A, 761-774

Mouchet A., E. Deleersnijder and F. Primeau, 2012, The leaky funnel model revisited, *Tellus*, 64A, 19131, doi: 10.3402/tellusa.v64i0.19131

Thiele G. and J.L. Sarmiento, 1990, Tracer dating and ocean ventilation, *Journal of Geophysical Research*, 95 (C6), 9377-9391
