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Three Essays on Social Choice and Public Economics

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Contents

A Mamma e Papà

General Introduction

Men of good fortune
often cause empires to fall
While men of poor beginnings
often can't do anything at all
(Lou Reed)

And the whole earth was of one language, and of one speech.
(Genesis 11:1)

The main goal of a substantial part of this thesis is to understand how particular requirements of equality of opportunity can affect the design of a system of health-care subsidies.

Equality of opportunity is a major concern in many developed and developing countries. Economists and social philosophers (for a survey see Fleurbaey 2008 [20]) seem to have reached some consensus about the fact that equality of opportunity is not just about "levelling the playing field". More subtle factors need to be examined. The basic idea is that individuals are or should be held responsible, to some degree, for their achievements. Differences in agents' outcomes might raise both from differences in characteristics they should not be held responsible for, and from differences in characteristics they should be held responsible for. From a point of view of redistributive justice, society should only try to reduce inequalities deriving from differences of the former type.

The implementation of appropriate economic policies seems to be a necessary condition in order to reach such an objective. For example, people might have different earnings either because of different innate skills (even if they work equally hard) or because they put a different effort in their job (but eventually they have equal innate skills). Should we try to reduce these inequalities indiscriminately? Fleurbaey and Maniquet [25] show that tax schemes that maximize the basic income and exhibit a zero marginal tax rate on incomes below the minimum wage (the minimum wage being defined as the earnings of agents working full time at the smallest wage rate) would guarantee a natural social protection to individuals with poor productive skills that work hard. This is just an example of how public policy can affect the way in which personal responsibility relates to unequal achievements. Individual choices of labour and earnings will have different consequences in terms of inequalities in disposable incomes with different tax schedules. Chapter 1 and chapter 2 are meant to provide further insights to the relation between personal responsibility and public redistribution. More precisely, we will investigate the redistributive implications of responsibility in the domain of health care provision. We intend to provide simple criteria for evaluating the social welfare consequences of various distribution mechanisms and tax schedules that are meant to produce subsidies for the access to health care.

The economic theory of taxation (started by the seminal work of James Mirlees [42]) has put more effort on understanding the structure of the optimal tax (i.e. the best tax policy among those which are feasible under incentive constraints) than on the study of criteria for the comparison of arbitrary feasible taxes. Unfortunately, in practice, there are many economic and political obstacles to the implementation of the optimal tax. We think that knowing how to compare suboptimal policies is as relevant as knowing how to compute the optimal one.

The methodology adopted follows a definite path that goes from basic ethical principles to the tangible evaluation of public policies. The line of reasoning is rigorous. Once the relevant ethical principles are singled out we formulate precise requirements (axioms) that embody these principles. These requirements are used to single out a specific way to measure individual well being and a specific way to aggregate it. This process gives *(*fair) social preferences that we can use for the evaluation of public policies.

The social preferences we propose here can be viewed, in a sense, as refinements of the traditional social welfare functions which one finds in welfare economics. The refinement goes in the direction of enriching the way collective decisions are taken with ethical considerations which are directly drawn form political philosophy.

This alternative approach has been already applied to several economic environments (a wide and detailed description of the methodology and the possible applications is provided by Fleurbaey and Maniquet [28]): for example, Bossert, Fleurbaey and Van De Gaer [8] have already discussed some issues that arise if compensation schemes are to be applied in a second-best framework. Fleurbaey and Maniquet [24][25][27]), and Luttens and Ooghe [38] consider a model where agents have unequal production skills and different preferences over consumption and labor. They propose ways to rank social alternatives which satisfy properties of compensation for inequalities in skills and equal access to resources for all preferences. Once an ordering of the social alternative is available, they use it to devise the main features of the optimal tax schedule.

In chapter 1 and chapter 2 we will mostly focus on social preferences embodying two particular ethical principles. Following the previous example, the freedom to choose one's labour time automatically entails that differential earnings will follow from personal, responsible choices. Income inequalities exclusively deriving from a different exercise of individual responsibility should then be deemed acceptable by society. This is the so-called principle of Natural Reward (Fleurbaey [17]). On the other hand, different individuals with the same willingness to work might end up with different incomes because of their different innate productivities. Income inequality derives in this case from differences in characteristics they should not be held responsible for (to some extent, at least). Transfers reducing such inequalities are hence acceptable, if not desirable. This is the so-called principle of compensation (Fleurbaey [17]). It is worth to stress that our purpose is not to defend a single view of social welfare, but to clarify the link between some particular fairness principles and concrete policy evaluations.

In chapter 1 we consider a fair division model concerning compensation among individuals characterized by different, non-transferable, personal endowments. For instance, one could think of the problem of a social planner willing to provide assistance to people with disabilities by distributing among them a given amount of available resources. We propose a way to rank alternatives that takes into account the need to compensate individuals for differences in their personal endowments and, to some extent, the need of treating neutrally agents with the same personal endowments. Finally, we use these social preferences to derive the appropriate public policy. Notice that, since we rely on an ordering of all the conceivable allocation, this exercise is still possible even if incentive constraints arise. Our main finding is that when society decides, what is the *fair* compensation for any given disability level, it should take the point of view of the individuals that are less willing to accept a bad condition. The fact that the compensation scheme we propose grants maximal priority to agents with a worse condition, together with the relative scarcity of resources, also implies that it may be desirable to give a monetary compensation only to people whose level of disability is above a certain threshold.

In chapter 2 we put forward a more sophisticated model: agents differ in their preferences about consumption labor and health, in their (health-dependent) earning ability, and in their health disposition. We explores the possibility of integrating the individual health expenditure into the tax system. Health expenditure has become a hot topic both in developed and developing countries. In many countries there is a widespread support for public intervention in the health care delivery system on redistributive and efficiency grounds. In spite of this it is not entirely clear whether and how health subsidies should be related to the income tax system. Our final objective is to devise criteria for the evaluation of such tax schemes, given the particular fairness principles we propose. On one side, we consider desirable to reduce inequalities in consumption deriving from factors that do not depend on individuals' responsibility (namely, their earning ability and their health disposition). On the other side, redistribution should be precluded at least when all agents in the economy have equal physical characteristics. We first propose, on the basis of such principles, a particular social welfare function. Finally we derive a simple criterion for the comparison of tax policies: we prove that, for any level of health expenditure, a tax reform should always benefit agents with the lowest earning ability first. We also prove that, as far as the optimal tax policy is concerned, the consumption granted to hardworking-poor agents (unskilled agents who work full time) who decide to be healthy but have a different health dispositions, should be equalized. Moreover, within each sub-population defined by the health expenditure level, hardworking-poor agents are those who will be granted the greatest subsidy (i.e. the smallest tax).

In chapter 3 we depart from the theme underlying the previous two. This work, based on a project co-authored with Efthymios Athanasiou and Santanu Dei, is meant to contribute to the already vast literature on mechanism design. What can be considered the most original contribution of this chapter is, in a sense, the model. We study communication among individuals. Individuals attach value to communication, this is as apparent today, as it has ever been. The benefits notwithstanding, communication does not come for free. The cost of communication proceeds from the presence of a multitude of different languages. There is hardly one individual that speaks all of the approximately 6000 languages that are spoken today, while there are plenty that speak exclusively one of them. In our model, the benefit from communication stems from the number of people one may communicate with, multiplied by a real number that constitutes the individual's willingness to communicate. The costs are determined by the language one speaks relative to the language one learns. Utility is separable in costs and benefits and transferable. An allocation is a list of linguistic assignments and monetary transfers. A mechanism is a function that, given the primitives of the economy decides the *optimal* allocation that is, it says which language each agent should learn and the subsidy she will receive (or the tax she will have to pay).

In such a framework we study mechanisms that each satisfies three of the following requirements (we prove in fact that no mechanism satisfies all of them):

1. *Assignment Efficiency* : the sum of net benefits should be maximized.
2. *Strategy-Profness* : all agents, if asked, should have a dominant strategy to reveal their willingness to communicate truthfully.
3. *Individual Rationality* : no agent should enjoy a utility level that is lower than the level of utility he would enjoy if he was not a constituent of the economy.
4. *Feasibility* : any mechanism should rely exclusively on the resources generated within the economy (no outside funding should be permitted).

We characterize a class of Groves' mechanisms based on its performance as regards the budget deficit it generates. We find that Groves' mechanism suffer from certain defects, especially when the population of individuals is small. For instance, within the class of mechanisms satisfying all properties except *Feasibility*, the mechanism that minimizes the deficit may still generate a deficit larger than the social value of the assignment efficient communication structure. Conversely, insisting on *Feasibility* at the expense of *Individual Rationality*, at some instances, may mean that as many as all the individuals in the economy face a negative utility.

As a means of circumventing these issues we propose a mechanism that is outside the Groves' family of mechanisms. However, there is a price to pay. Any such mechanism will need to rely on the ability of a planner to exclude individuals by appropriately selecting the assignment of languages. Under this regime, individuals often will have an incentive to unilaterally deviate and learn more languages than those the allocation prescribes for them.

On the positive side we find that all these difficulties vanish in 'large economies'. Largeness is meant in the replication sense. Therefore, a large economy is far from an economy consisting of an infinity of individuals. Indeed, a 'large economy' may not be large at all. Under mild assumptions, we show that any economy replicated a finite number of times yields an environment in which all the mechanisms we propose come close to satisfying the property they miss.

Chapter 1

A fair solution to the compensation problem

1.1 Introduction

We study[1] the fair allocation of a given amount of a one-dimensional, divisible resource (e.g., money) among a finite population of individuals to compensate them for their differences in a non-transferable, personal endowment. Such a personal endowment can be either an innate talent or a handicap and can be exemplified by features of human capital like health condition, bodily characteristics or social background.

For instance, one could think of the problem of a social planner willing to provide assistance to people with disabilities by distributing among them a given amount of available resources. In several countries, doctors evaluate the mental and physical condition of individuals with handicaps. An index is thus computed which determines to which extent the applicant is able to work. Some benefits are then assigned as a function of this parameter.

Recent theories of justice agree that inequalities in agents'outcomes might arise both from differences in characteristics they *should not* be held responsible for and from differences in characteristics they *should* be held responsible for. They all arrive at the common conclusion that society should only try to reduce inequalities deriving from differences of the former type[2].

We will assume that each individual is endowed with two characteristics: her talent t, for which she is not responsible and her preferences R for which she is[3], since they embody her subjective evaluation of her condition (namely the amount of resources she receives and her talent).

The purpose of our analysis is to look for a distribution mechanism that, on one side, allocates a given amount of available external resources M in order to correct inequalities deriving from differences in talent solely. This idea goes under the name of principle of *compensation* (Fleurbaey [16], [17]). In performing such a compensation we cannot disregard the way individuals evaluate their condition: if all the individuals in a society deem equivalent being endowed with a given level of talent, rather than another one, then

[1] This chapter is based on Valletta [55]

[2] See for instance Rawls [47], Dworkin [15], Arneson [1], Cohen [9], van Parijs [54], Roemer [49].

[3] Modeling responsibility by preferences it is not free of controversy. Cohen [9] and Roemer [49] have argued that individuals should be held responsible only for what lies within their control. According to this definition individual preferences do not necessarily belong to do domain of characteristics for which an individual should be held responsible. For an extensive discussion about the compensation-responsibility cut see Fleurbaey [20].

differences in talents do not need to be compensated.

On the other side, we endorse the idea that differences due to characteristics that fall into the domain of responsibility should be treated neutrally: this means that, in our case, the distribution mechanism should give the same amount of resources to individuals with the same talent regardless of the way they (subjectively) evaluate their condition. This is the so called principle of *equal resources for equal talent* (Fleurbaey [16], [17]).

The fact that the two principles conflict with each other seems rather intuitive (Bossert [7], Fleurbaey [16], [17]). The latter principle prescribes a distribution of resources that does not depend on individual preferences. On the other hand preferences affect the need for a monetary compensation. Hence fully compensating agents within each preference class may lead to a distribution of resources that treats differently agents with the same talent.

To solve this problem, the literature has so far proposed allocation rules (for a complete survey of the subject see Fleurbaey and Maniquet [22]). An allocation rule specifies the set of optimal allocations as a function of the parameters of the problem: the population's profile of talents, the population's profile of preferences and the available resources. Such allocation rules can be enforced only on the basis of a full knowledge of the population's profiles of characteristics. The presence of informational constraints typically renders impossible to implement the allocation dictated by the allocation rule.

In order to overcome this difficulty we move away from this approach and we look for social ordering functions specifying a complete ranking of all the allocations as a function of the parameters of the problem. Once an ordering of the allocations is available, it is always possible to produce a precise policy recommendation, even if the social planner faces informational constraints (or any other kind of constraint).

This alternative approach has been already applied to several economic environments: Bossert, Fleurbaey and Van de gaer [8] have already discussed some issues that arise if compensation schemes are to be applied in a second-best framework. In a very recent contribution Fleurbaey [20] pushes this analysis further by providing the characterization of several social ordering functions and some insights about incentive-compatible policies. Fleurbaey and Maniquet [24], [25] and Luttens and Ooghe [38] consider a model where agents have unequal production skills and different preferences and they characterize social ordering functions which satisfy properties of compensation for inequalities in skills and equal access to resources for all preferences. Maniquet and Sprumont [40], [41] construct orderings of allocations in economies with one private good and one partially excludable nonrival good. Fleurbaey and Maniquet [26] construct social ordering functions in the canonical economic problem of dividing a bundle of commodities among individuals with different preferences. Maniquet and Sprumont [40], [41] construct orderings of allocations in economies with one private good and one partially excludable nonrival good. Fleurbaey and Maniquet [26].

We propose a way to rank alternatives based on efficiency, fairness and robustness properties. It is explained below how it is possible to embody the principle of compensation and the principle of equal resources for equal talent into a variety of specific, precise axioms. It will be clear soon that it is not possible to find a social ordering function that satisfies both principles. We show how axioms embodying the principle of compensation and axioms *weakly* embodying the principle of equal resources for equal talent together with efficiency and separability conditions (with some variants, separability conditions state that indifferent agents should not matter in the social evaluation of two alternatives) single out a specific welfare representation of agents' preferences and a specific way of ranking alternatives.

Maximizing such an ordering, under the relevant incentive constraints, gives us a specific suggestion about the way resources should be allocated. Our main finding is that when society decides, for any level of talent, what is the *fair* compensation, it should take the point of view of the individuals that are less willing to accept a low talent.

The fact that the compensation scheme we propose grants maximal priority to agents with lower talent, together with the relative scarcity of resources, also implies that it may be desirable to give a monetary compensation only to people whose level of talent is below a certain threshold.

The paper is organized as follows. Section 2 introduces the model and the relevant notation. Section 3 describes the requirements imposed on the social preferences and describes logical relations between them. Section 4 deals with social orderings. Section 5 studies the optimal allocation of resources under informational constraints. Section 6 concludes. The Appendix provides the proofs.

1.2 Model and Main Definitions

The model we analyze is derived from Fleurbaey [16], [17]. We consider a set of economies with a finite set of agents $N \subseteq \mathbb{N}$. A given amount $M \in \mathbb{R}_{++}$ has to be shared among them through monetary transfers of amount $m_i \in \mathbb{R}_+$, for all $i \in N$. Each agent in the economy is characterized by a fixed, non transferable talent $t_i \in T$. Let $T = [t^L, t^H] \subset \mathbb{R}$ be the set of possible talents. Moreover each agent is endowed with a personal preference ordering R_i over pairs $(m, t) \in \mathbb{R}_+ \times T$. A pair (m, t) will be called hereafter an extended bundle. This means that individuals are assumed to be able to compare extended bundles of individuals with a talent different from their own. In this framework one can think of preferences as to valuation functions: each individual is able to give a valuation of her own condition and compare it with someone else's. Even if individuals cannot actually choose between different money-talent pairs, still they are able to express a judgement about different possible conditions: one might consider the condition of other agents better or worse than her own (i.e., one might think she deserves a higher or lower compensation than what she actually receives). The individual preference ordering R_i is assumed to be continuous and strictly monotonic with respect to m_i and t_i, strict preference and indifference will be respectively denoted by P_i and I_i. Let $\mathcal{R}$ denote the set of such preferences. We will say that individual preferences $R \in \mathcal{R}$ exhibit a higher aversion to lack of talent than $R' \in \mathcal{R}$ ($R \succ_A R'$) if they satisfy the following property: for any $t, t' \in T$ and for any $m, m' \in \mathbb{R}^N_+$ we have

$$t' > t \textit{ and } (m', t')R'(m, t) \Longrightarrow (m', t')P(m, t)$$
$$t' < t \textit{ and } (m', t')R\ (m, t) \Longrightarrow (m', t')P'(m, t)$$

that is, any pair of indifference curves, one for R and one for R', cross at most once in the (m, t) space.

An *economy* is denoted by $e = (t_N, R_N, M)$, where $t_N = (t_i)_{i \in N}$ is the population's profile of talents, $R_N = (R_i)_{i \in N}$ is the population's profile of individual preferences and M is the available social endowment. Let us stress that the purpose of the paper is neither to establish the optimal amount M nor to establish an optimal way to tax more talented individuals in order to compensate less talented ones. This would require a more involved model that would also take into account individual productive skills and that would study the optimal tax system. This is beyond the scope of this paper. Here we do not have a market structure and the amount M is exogenously determined. Indeed in many countries a compensation is granted to disabled people independently of redistributive issues.

For any $e \in \mathcal{D}$, let T^e and $\mathcal{R}^e$ denote, respectively, the set of different levels of talents and the set of individual preference orderings we can find in e. The domain of all economies satisfying the above assumptions will be denoted by $\mathcal{D}$. An *allocation* is a vector $m_N = (m_i)_{i \in N} \in \mathbb{R}_+^N$.

The first concern of this paper is to construct a social ordering of (feasible and non feasible) allocations for all economies in the domain. A social ordering, for any given $e \in \mathcal{D}$, is a complete and transitive ranking defined over allocations. A *social ordering function* $\overline{R}$ associates every admissible economy $e = (t_N, R_N, M) \in \mathcal{D}$ with a social ordering $\overline{R}(e)$. So for an economy $e = (t_N, R_N, M) \in \mathcal{D}$ and two allocations m_N and $m'_N \in \mathbb{R}_+^N$ we write $m_N \overline{R}(e) m'_N$ to denote that m_N is (socially) at least as good as m'_N. Strict social preference and indifference will be respectively denoted by $\overline{P}(e)$ and $\overline{I}(e)$.

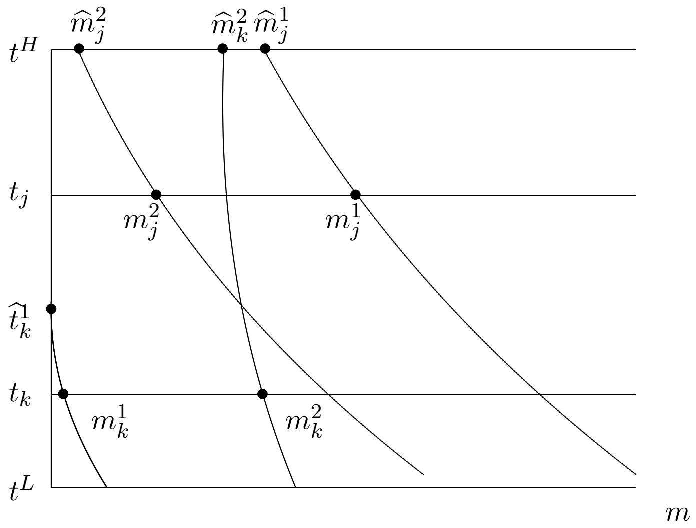


Figure 1.1:

To illustrate our approach consider the simple economy depicted in figure 1.1. We only have two agents, j and k, with different talents, t_j and t_k. We consider two allocations, (m_j^1, m_k^1) and (m_j^2, m_k^2) and we want to rank them. The social ordering function we propose in this paper works in the following way. For each $i \in \{j, k\}$, $q \in \{1, 2\}$ we consider the level of resource $\widehat{m}_i^q \geq 0$ which would make the agents accept the level of talent t^H instead of their current situation (m_i^q, t_i). As we can see from the figure, such a value does not exist for agent k at (m_j^1, m_k^1). In order to evaluate the situation of agent k, we then consider the lowest level of talent $\widehat{t}_k^1$ that would make her accept a null transfer instead of her current situation. More formally, $\widehat{t}_k^1$ is the level of talent such that $(0, \widehat{t}_k^1) I_k (m_k^1, t_k)$. To summarize, for each agent i we can have a continuous numerical representation of her preferences over pairs $(m, t) \in \mathbb{R}_+ \times T$:

$$\widehat{E}(m_i, t_i, R_i) = \begin{cases} \widehat{m}_i & \text{if } \widehat{m}_i \text{ exists} \\ \widehat{t}_i - t^H & \text{otherwise.} \end{cases}$$

Finally the vectors of such welfare representations are compared using the maximin rule: in our example we have that (m_j^2, m_k^2) is strictly better than (m_j^1, m_k^1) since $min\{\widehat{E}_j^2, \widehat{E}_k^2\} > min\{\widehat{E}_j^1, \widehat{E}_k^1\}$.

A social ordering function $\overline{R}$ is an $\widehat{E}$*-maximin* function if, for each economy, it prescribes an ordering of the allocations that is consistent with the application of the maximin criterion to the vectors of welfare levels generated by the function $\widehat{E}(.)$ at these allocations. More precisely, for any $e \in \mathcal{D}$ and m_N, $m'_N \in \mathbb{R}^+_N$

$$min_i \widehat{E}(m'_i, t_i, R_i) > min_i \widehat{E}(m_i, t_i, R_i) \Rightarrow m'_N \overline{P}(e) m_N.$$

Equal Well-being for Equal Preferences For all $e \in \mathcal{D}$, $z, z' \in Z$, if there exist $i, j \in N$ and some $\Delta > 0$, such that $R_i = R_j$, $l_i = l_j = l'_i = l'_j$, $h_i = h_j = h'_i = h'_j$ with $z_k = z'_k$ for all $k \neq i, j$,

$$c_i - \Delta = c'_i > c'_j = c_j + \Delta$$

then $z' \overline{P}(e) z$; if otherwise

$$c'_i = c_j \text{ and } c_i = c'_j$$

then $z' \overline{I}(e) z$.

1.3 Axioms

This section defines the properties we want to impose on our social ordering function. We start with a basic efficiency condition. The Strong Pareto axiom takes a very intuitive form here since talents are fixed and resources are one-dimensional. Notice in fact that all allocations exhausting the resources are efficient.

Strong Pareto. *For all $e \in \mathcal{D}$ and $m_N, m'_N \in \mathbb{R}^N_+$, if $m'_i \geq m_i$ for all $i \in N$, then $m'_N \overline{R}(e) m_N$; if, in addition, $m'_i > m_i$ for some $i \in N$, then $m'_N \overline{P}(e) m_N$.*

The second axiom we introduce is a property of robustness of the social ordering with respect to changes in the set of agents. We require that adding or removing agents who receive the same transfer in two different allocations should not modify the social ordering. This axiom was introduced by Fleurbaey and Maniquet [21] and it is reminiscent of the Separability condition, quite familiar in social choice, due to d'Aspremont and Gevers [3] and of the Consistency condition, widely used in the theory of fair allocation (see Thomson [53]), except that it does not require to delete the resources consumed by the removed agents from the social endowment.

Separation. *For all $e \in \mathcal{D}$ and $m_N, m'_N \in \mathbb{R}^N_+$, if there is $i \in N$ such that $m_i = m'_i$, then*

$$m'_N \overline{R}(e) m_N \Longleftrightarrow m'_{N\setminus\{i\}} \overline{R}(t_{N\setminus\{i\}}, R_{N\setminus\{i\}}, M) m_{N\setminus\{i\}}.$$

We can now turn to the fairness conditions. We first focus on the principle of compensation. Assume there are two agents, j and k, with equal preferences and possibly a different talent. Assume also that they both consider agent j's extended bundle strictly better than agent k's. Assume finally that the monetary compensation received by agent j is reduced by a given amount and the one received by agent k is increased by the same amount but they both still consider j's extended bundle to be better than k's. The allocation we obtain must be, according to the principle of compensation, socially preferred to the former one.

Equal Preferences Transfer. *For all $e \in \mathcal{D}$ and $m_N, m'_N \in \mathbb{R}^N_+$, if there exist $j, k \in N$ and $\Delta \in R_+$ such that $R_j = R_k$,*

$$m'_j = m_j - \Delta,\ m'_k = m_k + \Delta \text{ and } (m'_j, t_j) P_j (m'_k, t_k),$$

with $m_i = m'_i$ *for all* $i \neq j, k$, *then* $m'_N \overline{P}(e) m_N$.

The principle implies that, ideally, two agents with equal preferences and different talents should end up on the same indifference curve (Fleurbaey [17]). Here we deal with social orderings so the axiom compares two different allocations. The aim is to reduce inequalities exclusively deriving from differences in talent.

We also consider a quite natural strengthening of this axiom: instead of requiring the preferences of the agents to be identical, we simply require that agents' indifference curves through their extended bundles in m_N and m'_N are nested. For each $i \in N$ let $L((m_i, t_i), R_i)$ and $U((m_i, t_i), R_i)$ denote respectively the (closed) lower contour set and the (closed) upper contour set of R_i at (m_i, t_i).

Nested Preferences Transfer. *For all* $e \in \mathcal{D}$ *and* $m_N, m'_N \in \mathbb{R}^N_+$ *if there exist* $j, k \in N$ *and* $\Delta \in R_+$ *such that*

$$U((m'_j, t_j), R_j) \bigcap L((m'_k, t_k), R_k) = \emptyset,$$

$m'_j = m_j - \Delta$ *and* $m'_k = m_k + \Delta$, *with* $m_i = m'_i$ *for all* $i \neq j, k$, *then* $m'_N \overline{P}(e) m_N$.

The rationale of this axiom is that, in both allocations, one of the two agents envies the other (not only at her actual talent but also at any hypothetical level of talent she may have), so performing the transfer actually reduces resources inequality.

Let us now turn to the principle of equal resources for equal talent. Assume there are two agents with the same talent but possibly different preferences who receive a different monetary compensation: performing an equalizing transfer that reduces such an inequality should be considered a social improvement.

Equal Talent Transfer. *For all* $e \in \mathcal{D}$ *and* $m_N, m'_N \in \mathbb{R}^N_+$, *if there exist* $j, k \in N$ *and* $\Delta \in R_+$ *such that* $t_j = t_k$ *and*

$$m_k + \Delta = m'_k < m'_j = m_j - \Delta,$$

with $m_i = m'_i$ *for all* $i \neq j, k$, *then* $m'_N \overline{P}(e) m_N$.

The principle implies that, ideally, two agents with equal talent should receive the same transfer (Fleurbaey [17]) but here we are interested in comparing two different allocations. The aim is to reduce the inequality between agents with equal talent that are treated differently.

The fairness conditions presented so far are an adaptation to this framework of the Pigou-Dalton principle of transfer, widely used in income inequality theory. Our first result shows that the requirements we have presented so far, expressing the two different ethical principles, are incompatible.

Lemma 1 *On the domain* $\mathcal{D}$, *no social ordering function satisfies* Equal Preferences Transfer *and* Equal Talent Transfer.

We propose now an alternative way of capturing the ethical goals of compensation and equal resources for equal talent that is inspired by Suppes' grading principles (Suppes [51]) and was introduced in the literature by Maniquet [39]. The first property, called Equal Preferences Permutation, requires that permuting the outcomes of two agents having the same preferences but possibly different talents gives us two socially equivalent allocations. Indeed, according to the principle of compensation there is no reason why society should prefer one of the two allocations: agents with the same responsibility characteristics should be treated anonymously.

Equal Preferences Permutation. *For all $e \in \mathcal{D}$ and $m_N, m'_N \in \mathbb{R}^N_+$, if there exist $j, k \in N$ such that $R_j = R_k$,*

$$(m_j, t_j) I_j (m'_k, t_k) \text{ and } (m_k, t_k) I_j (m'_j, t_j),$$

with $m_i = m'_i$ for all $i \neq j, k$, then $m_N \overline{I}(e) m'_N$.

The next axiom requires, according to the principle of equal resources for equal talent, that permuting the transfers of two agents having the same talent but possibly different preferences does not alter the value of the allocation in the social ranking. Again the justification is clear: as those agents have the same talent, they should be treated anonymously. By permuting the transfers they receive, we obtain an equally good, or equally bad, allocation.

Equal Talent Permutation. *For all $e \in D$ and $m_N, m'_N \in \mathbb{R}^N_+$, if there exist $j, k \in N$ such that $t_j = t_k$,*

$$m_k = m'_j \text{ and } m_j = m'_k,$$

with $m_i = m'_i$ for all $i \neq j, k$, then $m_N \overline{I}(e) m'_N$.

The next result shows that the incompatibility between compensation and responsibility extends beyond Lemma 1 since it has nothing to do with the degree of inequality aversion exhibited by the axioms that were involved there: also requirements like Equal Preferences Permutation and Equal Talent Permutation, which are consistent with any degree of inequality aversion (even a negative one), are in fact incompatible if combined with Strong Pareto.

Lemma 2 *On the domain $\mathcal{D}$, no social ordering function satisfies* Strong Pareto*,* Equal Preferences Permutation *and* Equal Talent Permutation*.*

Lemma 1 and 2 are introductory results meant to show that the underlying incompatibility between the principle of compensation and the principle of equal resources for equal talent still persists when we deal with social ordering functions. The impossibility of finding a social ordering function satisfying the two principles has already been pointed out, using the same logic, by Fleurbaey and Maniquet [23] in a model where agents have unequal skills and heterogeneous preferences over consumption and leisure.

Here we analyze a compensation minded social ordering function because we consider the principle of compensation more compelling than the principle of equal resources for equal talent. Nonetheless, as we will see later, introducing incentive-compatibility constraints considerably shapes the incompatibility between the two principles.

Interestingly, if we decide to give priority to the principle of compensation, then, combining together Nested Preferences Transfer with axioms that are essentially neutral with respect to inequality aversion, determines an extreme form of inequality aversion described by the following axiom.

Nested Preferences Priority. *For all $e \in \mathcal{D}$ and $m_N, m'_N \in \mathbb{R}^N_+$, if there exist $j, k \in N$ such that*

$$(m_j, t_j) P_j (m'_j, t_j) P_j (m'_k, t_k) P_k (m_k, t_k)$$
$$U((m'_j, t_j), R_j) \bigcap L((m'_k, t_k), R_k) = \emptyset,$$

with $m_i = m'_i$ *for all* $i \neq j, k$, *then* $m'_N \overline{P}(e) m_N$.

This property represents an extreme strengthening of Nested Preferences Transfer. It states that an inequality reduction is always socially desirable, provided that indifference curves are nested at the bundles, even when the gain from m_k to m'_k is very small and the loss from m_j to m'_j is quite large.

Lemma 3 *On the domain* $\mathcal{D}$, *if a SOF satisfies* Nested Preferences Transfer, Equal Preferences Permutation *and* Separation *then it satisfies* Nested Preferences Priority.

Results similar to this one can already be found in Fleurbaey and Maniquet [24] and Maniquet and Sprumont [40], [41]. They all show, in different economic environments, that combining together fairness conditions that exhibit a limited aversion to inequality with robustness and efficiency conditions leads to an infinite aversion to inequality. Let us stress that we reach a similar conclusion dealing with a one-dimensional resource. This makes the contrast with the income inequality measurement framework, where combining the Pigou-Dalton transfer principle with robustness requirements does not necessarily lead to such extreme consequences (Chakravarty [10]), even stronger.

On the other hand, if we decide to give priority to the principle of equal resources for equal talent, we cannot find anything similar. In this case, in fact, we would "fall" in a purely one-dimensional framework were we are eventually free to choose the degree of inequality aversion of the social ordering function[4].

In the last part of this section we propose two alternative weakenings of Equal Talent Transfer that represent somehow the strongest responsibility axioms that would not clash with Equal Preferences Transfer. We focus on cases where the inequality in external resources between agents with equal talent looks particularly undesirable.

The first weakening we propose takes somehow into account also individual preferences. Consider a couple of agents with the same talent who receive a different amount of resources. Moreover assume that the agent who receives less has an higher aversion to lack of talent than the other. It seems obvious that such a situation is less acceptable than the symmetric one where the "unconcerned" agent is the one who receives less. We say that it is socially desirable to perform a transfer of resources between two agents with equal talent only if the recipient's preferences exhibit a higher degree of aversion to lack of talent than the donor's ones.

Equal Talent Transfer to the Unhappy. *For all* $e \in \mathcal{D}$ *and* $m_N, m'_N \in \mathbb{R}^N_+$, *if there exist* $j, k \in N$ *and* $\Delta \in R_+$ *such that* $t_j = t_k$, $R_k \succ_A R_j$ *and* $m_k + \Delta = m'_k < m'_j = m_j - \Delta$, with $m_i = m'_i$ *for all* $i \neq j, k$, then $m'_N \overline{P}(e) m_N$.

The second weakening focuses on cases where one of the two agents receives a monetary compensation while the other does not. In such cases we propose a minimal redistribution requirement: it must be always possible to slightly reduce the difference in the way they are treated (i.e., it must be always possible to transfer *something* to the agent who did not receive anything in the first place) and the allocation we obtain is socially preferred to the

[4]We could, for example, compare allocations just by computing their Gini index. This would satisfy Strong Pareto and Equal Talent Transfer (but it would violate Equal Preferences Transfer).

former one.

Minimal Equal Talent Transfer. *For all $e \in \mathcal{D}$ and $m_N \in \mathbb{R}_+^N$, if there exist $j, k \in N$ such that $t_j = t_k$, $m_j = 0$, $m_k > 0$ then there exists at least one $\varepsilon \in \mathbb{R}_{++}$ and a corresponding allocation $m'_N \in \mathbb{R}_+^N$ such that: $\varepsilon = m'_j < m'_k = m_k - \varepsilon$, $m_i = m'_i$ for all $i \neq j, k$, and $m'_N \overline{P}(e) m_N$.*

Let us stress again that the two axioms are logically independent but they are both implied by Equal Talent Transfer.

1.4 Social Orderings

In this section we show how combining together the axioms presented in the previous section determines a specific way of ranking the social alternatives.

Theorem 1 *Every Social Ordering Function, defined on the domain $\mathcal{D}$, satisfying* Strong Pareto*,* Nested Preferences Transfer*,* Equal Preferences Permutation*,* Equal Talent Transfer to the Unhappy*,* Minimal equal Talent Transfer *and* Separation *is a $\widehat{E}$-maximin function.*

The fact that no axiom is redundant, is shown in the appendix.

The combination of the axioms singles out a particular welfare representation of individual preferences, that is $\widehat{E}(.)$. The corresponding welfare-level vectors need then to be ranked according to the maximin criterion: a way of ranking alternatives that gives priority to agents with a low $\widehat{E}(.)$, that is, either agents with a low m_i or agents who dislike their level of talent.

The theorem does not provide a full characterization of social preferences because it does not not tell us how to rank two allocations for which $min_i \widehat{E}(m'_i, t_i, R_i) = min_i \widehat{E}(m_i, t_i, R_i)$. Nonetheless, as we will see in the next section, we can still devise a criterion to evaluate different compensation policies and, in particular, to single out the optimal one. Moreover, the refinement of the proposed social ordering obtained by applying the leximin criterion[5] to the welfare-level vectors gives us a complete ranking of the social alternatives and, as can be easily checked, satisfies all the involved axioms.

Some further discussion will help to understand how the axioms lead to the characterization result. We could in fact think of different ways to rank alternatives. For example, we might measure welfare differently keeping the same ranking criterion: take two allocations m_N and m'_N and order the extended bundles according to a reference preference $\widetilde{R} \in \mathcal{R}$ in such a way that $(m_i, t_i)\widetilde{R}(m_{i+1}, t_{i+1})$ and $(m'_i, t_i)\widetilde{R}(m'_{i+1}, t_{i+1})$ for all $i \in (1, ..., n-1)$ then, m'_N is socially (strictly) preferred to m_N if $(m'_n, t_n)\widetilde{P}(m_n, t_n)$[6]. This social ordering function is to some extent dual to the $\widehat{E} - maximin$ function. It fails to satisfy the compensation minded axioms (except for agents whose individual preferences are equal to the reference one) and on the contrary satisfies the responsibility minded ones (this is indeed what we would have obtained if we had decided to give priority to the principle of equal resources for equal talent).

[5]Let $\geq_L$ denote the usual leximin ordering of $\mathbb{R}_+^n$: for any $w, w' \in \mathbb{R}_+^n$, $w' \geq_L w$ if and only if the smallest coordinate of w' is greater than the smallest coordinate of w or they are equal but the second smallest coordinate of w' is greater than the second smallest coordinate of w and so on.

[6]Notice that the agents with the worst-off extended bundles with respect to $\widetilde{R}$, at the two allocations, are not necessarily the same.

On the other hand, we might keep the welfare representation proposed in the first place but drop the maximin criterion. We could, for example, take the welfare-level vectors underlying two different allocations and calculate their Gini index. The allocation with the smallest Gini index will be socially preferred to the other one. This ranking criterion is, to some extent, compensation minded but it is not necessarily consistent with an infinite degree of inequality aversion.

Finally, the social ordering function we propose share some similarities with the *Full Health Equivalent Maximin* function proposed by Fleurbaey [18]. In a model where individuals can chose consumption and health he proposes to maximin the lowest level of full health equivalent consumption occurring in society. The full health equivalent consumption is the smallest level of consumption which an agent would accept in replacement of her actual bundle provided that she is fully healthy. Fleurbaey characterizes his criterion with four axioms: Pareto, Equal Preferences Transfer, a weaker version of Equal Talent Transfer[7] and Hansson Independence[8].

In Fleurbaey's model individuals can freely choose the level of health. On the contrary in our model the talent is a given parameter that cannot be changed and hence it is a parameter that defines the economy. This is why, in our framework, the axioms proposed by Fleurbaey do not necessarily single out a welfare representation that is independent of the particular profile of talents in a given economy. For example, for each $i \in N$ and $m_i \in \mathbb{R}_+$ consider $\widetilde{m}_i \geq 0$ such that $(m_i, t_i) I_i (\widetilde{m}_i, t^m)$, where t^m is the maximal element of T^e. If such a value does not exist then consider the hypothetical level of talent $\widetilde{t}_i$ such that $(0, \widetilde{t}_i) I_i (m_i, t_i)$. For each $i \in N$ consider the function

$$\widetilde{E}(m_i, t_i, R_i) = \begin{cases} \widetilde{m}_i & \text{if } \widetilde{m}_i \text{ exists} \\ \widetilde{t}_i - t^m & \text{otherwise.} \end{cases}$$

Finally compare social alternatives by applying the maximin criterion to the vectors of such welfare representation of individual preferences. Such a way to rank alternatives still satisfies all the axioms proposed by Fleurbaey but the welfare representation of individual situations is profile dependent.

1.5 Incentive Compatible Allocations

In his seminal contributions Fleurbaey [17] characterizes two families of allocations rules that display dual properties with respect to compensation and responsibility.

$\widetilde{t}$-Egalitarian Equivalent. Choose a feasible allocation $m_N \in \mathbb{R}^+_N$ such that there exist $\widetilde{m} \in \mathbb{R}_+$, $\forall i \in N$, $(m_i, t_i) I_i (\widetilde{m}, \widetilde{t})$, or $m_i = 0$ and $(0, t_i) R_i (\widetilde{m}, \widetilde{t})$.

where $\widetilde{t} \in T$ is a reference talent chosen arbitrarily.

$\widetilde{R}$-Conditional Equality. Choose a feasible allocation $m_N \in \mathbb{R}^+_{\mathbb{N}}$ such that $\forall i, j \in N$, $(m_i, t_i) \widetilde{I} (m_j, t_j)$ *or* $m_i = 0$ and $(0, t_i) \widetilde{R} (m_j, t_j)$.

where $\widetilde{R} \in \mathcal{R}$ is a reference preference ordering chosen arbitrarily.

[7]The transfer is performed only between agents with full health.

[8]The social ranking of two allocations should remain unaffected by a change in individual preferences which does not modify agents' indifference curves at the bundles they receive in these two allocations.

The first one is a family of compensation minded allocation rules the second one is a family of responsibility minded allocation rules.

It is easy to see that the the $\widehat{E} - maximin$ rationalizes (with some restrictions on the domain of preferences) a particular rule ($\widetilde{t} = t^H$) of the $\widetilde{t}-$Egalitarian Equivalent family of rules. At the optimum, we would obtain an allocation such that each agent is indifferent between the extended bundle she receives and a hypothetical extended bundle in which everyone receives the same transfer and everyone's talent is equal to t^H. Such an allocation would be achievable if the social planner knew the talent and the preferences of each agent. In our framework we can reasonably assume that the former is known (for example, doctors can establish the degree of disability of an individual) while individual preferences are typically not observable[9].

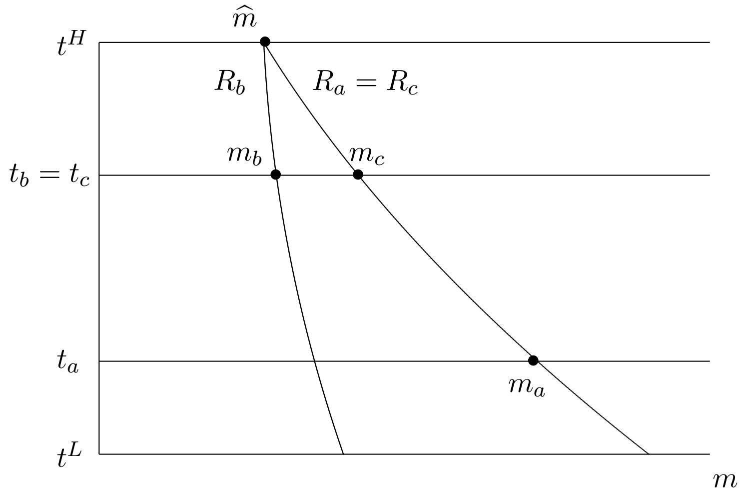


Figure 1.2: *Agent b has an incentive to misreport her preferences.*

Consider, for instance, the simple three-agent economy depicted in figure 1.2. At the optimum, we have full compensation between agent a and agent c. On the other hand, agent b and agent c, who have different preferences but equal talent, receive a different transfer: $m_c > m_b$. This clearly gives an incentive to agent b to misreport her preferences. We can however still use the ranking criterion we have proposed in order to choose the optimal policy available given the informational constraints.

From the previous example it is easy to understand that, if the social planner knows the distribution of individual preferences, an allocation $m_N \in \mathbb{R}_+^N$ is incentive-compatible if and only if

$$m_i = m_j \ for \ all \ i, j \in N \ with \ t_i = t_j.$$

Before we show the shape of the optimal incentive compatible allocation we need to introduce two more assumptions. The first one is nothing but the adaptation to this framework of the Mirrlees-Spence single crossing condition: for any couple of preferences ordering R and R' in $\mathcal{R}^e$ we either have that R exhibits a higher degree of inequality aversion to lack of talent than R' or vice versa. This means that it is possible, within each economy, to rank all the preference orderings with respect to the aversion to lack of talent they exhibit.

Assumption 1: For each $R, R' \in \mathcal{R}^e$, with $R \neq R'$, either $R \succ_A R'$ or $R' \succ_A R$.

[9] If the talent was not observable either, the only incentive-compatible allocation would trivially be the one that gives an equal split of the available resources to each agent.

Let R^H and R^L denote the preference orderings in $\mathcal{R}^e$ with, respectively, highest and lowest aversion to lack of talent.

The second assumption we introduce is a richness requirement that looks quite reasonable specially if the number of agents in the economy is considerably large: for any class of agents with a given talent $t \in T^e$ there is at least one agent with preferences R, for each $R \in \mathcal{R}^e$.

Assumption 2: For each $(t, R) \in T^e \times \mathcal{R}^e$ there exists $i \in N$ such that $R_i = R$ and $t_i = t$.

Consider an incentive-compatible allocation, by Assumption 2, for each $t \in T^e$, there is at least one agent with such a talent and with preferences equal to R^H. It is easy to check that, among the other agents with the same talent, she is the one with lowest $\widehat{E}(.)$ (for an example see figure 1.3, for simplicity we represent an economy where there are only two kinds of individual preferences: R^L and R^H). Since we are using the maximin criterion, we should equalize, whenever possible, the lowest $\widehat{E}(.)$ values across different classes of talent.

Theorem 2 *In any economy $e \in \mathcal{D}$, under Assumptions 1 and 2, if a Social Ordering Function satisfies* Strong Pareto*,* Nested Preferences Transfer*,* Equal Preferences Permutation*,* Equal Talent Transfer to the Unhappy*,* Minimal Equal Talent Transfer *and* Separation *then, the optimal incentive compatible allocation $(m_N^{t^H})$ distributes the available resources in the following way:*

$$\textit{for all } i, j \in N, (m_i^{t^H}, t_i) I^H (m_j^{t^H}, t_j) \textit{ or } m_i^{t^H} = 0 \textit{ and } (0, t_i) R^H (m_j^{t^H}, t_j).$$

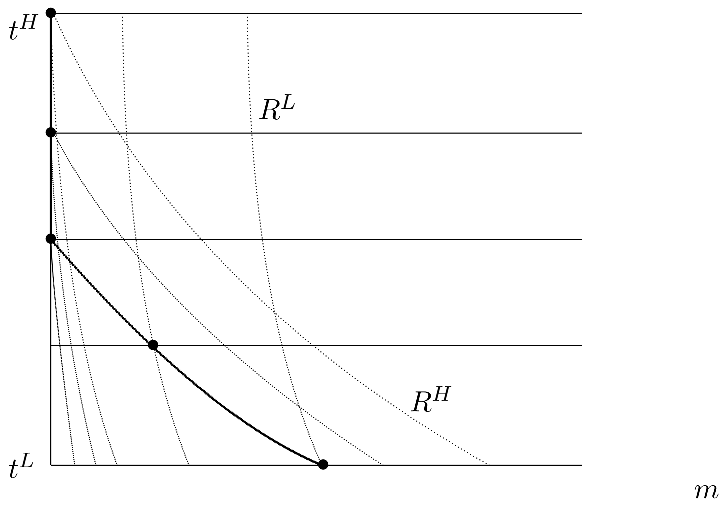


Figure 1.3: *Proof of Theorem 2.*

The resulting allocation is strongly affected by the incentive compatibility constraint: people with the same talent receive the same monetary transfer. The allocation we obtain by maximizing our (compensation minded) social ordering function under incentive-compatibility constraints is equivalent to a specific rule ($\widetilde{R} = R^H$) of the $\widetilde{R}$-Conditional Equality family of rules. Hence we finally obtain a first best efficient scheme where the principle of equal resources for equal talent is fully satisfied, nonetheless it shows us the highest level of priority that we can give to the principle of compensation given the informational constraints we have to face. Moreover resources are distributed taking into account the point of view of the individuals with highest aversion to lack of talent. Equality of extended resources is in fact pursued in the counterfactual situation were all agents have preferences over money-talent pairs equal to R^H .

Finally it may be impossible to fully compensate some inequalities in personal characteristics with the available resources (for example, the proportion of individuals with a low disability is typically higher than the proportion of agents with a severe disability). In that case, some inequalities persist and the better-off agents are left with no external resource, while only disadvantaged agents receive a strictly positive monetary compensation. This determines a talent threshold above which $m = 0$. Such a compensation path actually gives maximal priority to agents with a lower talent leaving a relatively big share of the population without any compensation.

This gives justification to several compensation schemes used in reality that choose to leave with no transfer people with a relatively high talent. For example, the scheme of benefits for people with disabilities used in Italy (but similar schemes are used in Belgium, France and USA) is illustrated in figure 1.4.[10] The vertical axis represents the degree of inability to work, whereas the horizontal axis represents bundles of benefits. The shape of the curve gives us a clear idea of how the compensation policy is designed: the bundle of benefits grows cumulatively as the level of handicap increases and agents with the same level of disability are equally treated. A significant part of the applicants, people with a level of disability between 0 and 32%, does not receive any kind of compensation. People with a level of disability between 33% and 73% are entitled to some monetary and non-monetary benefits that cumulate depending on the severity of their condition (i.e., specific medical supplies, priority in the unemployment, free drugs). People with a disability between 74% and 99% also receive a monthly monetary subsidy but only until their retirement age and provided their earnings are below a certain threshold. Individuals with a disability of 100% receive the same amount of money permanently and independently of their wealth condition.

Figure 1.4: *The Italian social security scheme of benefits for people with disabilities.*

[10]The website http://www.disabili.it gives a precise description of the compensation scheme, the figure is mine and it has only a descriptive purpose.

1.6 Conclusion

In the first part of this paper we have examined how axioms embodying the principle of compensation and axioms weakly embodying the principle of equal resources for equal talent lead to a particular way of evaluating individual welfare and single out particular social preferences. Such social preferences grant absolute priority to agents who dislike their level of talent and to agents who receive a low compensation. The measure of individual situations obtained here (that is, $\widehat{E}(.)$) is derived from the fairness principles and the analysis did not require any other information about individual welfare than ordinal non-comparable preferences.

The second part of this paper has studied the implications of such social preferences for the evaluation of incentive-compatible policies. By maximizing a compensation minded social ordering function, under the relevant informational constraints, we anyway obtain a scheme that gives equal transfer to agents with equal talents. The fact that incentive constraints prevent the full expression of priority given to the principle of compensation does not render the principle irrelevant. On the contrary it tells us to what extent an implementable policy can take this ethical requirement into account.

Our main result is that resources should be allocated by giving maximal priority to agents with a low talent, they in fact receive the largest relative compensation one can think of.

This also implies that leaving a certain share of the population without a monetary compensation can be defended on normative grounds. In the scheme we propose such a share is endogenously determined. This result crucially depends on the particular weakening of Equal Talent Transfer we have chosen. For example, we might have opted for a different weakening entailing the arbitrary choice of a reference talent $\widetilde{t}$. Namely, to impose the full transfer condition only among agents with a talent equal to $\widetilde{t}$.

$\widetilde{t}$-Equal Talent Transfer. *For all* $e \in \mathcal{D}$, $m_N, m'_N \in \mathbb{R}^N_+$, *for at least one* $\widetilde{t} \in T$ *if there exist* $j, k \in N$ *and* $\Delta \in R_+$ *such that* $t_j = t_k = \widetilde{t}$ and $m_j - \Delta = m'_j > m'_k = m_k + \Delta$, *with* $m_i = m'_i$ *for all* $i \neq j, k$, *then* $m'_N \overline{P}(e) m_N$.

If we replaced, Equal Talent Transfer to the Unhappy with $\widetilde{t}$-Equal Talent Transfer in Theorem 1 we would obtain that the welfare representation of individual utility becomes a function of the parameter $\widetilde{t}$. Namely, for each $i \in N$

$$\widehat{E}(m_i, t_i, R_i, \widetilde{t}) = \begin{cases} \widehat{m}_i & \text{if } \widehat{m}_i \text{ exists} \\ \widehat{t}_i - \widetilde{t} & \text{otherwise} \end{cases}$$

Moreover, it would be possible to adapt the proof to theorem 1 too see that on the domain $\mathcal{D}$, for any m_N, $m'_N \in \mathbb{R}^N_+$, $m'_N \overline{P}(e) m_N$ if $min_i \widehat{E}(m'_i, t_i, R_i, \widetilde{t}) > min_i \widehat{E}(m_i, t_i, R_i, \widetilde{t})$.

The main drawback of this social ordering function is that the social ordering would depend on the (arbitrary) choice of the reference talent. This would give the social planner the freedom of choosing the level of priority to be given to agents with a severe disability. Among the agents with a talent higher than $\widetilde{t}$ the worst-off would be those with preferences equal to R^L. On the contrary, among the agents with a talent lower than $\widetilde{t}$ the worst-off would be those with preferences equal to R^H. By equalizing the lowest $\widehat{E}(.)$ values across different classes we would obtain schemes that look like those depicted in figure 1.5. Notice that the lower is the reference talent chosen by the social planner the smaller is the compensation received by people with a severe disability, the smaller is the share of population left without compensation.

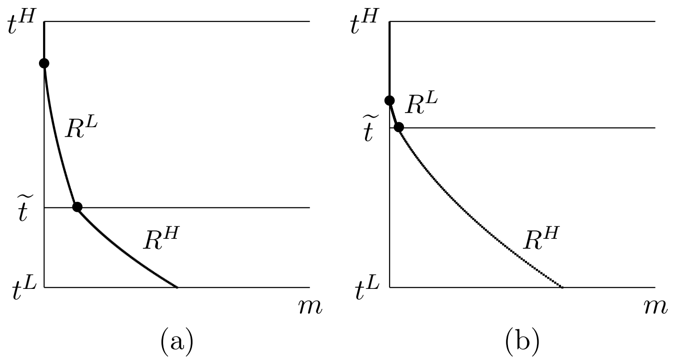


Figure 1.5: *A different reference talent determines a different compensation policy.*

Finally let us briefly discuss the assumptions underlying the analysis. First it is worth stressing that all the results obtained can be generalized to a framework without the non-negativity constraint on the monetary transfer. In particular, the first condition in the definition of $\widehat{E}(.)$ would always be satisfied. Moreover the compensation scheme we propose would eventually ask people with higher talent to pay a tax rather than receiving no transfer but still, the distribution of resources would be performed taking into account the point of view of agents with highest aversion to lack of talent. More precisely we would have that the optimal incentive compatible allocation $(m_N^{t^H})$ distributes the available resources in the following way:

$$for\ all\ i, j \in N, (m_i^{t^H}, t_i) I^H (m_j^{t^H}, t_j).$$

In Fleurbaey [17] no particular structure is assumed on the set T. Here Lemma 1,2 and 3 would still hold under the same assumption too (and this would be the most general domain in which they hold). On the other hand assuming that T has at least some ordinal structure plays a fundamental role in the way we built the weakening of Equal Preferences Transfer. In particular it would not be possible to talk about aversion to lack of talent without imposing a structure on T. As a consequence we obtain a welfare representation of individual preferences that also relies on such a structure.

1.7 Appendix

Proof of Lemma 1. Take the economy $e = (t, t', t', t, R', R', R, R, M) \in \mathcal{D}$, the extended bundles (m^1, t), (m^2, t), (m^3, t), (m^4, t), (m^5, t'), (m^6, t'), (m^7, t'), (m^8, t') and $\Delta \in \mathbb{R}_{++}$ with:

1. $t \neq t'$
2. $(m^1, t) P (m^8, t')$
3. $(m^5, t') P' (m^4, t)$
4. $m^4 - \Delta = m^3 > m^2 = m^1 + \Delta$

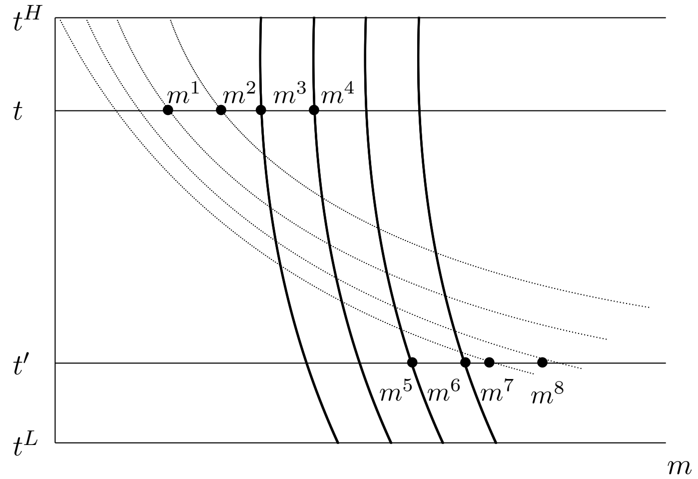


Figure 1.6: *Equal Preferences Transfer and Equal Talent Transfer are incompatible.*

5. $m^8 - \Delta = m^7 > m^6 = m^5 + \Delta$

We give a graphical example of such an economy in figure 1.6. By *Equal Preferences Transfer* and conditions 2, 4 and 5, $(m^1, m^8, m^5, m^3)\ \overline{P}(e)\ (m^2, m^7, m^5, m^3)$. By *Equal Talent Transfer* and condition 5, $(m^1, m^7, m^6, m^3)\ \overline{P}(e)\ (m^1, m^8, m^5, m^3)$. By *Equal Preferences Transfer* and conditions 3, 4 and 5, $(m^1, m^7, m^5, m^4)\ \overline{P}(e)\ (m^1, m^7, m^6, m^3)$. By *Equal Talent Transfer* and condition 4, $(m^2, m^7, m^5, m^3)\ \overline{P}(e)\ (m^1, m^7, m^5, m^4)$. Finally, by transitivity, $(m^1, m^7, m^6, m^3)\ \overline{P}(e)\ (m^1, m^7, m^6, m^3)$, which yields the desired contradiction. ■

Proof of Lemma 2. Take the economy $e = (t, t, t', t', R, R', R', R, M) \in \mathcal{D}$, the extended bundles (m^1, t), (m^2, t), (m^3, t), (m^4, t), (m^5, t), (m^6, t), $(m^{1'}, t')$, $(m^{2'}, t')$, $(m^{3'}, t')$, $(m^{4'}, t')$, $(m^{5'}, t')$, $(m^{6'}, t')$ with:

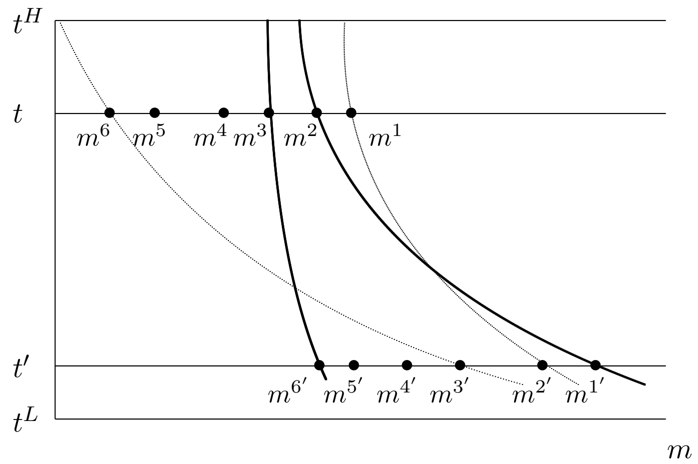


Figure 1.7: *Strong Pareto, Equal Preferences Permutation and Equal Talent Permutation are incompatible.*

1. $t \neq t'$

2. $m^1 > m^2 > m^3 > m^4 > m^5 > m^6$

3. $m^{1'} > m^{2'} > m^{3'} > m^{4'} > m^{5'} > m^{6'}$

4. $(m^1, t)I(m^{2'}, t')$ *and* $(m^6, t)I(m^{3'}, t')$

5. $(m^2, t)I'(m^{1'}, t')$ *and* $(m^3, t)I'(m^{6'}, t')$

A graphical example of such an economy is given in figure 1.7. By *Strong Pareto* and conditions 2 and 3, $(m^1, m^4, m^{3'}, m^{5'})\ \overline{P}(e)\ (m^2, m^5, m^{4'}, m^{6'})$. By *Equal Talent Permutation* and condition 3, $(m^4, m^1, m^{3'}, m^{5'})\ \overline{I}(e)\ (m^1, m^4, m^{3'}, m^{5'})$. By *Equal Preferences Permutation* and condition 4, $(m^4, m^6, m^{2'}, m^{5'})\ \overline{I}(e)\ (m^4, m^1, m^{3'}, m^{5'})$. By *Equal Talent Permutation* and condition 3, $(m^4, m^6, m^{2'}, m^{5'})\ \overline{I}(e)\ (m^4, m^6, m^{5'}, m^{2'})$. By *Strong Pareto* and conditions 2 and 3, $(m^3, m^5, m^{4'}, m^{1'})\ \overline{P}(e)\ (m^4, m^6, m^{5'}, m^{2'})$. By *Equal Preferences Permutation* and condition 5, $(m^2, m^5, m^{4'}, m^{6'})\ \overline{I}(e)\ (m^3, m^5, m^{4'}, m^{1'})$. Finally by transitivity, we have $(m^2, m^5, m^{4'}, m^{6'})\ \overline{P}(e)\ (m^2, m^5, m^{4'}, m^{6'})$, a contradiction. ■

Proof of Lemma 3. Let $\overline{R}$ satisfy *Nested Preferences Transfer*, *Equal Preferences Permutation* and *Separation*. Let $e = (t_N, R_N, M) \in \mathcal{D}$, $m_N, m'_N \in \mathbb{R}^N_+$, $j, k \in N$ be such that:

$$(m_j, t_j)P_j(m'_j, t_j)P_j(m'_k, t_k)P_k(m_k, t_k),$$

$$U(m'_j, R_j) \bigcap L(m'_k, R_k) = \emptyset,$$

and for all $i \neq j, k$, $m_i = m'_i$. We want to prove that $m'_N \overline{P}(e) m_N$. Let $\triangle_j = (m_j - m'_j)$ and $\triangle_k = (m'_k - m_k)$. Let $b, c \in \mathbb{N}_{++} \setminus N$, $R_b, R_c \in \mathcal{R}$, t_b, t_c, $m^1_b, m^2_b, m^1_c, m^2_c \in \mathbb{R}_+$ and $q \in \mathbb{N}_{++}$, be defined in such a way that: $R_b = R_c$; $(m^1_b, t_b)I_b(m^1_c, t_c)$ and $(m^2_b, t_b)I_b(m^2_c, t_c)$; $m^1_b = m^2_b + \frac{\triangle_k}{q}$ and $m^1_c = m^2_c + \frac{\triangle_j}{q}$; $(m^2_b, t_b)P_b(m'_k, t_k)$, $U((m^2_b, t_b), R_b) \bigcap L((m'_k, t_k), R_k) = \emptyset$, $(m'_j, t_j)P_j(m^1_b, t_b)$ and $U((m'_j, t_j), R_j) \bigcap L((m^1_b, t_b), R_b) = \emptyset$. Let $e' = (t_N, t_b, t_c, R_N, R_b, R_c, M) \in \mathcal{D}$ (an example of such a construction is given in figure 1.8).

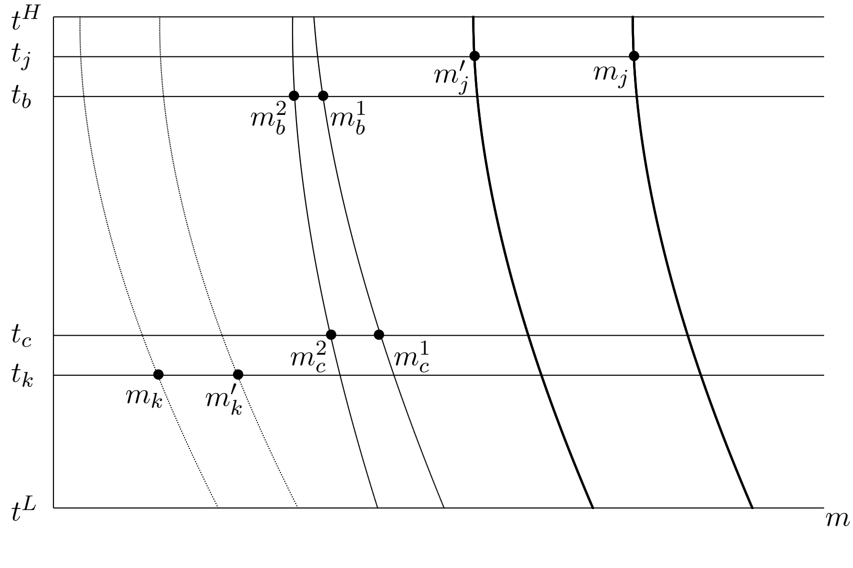


Figure 1.8: *From Nested Preferences Transfer to Nested Preferences Priority.*

By *Nested Preferences Transfer*,

$$(m_{N\setminus\{k\}}, m_k + \tfrac{\triangle_k}{q}, m_b^2, m_c^1)\overline{P}(e')(m_N, m_b^1, m_c^1).$$

By *Equal Preferences Permutation*,

$$(m_{N\setminus\{k\}}, m_k + \tfrac{\triangle_k}{q}, m_b^1, m_c^2)\overline{I}(e')(m_{N\setminus\{k\}}, m_k + \tfrac{\triangle_k}{q}, m_b^2, m_c^1).$$

By *Nested Preferences Transfer*,

$$(m_{N\setminus\{k,j\}}, m_k + \tfrac{\triangle_k}{q}, m_j - \tfrac{\triangle_j}{q}, m_b^1, m_c^1)\overline{P}(e')(m_{N\setminus\{k\}}, m_k + \tfrac{\triangle_k}{q}, m_b^1, m_c^2).$$

By transitivity,

$$(m_{N\setminus\{k,j\}}, m_k + \tfrac{\triangle_k}{q}, m_j - \tfrac{\triangle_j}{q}, m_b^1, m_c^1)\overline{P}(e')(m_N, m_b^1, m_c^1).$$

Replicating the argument q times yields

$$(m_{N\setminus\{k,j\}}, m_k + \triangle_k, m_j - \triangle_j, m_b^1, m_c^1)\overline{P}(e')(m_N, m_b^1, m_c^1).$$

Since $m_j - \triangle_j = m'_j$ and $m_k + \triangle_k = m'_k$, by replacing into the former equation one obtains

$$(m'_N, m_b^1, m_c^1)\overline{P}(e')(m_N, m_b^1, m_c^1).$$

Finally, by *Separation*,

$$m'_N\overline{P}(e)m_N,$$

the desired result. ■

Proof of Theorem 1. Let $\overline{R}$ be a social ordering function that satisfies *Strong Pareto*, *Nested Preferences Transfer*, *Equal Preferences Permutation*, *Equal Talent Transfer to the Unhappy*, *Minimal Equal Talent Transfer* and *Separation.* By Lemma 3, the social ordering function $\overline{R}$ also satisfies *Nested Preferences Priority.* For the sake of clarity the remaining of the proof is divided in two steps.
Step 1. Consider two allocations m_N and m'_N and two different agents j and k such that for all $i \neq j, k$, $m_i = m'_i$. Let

$$\widehat{E}_m = min\{\widehat{E}(m_j, t_j, R_j), \widehat{E}(m'_j, t_j, R_j), \widehat{E}(m'_k, t_k, R_k).\}$$

Without loss of generality, let us assume that $\widehat{E}(m_k, t_k, R_k) < \widehat{E}_m$. We want to prove that $m'_N\overline{P}(e)m_N$. If we also have $m'_j \geqslant m_j$ then by *Strong Pareto* we immediately get the desired result. Consider now the case $m_j > m'_j$ and assume, by contradiction, that $m_N\overline{R}(e)m'_N$. We have to examine three possible cases.
•*Case 1*: $\widehat{E}(m_k, t_k, R_k) < 0$.
Let $b, c \in \mathbb{N}_{++} \setminus N$, $R_b, R_c \in \mathcal{R}$, $t_b, t_c \in T$, $m_b, m_c, m''_k \in \mathbb{R}_+$ be defined in such a way that: (a graphical example of the following construction is given in figure 1.9):

1. $\widehat{t}_k < t_b = t_c < t^H$ if $\widehat{m}_m$ exists, $\widehat{t}_k < t_b = t_c < \widehat{t}_m$ otherwise
2. $R_k = R_b$ and $R_j = R_c$
3. $0 = m_b < m_c$ and $(m'_j, t_j)P_j(m_c, t_c)$
4. $m_k < m''_k < m'_k$ and $(m_b, t_b)P_k(m''_k, t_k)$.

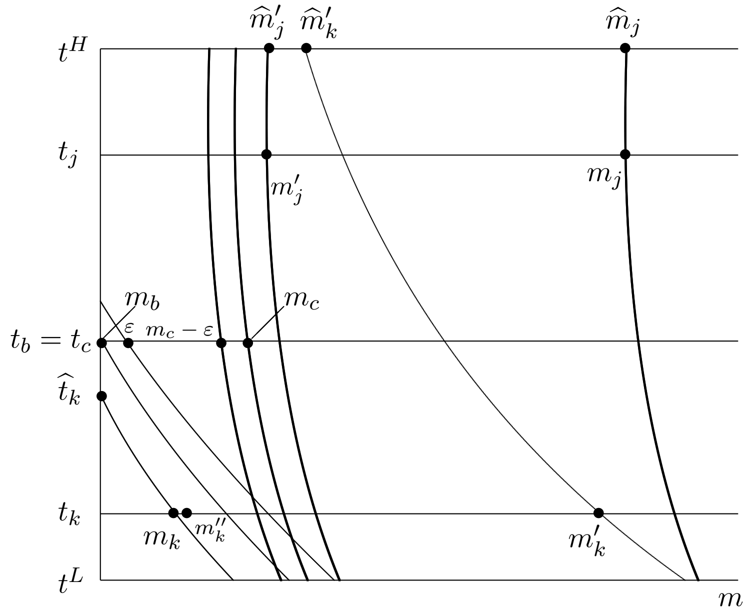


Figure 1.9: *Case 1.*

Let $e' = (t_j, t_k, R_j, R_k, M)$, $e'' = (t_j, t_k, t_b, t_c, R_j, R_k, R_b, R_c, M) \in \mathcal{D}$. Since we have assumed $m_N \overline{R}(e) m'_N$ then, by *Separation*, we have $(m_j, m_k)\ \overline{R}(e')\ (m'_j, m'_k)$. Take any ε such that $0 < \varepsilon < \frac{m_c}{2}$. By *Separation* again $(m_j, m_k, m_b + \varepsilon, m_c - \varepsilon)\ \overline{R}(e'')\ (m'_j, m'_k, m_b + \varepsilon, m_c - \varepsilon)$. By *Nested Preferences Priority* and conditions 2 and 3, $(m'_j, m_k, m_b + \varepsilon, m_c)\ \overline{P}(e'')\ (m_j, m_k, m_b + \varepsilon, m_c - \varepsilon)$. By *Nested Preferences Priority* and conditions 2 and 4, $(m'_j, m''_k, m_b, m_c)\ \overline{P}(e'')\ (m'_j, m_k, m_b + \varepsilon, m_c)$. By *Strong Pareto* and condition 4, $(m'_j, m'_k, m_b, m_c)\ \overline{P}(e'')\ (m'_j, m''_k, m_b, m_c)$. By *Transitivity*, $(m'_j, m'_k, m_b, m_c)\ \overline{P}(e'')\ (m'_j, m'_k, m_b + \varepsilon, m_c - \varepsilon)$. This is true for each ε such that $0 < \varepsilon < \frac{m_c}{2}$ hence, given condition 1 and 3, it represents a violation of *Minimal Equal talent Transfer* and yields the desired contradiction so that $m'_N \overline{P}(e) m_N$

•*Case 2*: $\widehat{E}(m_k, t_k, R_k) \geq 0$ and $R_k \succ_A R_j$.

Let $b, c \in \mathbb{N}_{++} \setminus N$, $R_b, R_c \in \mathcal{R}$, $t_b, t_c \in T$, $m_b, m'_b, m_c, m'_c, m''_k \in \mathbb{R}_+, \Delta \in \mathbb{R}_{++}$ be defined in such a way that (see figure 1.10):

1. $t_b = t_c = t^H$
2. $R_k = R_b$ and $R_j = R_c$
3. $\widehat{E}(m_k, t_k, R_k) < m'_b < m'_c < \widehat{E}_m$ and $m'_b + \Delta = m_b \leq m_c = m'_c - \Delta$
4. $m_k < m''_k$ and $(m_b, t_b) P_k (m''_k, t_k)$.

Let $e' = (t_j, t_k, R_j, R_k, M)$, $e'' = (t_j, t_k, t_b, t_c, R_j, R_k, R_b, R_c, M) \in \mathcal{D}$. As in the previous proof we assume, $m_N \overline{R}(e) m'_N$. By applying *Separation* twice we have $(m_j, m_k)\ \overline{R}(e')\ (m'_j, m'_k)$ and $(m_j, m_k, m_b, m_c)\ \overline{R}(e'')\ (m'_j, m'_k, m_b, m_c)$. By *Nested Preferences Priority* and conditions 2, 3 and 4, $(m'_j, m_k, m_b, m'_c)\ \overline{P}(e'')\ (m_j, m_k, m_b, m_c)$ and $(m'_j, m''_k, m'_b, m'_c)\ \overline{P}(e'')\ (m'_j, m_k, m_b, m'_c)$. By *Strong Pareto* and condition 4, $(m'_j, m'_k, m'_b, m'_c)\ \overline{P}(e'')\ (m'_j, m''_k, m'_b, m'_c)$. By *Transitivity*, $(m'_j, m'_k, m'_b, m'_c)\ \overline{P}(e'')\ (m'_j, m'_k, m_b, m_c)$, which, given conditions 1 and 2, is a violation of *Equal Talent Transfer to the Unhappy* since, by construction, $R_b \succ_A R_c$. This yields the desired contradiction so that $m'_N \overline{P}(e) m_N$.

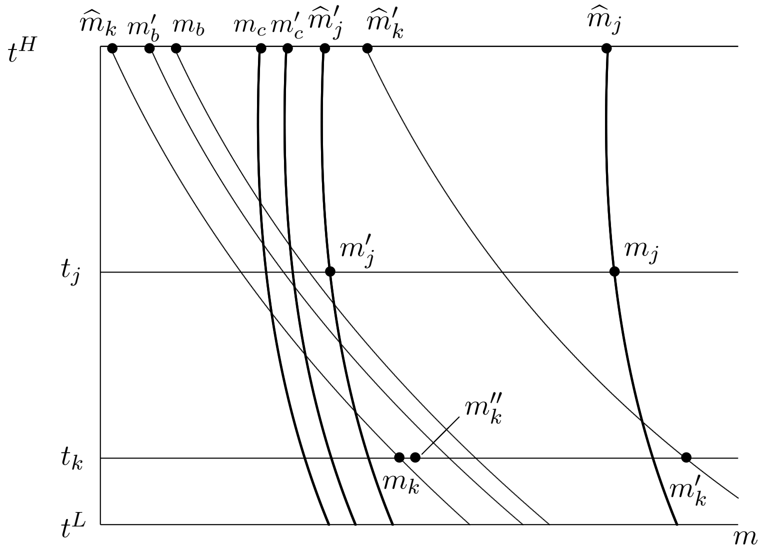


Figure 1.10: *Case 2.*

•*Case 3*: $\widehat{E}(m_k, t_k, R_k) \geq 0$. and $R_j \succ_A R_k$.
If we have $\widehat{E}(m'_k, t_k, R_k) < \widehat{E}(m'_j, t_j, R_j)$ then directly by *Nested Preference Transfer* we get the desired result. On the other hand, assume $\widehat{E}(m'_k, t_k,\ R_k) \geq \widehat{E}(m'_j, t_j, R_j)$

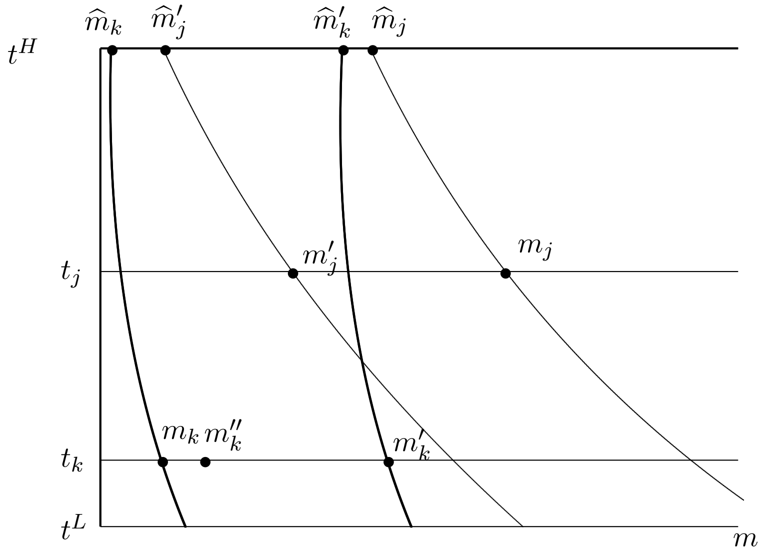


Figure 1.11: *Case 3.*

and $m_N \overline{R}(e) m'_N$. Consider m''_k such that $m_k < m''_k < m'_k$, $(m'_j, t_j) P_j\ (m''_k, t_k)$ and $U((m'_j, t_j), R_j) \bigcap L((m''_k,\ t_k), R_k) = \emptyset$. Let $e' = (t_j, t_k, R_j, R_k, M)$. By *Separation* we have $(m_j, m_k)\ \overline{R}(e')(m'_j, m'_k)$. By *Nested Preferences Priority*, $(m''_k, m'_j)\overline{P}(e')\ (m_k, m_j)$. By *Strong Pareto*, $(m'_k, m'_j)\ P(e')(m_k, m_j)$ which yields the desired contradiction.

Step 2. Take now two allocations m_N and m'_N such that

$$min_i \widehat{E}(m'_i, t_i, R_i) > min_i \widehat{E}(m_i, t_i, R_i).$$

Then, by monotonicity of preferences, one can find two allocations $\overline{m}_N$, $\overline{m}'_N$ such that for all i, $m_i < \overline{m}_i$, $\overline{m}'_i < m'_i$, and there exists i_0 such that for all $i \neq i_0$

$$\widehat{E}(\overline{m}_i, t_i, R_i) > \widehat{E}(\overline{m}'_i, t_i, R_i) > \widehat{E}(\overline{m}'_{i_0}, t_{i_0}, R_{i_0}) > \widehat{E}(\overline{m}_{i_0}, t_{i_0}, R_{i_0}).$$

Let $Q = N \backslash \{i_0\}$ and consider a sequence of allocations $(\overline{m}^q_N)_{1 \leq q \leq |Q|+1}$ such that

$$\begin{array}{l} \overline{m}^q_i = \overline{m}'_i \text{ for all } i \in Q \text{ such that } i < q \\ \overline{m}^q_i = \overline{m}_i \text{ for all } i \in Q \text{ such that } i \geq q \end{array}$$

while

$$\overline{m}'_{i_0} = \overline{m}^{|Q|+1}_{i_0} > \overline{m}^{|Q|}_{i_0} > ... > \overline{m}^1_{i_0} = \overline{m}_{i_0}.$$

This entails that for all $q \in Q$

$$\widehat{E}(\overline{m}^q_q, t_m, R_m) > \widehat{E}(\overline{m}^{q+1}_q, t_m, R_k) > \widehat{E}(\overline{m}^{q+1}_{i_0}, t_{i_0}, R_{i_0}) > \widehat{E}(\overline{m}^q_{i_0}, t_{i_0}, R_{i_0}),$$

while for all $q \in Q$, and all $i \neq i_0, q$, $\overline{m}^q_i = \overline{m}^{q+1}_i$. By Step 1, $\overline{m}^{q+1}_N P(e) \overline{m}^q_N$ for all $q \in Q$. By *Strong Pareto*, $m'_N P(e) \overline{m}^{|Q|+1}_N$ and $\overline{m}^1_N P(e) m_N$. By transitivity, $m'_N P(e) m_N$. ■

Examples of social ordering functions satisfying all the axioms but:

1. *Strong Pareto*. Compare the welfare-level vectors using the lexicographic minimax criterion: a distribution is at least as good as another one if its maximum is lower, or they are equal and the second highest value is lower, or... or they are all equal.

2. *Nested Preferences Transfer*. Let $\overline{R}$ be defined by: for all $e \in \mathcal{D}$, $m_N, m'_N \in \mathbb{R}^+_N$, if

$$(\widehat{E}(m'_i, t_i, R_i))_{i \in N} >_L (\widehat{E}(m_i, t_i, R_i))_{i \in N}$$

and there exist $k, k' \in N$ with $t_k \neq t_{k'}$ such that

$$0 < \widehat{E}(m_k, t_k, R_k) < \widehat{E}(m'_k, t_k, R_k) < \widehat{E}(m'_{k'}, t_{k'}, R_{k'}) <$$

$$\widehat{E}(m_{k'}, t_{k'}, R_{k'}),$$

with $\widehat{E}(m'_i, t_i, R_i) = \widehat{E}(m_i, t_i, R_i)$ for all $i \neq k, k'$, then $m'_N \overline{I}(e) m_N$. In all other cases $m'_N \overline{P}(e) m_N$. If

$$(\widehat{E}(m'_i, t_i, R_i))_{i \in N} =_L (\widehat{E}(m_i, t_i, R_i))_{i \in N}$$

then $m'_N \overline{I}(e) m_N$.

3. *Equal Preferences Permutation*. Let $\overline{R}$ be defined by: for all $e \in \mathcal{D}$, $m_N, m'_N \in \mathbb{R}^+_N$, if

$$(\widehat{E}(m'_i, t_i, R_i))_{i \in N} >_L (\widehat{E}(m_i, t_i, R_i))_{i \in N}$$

or

$$(\widehat{E}(m'_i, t_i, R_i))_{i \in N} =_L (\widehat{E}(m_i, t_i, R_i))_{i \in N}$$

and there exist $k, k' \in N$ with $t_{k'} \neq t_k$ such that $\widehat{E}(m_k, t_k, R_k) < \widehat{E}(m_i, t_i, R_i)$ for all $i \neq k$; $\widehat{E}(m'_{k'}, t_{k'}, R_{k'}) < \widehat{E}(m'_i, t_i, R_i)$ for all $i \neq k'$, then $m'_N \overline{P}(e) m_N$. In all other cases $m'_N \overline{I}(e) m_N$.

4. *Equal Talent Transfer to the Unhappy and Minimal Equal Talent Transfer.* Let $\overline{R}$ be defined by: for all $e \in \mathcal{D}$, $m_N, m'_N \in \mathbb{R}^+_N$, if

$$(\widehat{E}(m'_i, t_i, R_i))_{i \in N} >_L (\widehat{E}(m_i, t_i, R_i))_{i \in N}$$

and there exist $k, k' \in N$ with $R_{k'} \neq R_k$ such that

$$\widehat{E}(m_k, t_k, R_k) < \widehat{E}(m'_k, t_k, R_k) < \widehat{E}(m'_{k'}, t_{k'}, R_{k'}) <$$
$$\widehat{E}(m_{k'}, t_{k'}, R_{k'}),$$

with $\widehat{E}(m'_i, t_i, R_i) = \widehat{E}(m_i, t_i, R_i,)$ for all $i \neq k, k'$ and $U((m'_{k'},\ t_{k'}), R_{k'}) \bigcap L((m'_k,$ $t_k), R_k) \neq \emptyset$, then $m'_N \overline{I}(e) m_N$. In all other cases $m'_N \overline{P}(e) m_N$. If

$$(\widehat{E}(m'_i, t_i, R_i))_{i \in N} =_L (\widehat{E}(m'_i, t_i, R_i))_{i \in N}$$

then $m'_N \overline{I}(e) m_N$.

5. *Separation.* See example page 11 paragraph 4.

Proof of Theorem 2. By Theorem 1 we know that if a Social Ordering Function satisfies the listed axioms then, for any $m'_N, m_N \in \mathbb{R}^N_+$,

$$min_i \widehat{E}(m'_i, t_i, R_i) > min_i \widehat{E}(m_i, t_i, R_i) \Longrightarrow m'_N \overline{P}(e) m_N$$

At $m^{t^H}_N$, given assumption 1 and 2, it holds true that the worst-off agents are (for a graphical example see figure 4) all $k \in N$ such that $R_k = R^H$ and $m_k > 0$:

$$min_i \{\widehat{E}(m^{t^H}_i, t_i, R_i)\}_{i \in N} = \widehat{E}(m^{t^H}_k, t_k, R^H).$$

Assume, by contradiction, that there exists an incentive compatible allocation m'_N such that $min_i \widehat{E}(m'_i, t_i, R_i) > min_i \widehat{E}(m^{t^H}_i, t_i,\ R_i)$. This implies that, for all $j \in N$ such that $R_j = R^H$ and $m'_j > 0$, $m'_j > m^{t^H}_j$. By incentive compatibility we should then increase by the same amount the transfer received by all the other agents having the same level of talent but different preferences. This contradicts that, by construction, $m^{t^H}_N$ is an allocation exhausting the available resources. ■

Chapter 2

Health, Fairness and Taxation.

2.1 Introduction

In this chapter we study the joint taxation of income and health expenditure. The literature has provided several possible reasons for public intervention in the domain of health care provision. Poterba [46] summarizes them in the following list. First, adverse selection and moral hazard problems might lead to market failures in the health insurance sector. Second, the existence of externalities (associated, for instance, to the risk of contagion of some illnesses) might induce agents to sub-optimal choices regarding health expenditure. Third, agents' irrationality might lead to a wrong perception of the the risks and the consequences of illness. Fourth, equity considerations might lead to think that basic health care must be accessible to all and that a system of health subsidies can be used as a mean for redistributing income across agents.

Our contribution is meant to provide some insights to the last item of such a list. Indeed, health can be considered a consumption good with a different price (due to a different health disposition) for different agents and they should not be held responsible for such a difference. Moreover, health affects the earning ability of different agents in a different way. Agents should not be held responsible for such a difference either. We are interested in equalizing opportunities (in term of resources) to the access to health care trough an adequate subsidy scheme. An ex post viewpoint is adopted in order to deal with uncertainty. Indeed, we are interested in the normative considerations that might lead to transfers from agents with a good health disposition to agents with a bad one. By taking an ex post viewpoint, we take account of all differences between individuals which may justify transfers, whether such differences come from fixed characteristics or from random events.

We propose a model where individuals differ in their health-dependent earning ability and in their health disposition, namely, they have to face unequal costs in order to be in good health. Moreover, they have heterogeneous preferences over consumption, health and labor. As a consequence, they typically make different choices about their labor time and their health status. In redistributing resources through an income tax a social planner typically face distortions due to asymmetry of information. Such an efficiency loss is typically justified by the potential improvement in the fairness of the distribution of resources. We deal with this trade-off by constructing social preferences that incorporate efficiency and fairness concerns.

The first ethical principle we adopt defends inequality reduction between individuals who have equal preferences but differ in their (health-dependent) earning ability and in their health disposition. Its main attempt is to reduce inequalities deriving from factors that do not depend on individual responsibility. The second one is oriented toward the acceptance of inequalities deriving from a different exercise of individual responsibility: there should not be redistribution across individuals at least when they all have equal earning ability and health disposition but possibly different preferences. In such a case, income inequalities would just reflect free choices from different preferences on an identical budget set, and such choices ought to be respected. These two requirements can be related to ethical principles that are well known in the fairness literature (see Fleurbaey and Maniquet [22] and Fleurbaey [20] for a detailed survey of the topic). Such ethical principles are embedded in axioms that are used to characterize a precise definition of social welfare.

Our main contribution is to provide a precise and simple criterion for the evaluation of tax policies. Different tax policies entail different social consequences and these are evaluated with the help of the social preferences we have defined. A tax policy here is a function defining a transfer of income depending *both* on the level of earnings and on the health expenditure. So two individuals with the same earnings might be subject to a different taxation (for example, one is subsidized and the other is taxed) solely because of a different health expenditure (due to different health care needs). Indeed, we are able to describe, under some richness assumption, which part of the budget set modified by the tax policy should be the focus of the social planner, and which modification of it leads to a social improvement. We prove that, when comparing two different tax policies, priority should always be given, for each level of health expenditure, to unskilled individuals first. Moreover, such a tool can be easily used by policy makers since little information about individual characteristics is needed for its application.

We consider this kind of contribution particularly useful. Typically, social reforms deal with the choice of a certain alternative belonging to a subset of suboptimal alternatives. Our criterion can be used for the comparison of any pair of tax policies, no matter how far from the optimum. In this sense, it could represent a better practical support to the social planner, than a theoretical characterization of the optimal tax scheme (usually out of reach). The criterion suggested in this paper is related to particular fairness principles, but the scope of our methodology is much larger in the sense that different principles might lead to different policy recommendations.

We also prove that, as far as the optimal tax policy is concerned, the consumption granted to hardworking-poor agents (unskilled agents who work full time) who decide to be healthy but have a different health dispositions, should be equalized. Moreover, within each sub-population defined by the health expenditure level, hardworking-poor agents are those who will be granted the greatest subsidy (i.e. the smallest tax).

The paper is organized as follows. Section 2 describes the relation to the literature, section 3 introduces the model and the relevant notation. Section 4 describes the requirements imposed on the social preferences. Section 5 deals with axiomatic derivation of social preferences. Section 6 proposes a way to compare different tax policies and describes some features of the optimal one. Section 7 concludes. The Appendix provides the proofs.

2.2 Relation to the literature

Our paper relates to different strands of literature. Indeed the idea that the joint taxation of income and health expenditure might lead to further redistribution than income tax alone is not new. Blomqvist and Horn [5], in a model where individuals differ in their labor productivity and in their probability of being ill, prove that subsidizing ill people (trough lump-sum transfers) can be an effective complement to linear taxation for redistributive purposes. Rochet [48] extends the analysis to a continuum of types and shows that the existence of a negative relation between productivity and morbidity (the probability to become ill) is a necessary and sufficient condition for full public health insurance to be optimal. Such a result is confirmed by Cremer and Pestieu [13] (who restrict their attention to a discrete distribution of types). Building on the same intuition Henriet and Rochet [33] explain the positive correlation between marginal tax rates (used as a proxy for social preferences toward redistribution) across different countries and the extent of public provision of health insurance. In these papers bad health is used as an imperfect signal of an agent's ex ante risk of falling ill, so that covering people against such a risk by mean of a public health insurance is welfare improving. Jack [37] considers a model where agents face heterogeneous prices for health improvement and where their health status affects their earning ability. In such a framework health expenditure could be either subsidized or taxed depending on the correlation between health status and income, and the elasticities of demand for health care and supply of labor. The papers belonging to this strand of literature tend to use, in a different way, health status or health expenditure as a signal of earning ability in order to relax, in some way, the incentive compatibility constraint. This might push further redistribution between skilled and unskilled agents and hence improve welfare. It is important to stress that these models typically rely on social welfare functions with little specification so that the tension toward redistribution is determined by some degree of concavity in the utilitarian objective. So it is natural to think that, with a more precise specification of the normative principles behind redistribution on could also obtain more precise conclusions. Moreover, individual preferences are usually assumed to be homogeneous: this makes the exercise of defining the social welfare function easier but undoubtedly diminishes the realism of the model.

We build axiomatically a precise definition of social welfare for an heterogeneous (in three dimensions) population. The axioms used to perform such an exercise are derived from fairness criteria that are specific to the problem we are dealing with. Such a methodology was introduced by Fleurbay and Maniquet [21] and has been used for the evaluation of public policies in several frameworks already (a wide and detailed description of the methodology and the possible applications is provided by Fleurbaey and Maniquet [28]).

We mostly build on Fleurbaey [18] and Valletta [55]. The former one in particular, studies how to define social objectives for the allocation of health and consumption in a model where individuals differ in their preferences over health and consumption, earning ability, and health disposition. The paper proposes to maximin the lowest level of full health equivalent consumption occurring in society. The full health equivalent consumption is the sincere answer to the question: how much would you need to be indifferent between your current bundle and being fully healthy? The main drawback of the paper is that agents offer labor inelastically so that the distortions that social security contributions might cause to the price of production factors, specially labor, cannot be taken into account by the model. We extend the analysis by adding labor, as a new dimension and we propose an alternative social objective. One could in fact think of applying the maximin criterion to the full-health leisure equivalent incomes, namely, the amount of income that would be sufficient for an individual if her health problems disappeared and she no longer had to work. As it will be clearer later this might lead to some ambiguous redistributive effects. In particular, redistribution might go, from the hardworking-poor to the lazy-rich (a skilled agent who decides to work little). Moreover such a way to rank alternatives might lead to redistribution even in a situation were all agents are equally productive and have the same health disposition.

On the other hand, Fleurbaey and Maniquet [24], [25] propose an array of social preferences and an array of criteria for the welfare evaluation of tax policies over the allocation of consumption and labor (ignoring health) based on normative criteria that have to be thought anew if a further dimension like health is added. A common features of the (optimal) tax policies they propose is that hardworking-poor agents should be granted the smallest tax burden of the whole population. But still, imposing the same tax policy to individuals with different health expenditure might lead, *de facto*, to income inequalities that are particularly undesirable among low income earners. Indeed, the main reason it might be desirable to integrate health care expenditure into a redistributive tax system is that even people with the same earning ability might have to face very different health-care costs.

Finally, our paper also relates, to some extent, to the literature about commodity taxation. In their seminal contribution Atkinson and Stiglitz [4] showed that within a population of individuals who differ only in their labor productivity, if preferences are separable between labor and consumption of other goods, then commodity taxation cannot increase welfare above the level obtained with an optimal income tax alone. Many studies have studied the robustness of this result (see Boadway and Pestieau [6] for an overview), Fleurbaey [19], in particular, has proven (using an approach similar to ours) that, in presence of heterogeneous preferences and non-linear commodity taxation, the hard-working poor agents should be submitted to a uniform or null commodity tax. The fact that, in our paper, agents have to face unequal costs in order to be in good health leads to a quite different result: the optimal tax should always tax less (subsidize more) hardworking poor agents with a bad health disposition if compared to hardworking poor agents with a good health disposition.

2.3 The Model

We consider a set of economies with a finite set of agents $N \subseteq \mathbb{N}$. There are three goods: consumption, labor and health. A bundle, for agent $i \in N$, is a triple $z_i = (c_i, l_i, h_i)$, where c_i is consumption, l_i is labor, and h_i is health status. In particular, $c_i \in \mathbb{R}_+$ will be interpreted here as the expenditure on ordinary consumption goods, excluding health expenditure (medical treatments and drugs). As usual for this kind of analysis, $l_i \in [0, 1]$. For the sake of simplicity we assume that the choice of health is dichotomous. Each individual can either decide to keep a basic level of health (for simplicity we normalize such an amount to zero) or she can decide to face the necessary costs in order to be fully healthy (that is, $h = 1$). A more general model, describing health as a multidimensional, continuos variable, would be certainly more realistic and most of the results, in principle, would still hold. We think this very stylized description of reality better serves the purpose of the paper since it allows to describe with much more precision which direction redistribution should take among certain groups of agents. To sum up, each agent chooses her own bundle within the consumption set $Z_i \subset \mathbb{R}_+ \times [0, 1] \times \{0, 1\}$. An allocation describes each agent's bundle, and will be denoted by $z = (z_i)_{i \in N}$. Finally, let Z denote the set of all allocations.

Agents have three characteristics: their personal preferences, their earning ability and their health disposition.

For each agent $i \in N$, preferences are denoted R_i and $z'_i R_i z_i$ (resp $z'_i P_i z_i$, $z'_i I_i z_i$) means that bundle z'_i is weakly preferred (resp. strictly preferred, indifferent) to bundle z_i. Let $R = (R_i)_{i \in N}$ denote the population profile of preferences[1]. We restrict our attention to preferences which are continuous, strictly monotonic (increasing in c_i and h_i, decreasing in l_i) and convex.

[1]Given a list of objects $a = (a_i)_{i \in N}$, a_{-i} will denote the list $(a_1, ..., a_{i-1}, a_{i+1}, ..., a_N)$ and, more in general, a_{-S} will denote the original list excluding all the elements belonging to $S \subseteq N$.

The earning ability of each agent (namely, her marginal productivity of labor) is assumed to be an increasing function of health: being healthy increases productivity. To keep things simple we assume there are only two types of agents: unskilled and skilled ones. So, for each $i \in N$, either $w_i(h_i) = w^u(h_i)$ or $w_i(h_i) = w^s(h_i)$ with $w^u(0) < w^s(0)$ and $w^u(1) < w^s(1)$. The marginal productivity of labor is measured in consumption units so that for any l_i, $w_i l_i$ is the agent's pre-tax income (earnings). Let $w = (w_i(.))_{i \in N}$ denote the population profile of marginal-productivity functions.

Every individual $i \in N$ is also characterized by a mapping $m_i(h_i)$ describing how much of medical expenses must be made in order to bring her to health status h_i. We assume there are only two types of agents: agents with a good health disposition and agents with a bad health disposition. So, for each $i \in N$ either

$$m_i(h_i) = m_i^g(h_i) = \begin{cases} 0 & \text{if } \ h_i = 0 \\ m^g & \text{if } \ h_i = 1 \end{cases}$$

or

$$m_i(h_i) = m_i^b(h_i) = \begin{cases} 0 & \text{if } \ h_i = 0 \\ m^b & \text{if } \ h_i = 1 \end{cases}$$

with $m^g < m^b$. Such a mapping captures all factors that determine the medical costs, including not only purely medical features but also social and economic characteristics which influence each agent's health. Let $M = \left\{0, m^g, m^b\right\}$ denote the set of all possible health expenditure levels and $m = (m_i(.))_{i \in N}$ the population profile of health disposition mappings.

Again, the choice of only two types of agents, both for the description of earning ability and health disposition may look too restrictive. It would be possible to describe infinite types by mean of infinite functions. Most of the results that follow would still hold but it would be impossible to devise sharp conclusions about the direction redistribution should take among different types of agents. An economy is denoted by $e = (R, w(.), m(.))$. Let $\mathcal{D}$ denote the set of economies complying with our assumptions.

The first problem addressed here is the definition of social preferences over allocations. Social preferences will allow us to compare allocations in terms of fairness and efficiency. They will be formalized as a complete ordering over all the (feasible and not feasible) allocations and they will be denoted by $\overline{R}$ for the weak preferences, with related strict preferences $\overline{P}$ and indifference $\overline{I}$. In other words, $z'\overline{R}z$ means that the allocation z' is (socially) at least as good as z, $z'\overline{P}z$ means that it is strictly better, and $z'\overline{I}z$ that they are equivalent. The problem is to define social preferences for all cases that may occur. Formally, a social ordering function (SOF) is a mapping from the set of economies to the set of complete orderings over allocations. So, for each $e \in \mathcal{D}$, we write $\overline{R}(e)$, $\overline{P}(e)$, and $\overline{I}(e)$ in order to express the fact that particular social preferences are specific to the economy $e \in \mathcal{D}$.

An allocation is feasible if

$$\sum_{i=1}^{n} c_i + \sum_{i=1}^{n} m_i(h_i) \leq \sum_{i=1}^{n} w_i(h_i) l_i.$$

In absence of redistribution, the budget set of each agent $i \in N$ is equal the possible combinations of consumption, health and labor that are attainable for her given her earning ability and her health disposition. More formally:

$$B^*(w_i(.), m_i(.)) = \{(c, l, h) \in Z_i | c + m_i(h) \leq +w_i(h) l\}.$$

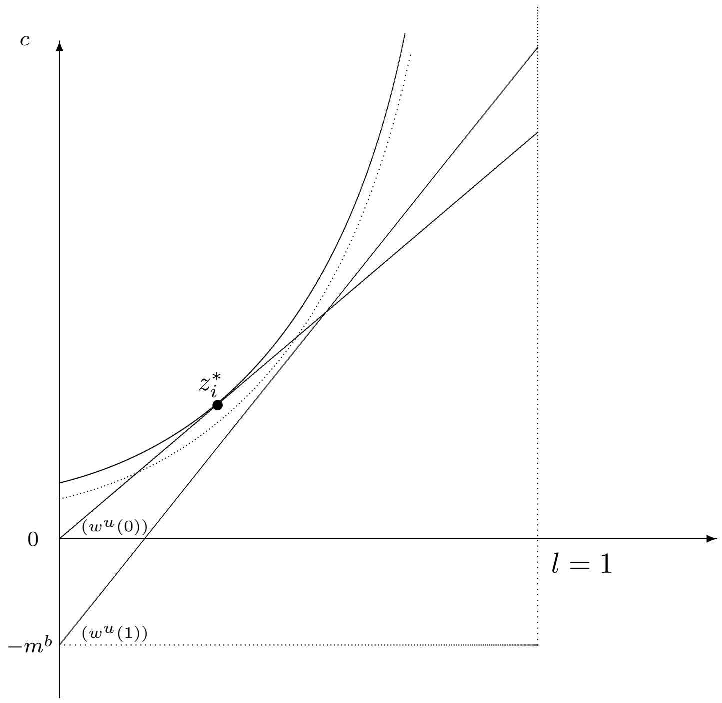


Figure 2.1: The main features of the model.

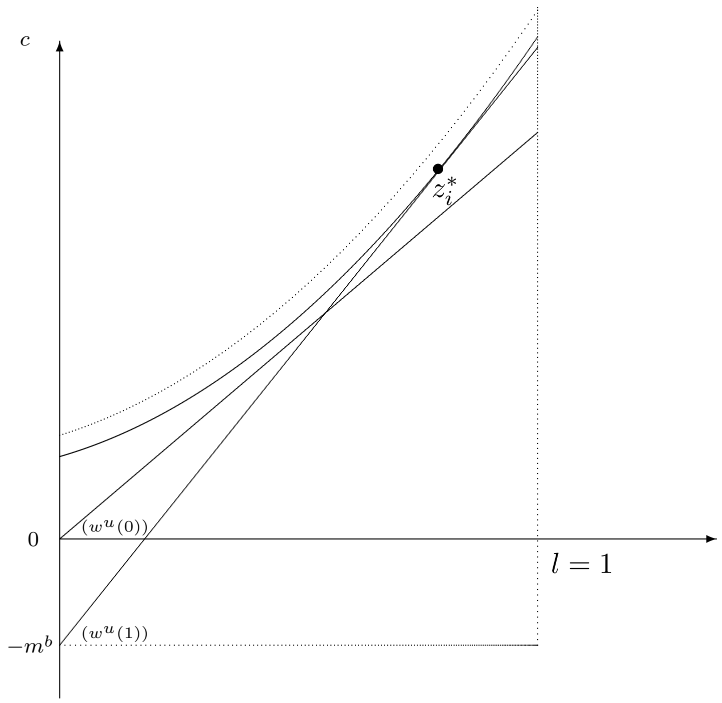


Figure 2.2: The main features of the model.

A $laissez-faire$ allocation z^* is such that for every agent $i \in N$, z_i^* is the best for R_i over $B^*(w_i(.), m_i(.))$. On the other hand, an agent $i \in N$ is said to be taxed (resp., subsidized) when $c_i + m_i(h_i) < w_i(h_i)l_i$ (resp. $c_i + m_i(h_i) > w_i(h_i)l_i$). In the first best context income redistribution could be performed through lump-sum transfers. Such transfers will be denoted, for each agent $i \in N$, as $t_i \in \mathbb{R}$. This means that agent i's first-best budget set is then:

$$B(t_i, w_i(.), m_i(.)) = \{(c, l, h) \in Z_i | c + m_i(h) \leq t_i + w_i(h)l\}.$$

It is important to notice that our model allows to capture the fact that agents are held partially responsible for their marginal productivity of labor, given their health status choices. Indeed agents might choose an health status $h = 1$ just because they care about their health condition, because they care about consumption (and a good health status increases her earning ability, agents that like to consume a lot are more likely to be healthy than others), or both.

Figure 2.1 and 2.2 describe this feature of the model with an example. Consider an agent $i \in N$ such that $w_i(.) = w^u(.)$ and $m_i(.) = m^b(.)$. Depending on her preferences, such an agent chooses, at $laissez-faire$, a bundle $z_i^* = (c_i, l_i, h_i)$ on $B^*(w_i^u(.), m^b(.))$. In figure 2.1, agent i's preferences are such that she chooses a bundle z_i^* such that $h_i = 0$. This means that she is choosing some pair (c, l) on the line with slope $w^u(0)$ and origin in 0. The darker indifference curve depicts all the pairs (c, l) such that $(c, l, 0)I_i(c_i, l_i, h_i)$. Moreover, the dotted curve depicts all the pairs (c, l) such that $(c, l, 1)I_i(c_i, l_i, h_i)$. As a whole, the two indifference curves represent the set $I(z_i) = \{z_i' \in Z_i | z_i' I_i z_i\}$. On the other hand, figure 2.2 represents a different choice due to different preferences. Agent i decides to pay the amount m^b in order to be healthy and her productivity is increased from $w^u(0)$ to $w^u(1)$. This means that she is choosing some pair (c, l) on the line with slope $w^u(1)$ and origin in $-m^b$.

2.4 Fairness Requirements

The first requirement is grounded on the ethical principle that inequalities in outcome only due to differences in characteristics for which agents cannot be held responsible (namely, their health disposition and their marginal-productivity function) are not acceptable. The first part of the requirement is an adaptation to our framework of the Pigou-Dalton transfer principle. The main difference is that the transfer principle is applied here in a much more restrictive way. In fact, we do not formulate the reduction of inequalities only in terms of disposable income. We consider a transfer desirable only in a case when there is absolute consensus that the donor has a better situation than the receiver. Assume there are two agents, i and j, with equal preferences and possibly a different health disposition and a different earning ability. Assume also that, at a given allocation, they have the same quantity of labor and the same health status but agent i can consume strictly more than agent j. Assume finally that agent i's consumption is reduced by a given amount while agent j's consumption is augmented by the same amount, but still $c_i > c_j$. The allocation we obtain must be socially preferred to the former one (see Figure 2.3). Otherwise, assume that, at a given allocation, they still have the same quantity of labor, the same health condition and agent i can consume strictly more than agent j. Consider the allocation obtained just by permuting their respective levels of consumption. There is no reason why society should prefer one of the two allocations: agents with the same responsibility characteristics (namely, with equal preferences) should be treated anonymously.

Equal Well-being for Equal Preferences For all $e \in \mathcal{D}$, $z, z' \in Z$, if there exist $i, j \in N$ and some $\Delta > 0$, such that $R_i = R_j$, $l_i = l_j = l'_i = l'_j$, $h_i = h_j = h'_i = h'_j$ with $z_k = z'_k$ for all $k \neq i, j$,

$$c_i - \Delta = c'_i > c'_j = c_j + \Delta$$

then $z'\overline{P}(e)z$; if otherwise

$$c'_i = c_j \text{ and } c_i = c'_j$$

then $z'\overline{I}(e)z$.

The second ethical requirement goes in the direction of limiting redistribution: inequalities

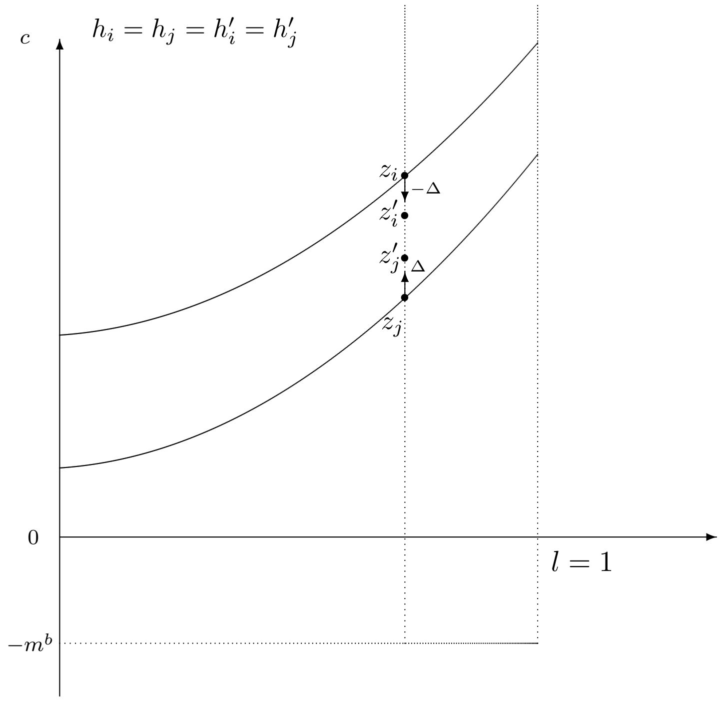


Figure 2.3: Equal Well-being for Equal Preferences

due to differences in preferences might be acceptable since individuals should, at least to some extent, be held responsible for their goals. If all agents had the same health disposition and the same earning ability then they should be let free to choose a different amount of labor, a different health condition and hence, indirectly, a different productivity. The differences in consumption eventually deriving from such choices would be exclusively due to their differences in preferences. In our view there is nothing objectionable about such inequalities or, to put it differently, society should treat neutrally individual preferences and choices. The idea that, at least when all agents have the same health disposition and the same earning ability, redistribution should not depend on preferences, clearly pushes in the direction of reducing the inequality of budgets, in particular, the inequality in lump sum transfers.

Uniform Circumstances Neutrality: For all $e \in \mathcal{D}$, $z, z' \in Z$, if for all $i, j \in N$, $w_i(.) = w_j(.)$, $m_i(.) = m_j(.)$, and if there exist $m, n \in N$ and some $\Delta > 0$ such that , $z_m \in \max|_{R_m} B(t_m, w_m(.), m_m(.))$, $z'_m \in \max|_{R_m} B(t'_m, w_m(.), m_m(.))$, $z_n \in \max|_{R_n} B(t_n, w_n(.), m_n(.))$, $z'_n \in \max|_{R_n} B(t'_n, w_n(.), m_n(.))$, with $z_k = z'_k$ for all $k \neq m, n$,

$$t_m - \Delta = t'_m > t'_n = t_n + \Delta$$

then $z'\overline{P}(e)z$; if otherwise

$$t'_m = t_n \text{ and } t_m = t'_n$$

then $z'\overline{I}(e)z$.

Notice that the first part of the requirement implies that the $laissez-faire$ allocation (i.e., no redistribution) should be the social optimum in this particular case of uniform earning ability and uniform health disposition.

The remaining requirements are basic conditions derived from the social choice literature. First, we want to take into account efficiency, for this it is essential to have a social ordering function satisfying the standard Pareto condition. Such a social ordering will never lead to the selection of allocations that are inefficient.

Strong Pareto: For all $e \in \mathcal{D}$, $z, z' \in Z$ if, for all $i \in N$, $z'_i R_i z_i$ then $z'\overline{R}(e)z$. If moreover, for some $j \in N$, $z'_j P_j z_j$ then $z'\overline{P}(e)z$.

Second, we introduce a condition that, like Arrow's condition of independence of irrelevant alternatives [2], limits the amount of information about individual preferences that may be used in the comparison of two allocations. Arrow's independence of irrelevant alternatives requires social preferences over two allocations to depend only on individual preferences over these two allocations. Fleurbaey and Maniquet [21], and Fleurbaey, Suzumura and Tadenuma [29], have argued that such an informational requirement is excessively strong, specially in an economic environment like ours. In line with Hansson [35], Pazner [45] and Fleurbaey and Maniquet [24] we propose a weaker requirement still consistent with the idea that as little information as possible should be used to build social preferences. Indeed we require social preferences over two allocations to depend only on individual indifference curves at these two allocations. More precisely, the social ranking of two allocations should be the same in two different profiles of preferences when agents' indifference curves through the bundles they are assigned in these allocations are the same.

Independence: For all $z, z' \in Z$, $e, e' \in \mathcal{D}$, with $e = (R, w, m)$ and $e' = (R', w, m)$, if for all $i \in N$ and $q \in Z$,

$$z_i I_i q \Longleftrightarrow z_i I'_i q$$
$$z'_i I_i q \Longleftrightarrow z'_i I'_i q$$

then

$$z'\overline{R}(e)z \Longleftrightarrow z'\overline{R}(e')z.$$

Third, we introduce a robustness condition following the idea that a social ordering function should provide similar solutions to similar problems. More precisely, adding or removing agents who receive the same bundle in two different allocations should not modify the social ordering. This axiom was introduced by Fleurbaey and Maniquet [21] and it is reminiscent of the Separability condition, quite familiar in social choice, due to d'Aspremont and Gevers [3] and of the Consistency condition, widely used in the theory of fair allocation (see Thomson [53]), except that it does not require to delete the resources consumed by the removed agents from the social endowment. As it will be clearer in the next section, there is some tension between this condition, if considered in its full force, and Uniform Circumstances Neutrality. This is why we propose a weaker version of it that applies only to specific sub-populations: for any $S \subseteq N$ let $\overline{m}_S(.) = \frac{1}{|S|}\sum_{j\in S} m_j(.)$ and t $\overline{w}_S(.) = \frac{1}{|S|}\sum_{j\in S} w_j(.)$.

Separation: For all $e \in \mathcal{D}$ and for all $z, z' \in Z$, if there is $S \subseteq N$ such that $\overline{m}_S = \overline{m}$, $\overline{w}_S = \overline{w}_N$ and, for all $j \in S$, $z_j = z'_j$ then

$$z'\overline{R}(e)z \iff z'_{-S}\overline{R}(R_{-S}, w_{-S}(.), m_{-S}(.))z_{-S}.$$

2.5 Social Preferences

The fairness conditions introduced above do not convey, per se, a strong aversion to inequality. Nevertheless, combining together the requirements we have mentioned, entails an infinite aversion to inequality and forces social preferences to rely on the leximin criterion. Let $\geq_{lex}$ denote the usual leximin ordering of $\mathbb{R}^N_+$: for any $v, v' \in \mathbb{R}^N_+$, $v' \geq_{lex} v$ if and only if the smallest coordinate of v' is greater than the smallest coordinate of v or they are equal but the second smallest coordinate of v' is greater than the second smallest coordinate of v and so on. Moreover, the leximin criterion needs to be applied to a precise evaluation of individual situations that is also singled out by the listed axioms. Let us first define the notion of implicit transfer associated with an agent's indifference curve, an hypotetical health disposition $\widetilde{m}(.)$, and an hypothetical earning ability $\widetilde{w}(.)$. It is the (hypothetical) lump sum transfer that would leave the agent indifferent between her current bundle and being free to choose her labor time and her health expenditure from her budget set if her earning ability was equal to $\widetilde{w}(.)$ and her health disposition was equal to $\widetilde{m}(.)$. Formally, for $z_i \in Z$, and $R_i \in \mathcal{R}$,

$$IT_i(z, \widetilde{w}(.), \widetilde{m}(.), R_i) = t \Leftrightarrow z_i I_i \max|_{R_i} B(t, \widetilde{w}(.), \widetilde{m}(.))$$

This expression (we will also use the shorter notation $IT_i(z_i)$), viewed as a function of z_i, corresponds to a particular (money-metric) utility function. Let us stress that this measure of individual well-being does not require any information about individuals' subjective utility, it relies in fact only on ordinal non-comparable preferences about consumption, labor and health. The social preferences we present here use implicit transfers obtained with a specific reference earning ability, $\widetilde{w}(.) = \overline{w}_N(.)$, and a specific health disposition, $\widetilde{m}(.) = \overline{m}_N(.)$.

Average Talents Egalitarian Equivalent Leximin SOF(ATEE). For all $e \in \mathcal{D}$, $z, z' \in Z$,

$$z'\overline{R}(e)z \Leftrightarrow (IT(z'_i, \overline{w}_N(.), \overline{m}_N(.), R_i))_{i\in N} \geq_{lex} (IT(z_i, \overline{w}_N(.), \overline{m}_N(.), R_i))_{i\in N}$$

Such social preferences give priority to agents with the lowest $IT_i(z_i)$. Moreover, since $IT_i(z_i)$ is a particular representation of individual preferences, such a way to rank alternatives satisfies Strong Pareto.

Theorem 1: *On the domain $\mathcal{D}$ a social ordering function satisfies* Strong Pareto, Equal Well-being for Equal Preferences, Uniform Circumstances Neutrality, Independence *and* Separation, *if and only if it is an Average Talents Egalitarian Equivalent Leximin function.*

The basic idea is that if all agents had the same health disposition and the same earning

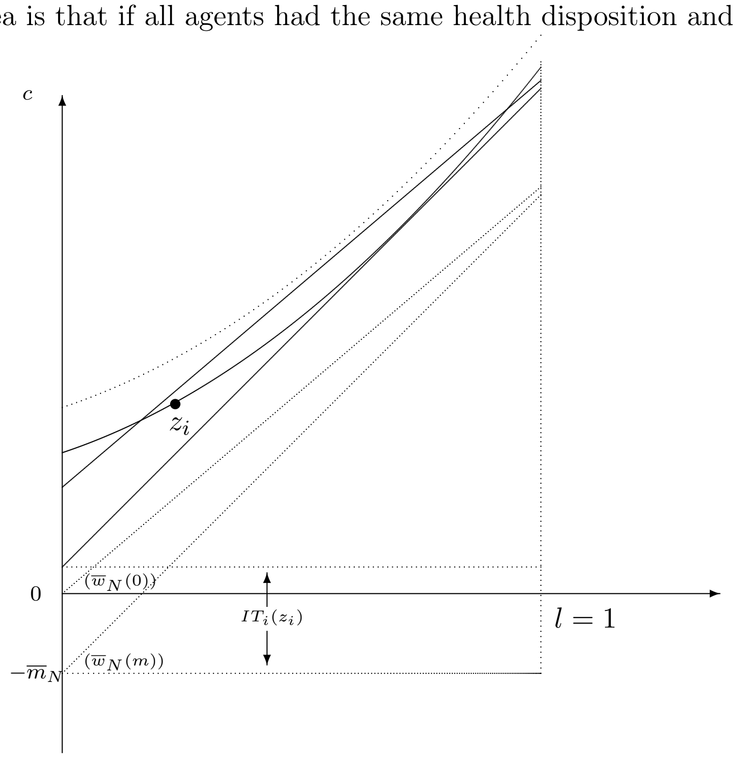


Figure 2.4: The IT well-being index

ability it would be nice to reduce inequalities in implicit transfers. Indeed, we put ourselves in the counterfactual situation where these functions are the same for all the agents and, in particular, they are equal to the average of the actual functions. In such a situation all agents would have a budget set shaped in the same way, independently of their preferences. The agents' earning ability and health disposition are not the same in reality, but the social objective pushes toward an allocation that, would equalize well-being (according to the particular metric we are using) in an ideal equal disposition, equal earning ability situation.

The notion of implicit transfer is illustrated in figure 2.4. It is the hypothetical lump-sum transfer that would make agent i indifferent between the bundle she is receiving, z_i, and being able to choose any bundle in $B(IT(z_i), \overline{w}(.), \overline{m}(.))$. The two dotted straight lines represent such a budget set, at $laissez-faire$. Indeed, depending on her choice about health status, her health expenditure could be either 0 or $\overline{m}_N(1)$ and her marginal productivity could be either $\overline{w}_N(0)$ or $\overline{w}_N(1)$. Consider the bundle z_i (where, for example, $h_i = 1$), what is the measure of $IT_i(z_i)$? We just have to slide upward the whole[2] manifold until one of the two lines touches (look at the two dark straight lines in the figure) one of the two indifference curves (we also need to consider the set of bundles with $h_i = 0$ that are indifferent, for agent i, to z_i, namely, the dotted indifference curve in the figure).

It is easy to check that the ATEE social ordering function satisfies Equal Well-being for Equal Preferences and Uniform Circumstances Neutrality. As far as the former axiom is concerned, whether an agent i's characteristics, $m_i(.)$ and $w_i(.)$, are actually equal to $\overline{m}_N(.)$ and $\overline{w}_N(.)$ or not is, in essence, irrelevant. What matters is her level of satisfaction. Therefore, assessing her relative satisfaction by measuring her implicit transfer in terms

[2]We have to slide the two dotted straight lines upward and, in doing so, we have to keep their vertical distance at the origin, $\overline{m}(1)$, unchanged.

of $\overline{m}_N(.)$ and $\overline{w}_N(.)$ and trying to equalize (permuting) these values complies with this axiom. Regarding the latter axiom, if we were facing the situation where all the agents have identical earning ability and health disposition, then the hypothetical budget set of each agent would coincide with the real one. So, equalizing (permuting) the implicit transfers across agents amounts to equalizing (permuting) their actual lump sum transfers.

The reader should also notice two more facts. First, for the characterization of the social preferences we only need to use the second part of Uniform Circumstances Neutrality. Second, if agents had identical earning ability and health disposition, the Average Talents Egalitarian Equivalent Leximin SOF would optimally require a *laissez-faire* policy.

It is important to stress that the choice of the average earning ability and of the average health disposition as reference functions is actually dictated by the axioms we have chosen. It is clear that the way we have formulated the separability axiom (that is, the fact that it only applies to specific sub-populations of individuals) plays a crucial role in the determination of the reference earning ability and the reference health disposition. But a more general formulation of the axiom would conflict with Uniform Circumstances Neutrality. Indeed, the latter axiom implies that the choice of the references earning ability and of the reference health disposition needs to be profile dependent. In fact, whenever all the agents in the economy have the same health disposition, $m(.)$, and the same earning ability, $w(.)$, then, necessarily, we must have $\widetilde{m}(.) = m(.)$ and $\widetilde{w}(.) = w(.)$. But this also means that the utility representation has to depend on the particular profile of talents in a given economy. Hence, we need to weaken the separability condition in such a way that, adding or removing agents who receive the same transfer in two different allocations, does not change the utility representation we are using (because this might modify the social ordering). This is achieved by applying the separability requirement only to sub-populations whose average health disposition and whose average earning ability is equal to the population's average[3]. Removing them does not change $\overline{m}_N$ and $\overline{w}_N(.)$ so that also the function $IT(.)$ does not change.

2.6 Taxation

In this section we use the particular notion of social welfare described before in order to derive a precise and simple criterion for the comparison of tax policies given the social consequences they determine.

We assume that, in the second best context, only earned income, $y_i = w_i(h_i)l_i$, is observable. We also assume that health expenditure is observable, $m_i = m_i(h_i)$. This means that, at least for those agents who decide to purchase health care, also the health disposition is observable. Notice that health expenditure univocally determines the health status. So, with a slight abuse of notation, in what follows, we will define the earning ability as a function of the health expenditure. In particular, for each $i \in N$, $w_i(m)$ will denote the marginal productivity of an agent whose health expenditure is equal to m (notice that $w_i(m^g) = w_i(m^b)$).

A tax policy is a function defining a transfer of income depending on the level of earnings and on the health expenditure. Namely, redistribution is made via a tax function $T(y, m)$. The tax turns into a subsidy when $T(y, m) < 0$. Individuals are free to choose their labor time and their health status in the budget set modified by the tax function. The

[3]Actually we had some degrees of freedom. For example, we could have used the geometric average instead of the arithmetic one. This would have led indeed to a different, less appealing, utility representation. The characterization of all the possible utility representations compatible with Equal Circumstances Neutrality and some Separation condition is left for future research.

government is assumed to know the distribution of types in the population but ignores the characteristics of any particular agent. Under the tax policy T, agent's i budget constraint is

$$c_i + m_i \leq w_i(h_i)l_i - T(w_i(h_i)l_i, m_i).$$

In what follows we will focus on the consumption, earning, health expenditure space agent's i budget constraint becomes

$$c_i + m_i \leq y_i - T(y_i, m_i).$$

A graphical representation is given in Figure 2.5. The budget set of each agent is composed of two different consumption curves depending on their health disposition. Agents with a good health disposition who decide (not) to acquire health choose their earning-consumption pair on the curve $y - m^g - T(y, m^g)$ $(y - T(y, 0))$. Among them, those who choose an earning-consumption pair that lies above the 45 degree line with origin in $-m^g$ (0) are subsidized. The unskilled ones who decide to work full time end up with a pre-tax income equal either to $w^u(0)$ or $w^u(m^g) - m^g$. Things work similarly for agents with a bad health disposition. Moreover, between two agents with the same health disposition, the budget set of the skilled ones is always bigger for inclusion if compared to that of the unskilled ones.

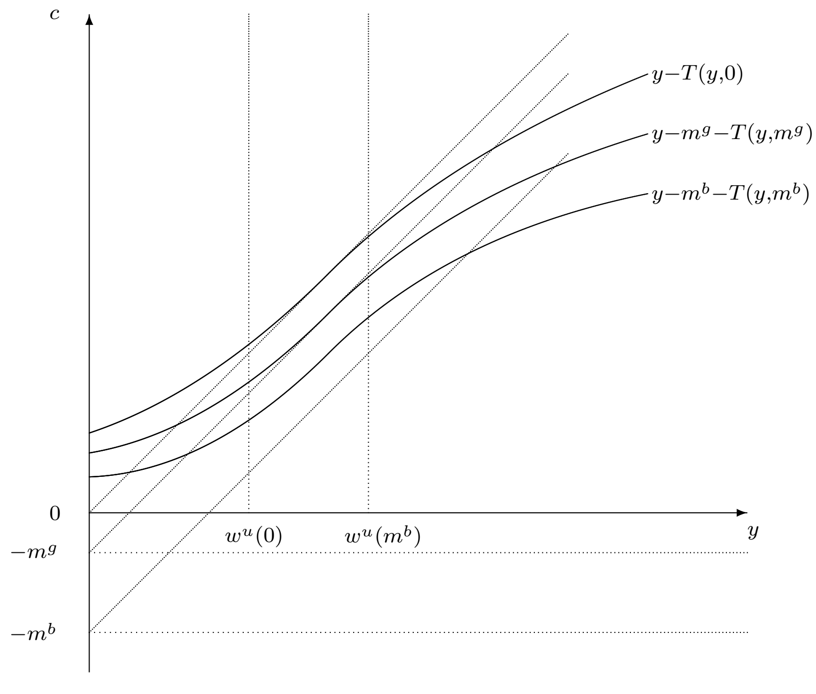


Figure 2.5: The budget set modified by the tax policy.

Notice that, at *laissez-faire*, the difference in income between hardworking-poor agents with good health disposition and hardworking-poor agents with a bad health disposition, if they decide to purchase health care, is exactly equal to $m^g - m^b$. That is, even if they have the same marginal productivity, the same concern for health and they put maximal effort into work they might end up facing a considerable difference in their consumption level exclusively due to their different health disposition.

In the (c, y, m)-space agent i's preferences over consumption, earnings and health expenditure, R_i^*, are defined, given her earning ability $w_i(h)$ and her health disposition $m_i(h)$ by

$$(y, c, m)R_i^*(y', c', m') \Leftrightarrow (\tfrac{y}{w_i(h)}, c, h)R_i(\tfrac{y'}{w_i(h')}, c', h')$$

Notice that for agents with a good (bad) health disposition, choosing between $h = 0$ and $h = 1$ in the in the (c, l, h) space amounts to choose between $m = 0$ and $m = m^g$ ($m = m^b$) in the (c, y, m) space.

An allocation $z \in Z$ is incentive compatible if and only if no agent envies the (earnings-consumption-health expenditure) bundle of any other agent provided that such a bundle is feasible to her: for all $i, j \in N$

$$(y_i, c_i, m_i)R_i^*(y_j, c_j, m_j) \textit{ with either } m_j = m_i \textit{ or } m_j = 0$$

or

$$y_j > w_i(m_j)$$

The first part of the statement says that potentially any agent can mimic any other agent who decides not to purchase health care but if two agents both buy health care then one can mimic the other only if they have the same health disposition. The second part of the condition follows from our assumption that labor time is bounded above and from the fact that the maximal amount that can be earned by working full time, depends on the health status. The informational structure of the problem forces us to focus, for any $e \in \mathcal{D}$, on the set of feasible incentive-compatible allocations. Let $\hat{Z}(e)$ denote such a set.

Each agent $i \in N$ is free to chose any point (y_i, c_i, m_i) maximizing her satisfaction within her budget constraint modified by a given tax policy T: this will inevitably lead to an envy-free allocation. Building on such an intuition, Flaurbaey and Maniquet [28] have proved that any incentive-compatible allocation can be generated by a tax function such that the locus of points

$$S(T) = \{(y, c, m) \in \mathbb{R}_+ \times \mathbb{R}_+ \times M | c + m = y - T(y, m)\}$$

lies nowhere above the envelope curve of the indifference curves in the (y, c, m)-space, and intersects this envelope curve at all points (y_i, c_i, m_i) for each $i \in N$. Notice that, monotonicity of individual preferences, there is not a loss of generality in restricting our attention to tax functions T such that, for each $m \in M$, the consumption function $y - m - T(y, m)$, is non-decreasing in y.

The budget constraint for the redistribution agency is then:

$$\textstyle\sum_{i=1}^N T(y_i, m_i) \geq 0.$$

We will say that a tax function is feasible if it satisfies the above constraint when all agents choose their labor time by maximizing their satisfaction over their budget set.

The first problem we want to tackle is the evaluation of tax policies. That is, for any given (non optimal) tax function, which part of it we should change first in order to obtain a reform that improves social welfare (given the social ordering function we are using)? Actually, the fact that we are using a social ordering function of the leximin type gives us a certain advantage in answering to this question. Indeed, for a given tax policy we have to find out who is the worst-off, according to IT(.) well-being index, at the allocation supported by such a tax policy. Once we know that, we also know who should benefit from an eventual reform. To put it differently, between two different tax policies we will recommend the one that grants a higher well-being level to the worst-off agent.

We can assume, without loss of generality that, given the tax function T, all the points along the consumption function are relevant, that is, each point of these curves hits the indifference curve of at least one agent. If there is a *small* set of agents this is obtained by assuming that the graph of the consumption function coincide with the envelope of the indifference curves of all agents.

Finally we assume that pre-tax income is not too informative of one's type The idea is that, within each group of agents induced by health expenditure, if we look at the interval $[0, w^u(m)]$, it is impossible to identify skilled agents and distinguish them, on the basis of their preferences, from unskilled agents. To put it differently, for any $m \in M$ and for any agent choosing a certain level of earning in the interval $[0, w^u(m)]$, there exist an unskilled agent with the same preferences in the (y, c, m) space over this same range of earnings.

Assumption 1: for any $m \in M$ and for any $i \in N$ such that $w_i = w^s(m)$, there exists $j \in N$ such that $w_j = w^u(m)$ and $R_j^* = R_i^*|_{[0,w_i(m)]\times\mathbb{R}_+\times M}$.

Under such an assumption it becomes relatively easier to spot the worst-off agent when a given tax policy is enforced. To understand why, let us focus on the agents that, at a given tax policy T, have chosen not to buy health care (that is, $m = 0$). Among them let us first consider the unskilled ones. Assumption 1 grants that over $y \in [0, w^u(m)]$ it is the envelope curve of agents from such sub-population. The minimum value of $IT_i(z_i)$ among them is given by

$$\min\left\{t \in \mathbb{R} | \exists l \in [0,1], t + \overline{w}_N(0)l = w^u(0)l - T(w^u(0)l, 0)\right\}$$

$$= \min\{y - T(y,0) - \frac{\overline{w}_N(0)}{w^u(0)} y | y \in [0, w^u(0)]\}$$

$$= \min\{c - \frac{\overline{w}_N(0)}{w^u(0)} y | y, c \in S(T), y \in [0, w^u(0)]\}$$

Let us consider the skilled ones, the worst-off among them has a level of well being equal to

$$\min\{c - \frac{\overline{w}_N(0)}{w^s(0)} y | y, c \in S(T), y \in [0, w^s(0)]\}$$

Since $y - T(y,0)$ is nondecreasing in y and $w^u(0) < w^s(0)$ then necessarily the former value is bigger than the latter one. So an unskilled agent is always worse-off than a skilled one. In the same way we can find the worst-off among agents whose health expenditure is respectively m^g and m^b. Again, by the same token, it will be an unskilled agent. This leads to the following

Theorem 2: Let z be an incentive-compatible allocation obtainable with the tax policy T, then, under Assumption 1,

$$\min_i IT_i(z_i) = \min_{(y,c,m)\in S(T),\ y\le w^u(m)} \begin{cases} c - \frac{\overline{w}_N(m)}{w^u(m)} y & \text{if m=0} \\ c - \frac{\overline{w}_N(m)}{w^u(m)} y + \overline{m} & \text{otherwise} \end{cases}$$

where $\overline{m} = \frac{m^g + m^b}{2}$ is the average expenditure for purchasing health care. Figure 2.6 gives a graphical representation of the previous result. The dashed segments with origin in 0 and $-\overline{m}$ are the graphical representation of the IT(.) well-being index for an unskilled

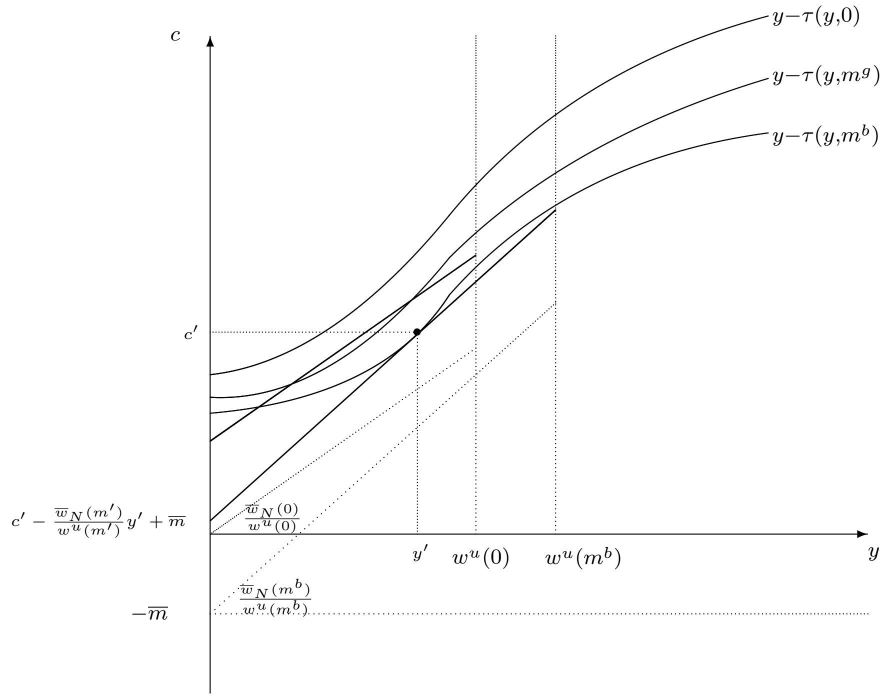


Figure 2.6:

agent when such an index is equal to zero. We slide vertically the two segments (keeping the vertical distance at the origin unchanged) until one of them touches the consumption function in some $y' \in [0, w^u(m)]$. By assumption 1, for each $m \in M$, the part of the envelope curve that lies within the interval $[0, w^u(m)]$, is also the envelope curve of agents with productivity $w^u(.)$. So, at z, there is an agent i with productivity $w^u(.)$ who chooses the bundle $z_i = (y', c', m')$ such that

$$IT_i(z_i) = c' - \frac{\overline{w}_N(m')}{w^u(m')}y' + \overline{m}$$

As explained earlier, all the other members of the population will necessarily have a higher level of welfare.

This result is very important because it gives us an easy way to compare two different tax policies T and T'. We just have to apply the previous formula in order to obtain $min_i IT_i(z_i)$ and $min_i IT_i(z'_i)$ where z and z' are the incentive compatible allocations obtainable respectively from T and T'. If for example

$$\min_i IT_i(z'_i) > \min_i IT_i(z_i)$$

then, by Theorem 1, we know that z is socially preferred to z' so that T should be preferred to T'.

What is also very important about this result is that the deriving comparison criterion requires little knowledge of the population's characteristics. The social planner only needs to know $w^u(.)$. There is no need, for example, to elicit particular information about people preferences, the computation only depends on the function T. The knowledge of the distribution of individual characteristics is necessary only to establish wether a tax policy is feasible. The criterion we propose would in any case rank all feasible and unfeasible tax policies without discrimination.

The following theorem provides more information about the optimal tax. Let $c_0^u = w^u(0) - T(w^u(0), 0)$ denote the consumption granted by the tax policy T to the unskilled agent who works full time and whose health expenditure is 0. In the same fashion let $c_{m^g}^u = w^u(m^g) - m^g - T(w^u(m^g), m^g)$ and $c_{m^b}^u = w^u(m^b) - m^b - T(w^u(m^b), m^b)$.

Theorem 3: If z^* is an optimal (incentive-compatible) allocation for social preferences satisfying Equal Well-being for Equal Preferences, Uniform Circumstances Neutrality, Strong Pareto, Independence and Separability, then, under Assumption 1, it can be obtained with a tax function T^* which, among all feasible tax functions, maximizes the net income of the hardworking-poor, c_0^u, $c_{m^g}^u$ and $c_{m^b}^u$ under the constraints that:

i) $c_{m^g}^u = c_{m^b}^u = c_0^u + \overline{w}_N(m^b) - \overline{w}_N(0) - \overline{m}$

ii) for $m = 0$ and
a) $y \geq 0, \quad T^*(y, 0) \geq T^*(w^u(0), 0),$
b) $y \leq w^u(0), \quad y - T^*(y, 0) \geq w^u(0) - T^*(w^u(0), 0) + \frac{\overline{w}_N(0)}{w^u(0)} y - \overline{w}_N(0)$

iii) for $m \neq 0$ and
a) $y \geq 0, \quad T^*(y, m) \geq T^*(w^u(m), m),$
b) $y \leq w^u(m), \quad y - T^*(y, m) \geq w^u(m) - T^*(w^u(m), m) + \frac{\overline{w}_N(m)}{w^u(m)} y - \overline{w}_N(m) + m - \overline{m}$

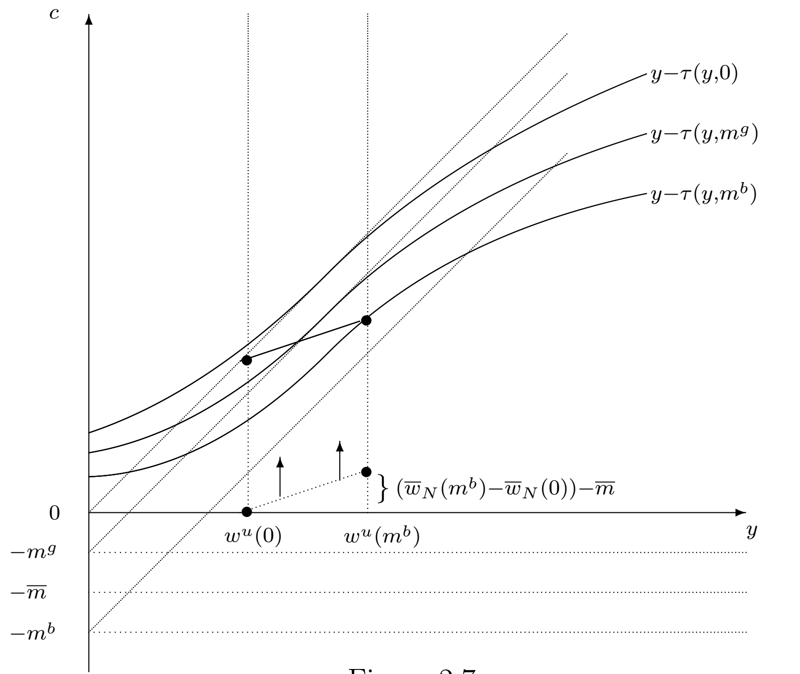


Figure 2.7:

The theorem says that there is not welfare loss in focusing on tax policies such that the values c_0^u, $c_{m^g}^u$ and $c_{m^b}^u$ are maximized and, such that, the constraints listed at the end of the statement are satisfied. The first constraint means that the well-being of the hard-working poor agents, for each level of health expenditure, should be equalized. This also means, on the one hand, that the consumption granted to unskilled agents who work full time and whose health expenditure is different from zero should optimally be equalized. This is a the clear consequence of the fact that we are using social preferences aiming at the reduction of inequalities solely due to differences in characteristics for which agents are not responsible. On the other hand, the consumption granted to unskilled agents who work full time and decide to buy health care might be higher than the consumption granted to unskilled agents who work full time and decide not to buy health care. This happens if and only if $(\overline{w}_N(m^b) - \overline{w}_N(0)) - \overline{m} \geq 0$. That is, if and only if, the average increase in productivity deriving from being healthy is higher than the average expenditure necessary to be healthy. This result seems more in line with the responsibility perspective. Indeed, to some extent individuals should be able to benefit from the increase in wealth deriving from their choice to be healthy. The other two constraints confirm the fact that the social preferences singled out by our requirements lead to focus on the hardworking poor. At the optimum, in fact, they should get the greatest absolute amount of subsidy within each of the sub-populations that are determined by health expenditure ($ii - a$, $iii - a$). Notice that actually, if $(\overline{w}_N(m^b) - \overline{w}_N(0)) - \overline{m} > 0$, than the first constraint becomes non-binding in the sense that nobody will choose the point $(c_0^u, w^(0), 0$ and points that are sufficiently close to it. This choice would be dominated by the choice of a point on other parts of the budget set involving a positive health expenditure that always provide a strictly higher level of consumption (and the same labor supply). This also implies that the optimal tax scheme can be obtained with a tax function that maximizes the net income of the hardworking-poor agent who buy health care. Finally, for each $m \in M$ and $y \leq w(m)$, the average marginal tax rate is non-positive.

An important consequence of Theorem 3 is that, if we restrict our attention to a particular family of tax functions, we can use a much easier way to rank tax policies than the one proposed in theorem 2. If in fact, for each $m \in M$ and $y \in [0, w^u(m)]$ the slope of $y - T(y, m)$ is smaller than $\overline{w}_N(m)/w^u(m)$ then, the worst off within each group of agents choosing a different health expenditure, will always be the hardworking poor. This means that, when we compare two different tax policies we just have to look at the level of consumption that they grant to agents with earnings equal to $w^u(0)$ and $w^u(m)$ keeping in mind that, at the optimum, $c^u_{m^g} = c^u_{m^b} = c^u_0 + \overline{w}_N(m) - \overline{w}_N(0) - \overline{m}$. A graphical way to proceed (see figure 2.7) is to take the dotted segment line which is drawn on the figure and slide it upward until it either its left extreme touches the $y - T(y, 0)$ budget curve or, the right extreme touches one of the other two budget curves. The higher the position of the segment the better.

2.7 Conclusion

The main contribution of this paper consists of providing a useful and simple criterion for the comparison of tax policies that have as an argument not only the earned income y but also the health expenditure m. This exercise relies on social preferences that satisfy the Pareto principle and fairness requirements and that would always recommend no redistribution as soon as agents in the economy have identical health disposition and earning ability. The only information about individuals that is taken into account by such social preferences is their ordinal non-comparable preferences. No other information about subjective utility is needed for the evaluation of social alternatives. Such social preferences are used here in a second best framework. The only data to perform the analysis is the knowledge of $w^u(.)$ and the distribution of individual preferences. Our main result is that, any tax reform should always benefit unskilled individuals first, for any level of health expenditure. This also means that, taking health into account, leads to further redistribution specially between hardworking-poor agents with a different health disposition and the same concern for their health condition.

Finally, let us briefly discuss the assumptions underlying the analysis.

First, the fact that we only allow for a dichotomous choice of health status and we only allow for two types of earning ability is mostly meant to simplify the exposition. Theorem 1 would still hold in a more general model allowing for a continuum of health statuses and a continuum of earning ability types. Theorem 2 would also still be adaptable to such a framework. Indeed the underlying policy recommendation would still push toward giving priority to unskilled agents (namely, agents with the lowest earning ability). In fact, It would still be true that, for any m, the agent with the lowest $w(m)$ is the worst off. On the other hand it might be more difficult to provide a clear-cut result about the optimal tax scheme. This might be object of future research.

Third, Assumption 1 is a generalization of the No-identification assumption used in Fleurbaey and Maniquet [24] and [25]. Even if it is a strong assumption it is only meant to rule out the design of tax functions that present unrealistically high tax rates over some interval $[y, y']$ only because the policy maker is aware of the fact that only agents able to earn a much greater income would actually decide to pick up their own income in such an interval.

Appendix

The following lemma will be useful for the proof of Theorem 1.

Lemma 1 If a Social Ordering Function satisfies Strong Pareto, Equal Well-being for Equal Preferences and Independence, then for any pair of allocations $z, z' \in Z^N$ and any $i, j \in N$ with identical preferences R, such that

$$z_i P z_i' P z_j' P z_j$$

and $z_k = z_k'$ for each $k \neq j \neq i$, one has $z' \overline{P}(e) z$.

Proof of Lemma 1: It is similar to the proof of Lemma 1 in Fleurbaey and Maniquet [24]. ■

Proof of Theorem 1: Let $\overline{R}$ be a social ordering function that satisfies the listed axioms. For the sake of clarity the remaining of the proof is divided in three steps.

Step 1. Consider two allocations z and z' and two different agents j and k such that

$$IT_j(z_j) > IT_j(z_j') > IT_k(z_k') > IT(z_k)$$

and for all $i \neq j, k$, $z_i = z_i'$. We want to prove that $z' \overline{P}(e) z$. Suppose, by contradiction, that $z \overline{R}(e) z'$. Introduce two agents a, b such that $w_a(.) = w_b(.) = \overline{w}_N(.)$, $d_a = d_b = \overline{d}$, $R_a = R_k$ and $R_b = R_j$. Let $e' = (R, R_a, R_b, w, w_a(.), w_b(.), d, d_a, d_b)$ and $e'' = (R_a, R_b, w_a(.), w_b(.), d_a, d_b)$. Finally let z_a, z_b, z_a', z_b' be such that:

1. $IT_j(z_j') > IT_b(z_b') = IT_a(z_a) > IT_b(z_b) = IT_a(z_a') > IT_k(z_k')$,

2. $z_a \in \max|_{R_a} B(IT_a(z_a), w_a(.), d_a)$, $z'_a \in \max|_{R_a} B(IT'_a(z_a), w_a(.), d_a)$,

3. $z_b \in \max|_{R_b} B(IT_b(z_b), w_b(.), d_b)$, $z'_b \in \max|_{R_b} B(IT'_b(z_b), w_b(.), d_b)$.

By *Separation*,

$$(z, z_a, z_b)\overline{R}(e')(z', z_a, z_b)$$

By *Lemma 1*

$$(z_{-k}, z'_k, z'_a, z_b)\overline{P}(e')(z, z_a, z_b)$$

and

$$(z_{-\{j,k\}}, z'_j, z'_k, z'_a, z'_b)\overline{P}(e')(z_{-k}, z'_k, z'_a, z_b)$$

By Transitivity,

$$(z_{-\{j,k\}}, z'_j, z'_k, z'_a, z'_b)\overline{P}(e')(z', z_a, z_b)$$

By *Separation*

$$(z'_a, z'_b)\overline{P}(e'')(z_a, z_b)$$

Which, by *Uniform Circumstances Neutrality*, yields the desired contradiction.

Step 2. Consider two allocations z and z' and two different agents j and k such that

$$IT_k(z_k) = IT_j(z'_j) > IT_k(z'_k) = IT_j(z_j)$$

and for all $i \neq j, k$, $z_i = z'_i$. We want to prove that $z'\overline{I}(e)z$. Suppose, by contradiction, that $z\overline{P}(e)z'$. Introduce two agents a, b such that $w_a(.) = w_b(.) = \overline{w}_N(.)$, $d_a = d_b = \overline{d}$, $R_a = R_k$ and $R_b = R_j$. Let $e' = (R, R_a, R_b, w, w_a(.), w_b(.), d, d_a, d_b)$ and $e'' = (R_a, R_b, w_a(.), w_b(.), d_a, d_b)$. Finally let $z_a, z_b, z'_a, z'_b, z''_j, z''_k, z''_a, z''_b$ be such that:

1. $IT_a(z'_a) = IT(z_b) = IT_k(z_k)$, $IT_a(z_a) = IT_b(z'_b) = IT_k(z'_k)$

2. $z''_k I_k z_k, z''_a I_a z_a$ and $l''_k = l''_a$
   $z''_j I_j z_j, z''_b I_b z_b$ and $l''_j = l''_b$

3. $z_a \in \max|_{R_a} B(IT_a(z_a), w_a(.), d_a)$, $z'_a \in \max|_{R_a} B(IT'_a(z_a), w_a(.), d_a)$,

4. $z_b \in \max|_{R_b} B(IT_b(z_b), w_b(.), d_b)$, $z'_b \in \max|_{R_b} B(IT'_b(z_b), w_b(.), d_b)$.

By *Separation*,

$$(z, z_a, z_b)\overline{P}(e')(z', z_a, z_b).$$

By *Pareto Indifference* (which is implied by *Strong Pareto*) and *Equal Well-being for Equal Preferences*

$$(z_{-k}, z''_a, z''_k, z_b)\overline{I}(e')(z, z_a, z_b)$$

Again, by *Pareto Indifference* and *Equal Well-being for Equal Preferences*,

$$(z_{-\{j,k\}}, z''_a, z''_b, z''_k, z''_j)\overline{I}(e')(z_{-k}, z''_a, z''_k, z_b).$$

By *Pareto Indifference*,

$$(z_{-\{j,k\}}, z'_k, z'_j, z'_a, z'_b)\overline{I}(e')(z_{-\{j,k\}}, z''_a, z''_b, z''_k, z''_j).$$

By *Transitivity*,

$$(z_{-\{j,k\}}, z'_k, z'_j, z'_a, z'_b)\overline{P}(e')(z', z_a, z_b).$$

By *Separation*,

$$(z'_a, z'_b)\overline{P}(e'')(z_a, z_b).$$

The desired contradiction.

Step 3. The rest of the proof derives from standard characterizations of the leximin criterion (Hammond [34]). ■

Proof of Theorem 2: Consider a tax function T defined by the fact that $y-m-T(y,m)$ is continuos in y, nondecreasing in y and the it coincides with the envelope curve of the population's indifference curve in (y,c,m) space, at the incentive-compatible allocation generated by T.

By Assumption 1, for all $m \in M$, over $y \in [0, w^u(m)]$ it is the envelope curve of agents from the $w^u(.)$ sub-population. Let $N_0^u = \{i \in N | m_i = 0 \text{ } and \text{ } w_i = w^u(0)\}$. The minimum value of $IT_i(z_i)$ within this sub-population is given by

$$IT_0^u = \min_{i \in N_0^u} IT_i(z_i) = \min\left\{t \in \mathbb{R} | \exists l \in [0,1], t + \overline{w}_N(0)l = w^u(0)l - T(w^u(0)l, 0)\right\}$$

$$= \min\{y - T(y,0) - \frac{\overline{w}_N(0)}{w^u(0)}y | y \in [0, w^u(0)]\}$$

$$= \min\{c - \frac{\overline{w}_N(0)}{w^u(0)}y | y, c \in S(T), y \in [0, w^u(0)]\}$$

Let now $N_0^s = \{i \in N | m_i = 0 \text{ } and \text{ } w_i = w^s(0)\}$. The minimum value for $IT_i(z_i)$ within this sub-population is given by

$$IT_0^s = \min_{i \in N_0^s} IT_i(z_i) = \min\{c - \frac{\overline{w}_N(0)}{w^s(0)}y | y, c \in S(T), y \in [0, w^s(0)]\}$$

Since $y - T(y,0)$ is nondecreasing in y and $w^u(0) < w^s(0)$ then $IT_0^u < IT_0^s$. Let now $N_{m^b}^u = \{i \in N | m_i = m^b \text{ } and \text{ } w_i = w^u(m^b)\}$. Again, by Assumption 1, the minimum value of $IT_i(z_i)$ within this sub-population is given by

$$IT_{m^b}^u = \min_{i \in N_{m^b}^u} IT_i(z_i) =$$

$$\min\left\{t \in \mathbb{R} | \exists l \in [0,1], t + \overline{w}_N(m^b)l + \overline{m} = w^u(m^b)l - m^b - T(w^u(m^b)l, m^b)\right\} =$$

$$\min\{c - \frac{\overline{w}_N(m^b)}{w^u(m^b)}y + \overline{m} | y, c, m \in S(T), y \in [0, w^u(m^b)]\}$$

Let now $N_m^s = \{i \in N | m_i \neq 0 \text{ } and \text{ } w_i = w^s(m^b)\}$. The minimum value of $IT_i(z_i)$ within this sub-population is given by

$$IT_{m^b}^s = \min_{i \in N_{m^b}^s} IT_i(z_i) = \min\{c - \frac{\overline{w}_N(m^b)}{w^s(m^b)}y + \overline{m} | y, c, m \in S(T), y \in [0, w^s(m^b)]\}$$

Since $y - m^b - T(y, m^b)$ is nondecreasing in y and $w^u(m^b) < w^s(m^b)$ then $IT^u_{m^b} < IT^s_{m^b}$. In the same way we can compute $IT^u_{m^g}$ and $IT^s_{m^g}$. Finally we just need to compare the well-being levels of the worst-off agents within each category. ■

Proof of Theorem 3: Consider a tax function T^* such that, for any $m \in M$, the income function $y - m - T^*(y, m)$ is continuos and nondecreasing in y and such that its graph is the envelope curve of the population's indifference curves in (y, c, m) space, at the optimal allocation z^*. By Theorem 2 we know that, the minimum value of $IT_i(z_i^*)$ over the population of agents for whom $m = 0$ is given by:

$$IT^u_0 = \min\{y - T^*(y, 0) - \frac{\overline{w}_N(0)}{w^u(0)} y | y \in [0, w^u(0)]\}$$

Analogously, if we consider the sub-populations of agents for whom $m \neq 0$, we have

$$IT^u_m = \{y - m - T^*(y, m) - \frac{\overline{w}_N(m)}{w^u(m)} y + \overline{m} | y \in [0, w^u(m)]\}$$

The optimality of z^* and the fact that the health disposition is observable implies:

$$IT^u_0 = IT^u_{m^g} = IT^u_{m^b}. \tag{2.7.1}$$

Optimality also implies that z^* is socially preferred to the allocation z' that we would obtain at $laissez - faire$. At such an allocation, for $m = 0$

$$\min\left\{y - \frac{\overline{w}_N(0)}{w^u(0)} y | y \in [0, w^u(0)]\right\} = w^u(0) - \overline{w}_N(0) \leq IT^u_0$$

and for $m \neq 0$,

$$\min\left\{y - m - \frac{\overline{w}_N(m)}{w^u(m)} y + \overline{m} | y \in [0, w^u(m)]\right\} = w^u(m) - \overline{w}_N(m) - m + \overline{m} \leq IT^u_m.$$

Consider now an alternative tax policy T^{**} which is obtained from T^* in the following way

$$T^{**}(y, m) = \begin{cases} \max\{T^*(y, 0), -IT^u_0 - w^u(0) + \overline{w}_N(0)\} & \text{if m=0} \\ \max\{T^*(y, m), -IT^u_m - w^u(m) + \overline{w}_N(m) + m - \overline{m}\} & \text{otherwise} \end{cases}$$

Such a tax policy it is still feasible because it cuts subsidies greater than a certain constant. Moreover, for $m \neq 0$,

$$\min\left\{y - m - T^{**}(y, m) - \frac{\overline{w}_N(m)}{w^u(m)} y + \overline{m} | y \in [0, w^u(m)]\right\} =$$

$$= \min\left\{\begin{array}{l} \min\left\{y - m - T^*(y, m) - \frac{\overline{w}_N(m)}{w^u(m)} y + \overline{m} | y \in [0, w^u(m)]\right\}, \\ \min\left\{y + IT^u_m + (\overline{w}_N(m) - w^u(m)) - \frac{\overline{w}_N(m)}{w^u(m)} y | y \in [0, w^u(m)]\right\} \end{array}\right\}$$

$$= \min\left\{\begin{array}{l} \min\left\{y - m - T^*(y, m) - \frac{\overline{w}_N(m)}{w^u(m)} y + \overline{m} | y \in [0, w^u(m)]\right\}, \\ IT^u_m \end{array}\right\}$$

$= IT_m^u$.

Following the same argument, for $m = 0$,

$$\min \left\{ y - T^{**}(y,0) - \frac{\overline{w}_N(0)}{w^u(0)} y | y \in [0, w^u(0)] \right\} = IT_0^u$$

This means that T^{**} is an optimal tax policy too. Given what we have just proven and the way we have have constructed T^{**} it follows that

(a) for $m = 0$ and
$y \geq 0, \qquad T^{**}(y,0) \geq -IT_0^u - (\overline{w}_N(0) - w^u(0))$
$y \leq w^u(0),\ y - T^{**}(y,0) - \frac{\overline{w}_N(0)}{w^u(0)} y \geq IT_0^u$.

Hence, for $y \leq w^u(0)$

$$-IT_0^u - (\overline{w}_N(0) - w^u(0)) \leq T^{**}(y,0) \leq -IT_0^u + y - \frac{\overline{w}_N(0)}{w^u(0)} y$$

(b) for $m \neq 0$ and
$y \geq 0, \qquad T^{**}(y,m) \geq -IT_m^u - (\overline{w}_N(m) - w^u(m) + m - \overline{m})$
$y \leq w^u(m),\ y - T^{**}(y,m) - m - \frac{\overline{w}_N(m)}{w^u(m)} y + \overline{m} \geq IT_m^u$.

Hence, for $y \leq w^u(m)$

$$-IT_m^u - (\overline{w}_N(0) - w^u(0) + m - \overline{m}) \leq T^{**}(y,m) \leq -IT_m^u + y - m - \frac{\overline{w}_N(m)}{w^u(m)} y + \overline{m}$$

So, for $m = 0$ and $y = w^u(0)$, from (a) we obtain that $T^{**}(y,0) = -IT_0^u - (\overline{w}_N - w^u(0))$ so that

(a') for $m = 0$ and
$y \geq 0, \qquad T^{**}(y,0) \geq T^{**}(w^u(0),0)$,
$y \leq w^u(0),\ y - T^{**}(y,0) \geq \frac{\overline{w}_N(0)}{w^u(0)} y - T^{**}(w^u(0),0) + w^u(0) - \overline{w}_N(0)$

Similarly, for $m \neq 0$ and $y = w^u(m)$, from (b) we obtain that $T^{**}(y,m) = -IT_0^u - (\overline{w}_N - w^u(0) + m - \overline{m})$ so that

(b') for $m \neq 0$ and
$y \geq 0, \qquad T^{**}(y,m) \geq T^{**}(w^u(m),m)$,
$y \leq w^u(m),\ y - T^{**}(y,m) \geq \frac{\overline{w}_N(m)}{w^u(m)} y - T^{**}(w^u(m),m) + w^u(0) - \overline{w}_N(0) + m - \overline{m}$ Moreover, the fact that

$$IT_0^u = w^u(0) - T^{**}(w(0),0) - \overline{w}_N(0) = c_0^u - \overline{w}_N(0)$$

and

$$IT_m^u = w^u(m) - m - T^{**}(w(m),m) - \overline{w}_N(m) + \overline{m} = c_m^u - \overline{w}_N(m) + \overline{m}$$

together with equation (6.1) implies that an optimal tax policy maximizes c_0^u, $c_{m^g}^u$ and $c_{m^b}^u$ under the constraint that

$$c_{m^g}^u = c_{m^b}^u = c_0^u + \overline{w}_N(m) - \overline{w}_N(0) - \overline{m}$$

■

Chapter 3

How to share benefits deriving from communication.

3.1 Introduction

Individuals attach value to communication. This value may be inherent to communication per se, as human beings are "social animals". More importantly, however, the ability to communicate serves as a means for one's advancement professionally, socially and otherwise. This is as apparent today, as it has ever been. The benefits notwithstanding, communication does not come for free. The cost of communication proceeds from the presence of a multitude of different languages. There is hardly one individual that speaks all of the approximately 6000 languages that are spoken today, while there are plenty that speak exclusively one of them.

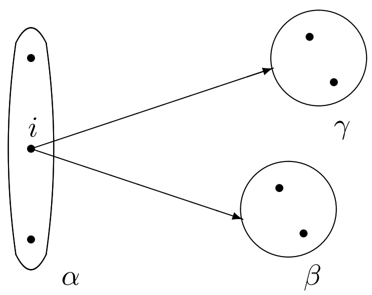


Figure 3.1:

The model we propose[1] is the simplest one that adheres to these observations. The benefits of communication stem from the number of people one may communicate with, multiplied by a non-negative real number that encompasses an individual's willingness to communicate (this is more or less the object for which Selten and Pool[50] coin the term 'communicative benefit'). The costs are determined by the language one speaks relative to the language one learns. Utility is separable in costs and benefits, and transferable. Figure 3.1 elaborates. Nodes denote individuals. The set of nodes is partitioned in three language

[1]This chapter is based on a paper co-authored with Efthymios Athanasiou and Santanu Dei

groups, α, β, γ. Language α is native to individual i. An action will be depicted by an arrow stemming from a node and pointing to a set of nodes. Individual i learns languages β and γ. Individual i, therefore, speaks with four individuals, aside from those with whom he shared the same language in the first place. His benefit will be his personal willingness to communicate times four. Suppose $c_{\alpha\beta}, c_{\alpha\gamma}$ is the cost of learning β and γ respectively, if one speaks α natively. Individual i faces a cost equal to $c_{\alpha\beta} + c_{\alpha\gamma}$.

The model depicted in Figure 1 will be the backdrop against which we will formulate our discussion. We will be interested in mechanisms. Roughly speaking, these objects associate a social outcome to the various values the primitives of the model may take. In particular, our main concern will be to examine the extent to which the following four properties can be attributed to such mechanisms:

1. *Assignment Efficiency* : the sum of net benefits should be maximized.
2. *Strategy-Profness* : all individuals, if asked, should have a dominant strategy to reveal their willingness to communicate truthfully.
3. *Individual Rationality* : no individual should enjoy a utility level that is lower than the level of utility he would enjoy if he was not a constituent of the economy.
4. *Feasibility* : any mechanism should rely exclusively on the resources generated within the economy, i.e. no outside funding should be permitted.

It turns out that *no mechanism satisfies all of the above requirements.* We proceed to examine mechanisms that each satisfy three of the properties above. Figure 3.2 sketches a plan of the paper. Each triangle corresponds to a section of the paper[2].

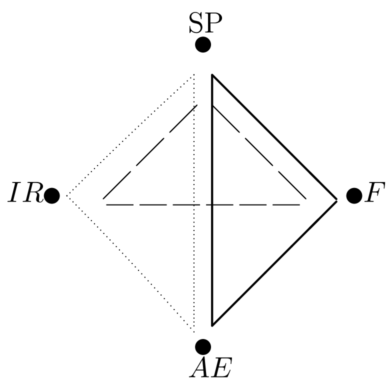


Figure 3.2: A plan of the paper.

This project directly embraces two well established strands of the literature and indirectly relates to one more. Holmström[36] shows that while investigating the thin and thick lined triangles we will inevitably be considering Groves mechanisms. The discussion of the dashed lined triangle follows the tradition of axiomatic cost sharing. In its aim, rather than in terms of the actual mechanism proposed, we may associate it with Moulin

[2]The reader may wonder: what of the missing triangle? It is not very fruitful to drop Strategy-Proofness. If we were to do so, we could propose the Equal Split allocation. All individuals share the same portion of the Assignment Efficient pie. This would satisfiy a long list of desirable properties alongside the ones we have already proposed. However, by failing only Strategy-Proofness, its practical appeal becomes questionable.

and Shenker [44]. Aside from the agenda we pursue in this paper, a parallel literature deals with decentralized outcomes that may arise in situations similar to the ones we explore. In their seminal contribution Selten and Pool [50] introduce a general model of language acquisition. They show that an equilibrium of the multi-country multilingual language acquisition model exists. The characterization of an equilibrium is then studied by Church and King [11]. Recently, Ginsburgh et al. [31] and Gabszewicz et al. [30] study qualitative properties of such equilibria in the context of bilingual societies. In our model one may rationalize different Nash Equilibria, exhibiting both one-sided as well as multi-sided learning. However, the efficient outcome does not generically come about as a Nash Equilibrium.

Our proposal involves three mechanisms. Each can be criticized from the point of view of the property it fails. In economies consisting of few individuals the antithesis can be pretty stark. For instance, within the class of mechanisms satisfying all properties except *Feasibility*, the mechanism that minimizes the deficit may still generate a deficit larger than the social value of the assignment efficient communication structure. Conversely, insisting on *Feasibility* at the expense of *Individual Rationality*, at some instances, may mean that as many as all the individuals in the economy face a negative utility. These phenomena arise solely in economies comprising few individuals. In those cases the mechanisms that fail *Assignment Efficiency* seem to have the edge. However, there is a price to pay. Any such mechanism will need to rely on the ability of a Planner to exclude individuals by appropriately selecting the assignment of languages. Under this regime, individuals often will have an incentive to unilaterally deviate and learn more languages than those the allocation prescribes for them. We are all accustomed to institutions that demand of us a certain skill set and that, moreover, control for our mastery over it. However, it is unnatural for an institution to act so as to prevent learning. Any mechanism that violates *Assignment Efficiency* will need to have this feature. On the positive side we find that all these difficulties vanish in 'large economies'. Largeness is meant in the replication sense. Therefore, a large economy is far from an economy consisting of an infinity of individuals. Indeed, a 'large economy' may not be large at all. Under mild assumptions, we show that any economy replicated a finite number of times yields an environment in which all the mechanisms we propose come close to satisfying the property they miss.

In section 2 we present the model, followed by the introduction of our key axioms in section 3. Sections 4-6 deal respectively with each of the three triangles of Figure 3.1. Section 7 concludes. Some computational considerations along with some proofs are presented in the appendices.

3.2 The model

The finite set of individuals is denoted $N \subseteq \mathbb{N}$. The finite set of languages is denoted Λ. For each $i \in N$ the Λ-dimensional vector $l_i \in \{0,1\}^\Lambda$ is the comprehensive description of that individual's linguistic achievement[3]. For instance, if $\Lambda = \{\lambda, \mu, \nu\}$ and for some $i \in N$ we have $l_i = (1,0,1)$ we infer that i speaks languages λ and ν. Initially, each individual speaks one language, his native one. We denote $\bar{l}_i \in \{0,1\}^\Lambda$ the initial language endowment of each $i \in N$. Thus, in the previous example, if agent i's native language is λ then $\bar{l}_i = (1,0,0)$. Let $\gamma : N \to \Lambda$, determine agent i's native language. For each $i \in N$, define $N^{\gamma(i)}$ to be the set of agents whose native language is the same as i.

The amount of effort that one needs to exert in order to learn a foreign language depends

[3] Abusing notation we will denote by N, Λ both the sets and their cardinality.

on his native tongue. The $\Lambda \times \Lambda$ matrix C with typical element $c_{\lambda\mu} \in \mathbb{R}_+$ provides this information. In particular, $c_{\lambda\mu}$ is the cost of learning language μ if one's native language is λ. Our only assumptions are that $c_{\lambda\mu} = 0$, whenever $\lambda = \mu$ and $c_{\lambda\mu} < \infty$ for each $\lambda, \mu \in \Lambda$.

The benefit one derives from learning foreign languages is simply a linear function of the number of individuals one can communicate with as a result of his linguistic achievement. The marginal willingness to communicate is measured, for each individual, by the parameter $\theta_i \in \mathbb{R}_+$. Hence, for individual $i \in N$, the expression

$$\theta_i \left(\sum_{j \in N/N^{\gamma(i)}} min\{1, l_i l_j\} \right), \tag{3.2.1}$$

specifies his *gross benefit* from communication. It is important to stress that there is not inherent value in the linearity assumption. Assuming a decreasing marginal benefit from communication would not alter the nature of our results but it would come at the price of tedious calculations. Moreover, in our model agents either learn a language or not. Consequently, we do not allow for degrees of knowledge. All our results would hold if we were to qualify this assumption. Moreover, any mathematical difficulty the model presents would remain, even if we were to apply a convexification, namely letting l_i take values in an interval, say $[0, 1]$. In fact, the discreet nature of the problem proceeds from the min operator (equation 2.1). Even if two individuals share more than one language, the benefit each derives from communicating with the other remains unchanged relative to case of communication through a single language. Therefore, even if the parameter l_i was continuous, the problem of maximizing social welfare would have to be solved under multiple constraints.

The expression above does not include the benefit one derives from communicating with people with whom he shares the same mother tongue. This value is a constant. Whether the gross benefit accounts for it or not, has no bearing on our results. For instance, let $N = \{1, 2, 3\}$, $\Lambda = \{English, French, Italian\}$ and suppose that each individual has a different native language. Then, for each $i, j \in N$, $l_i l_j \in \{0, 1, 2, 3\}$, indeed i and j may have as many as three languages in common. By contrast, $min\{1, l_i l_j\} \in \{0, 1\}$. Utility is generated by the amount of communication alone. Agents do not care neither with whom they communicate, nor in what language they do so.

On the other hand, the disutility pertaining to action l_i, given $\bar{l}_i$ and C, is $\bar{l}_i C(l_i - \bar{l}_i)$. Hence, for each $i \in N$, the *net benefit* associated with $l_N = (l_i)_{i \in N}$ is

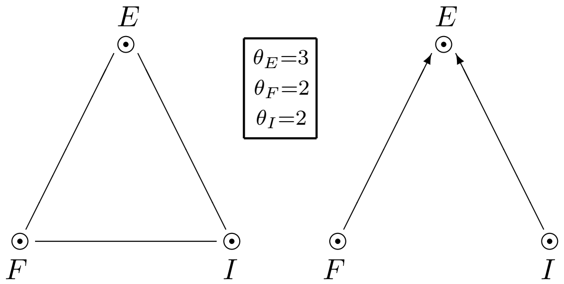
$$v\big(l_N; \theta_i, \bar{l}_i, C\big) = \theta_i \left(\sum_{j \in N/N^{\gamma(i)}} min\{1, l_i l_j\} \right) - \bar{l}_i C(l_i - \bar{l}_i).$$

To simplify things we will write, for each $i \in N$, $v_i(l_N; \theta_i)$ instead of $v\big(l_N; \theta_i, \bar{l}_i, C\big)$.

In order to better understand the model, consider the following example. Again we have three agents with different mother-tongues, say English, French and Italian. Figure 3.3 below summarizes their marginal utilities, the cost matrix C and the actions taken by the agents and the related network structure. Assume that both the French and the Italian learn English. Therefore, each individual has a language in common with the rest. The net benefit for the Italian is : $v(l_N, 2) = 2 \times (\#\ of\ links) - c_{IE}$=$2 \times 2 - 1$=3.

Each individual also consumes a transfer $t_i \in \mathbb{R}$. The final utility of each individual is then

$$u_i(l_N, t_i; \theta_i) = v_i(l_N; \theta_i) + t_i,$$



$$\mathbf{C}= \begin{array}{c} \begin{array}{ccc} E & F & I \end{array} \\ \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ F \\ I \end{array}$$

Figure 3.3: Explaining the notation. A straight line between two nodes signifies that the corresponding individuals have a language in common.

that is, preferences are quasi-linear.

Let $\bar{l}_N \equiv (\bar{l}_i)_{i \in N}$, $\theta_N \equiv (\theta_i)_{i \in N}$. An economy is denoted $e = (\theta_N, \bar{l}_N, C) \in \mathcal{E}$, where $\mathcal{E}$ is the set of economies complying with our assumptions.

An allocation is a list $(l_N, t_N) \equiv (l_i, t_i)_{i \in N}$ where l_i is a linguistic assignment for individual i and t_i is the transfer he receives. Let Z be the set of all allocations. In addition, for each $e \in \mathcal{E}$ and each $l_N \in \{0,1\}^{\Lambda \times N}$, let $\pi(l_N; e) = \sum_{i \in N} v_i\big(l_N; \theta_i\big)$ be the sum of net benefits generated by the allocation. A mechanism is a function φ defined over $\mathcal{E}$ that associates with each economy an allocation $(l_N, t_N) \in Z$.

3.3 Axioms

In this section we formally define the four core properties that motivate the discussion. We begin by *Assignment Efficiency.* For each $e \in \mathcal{E}$ let

$$\Sigma(e) = \{l_N \in \{0,1\}^{\Lambda \times N} \mid l_N \in argmax \sum_{i \in N} v_i(l_N; \theta_i)\} \tag{3.3.1}$$

be the set of all the linguistic assignments that maximize the sum of net benefits for a given economy. A mechanism is *Assignment Efficient* if, for each economy in the admissible domain, it selects an allocation whose linguistic assignment component maximizes the sum of net benefits. *Assignment Efficiency* differs from *Pareto Efficiency* in that it does not require transfers to sum up to zero.

Assignement – Efficiency For each $e \in \mathcal{E}$, if $(l_N, t_N) = \varphi(e)$ then $l_N \in \Sigma(e)$.

The next axiom is motivated by the fact that the social planner does not necessarily know the individuals' willingness to communicate. In fact, some individuals might find it profitable do behave strategically and misreport it. We require that each individual has a weakly dominant strategy to reveal his willingness to communicate truthfully.

Strategy – Proofness For each $e \in \mathcal{E}$, $\theta'_i \in \mathbb{R}_+$, $i \in N$,

$$u_i(\varphi_i(\theta_N, \bar{l}_N, C), \theta_i) \geq u_i(\varphi_i(\theta'_i, \theta_{N \setminus \{i\}}, \bar{l}_N, C), \theta_i).$$

Since the domain of preference profiles is convex (and hence smoothly connected) we know from Holmstrom [36] that a mechanism satisfies Assignment Efficiency and Strategy Proofness *if and only if* it belongs to the family of Groves mechanisms (see Groves [32]).

Such mechanisms determine a transfer composed of two parts. First, each agent receives the total net benefit obtained by all other agents at the assignment chosen by the mechanism. Second, each agent receives a sum of money that does not depend on his own (announced) willingness to communicate. Let h_i be a real-valued function defined on $\mathbb{R}_+^{N-1}$ such that for each $i \in N$ and $\theta_N \in \mathbb{R}_+^N$, h_i depends at most on $\theta_{N\setminus\{i\}}$. In other words, the function h_i does not depend on individual i's willingness to communicate.

The Groves Mechanism For each $e \in \mathcal{E}$, $(l_N, t_N) = \varphi^g(e)$ if and only if $l_N \in \Sigma(e)$ and, for each $i \in N$

$$t_i = \sum_{j \neq i} v_j(l_N; \theta_i) - h_i(\theta_{N\setminus\{i\}}).$$

In general, for Groves mechanisms, the sum of the transfers may be either positive or negative. Generically there is a waste. However, there is a particular difficulty pertaining to a deficit. A Planner will need to finance the Groves scheme using resources that are not generated within the economy. Any mechanism, be it Groves or not, by construction, is silent as to where these outside funds may be found. *Feasibility* requires of mechanisms to be self-sustainable.

Feasibility For each $e \in \mathcal{E}$ and each $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi(e)$,

$$\sum_{i \in N} t_i \leq 0.$$

Finally, as a minimal fairness property, we will require voluntary participation of individuals to the proposed scheme. In other words, we consider as unappealing, from an ethical point of view, mechanisms that have to rely on an authority coercing agents to interact. All individuals must enjoy a positive utility. Aside from the benefit they derive from communicating with their compatriots, a value that does not appear in their utility function as defined, none should suffer by participating in the proposed scheme.

Individual Rationality For each $e \in \mathcal{E}$ and each $i \in N$, $u_i(\varphi_i(e); \theta_i) \geq 0$.

3.4 An Individually Rational Groves Mechanism

In this section we will focus on the sub-class of Groves mechanisms that satisfy *Assignment-Efficiency*, *Strategy-Proofness* and *Individual Rationality*. It is well known that in order to obtain individual rationality one has to allow the center to run a deficit (Thomson [52]). We propose a mechanism which minimizes such a deficit.

For each $e \in \mathcal{E}$, $i \in N$ let e^i denote an economy that is otherwise identical to e, except for the fact that agent i's willingness to communicate has been set equal to zero, that is if $e = (\theta_N, \bar{l}_N, C)$ then $e^i = ((0, \theta_{N\setminus\{i\}}), \bar{l}_N, C)$. In addition, let $l_N^i \in \Sigma(e^i)$ denote an efficient linguistic assignment in such an economy.

The Minimal Deficit Mechanism (**MDM**) For each $e \in \mathcal{E}$, $(l_N, t_N) = \varphi^{md}(e)$ if and only if $l_N \in \Sigma(e)$ and, for each $i \in N$,

$$t_i = \sum_{j \neq i} v_j(l_N; \theta_i) - \sum_{j \neq i} v_j(l_N^i; \theta_j) - v_i(l_N^i; 0).$$

In order to calculate the transfers the Planner assesses the impact of each agent by setting his willingness to communicate equal to zero and then recalculating the optimal amount of communication. Consider the example depicted in Figure 3.4. The top-left side of the

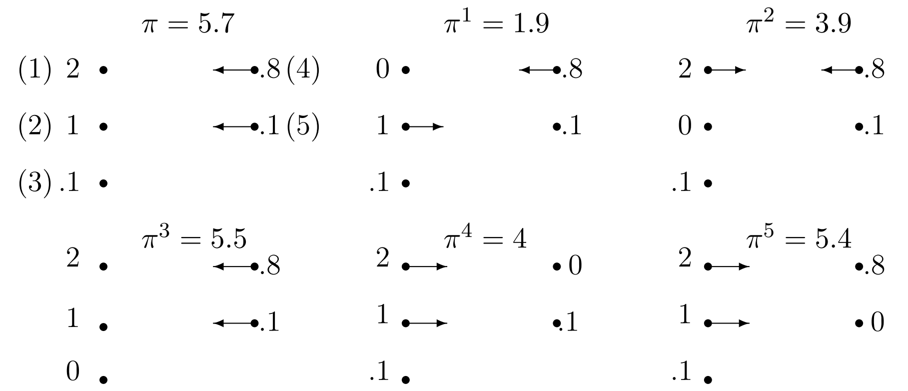


Figure 3.4:

figure describes the economy. The numbers in parentheses are a names of individuals.The number beside it represents their marginal willingness to communicate. On the left side we have agents whose mother-tongue is English on the right side agents whose mother-tongue is Italian. The learning costs are respectively $c_{EI} = 1.1$, $c_{IE} = 1.6$. At the efficient linguistic assignment both the Italians would be asked to learn English (as depicted by the arrows). Such an outcome would generate a sum of net benefits $\pi = 5, 7$. Then we set the willingness to communicate of each agent equal to zero and we compute again the efficient assignment. Interestingly a slight perturbation in the original problem can drastically change the efficient linguistic assignment. For each $i \in N$, π^i denotes the sum of net benefits at the optimal assignment when agent i's marginal willingness to communicate is set equal to zero. Once we have performed this exercise we can compute the transfers in the following way:

$t_1 = (\pi - v_1) - \pi^1 = -0.2$
$t_2 = (\pi - v_2) - \pi^2 = -0, 2$
$t_3 = (\pi - v_3) - \pi^3 = 0.0$
$t_4 = (\pi - v_4) - \pi^4 = 0.9$
$t_5 = (\pi - v_5) - \pi^5 = 1.6$

Agent 1 and 2 are taxed while agent 4 and 5 are subsidized. Such an allocation produces a deficit equal to 2.1.

A comparison with the Clarke mechanism (see Clarke [12]), which in our framework also belongs to the class of mechanisms satisfying Individual Rationality, seems particularly interesting. This mechanism corresponds to the case in which, for each $i \in N$, $h_i(\theta_{N\setminus\{i\}}) = \sum_{j\neq i} v_j(l'_N, \theta_j)$, where $l'_N \in \Sigma(\theta_{N\setminus\{i\}}, \bar{l}_{N\setminus\{i\}}, C)$ is the efficient linguistic assignment in the counterfactual situation where agent i is removed from the economy. In the canonical public good provision model whether an individual is removed from the economy or his valuation of the project is set to zero, amounts to the same effect. In our framework, the nature of the externality that each agent produces on the other agents is different. An individual is still a potential source of value for the rest even if his willingness to communicate is equal to zero. Removing him from the economy entails more than deducting his share of the pie. It has implications for all who may potentially communicate with him. Therefore, the component of the Clark mechanism that aims at balancing the budget not only falls short, it exacerbates the deficit relative to the MDM. In fact, the MDM is the mechanism that minimizes such a deficit.

Proposition 1 *For each $e \in \mathcal{E}$, any mechanisms satisfying Assignment Efficiency, Strategy-Profness and Individual Rationality generates at least as much deficit as the MDM..*

Proof. By Assignment Efficiency and Strategy-Proofness we need to compare our mechanism with other mechanisms belonging to the Groves family of mechanisms. Moreover, by Individual Rationality we need to have, for each $e \in \mathcal{E}$ and each $i \in N$

$$u_i(\varphi^g(e); \theta_i) = v_i(l_N; \theta_i) + t_i = \sum_{i\in N} v_i(l_N; \theta_i) - h_i(\theta_{N\setminus\{i\}}) \geq 0$$

with

$$\sum_{i\in N} v_i(l_N; \theta_i) = \sum_{i\in N} \left(\theta_i \left(\sum_{j\in N/N^{\gamma(i)}} min\{1, l_i l_j\} \right) - \bar{l}_i C(l_i - \bar{l}_i) \right)$$

where $l_N \in \Sigma(e)$. Hence, for any given profile $\theta_{N\setminus\{i\}} \in \mathbb{R}^{N-1}$ the component $\sum_{i\in N} v_i(l_N; \theta_i)$, which is the sum of net benefits at an efficient linguistic assignment, reaches its minimum value when agent i's willingness to communicate is equal to zero. Hence, in order to satisfy Individual Rationality we need to set, for each $e \in \mathcal{E}$, $i \in N$,

$$h_i(\theta_{N\setminus\{i\}}) \leq \sum_{j\neq i} v_j(l^i_N; \theta_j) + v_i(l^i_N; 0). \tag{3.4.1}$$

where ${}^i_N \in \Sigma(e^i)$. Moreover,

$$\sum_{i\in N} t_i = \sum_{i\in N} \sum_{j\neq i} v_j(l_N; \theta_j) - \sum_{i\in N} h_i(\theta_{N\setminus\{i\}})$$

By equation 3.4.1,

$$\sum_{i\in N} t_i \geq \sum_{i\in N} \sum_{j\neq i} v_j(l; \theta_j) - \sum_{i\in N} \sum_{j\neq N} v_j(l^i; \theta_j) - \sum_{i\in N} v_i(l^i_N; 0) \tag{3.4.2}$$

This means that, as soon a mechanism generates a deficit, it generates at least as much deficit as in the the right-end side of equation 3.4.2. Hence in order to minimize the deficit produced by the mechanism we need to set

$$h_i(\theta_{N\setminus\{i\}}) = \sum_{j\neq i} v_j(l^i_N; \theta_j) + v_i(l^i_N; 0)$$

■

We actually conjecture that, for any conceivable economy, any individually rational Groves mechanism runs a deficit. Namely, the right hand side of equation 3.4.2 is always non-negative. Consequently the sum of the transfers pertaining to the MDM would always be non-negative as well. While we are unable to prove the statement in general, first, simulations suggest that it is true and, second, we are able to show that in particular environments. We will focus on the case of *large economies*. We consider this case particularly interesting since, as pointed for example by Deb, Razzolini and Seo [14], some members of the Clarke-Groves family are asymptotically balanced as the number of agents increases in a well-behaved way, for instance by replication.

The previous result gives us a lower bound for the total amount of transfers generated by an assignment-efficient, strategy-proof and individually rational allocation. However the result is silent regarding the size of the deficit such a mechanism might generate. This parameter is crucial in assessing the significance of the MDM. The following result states that the deficit is, for each economy in the domain, less or equal to the total cost associated with an efficient linguistic assignment in that economy.

Proposition 2 *For each $e \in \mathcal{E}$ and each $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi^{md}(e)$,*

$$\sum_{i \in N} t_i \leq \sum_{i \in N} \bar{l}_i C(l_i - \bar{l}_i).$$

Proof. The statement derives from the fact that, for each $i \in N$

$$\theta_i \sum_{j \in N \setminus N^{\gamma(i)}} \min\{1, l_i l_j\} \geq \sum_{i \in N} v_i(l_N; \theta_i) - \sum_{j \neq i} v_j(l_N^i; \theta_j) - v_i(l_N^i; 0).$$

where $l_N \in \Sigma(e)$ and $l_N^i \in \Sigma(e^i)$. Suppose not. Then, just by rearranging the terms of the previous inequality one would obtain

$$\pi(l_N^i; e^i) = \sum_{j \neq i} v_j(l_N^i; \theta_j) + v_i(l_N^i; 0) < \sum_{i \in N} v_i(l_N; \theta_i) - \theta_i \sum_{j \in N \setminus N^{\gamma(i)}} \min\{1, l_i l_j\} = \pi(l_N; e^i),$$

which constitutes a contradiction, as, by assumption, $l_N^i \in \Sigma(e^i)$. Summing over $i \in N$ we obtain

$$\sum_{i \in N} \theta_i \sum_{j \in N \setminus N^{\gamma(i)}} \min\{1, l_i l_j\} \geq N\pi(l_N; e) - \sum_{i \in N} \pi(l_N^i; e^i).$$

The left-hand side of the previous equation represents the total gross benefit deriving from communication at l_N. A simple algebraic manipulation yields

$$\sum_{i \in N} \overline{l_i} C(l_i - \overline{l_i}) \geq (N-1)\pi(l_N; e) - \sum_{i \in N} \pi(l_N^i; e^i) = \sum_{i \in N} t_i.$$

■

The following results will enable us to make precise statements regarding both the size of the deficit as well as individual transfers in large economies. As mentioned earlier we adopt the replication view of largeness. Letting $\rho \in \mathbb{N}^*$, where $\mathbb{N}^* \equiv \mathbb{N}_+ \setminus \{0\}$, we may appeal to the following definition.

Definition 1 *For each $e \in \mathcal{E}$, $e_\rho \in \mathcal{E}$, comprising a set of agents N_ρ, is $\rho-$replica of e if and only if*

- *the set of languages and the cost matrix is the same in both economies,*
- *there exists a mapping* $\xi : N_\rho \to N$ *such that for each* $i \in N$, $|\xi^{-1}(i)| = \rho$, *and*
- *for each* $i \in N$ *and each* $j \in \xi^{-1}(i)$, $\theta_i = \theta_j$ *and* $\bar{l}_i = \bar{l}_j$.

Consider the three-agent, three-language economy depicted on the left side of Figure 3.5. Assume that $\theta_E > 0$, while $\theta_F = \theta_I = 0$. One can easily find a cost matrix such that the efficient linguistic assignment is as depicted on the left side of Figure 5. In fact, by appropriately choosing the cost matrix one may face a situation in which this communication pattern persists under replications. On the right side of Figure 5 we depict the 3-replica of the initial economy. Consider an economy e_3^i, for some individual i belonging to the english group. That individual will not learn any language at an efficient linguistic assignment. If he was to do so, he would add only a negative value, his cost, to the sum of net benefits. Therefore, if $l_N \in \Sigma(e_3)$ and $l_N^i \in \Sigma(e_3^i)$, for some individual i belonging to the english group, we conclude that $l_N \neq l_N^i$. The following assumption enable us to show that the efficient linguistic assignment in any large economy e remains unchanged when considering e^i, for any $i \in N$. The complexity we are stirring away from involves situations where an entire language group has zero willingness to communicate. Including such economies enriches the mathematical aspect of the problem without, however, advancing the discussion.

Assumption 1 *For each* $e \in \mathcal{E}$, *and each* $i \in N$, *there exists some* $j \in N^{\gamma(i)}$ *such that* $\theta_j \neq 0$.

We begin by identifying the first-order conditions associated with problem of determining the set of efficient linguistic assignments.

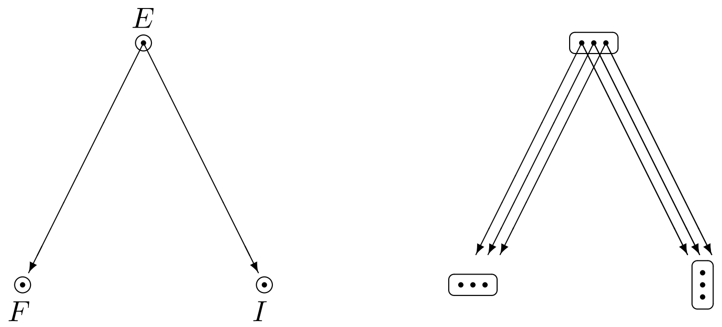


Figure 3.5: The role of Assumption 1

Lemma 4 *For each* $e \in \mathcal{E}$, *if* $l_N \in \Sigma(e)$ *then*

1. *for each* $i \in N$ *and each* $l_i' \in \{0,1\}^{|\Lambda|}$ *such that* $l_i' < l_i$

$$\theta_i \sum_{j \in N \setminus N^{\gamma(i)}} \left[\min\{1, l_j(l_i - l_i')\}\right] + \sum_{j \in N \setminus N^{\gamma(i)}} \theta_j \left[\min\{1, l_j(l_i - l_i')\}\right] - l_i' C(l_i - l_i') \geq 0, \tag{3.4.3}$$

2. *for each* $i \in N$ *and each* $l_i'' \in \{0,1\}^{|\Lambda|}$ *such that* $l_i'' > l_i$

$$\theta_i \sum_{j \in N \setminus N^{\gamma(i)}} \left[\min\{1, l_j(l_i'' - l_i)\}\right] + \sum_{j \in N \setminus N^{\gamma(i)}} \theta_j \left[\min\{1, l_j(l_i'' - l_j)\}\right] - l_i C(l_i'' - l_i) \leq 0, \tag{3.4.4}$$

Proof. If l_N is optimal for some economy $e \in \mathcal{E}$ then for each $i \in N$ unilaterally reducing the amount of communication (condition 3.4.3), or unilaterally increasing the amount of communication (condition 3.4.4) must decrease the sum of utilities. ■

The fact that, within each group determined by the mother tongue of each agent, the is always someone with some interest in communicating with other agents, guarantees first of all that after a certain number of replications the efficient communicative structure will be such that everyone communicates with everyone else. This observation will be key. In order to formally prove it, we will need to introduce a new piece of notation. For each $e \in \mathcal{E}$, let

$$L^f(e) \equiv \left\{l_N \in \{0,1\}^{\Lambda \times N} \;:\; l_i l_j \geq 1, \forall i, j \in N\right\}$$

define the set of linguistic assignments that ensure that all individuals have a language in common.

Lemma 5 *For each $e \in \mathcal{E}$, under assumption 1, there exists $\widehat{\rho} \in \mathbb{N}^*$ such that, for each $\rho > \widehat{\rho}$, and each $e_\rho \in \mathcal{E}$, $l_{N_\rho} \in \Sigma(e_\rho)$ only if $l_{N_\rho} \in L^f(e_\rho)$.*

Proof. Take any $e \in \mathcal{E}$ that satisfies Assumption 1. Suppose that for each $\rho \in \mathbb{N}^*$ and each $e^\rho \in \mathcal{E}$, $l_{N_\rho} \in \Sigma(e_\rho)$, $l_{N_\rho} \notin L^f(e_\rho)$. Then there exists $j \in N^\rho$, $l'_j \in \{0,1\}^\Lambda$ with $l'_j > l_j$ and $(l_{N_\rho \setminus \{j\}}, l'_j) \in \{0,1\}^{\Lambda \times N_\rho}$ such that

$$\theta_j \sum_{k \in N_\rho \setminus N_\rho^{\gamma(j)}} \left[\min\{1, l_k(l'_j - l_j)\}\right] + \sum_{k \in N_\rho \setminus N_\rho^{\gamma(j)}} \theta_k \left[\min\{1, l_k(l'_j - l_j)\}\right] - l_j C(l'_j - l_j) \quad (3.4.5)$$

$$\geq \rho[max\{0, \theta_j\}] + \rho[max\{0, \theta_k\}] - l_j C(l'_j - l_j),$$

with $k \in N_\rho \setminus N_\rho^{\gamma(j)}$. That is, since $l_{N_\rho} \notin L^f(e_\rho)$ there is always an agent $j \in N_\rho$, such that, by learning a new language is either able to achieve a sum of gross benefits of at least $\rho\theta_j$ or create some gross benefit for at least ρ agents (in another group) whose willingness to communicate is $\theta_k \neq 0$, or both. Morevorer by assumption 1 such a benefit is always stricltly positive. In addition, for some $\hat{\rho} \in \mathbb{N}^*$, some $j \in N_{\hat{\rho}}$ and some $l_{N_{\hat{\rho}}} \in \Sigma(e_{\hat{\rho}})$ such that $l_{N_{\hat{\rho}}} \notin L^f(e_{\hat{\rho}})$, it must hold true that

$$\hat{\rho}[max\{0, \theta_j\}] + \hat{\rho}[max\{0, \theta_k\}] - l_j C(l'_j - l_j) > 0.$$

This implies that also the left-hand side of equation 3.4.5 is strictly positive, but this, by condition 3.4.4, contradicts the fact that $l_{N_{\hat{\rho}}} \in \Sigma(e_{\hat{\rho}})$. ■

Lemma 5 proves that after a certain number of replications the efficient assignment will necessarily involves a situation where all agents have at least a language in common. Following the same argument, we can also argue that if $\rho \in \mathbb{N}^*$ is big enough such a communicative structure will persist regardless of small perturbations in the parameters of the problem.

Lemma 6 *For each $e \in \mathcal{E}$, under Assumption 1, there exists $\rho' \in \mathbb{N}^*$ such that, for each $\rho > \rho'$, each $e_\rho \in \mathcal{E}$, and each $i \in N_\rho$, $l_{N_\rho} \in \Sigma(e_\rho)$ if and only if $l_{N_\rho} \in \Sigma(e^i_\rho)$*

Proof. By lemma 5, there must exist some $\rho' \in \mathbb{N}^*$, such that, not only we have $l_{N_{\rho'}} \in \Sigma(e_{\rho'})$ only if $l_{N_{\rho'}} \in L^f(e_{\rho'})$, but also $l_{N_{\rho'}} \in \Sigma(e^i_{\rho'})$ only if $l_{N_{\rho'}} \in L^f(e^i_{\rho'})$ for each $i \in N$. Namely, after a certain number of replicas, by setting individual i's willingness to communicate equal to zero, for each $i \in N_{\rho'}$, the efficient linguistic assignment always achieves full communication. Moreover assume $l_{N_{\rho'}} \in \Sigma(e_{\rho'})$ and for some $i \in N_{\rho'}$, $l_{N_{\rho'}} \notin \Sigma(e^i_{\rho})$. Then there exists some $l'_{N_{\rho'}} \in \Sigma(e^i_{\rho'})$ such that $l'_{N_{\rho'}} \in L^f(e^i_{\rho'})$ and

$$\sum_{i \in N^{\rho'}} \bar{l}_i C(l'_i - \bar{l}_i) < \sum_{i \in N^{\rho'}} \bar{l}_i C(l_i - \bar{l}_i)$$

But then, since for each $i \in N_{\rho'}$, $L^f(e_{\rho'}) = L^f(e^i_{\rho'})$, $l'_{N_{\rho'}}$ would guarantee, in $e_{\rho'}$, the same gross benefit as $l_{N_{\rho'}}$ but at a lower total cost. This yields the desired contradiction. The *if* part of the statement can be proven using a similar argument. ■

It finally turns out that, under assumption 1, the transfer associated with the minimal deficit mechanism can be easily computed in *large economies*. Each individual is payed his own learning cost.

Proposition 3 *For each $e \in \mathcal{E}$, under assumption 1, there exists $\rho' \in \mathbb{N}^*$ such that, for each $\rho > \rho'$, each $e_\rho \in \mathcal{E}$, each $(l_{N_\rho}, t_{N_\rho}) \in Z$ such that $(l_{N_\rho}, t_{N_\rho}) = \varphi^{md}(e_\rho)$ and each $i \in N_\rho$*

$$t_i = \bar{l}_i C(l_i - \bar{l}_i)$$

Proof. By definition, for each $i \in N_\rho$

$$t_i = \sum_{j \neq i} v_j(l_{N_\rho}; \theta_i) - \sum_{j \neq i} v_j(l^i_{N_\rho}; \theta_j) - v_i(l^i_{N_\rho}; 0) \tag{3.4.6}$$

where $l_{N_\rho} \in \Sigma(e_\rho)$ and $l^i_{N_\rho} \in \Sigma(e^i_\rho)$. By lemma 6, there exists $\rho' \in \mathbb{N}^*$ such that for each $\rho > \rho'$ and each $i \in N_\rho$, $l_{N_\rho} \in \Sigma(e_\rho)$ if and only if $l_{N_\rho} \in \Sigma(e^i_\rho)$ hence equation 3.4.6 becomes

$$t_i = \sum_{j \neq i} v_j(l_{N_\rho}; \theta_i) - \sum_{j \neq i} v_j(l_{N_\rho}; \theta_j) - v_i(l_{N_\rho}; 0), \tag{3.4.7}$$

so that

$$t_i = -v_i(l_{N_\rho}; 0) = \bar{l}_i C(l_i - \bar{l}_i).$$

■

This means that the, in large economies, the deficit deriving from the implementation of the MDM equals the sum of the learning costs individuals have to face at the optimal assignment. Such a result might seem quite discouraging but in order to have a complete picture we should also consider the value generated in the economy. As in Moulin [43] we take as a measure of the efficiency loss of the mechanism the ratio between the deficit run by the mechanism and the sum of net benefits generated. More formally, for each $e \in \mathcal{E}$ and each $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi^{md}(e)$, define the ratio of the deficit over the sum of net benefits

$$L(e) = \frac{\sum_{i \in N} t_i}{\pi(l_N; e)} \tag{3.4.8}$$

In a large economy by Proposition 3, we have

$$L(e) = \frac{\sum_{i \in N_\rho} \left[\bar{l}_i C(l_i - \bar{l}_i)\right]}{\sum_{i \in N_\rho} \left[\theta_i \left(\sum_{j \in N_\rho / N_\rho^{\gamma(j)}} min\{1, l_i l_j\}\right) - \bar{l}_i C(l_i - \bar{l}_i)\right]}.$$

Some algebraic manipulation yields

$$L(e) = \frac{\rho \sum_i \left[\bar{l}_i C(l_i - \bar{l}_i)\right]}{\sum_{i \in N} \left[\rho \theta_i \sum_{\lambda \neq \gamma(i)} \rho |N^\lambda|\right] - \rho \sum_i \left[\bar{l}_i C(l_i - \bar{l}_i)\right]}, \text{ or}$$

$$L(e) = \frac{\sum_i \left[\bar{l}_i C(l_i - \bar{l}_i)\right]}{\rho \sum_{i \in N} \left[\theta_i \sum_{\lambda \neq \gamma(i)} |N^\lambda|\right] - \sum_i \left[\bar{l}_i C(l_i - \bar{l}_i)\right]}.$$

Hence as ρ grows larger, $L(e)$ tends to zero. However, it is important to note that in *small* economies the following paradoxical scenario might arise. Consider an economy comprising three individuals each speaking one of languages α, β, γ: $\theta_i = 1$ for each $i \in \{1, 2, 3\}$. Moreover, $c_{\beta\alpha} = c_{\gamma\alpha} = 2,9$ while the rest of the elements of the cost matrix are equal to 10. It is easy to compute that the ratio $L(e)$ in this economy will be greater than 1, that is the deficit pertaining to our mechanism is greater than the sum of net benefits.

Notice, finally, that the definition of a *full* structure we offer above is silent about the actual structure that prevails at the optimum. Typically, full communication will be achieved by convergence to a common language. This, however, is not the only possibility. The following example demonstrates. Consider the following three-agent economy:

	Θ
E	0,6
F	0,6
I	0,6

$C =$

	F	E	I
F	0	1	10
E	10	0	1
T	1	10	0

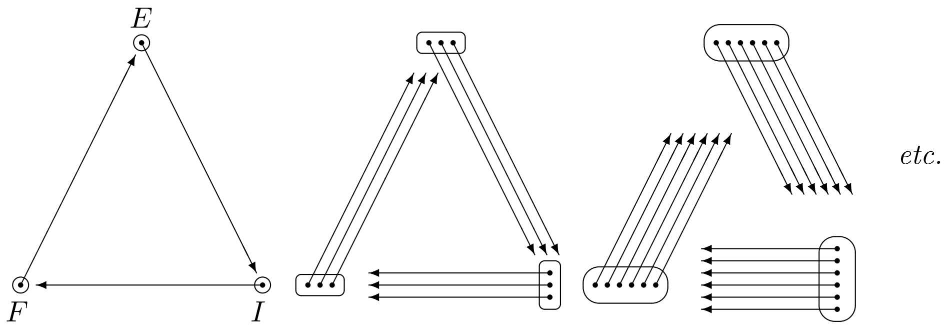


Figure 3.6: In 'large economies' convergence to common language is not necessarily the optimal structure.

Refer to Figure 6.6. In the original economy it would be optimal for the Italian to learn French, for the English to learn Italian and for the French to learn English. We would thus have a full network but not a convergence toward a common language. Interestingly, by replicating this economy, the optimal structure still follows the same path.

3.4.1 Largeness in bilingual societies

As mentioned earlier, we conjecture that, for any conceivable economy, any individually rational Groves mechanism runs a deficit. As it is possible to understand from the proof of proposition 1 this amounts to the following statements.

Conjecture 1 *For each $e \in \mathcal{E}$,*

$$\sum_{i \in N} \sum_{j \neq i} v_j(l_N; \theta_j) - \sum_{i \in N} \sum_{j \in N} v_j(l_N^i; \theta_j) - \sum_{i \in N} v_i(l_N^i; 0) \geq 0, \tag{3.4.9}$$

where $l_N \in \Sigma(e)$ and, for each $i \in N$, $l_N^i \in \Sigma(e^i)$.

So far we have managed to prove that such a condition holds if an economy is 'large enough' (that is, after a specific number of replications). By focusing our attention on bilingual economies we are able to determine a threshold for this parameter. Moreover, in Appendix 1, we provide an algorithm that solves the Pareto Efficiency problem for the two language case. The algorithm without per se dealing with the computational difficulties that may arise when the number of languages increases, nonetheless provides some indication that the problem might become eventually unmanageable from the computational point of view.

Consider the following simple two-language economy:

1. There are only 2 languages α and β.
2. Let $c_{\beta\alpha}$ be the cost of learning language α given that agent knows language β. Similarly, let $c_{\alpha\beta}$ be the cost of learning language β given that agent knows language α.
3. N is the set of agents, which can be divided into two disjoint sets N^α and N^β, let $N^\alpha = \{1, \ldots, m-1\}$, $N^\beta = \{m, \ldots, |N|\}$.
4. Within each language group and for each $i, j \in N^\alpha$, if $i \leq j$, then $\theta_i \geq \theta_j$. Similarly for N^β.

The spirit of the statement will be as follows. Fix any arbitrary non-negative real number T. Propositions 4 and 5 determine the number of individuals that need to have a willingness to communicate greater than T for Conjecture 1 to be true. The proofs can be found in Appendix 3. In what follows we assume without loss of generality that $c_{\alpha\beta} \leq c_{\beta\alpha}$.

Proposition 4 *For each positive real number T, let $L(T,e) = \{i \in N \,|\, \theta_i < T\}$ and $\zeta(T,e) = |L(T,e)|$. Let*

$$\bar{N}^\beta = inf_{T>0}\{\zeta(T,e) + 2 + \frac{c_{\alpha\beta}}{T} + \frac{c_{\beta\alpha}}{T}\}. \tag{3.4.10}$$

If $N^\beta \geq \bar{N}^\beta$, then conjecture 1 holds.

Proposition 5 *For each positive real number T, let $L(T,e) = \{i \in N \,|\, \theta_i < T\}$ and $\zeta(T,e) = |L(T,e)|$. Let*

$$\bar{N}^\alpha = inf_{T>0}\{\frac{c_{\beta\alpha}}{c_{\alpha\beta}}(\zeta(T,e)+1) + 1 + \frac{2c_{\beta\alpha}}{T}\} \tag{3.4.11}$$

If $N^\alpha \geq \bar{N}^\alpha$, then conjecture 1 holds.

The following example explains how to read the previous proposition. Consider a bilingual economy where $c_{\alpha\beta} = 1$, $c_{\beta\alpha} = 2$, $N^\alpha = \{1,2,3,4\}$, $N^\beta = \{5,6,7,8,9,10,11\}$ and $\theta_1 = 3$, $\theta_2 = 2$, $\theta_3 = 1$, $\theta_4 = 0$, $\theta_5 = 3$,$\theta_6 = 3$, $\theta_7 = 2$, $\theta_8 = 2$, $\theta_9 = 1.5$, $\theta_{10} = 1.5$, $\theta_{11} = 0.5$. The number of agents with θ_i less that 1, i.e., $L(1,e)$ is $\{4, 11\}$. Now $\zeta(1,e) + 2 + \frac{c_{\alpha\beta}+c_{\beta\alpha}}{1} = 7$. Therefore $\bar{N}^\beta \leq 7$. Since $|N^\beta| = 7$ by Proposition 4 the conjecture holds.

Three comments are in line. First, the domain restrictions 'almost' do not rely on preferences. The only consideration is about individuals that are entirely averse to the possibility of communication. Second, the domain restrictions do not require the economy to be large, unless both costs are enormous. Even in such a case though, one needs only one language group to be adequately large for the result to come through. Finally, the proof gives a rather precise picture of the pattern of language learning that is efficient, namely only one side learns.

3.5 A feasible Groves Mechanism.

In this section we drop Individual Rationality. Here we think of the social planner as an agent who is mandated to eliminate wasteful dissipation of money. He has the power to tax an agent even if he does not attach any value to communication. In the previous sections we have established that the deficit of the MDM in each economy is, in absolute terms, smaller than the total cost pertaining to the efficient linguistic assignment in that economy. Consider for each economy, the set of linguistic assignments that ensure full communication, $L^f(e)$. Select within this set the subset of linguistic assignments that are the least costly. That is, for each $e \in \mathcal{E}$, let

$$L^{f*}(e) \equiv \underset{l_N \in L^f(e)}{\operatorname{argmin}} \sum_{i \in N} \bar{l}_i C(l_i - \bar{l}_i) \subseteq L^f(e)$$

Moreover, for each $l_N \in L^{f*}(e)$, let

$$\underline{c} = \sum_{i \in N} \bar{l}_i C(l_i - \bar{l}_i).$$

Notice that in order to compute the value $\underline{c}$ one needs not know the profile of preferences. Moreover, for each economy the total cost pertaining to the efficient linguistic assignment in that economy will be weakly less than the value $\underline{c}$ for that economy. The mechanism we present below charges all individuals to the sum $\frac{\underline{c}}{N}$ on the top of what they were charged by the MDM.

The Translated Minimal Deficit Mechanism (TMDM) For each $e \in \mathcal{E}$, $(l_N, t_N) = \varphi^{tmd}(e)$ if and only if $l_N \in \Sigma(e)$ and, for each $i \in N$

$$t_i = \sum_{j \neq i} v_j(l_N; \theta_i) - \sum_{j \neq i} v_j(l_N^i; \theta_j) - v_i(l_N^i; 0) - \frac{\underline{c}}{N}$$

where $l_N^i \in \Sigma(e^i)$.

The following propositions shows that the TMDM is feasible.

Proposition 6 *For each $e \in \mathcal{E}$ and each $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi^{tmd}$*

$$\sum_{i \in N} t_i \leq 0. \tag{3.5.1}$$

Proof. From lemma 2 we know that, for each $e \in \mathcal{E}$ and each $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi^{md}(e)$,

$$\sum_{i \in N} \bar{l}_i C(l_i - \bar{l}_i) \geq \sum_{i \in N} t_i.$$

Hence, summing over $i \in N$, by definition of the TMDM, follows that for each $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi^{tmd}(e)$

$$\sum_{i \in N} t_i \leq \sum_{i \in N} \bar{l}_i C(l_i - \bar{l}_i) - \sum_{i \in N} \bar{l}_i C(l'_i - \bar{l}_i)$$

where $l'_N \in L^f(e)$. Moreover, for each $e \in \mathcal{E}$, and each $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi^{tmd}(e)$ necessarily we must have

$$\sum_{i \in N} \bar{l}_i C(l_i - \bar{l}_i) - \sum_{i \in N} \bar{l}_i C(l'_i - \bar{l}_i) \leq 0$$

otherwise it would be possible to have a linguistic assignement, l'_N, that grants an higher gross benefit at a lower cost then l_N. This would contradict the fact that $l_N \in \Sigma(e)$, hence

$$\sum_{i \in N} t_i \leq 0.$$

■

The TMDM, under assumption 1, is balanced in large economies. This is a direct consequence of its construction. The amount it charges individuals independently of the profile of preferences equals the deficit of the MDM in large economies.

Proposition 7 *For each $e \in \mathcal{E}$, under assumption 1, there exists $\rho' \in \mathbb{N}^*$ such that, for each $\rho > \rho'$, $e^\rho \in \mathcal{E}$, $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi^{tmd}(e^\rho)$,*

$$\sum_{i \in N^\rho} t_i = 0$$

Proof. This is an immediate consequence of Proposition 3. ■

Any mechanism in the class of feasible Groves mechanisms will have a component that charges individuals in a way that must not depend on the linguistic assignment it prescribes. In light of this fact, it is meaningful to require some symmetry in the way individuals are treated in an effort to ensure feasibility. We require of mechanisms to assign the same transfers to each individual in any economy where the efficient linguistic assignment requires no individual to learn any language.

Minimal Symmetry. For each $e \in \mathcal{E}$, if, for each $l_N \in \Sigma(e)$, $l_N = \bar{l}_N$ then $t_i = t_j$ for each $i, j \in N$.

The following proposition shows that there is no other feasible Groves mechanism that satisfies symmetry and perform strictly better than the TMDM. We prove this by showing that in a two-agent economy it is actually not possible to find an alternative mechanism that welfare dominates the TMDM[4].

Proposition 8 *If $N = 2$ and $\Lambda = 2$ then there is no solution $(l'_N, t'_N) = \varphi'(e)$ satisfying Assignment Efficiency, Strategy Proofness, Feasibility and Symmetry such that for each $(l_N, t_N) \in Z$ such that $(l_N, t_N) = \varphi^{md}(e)$,*

$$0 \geq \sum_{i \in N} t'_i \geq \sum_{i \in N} t_i \; for \; each \; e \in \mathcal{E}$$

with

$$0 \geq \sum_{i \in N} t'_i > \sum_{i \in N} t_i \; for \; some \; e \in \mathcal{E}$$

[4]Indeed there might be a mechanism that that is welfare equivalent for a two-agent economy but performs better than ours in all other economies. Our intuition is that this is not possible but we have not been able to prove yet a more general statement.

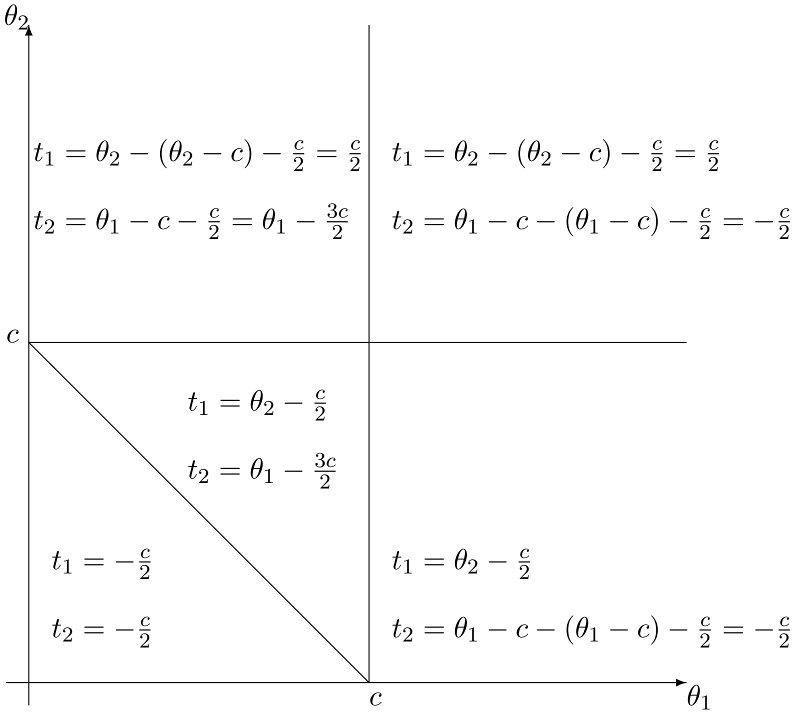


Figure 3.7: The transfers assigned by the TMDM in two-agent two-language economy

Proof. Assume, without loss of generality that $\Lambda = \{\alpha, \beta\}$, $\bar{l}_1 = (1, 0)$, $\bar{l}_2 = (0, 1)$ and $c = c_{\alpha\beta} < c_{\beta\alpha}$. By Assignment Efficiency and Strategy Proofness φ' belongs to the family of Groves-Clarke mechanisms.

If both $\theta_1 \geq \underline{c}$ and $\theta_2 \geq c$ then, as can be easily checked (see figure 3.7), by definition of TMDM, $t_1 + t_2 = 0$. Hence we need to impose

$$t'_1 + t'_2 = \theta_1 + \theta_2 - c - h_1(\theta_2) - h_2(\theta_1) = 0 \; for \; each \; \theta_1 \geq c \; and \; \theta_2 \geq c.$$

From this follows that

$$h_1(\theta_2) = \theta_2 + k_1 \; for \;\; all \; \theta_2 > \underline{c} \; and \; h_2(\theta_1) = \theta_1 + k_2 \; for \; all \; \theta_1 > c$$

with k_1 and $k_2 \in \mathbb{R}$ and $k_1 + k_2 = -c$.

If on the contrary, both $\theta_1 < c$ and $\theta_2 < c$ then the two agents are not asked to communicate and $t_1 + t_2 = -c$.

Assume then that for all $\theta_1 < c$ and $\theta_2 < c$.

$$0 \geq t'_1 + t'_2 \geq -c$$

Moreover for all $\theta_1 < c$ and $\theta_2 < c$, by symmetry we must have $t'_1 = t'_2$ which implies

$$t'_1 =\leq -\frac{c}{2} \; or, \; equivalently, \; h_1(\theta_2) = -\frac{c - \epsilon}{2} \; for \; all \; \theta_2 < c$$

$$t'_2 =\leq -\frac{c}{2} \; or, \; equivalently, \; h_2(\theta_1) = -\frac{c - \epsilon}{2} \; for \; all \; \theta_1 < c$$

with $\epsilon \in \mathbb{R}_+$. This entails that if $\theta_1 \geq c$ and $\theta_2 < c$ (the same argument can be symmetrically applied if $\theta_2 > c$ and $\theta_1 \leq c$)

$$t'_1 = \theta_2 - h_1(\theta_2) = \theta_2 - \frac{c - \epsilon}{2} \; for \; all \; \theta_2 < c$$

$$t'_2 = \theta_1 - c - h_1(\theta_2) = -c - k_2 \ for \ all \ \theta_1 \geq c.$$

If we assume without loss of generality (since $k_1 + k_2 = -\underline{c}$) that $k_2 \leq -\frac{c}{2}$ we obtain

$$t'_1 + t'_2 \geq \theta_2 - c + \frac{\epsilon}{2}$$

For values of θ_2 close enough to c the sum of the transfers would then be strictly positive contradicting Feasibility. ■

Two points should be made. First, insisting on Feasibility might have very severe consequences on Individual Rationality. Consider an economy comprising three individuals each speaking one of languages α, β, γ: $\theta_i = 1$ for each $i \in \{1, 2, 3\}$, while all the elements of the cost matrix are equal to 10. It is easy to verify that by Assignment Efficiency none of the individuals would be asked to learn any language by the TMDM. Nonetheless, each of them would be asked to pay a transfer $t_i = \frac{10}{3}$. As a consequence all the agents in the economy would suffer by participating in the proposed scheme. Second, the Clarke mechanism translated by an amount $\underline{c}$ is in deficit in large economies under assumption 1. This is interesting because as the economy grows larger such a mechanism can be thought of an approximation of the Translated Minimal Deficit Mechanism. This latter fact makes us believe that proposition 8 can be generalized to a wider class of economies.

3.6 Beyond Groves's Mechanism.

We will need to introduce some new pieces of notation. First, let us define for each $e \in \mathcal{E}$ and for each $X \subseteq N \cup \{\emptyset\}$ the set of linguistic assignments that ensure that all individuals except those in X have a language in common, namely

$$L^f(e|X) \equiv \big\{ l_N \in L(e) \ : \ l_i l_j \geq 1, \forall i, j \in N \setminus X \ \ and \ \ l_i l_j = 0, \ \forall i, j \in X \big\}.$$

Within this set we would like to identify the least costly linguistic assignments. Therefore, define for each $e \in \mathcal{E}$, for each $X \subseteq N \cup \{\emptyset\}$ the set

$$L^{f*}(e|X) \equiv \underset{l'_N \in L^f(e|X)}{\operatorname{argmin}} \sum_{i \in N} \bar{l}_i C(l'_i - \bar{l}_i) \subseteq L^f(e|X)$$

Obviously, for each $e \in \mathcal{E}$ and for each $X \subseteq N \cup \{\emptyset\}$, $L^{f*}(e|X) \subseteq L^f(e|X)$. Broadly speaking, the mechanism we introduce below selects the linguistic assignment that minimizes X subject to all individuals obtaining a utility greater than zero. It is more instructive to define the mechanism in terms of the algorithm that underlies it. Note that it relies on the ability of the game designer to enforce punishments, employing, however, means that are not arbitrary, but, rather, embedded in the mechanics of the model. We will come back to this point shortly.

The *Maximal Individually Rational Assignment Mechanism (MIRAM)* (φ^{mir}) selects for each $e \in \mathcal{E}$, an allocation by applying the following algorithm. Let individuals announce a profile of preferences $\tilde{\theta}_N \in \mathbb{R}^N_+$. Set $X^1 = \emptyset$, select any $l^1_N \in L^{f*}(e|X^1)$ and check whether the allocation l^1_N with associated transfer

$$t^1_i = \bar{l}_i C(l^1_i - \bar{l}_i) - \frac{\sum_{j \in N \setminus X^1} \bar{l}_j C(l^1_j - \bar{l}_j)}{|N \setminus X^1|}$$

satisfies

$$\tilde{\theta}_i |(N \setminus N^{\gamma(i)}) \setminus X^1| - \frac{\sum_{j \in N \setminus X^1} \bar{l}_j C(l^1_j - \bar{l}_j)}{|N \setminus X^1|} \geq 0, \ \text{for each } i \in N. \tag{3.6.1}$$

If the answer is yes, stop and select allocation (l^1_N, t^1_N). If the answer is no, define the set X^2 to be such that for each $i \in X^2$

$$\tilde{\theta}_i|(N \setminus N^{\gamma(i)}) \setminus X^1| - \frac{\sum_{j \in N \setminus X^1} \bar{l}_j C(l^1_j - \bar{l}_j)}{|N \setminus X^1|} < 0,$$

select some $l^2_N \in L^{f*}(e|X^1 \cup X^2)$ and associated transfer

$$t^2_i = \bar{l}_i C(l^2_i - \bar{l}_i) - \frac{\sum_{j \in N \setminus (X^1 \cup X^2)} \bar{l}_j C(l^2_j - \bar{l}_j)}{|N \setminus (X^1 \cup X^2)|}$$

and check whether

$$\tilde{\theta}_i|(N \setminus N^{\gamma(i)}) \setminus X^1 \cup X^2| - \frac{\sum_{j \in N \setminus (X^1 \cup X^2)} \bar{l}_j C(l^2_j - \bar{l}_j)}{|N \setminus (X^1 \cup X^2)|} \geq 0, \text{ for each } i \in N \setminus (X^1 \cup X^2). \tag{3.6.2}$$

If the answer is yes stop and select allocation (l^2_N, t^2_N), otherwise proceed as above to define X^3 and so on and so forth. The algorithm terminates at most at some stage r for which $X^1 \cup \cdots \cup X^r = N$. At that stage r, the selected allocation is $\bar{l}_N$ with associated transfer $t_i = 0$, for each $i \in N$.

First we prove that this procedure identifies a *Strategy-Proof* mechanism, if an individual mis-reports his true preferences he guarantees himself weakly less utility that he would otherwise enjoy.

Proposition 9 *For each $e \in \mathcal{E}$, the MIRAM satisfies Strategy-Proofness*

Proof. Consider $e \in \mathcal{E}$. Take any agent $i \in N$ whose real willingness to communicate is $\theta_i \in \mathbb{R}_+$. By announcing a willingness to communicate $\theta'_i \in \mathbb{R}_+$ such that $\theta'_i > \theta_i$, agent i would not change, at each step r, the decision taken by the algorithm if already

$$\theta_i|(N \setminus N^{\gamma(i)}) \setminus X^1 \cup ... \cup X^r| - \frac{\sum_{j \in N \setminus (X^1 \cup ... \cup X^r)} \bar{l}_j C(l^r_j - \bar{l}_j)}{|N \setminus (X^1 \cup ... \cup X^r)|} \geq 0 \tag{3.6.3}$$

with $l^r_N \in L^{f*}(e|X^r)$. If on the contrary, for some r, inequality 3.6.3 holds in the opposite direction, by announcing a higher willingness to communicate he might be assigned an allocation that gives him a negative utility. The argument work similarly if $\theta'_i < \theta_i$. ■

In addition, the MIRAM is by construction Feasible and Individually Rational. Moreover, it is easy to see that if all agents had, even if minimal, a strictly positive willingness to communicate, after a sufficiently big number of replications the algorithm presented above would always stop at the first step (that is, for a ρ big enough, the in equality in 3.6.1 always holds). This would lead to a situation where all agents have at least a language in common. As we know from the previous section this will constitute an efficient linguistic assignment.

Assumption 2 *For each $e \in \mathcal{E}$ and each $i \in N$, $\theta_i \in (0, \overline{\theta}]$ for some finite $\overline{\theta} \in \mathbb{R}_+$.*

Proposition 10 *For each $e \in \mathcal{E}$, under assumption 2, there exists $\rho' \in \mathbb{N}^*$ such that, for each $\rho > \rho'$, each $e_\rho \in \mathcal{E}$ and each $(l_{N_\rho}, t_{N_\rho}) \in Z$ such that $(l_{N_\rho}, t_{N_\rho}) = \varphi^{mir}(e^\rho)$, $l(l_{N_\rho}, t_{N_\rho}) \in \Sigma(e^\rho)$.*

Proof. By lemma 2 we know that for each $e \in \mathcal{E}$, for some $\rho'' \in \mathbb{N}^*$ and for each $\rho \geq \rho''$, $l_N \in \Sigma(e^\rho)$ is equivalent to $l_N \in L^f(e^\rho)$. Moreover, by definition, for each $e \in \mathcal{E}$, $L^f(e) = L^{f*}(e|\emptyset)$. Finally, for ρ big enough the *maximal individually rational assignment mechanism* terminates after one round, i.e. when $X^1 = \emptyset$ ■

Let us stress that the result above is independent of the way the cost is shared among conceding individuals at each stage r. In fact, any arbitrary way to share the cost would be in line with Proposition 10. This relies on the fact that the algorithm terminates at any stage only if all participants enjoy a positive level of utility and, after a certain number of replicas this is what we expect to happen. However, by charging all conceding individuals at each stage unequally, we might need a bigger number of replicas to guarantee a positive utility to the agents who are charged a higher share of the cost. On the contrary, by charging all conceding individuals at each stage equally we make sure that our mechanism will select an assignment efficient allocation after the minimum amount of replications. As encouraging as these observations may be, the mechanism can be criticized on certain grounds.

One may associate with each mechanism the cost of the institution that underlies it. Suppose that Bob is asked to learn spanish on top of his mother tongue. If he fails to do so it is easy to expose him; one needs only to present him with an exam. However, Bob finds it profitable to learn french as well as spanish. Any institution (and it would take a rather authoritarian one) would find it difficult to prevent him. Even if conceivably Bob could be banned from language schools teaching french, it would be hard, indeed costly, to monitor his every step, in order to make sure that his mother, who happens to speak the language, will not home-school him. The following property requires of mechanism to be immune to such individual deviations. No individual should find it profitable to learn more languages than is required of him.

Home-Schooling Proofness: For each $e \in \mathcal{E}$, each $(l_N, t_N) \in \varphi(e)$, each $i \in N$ and each $l'_i \in \{0,1\}^\Lambda$ such that $l'_i > l_i$

$$\theta_i \sum_{j \in N \setminus N^{\gamma(i)}} \left[\min\{1, l_j(l'_i - l_i)\}\right] - \bar{l}_i C(l'_i - l_i) < 0.$$

Unfortunately, the mechanism we propose fails this axiom. We can demonstrate that by means of an example. Refer to Figure 3.8. The economy comprises of two languages groups N^α, N^β. When setting $X^1 = \emptyset$ the planner finds that some individual $j \in N^\beta$ objects. Considering $X^2 = N \setminus \{j\}$ he finds that some individual $k \in N^\alpha$, also objects. This leaves the Planner with the option of setting $X^3 = N$. In such a situation any individual with a sufficiently high willingness to communicate, say agent $q \in N^\beta$, may find it profitable to "home-school" himself, in spite of what dictated by the mechanism.

This of course should come as shock. The following proposition explains.

Proposition 11 *There exists no mechanism φ that satisfies* Strategy Proofness, Individual Rationality, Feasibility *and* Home Schooling Proofness.

Proof. We construct a counter-example. Suppose that some mechanism φ satisfies the axioms. Consider an economy e consisting of two individuals, $N = \{1, 2\}$, speaking distinct languages. Let $\Lambda = \{\alpha, \beta\}$, $\bar{l}_1 = (1, 0)$ and $\bar{l}_2 = (0, 1)$. We have $\theta_1 = \theta_2 > c_{\alpha\beta} = c_{\beta\alpha} > 0$ and $c_{\alpha\beta} + c_{\beta\alpha} > \theta_1, \theta_2$. By *Home Schooling Proofness*, for each $(l_N, t_N) \in \varphi(e)$ either one of the following must be true:

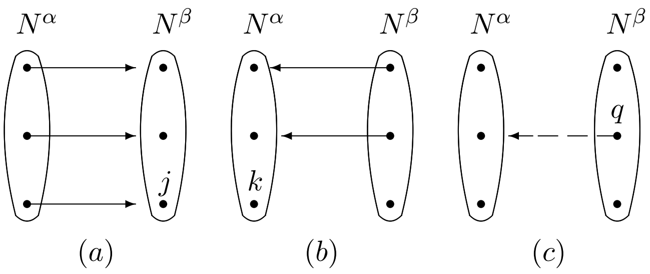


Figure 3.8: A violation of Home-Schooling Proofness.

1. $l_1 - \bar{l}_1 = (0,1)$ and $l_2 - \bar{l}_2 = (0,0)$, or
2. $l_1 - \bar{l}_1 = (0,0)$ and $l_2 - \bar{l}_2 = (1,0)$, or
3. $l_1 - \bar{l}_1 = (0,1)$ and $l_2 - \bar{l}_2 = (1,0)$.

Consider case (1), i.e. let there exist $(l_N, t_N) \in \varphi(e)$ such that $l_1 - \bar{l}_1 = (0,1)$ and $l_2 - \bar{l}_2 = (0,0)$. Suppose that $u_1(l_N, t_1; \theta_1) \geq \theta_1$ and $u_2(l_N, t_2; \theta_2) \geq \theta_2$. Therefore,

$$\begin{aligned} u_1(l_N, t_1; \theta_1) + u_2(l_N, t_2; \theta_2) &\geq \theta_2 + \theta_1 \\ \theta_1 - c_{\alpha\beta} + \theta_2 + t_1 + t_2 &\geq \theta_1 + \theta_2 \\ t_1 + t_2 &\geq c_{\alpha\beta} \end{aligned} \tag{3.6.4}$$

By *Feasibility*, $t_1 + t_2 \leq 0$ and hence inequality (3.6.4) constitutes a contradiction, as by assumption $c_{\alpha\beta} > 0$. The same reasoning applies for cases (2), (3). In conclusion, for each $(l_N, t_N) \in \varphi(e)$, either $u_1(l_N, t_1; \theta_1) < \theta_1$ or $u_2(l_N, t_2; \theta_2) < \theta_2$. Without loss of generality, suppose that for each $(l_N, t_N) \in \varphi(e)$, $u_1(l_N, t_1; \theta_1) < \theta_1$.

Consider economy e', to be one that is identical to economy e, except for the fact that $\theta'_1 = 0$. By *Individual Rationality* and *Feasibility*, since $c_{\alpha\beta} + c_{\beta\alpha} > \theta_2$, there cannot exist (l'_N, t'_N) such that $l'_1 = (1,1)$ and $l'_2 = (1,1)$. Therefore, by *Home Schooling Proofness*, for each $(l'_N, t'_N) \in \varphi(e')$, either $l'_1 = (0,1)$ and $l'_2 = (0,0)$, or $l'_1 = (0,0)$ and $l'_2 = (1,0)$. By *Individual Rationality*, in the former case $t'_1 \geq c_{\alpha\beta}$ and in the latter $t'_1 \geq 0$. This implies that from a profile of announcements (θ_1, θ_2) individual 1 can profitably deviate to the profile $(0, \theta_2)$ and obtain a utility level equal to θ_1. Thus, the mechanism φ violates *Strategy-Proofness*, a contradiction. ■

Assigment Efficiency implies *Home Schooling Proofness*. In fact, in lemma 4 we show that *Home Schooling Proofness* is one of the necessary conditions for a linguistic assignment to be assignment efficient. In light of this observation, the proposition above constitutes a formal proof of the fact that no mechanism satisfies the four core properties we put forward.

3.7 Conclusion

As a final note we raise the following issues.

1. The model we propose bears an alternative interpretation. The paradigm extends, beyond individuals that derive benefit from communication, to a wider class of entities. Examples are abundant. Historically in the Iberian Peninsula trains are running on tracks of a narrow gauge. Travelers from France to Spain (and vice versa) had to switch trains. Up until quite recently (and to a lesser extent still), academics would publish the output of their research in the native language of the institute they were

	MDM	TMDM	MIRAM
Assignment-Efficiency	$+$	$+$	$-^{a}$
Strategy-Proofness	$+$	$+$	$+$
Individual Rationality	$+$	-	$+$
Feasibility	$-^{b}$	$+^{c}$	$+$

Table 3.1: The table summarizes our results. Notice that in large economies: (a) MIRAM also satisfies Assignment Efficiency under assumption 2; (b) MDM still does not satisfy feasibility but the ratio between the deficit and the sum of net benefit generated by the allocation tends to zero; (c) the TDM is balanced.

affiliated with. In the entertainment industry, content is distributed through platforms that are often incompatible with each other. An HD-DVD player will not play Blu-Ray discs. A track bought on iTunes will not play on Windows Media Player. A Sony Playstation game will not run on Microsoft's Xbox. Therefore, one can speak more generally of platforms rather than languages.

2. One possible interpretation of our results is that they paint a very adverse picture. One may recognize that there is a range of situations where *Assignment Efficiency* comes either at too high a cost or is not a concern altogether. In the real world there many cases where sub-optimal levels communication prevail. Our results expose some the underlying reasons.

3. Similarly, the assumption that each individual speaks only one language natively has no real bearing on the discussion. Apart from complicating notation, dropping it would have no effect on the results.

3.8 Appendix 1. An Algorithm for the Two Languages Case.

In this section we provide an algorithm meant to solve the problem

$$P : \max \sum_{i \in N} \big(\theta_i \big(\sum_{j \in N/N^{\gamma(i)}} min\{1, l_i l_j\}\big) - \overline{l}_i C(l_i - \overline{l}_i)\big). \tag{3.8.1}$$

in the two languages case (i.e., $|\Lambda| = 2$). Each agent has at most 2 alternatives (he either learns the other language or not). There are n agents. Therefore, there are most 2^n candidate solutions. From this it follows immediately that

Lemma 7 *The problem (3.8.1) achieves an optimal solution.*

Lemma 8 *Let $i_1, i_2 \in N^{\alpha}$. If $\theta_{i_1} > \theta_{i_2}$ then there exists an optimal solution in which only one of the following can happen:*

1. Both agents i_1 and i_2 do not learn the language β,

2. If agent i_2 learns β then agent i_1 must learn β.

Proof. Assume by contradiction that there exists, an optimal solution $l\Sigma e$ in which agent i_2 learns β but i_1 does not learn β. (that implies $l_{i_1} = (1, 0)$ and $l_{i_2} = (1, 1)$)

To obtain a contradiction, we show that it is always possible to find a solution $\tilde{l}_N$ which is a better solution than l_N. Set

$$\tilde{l}_j = \begin{cases} l_j & \text{if } j \neq i_1, j \neq i_2 \\ (1,1) & \text{if } j = i_1 \\ (1,0) & \text{if } j = i_2 \end{cases} \tag{3.8.2}$$

Therefore, in $\tilde{l}$ we have now taught i_1 the language β and i_2 does not learn β.

Also note that the difference between the communication network of l_N and $\tilde{l}_N$ is that in l, i_2 is connected to all the members of N^β and in $\tilde{l}_N$, i_1 is connected to all the members of N^β. Moreover, if i_2 is connected to some agent of N^β in $\tilde{l}_N$ (because those agents from N^β learnt l_1), the same links are there between i_1 and agents of N^β in l_N. Let $S \subseteq N^\beta$, such that S and i_2 are not connected in $\tilde{l}_N$. Thus, $\pi(l_N;e)$ - $\pi(\tilde{l}_N;e) = (\theta_{i_1} - \theta_{i_2})|S| \geq 0$. ■

Lemma 9 *Let $\theta_{i^*} \geq \theta_i$ for each $i \in N^\alpha$. Suppose that agent i^* does not learn the language β in the optimal solution. The the optimal solution is*

$$\textit{For} \quad i \in N^\alpha \quad l_i = (1,0) \tag{3.8.3}$$

$$\textit{For} \quad i \in N^\beta \quad l_i = \begin{cases} (1,1) & \textit{if } |N^\alpha|\theta_i + \sum_{j \in N^\alpha} \theta_j - c_{\alpha\beta} > 0 \\ (0,1) & \textit{otherwise} \end{cases} \tag{3.8.4}$$

Proof. Implication (3.8.3) follows from lemma 8. Implication (3.8.4) follows from the fact that since no agent in N^α learns a new language, every time an agent from N^β learns α, the new links introduced are unique. Therefore, to create the optimal solution, we only check if the marginal return is positive. ■

The algorithm is given in Table 3.8 in based on lemmata 8 and 9 and gives the optimal solution to the two language case.

It can be verified that the algorithm will approximately do $|N^\alpha||N^\beta|$ arithmetic operations (like addition). This implies that at most $\frac{n^2}{4}$ operations will be done.

Table 3.2: A simple algorithm for two languages.

1. For each $i \in N^\alpha$ let $M_i = |N^\alpha|\theta_i + \sum_{j \in N^\alpha} \theta_j - c_{\alpha\beta}$. Similarly for N_2.
2. Sort the sets N^α and N^β such that if $N^\alpha := \{1, 2, 3.., m-1\}$ then $M_1 \geq M_{m-2} ... \geq M_m$. Similarly for N^β.
3. For $i \in N_1$, $i := 1$ to $m-1$
   (a) Either agent 1 learns β or no one in N^α learns β. Therefore, create two subproblems.
   (b) In first subproblem set $l_{i_1} = (1, 1)$. This completely determines agent 1's status. So remove agent 1 from problem and return to this algorithm with updated values of M_i's for agents in N^β and one less agent (that is, set the new value of $M_i := M_i - \theta_1 - \theta_i \; \forall i \in N^\beta$. Also set $|N^\alpha| = m-2$.]
   (c) In the second subproblem, we know that none of the agent in N^α learn β. In this case, the optimal solution is obtained using lemma 9. Save the solutions to this problems.
4. Among the m solutions obtained, pick the best one.

3.9 Appendix 2. Proofs for the Two Languages Case

With (I, J) we denote a structure where a set of I individuals learn language β, and a set of J individuals learn language α. With $z(I, J)$ we denote the value of such a structure. Similarly, define (I^i, J^i) and $z^i(I^i, J^i)$ where the superscript signifies the fact that θ_i has been set to zero. Let $\zeta(e) \in \mathbb{N}_+$ be the number of individuals in economy $e \in \mathcal{E}$ for whom $\theta_i = 0$.

The following lemmata are needed for the proofs of propositions 4 and 5.

Lemma 10 *For each $i \in N^\alpha$, $\pi(l_N^i; e^i) \leq \pi(l_N^1; e^1) + |N^\beta|(\theta_1 - \theta_i)$.*

Proof. Let (I^i, J^i) be an optimal solution corresponding to $\pi(l_N^i; e^i)$. There are three cases:

1. $i \in I^i$: Since in this economy θ_i is set to zero, this implies that $I^i = N^\alpha$ and $J^i = \emptyset$. Therefore
$$\pi(l_N^i; e^i) - z^1(N^\alpha, \emptyset) = -|N^\beta|\theta_i + |N^\beta|\theta_1$$
which implies
$$\pi(l_N^i; e^i) = z^1(N^\alpha, \emptyset) + |N^\beta|(\theta_1 - \theta_i) \leq \pi(l_N^1; e^1) + |N^\beta|(\theta_1 - \theta_i).$$
2. $i \notin I^i$, $1 \in I^i$: Then
$$\pi(l_N^i; e^i) = |N^\beta| \sum_{p \in I^i} \theta_p + |N^\alpha| \sum_{j \in J^i} \theta_j + |I^i| \sum_{j \in N^\beta \setminus J^i} \theta_j + |J^i| \sum_{p \in N^\alpha \setminus (I^i \cup \{i\})} \theta_p - |I^i| c_{\alpha\beta} - |J^i| c_{\beta\alpha}.$$

Now

$$z^1((I^i \setminus \{1\}) \cup \{i\}, J^i) = |N^\beta| \sum_{p \in (I^i \setminus 1 \cup \{i\})} \theta_p + |N^\alpha| \sum_{j \in J} \theta_j + |I^i| \sum_{j \in N^\beta \setminus J} \theta_j$$

$$+|J^i| \sum_{p \in N^\alpha \setminus (I^i \cup \{i\})} \theta_p - |I^i| c_{\alpha\beta} - |J^i| c_{\beta\alpha}.$$

Therefore,

$$\pi(l_N^i; e^i) - z^1((I^i \setminus \{1\}) \cup \{i\}, J^i) = -|N^\beta|\theta_i + |N^\beta|\theta_1.$$

Therefore,

$$\pi(l_N^i; e^i) \leq \pi(l_N^1; e^1) + |N^\beta|(\theta_1 - \theta_i).$$

3. $i \notin I^i$, $1 \notin I^i$: Then in this case $I^i = \emptyset$. In this case,

$$\pi(l_N^i; e^i) = |N^\alpha| \sum_{j \in J^i} \theta_j + |J^i| \sum_{p \in N^\alpha \setminus \{i\}} \theta_p - |J^i| c_{\beta\alpha}.$$

Now,

$$z^1(\emptyset, J^i) = |N^\alpha| \sum_{j \in J^i} \theta_j + |J| \sum_{p \in N^\alpha \setminus \{1\}} \theta_p - |J^i| c_{\beta\alpha}.$$

Therefore

$$\pi(l_N^i; e^i) - z^1(\emptyset, J^i) = |J^i|(\theta_1 - \theta_i).$$

Therefore,

$$\pi(l_N^i; e^i) \leq \pi(l_N^1; e^1) + |J^i|(\theta_1 - \theta_i) \leq \pi(l_N^1; e^1) + |N^\beta|(\theta_1 - \theta_i).$$

■

Lemma 11 *If* $1 \notin I^1$*, then* $z(I^1 \cup \{1\}, J^1) - \pi(l_N^i; e^i) \geq |N^\beta|\theta_i + \sum_{j \in N^\beta \setminus J^1} \theta_j - c_{\beta\alpha}$ $\forall i \in N^\alpha$.

Proof. Now $z(I^1 \cup \{1\}, J^1) - \pi(l_N^1; e^1) = |N^\beta|\theta_1 + \sum_{j \in N^\beta \setminus J^1} \theta_j - c_{\beta\alpha}$. Moreover from previous proposition we have that $\pi(l_N^i; e^i) \leq \pi(l_N^1; e^1) + |N^\beta|\theta_1 - |N^\beta|\theta_i$ and therefore $z(I^1 \cup \{1\}, J^1) - \pi(l_N^i; e^i) \geq |N^\beta|\theta_i + \sum_{j \in N^\beta \setminus J^1} \theta_j - c_{\beta\alpha}$.

■

Let $\tilde{J} \subseteq N^\beta$ be any subset. Note that similarly we can prove that if $m \notin J^m$, then $\sum_{j \in \tilde{J}} (\pi(l_N; e) - \pi(l_N^j; e^j)) \geq \sum_{j \in \tilde{J}} (z(I^m, J^m \cup \{m\}) - \pi(l_N^j; e^j)) \geq |N^\alpha| \sum_{j \in \tilde{J}} \theta_j + |\tilde{J}| \sum_{i \in N^\alpha \setminus I^m} \theta_i - |\tilde{J}| c_{\alpha\beta}$ where $\tilde{J}$ is any subset of N^β.

Lemma 12 *If* $I^1 = N^\alpha$ *then the conjecture is true.*

Proof. First note that it is easily verified that $\pi(l_N^i; e^i) = z^i(N^\alpha, \emptyset)$ $\forall i \in N^\alpha$. This is because using Proposition 10 we have that, $\pi(l_N^i; e^i) \leq \pi(l_N^1; e^1) + |N^\beta|(\theta_1 - \theta_i)$. However, $z^i(N^\alpha, \emptyset) = \pi(l_N^1; e^1) + |N^\beta|(\theta_1 - \theta_i)$. Therefore,

$$\sum_{i \in N^\alpha} (\pi(l_N; e) - \pi(l_N^i; e^i)) \geq |N^\beta| \sum_{i \in N^\alpha} \theta_i \tag{3.9.1}$$

Claim: $m \notin J^m$: If $m \in J^m$, we must that $\sum_{i\in N^\alpha \setminus I^m} \theta_i - c_{\alpha\beta} \geq 0$. This will imply that $J^m = N^\beta$. This is a contradiction since $|N_\alpha| c_{\beta\alpha} \leq |N^\beta| c_{\beta\alpha}$.

Let $\pi(l_N^m; e^m) = z^m(I^m, J^m)$. Therefore, using Proposition 11, we obtain that

$$\sum_{j\in J^m \cup \{m\}} (z(I^m, J^m \cup \{m\}) - \pi(l_N^j; e^j)) \geq$$

$$|N^\alpha| \sum_{j\in (J^m \cup \{m\})} \theta_j + |J^m \cup \{m\}| \sum_{i\in N^\alpha \setminus I^m} \theta_i - |J^m \cup \{m\}| c_{\beta\alpha} \tag{3.9.2}$$

Adding (3.9.1) and (3.9.2) we obtain

$$\sum_{i\in N^\alpha} (\pi(l_N; e) - \pi(l_N^i; e^i)) + \sum_{j\in J^m \cup \{m\}} (z(I^m, J^m \cup \{m\}) - \pi(l_N^j; e^j)) \geq |N^\beta| \sum_{i\in N^\alpha} \theta_i$$
$$+ |N^\alpha| \sum_{j\in (J^m \cup \{m\})} \theta_j + |J^m \cup \{m\}| \sum_{i\in N^\alpha \setminus I^m} \theta_i - |J^m \cup \{m\}| c_{\beta\alpha} \tag{3.9.3}$$

Claim: $\sum_{j\in N^\beta \setminus (J^m \cup \{m\})} \theta_j - c_{\alpha\beta} \leq 0$. Since the optimal solution of corresponding to the m^{th} economy is (I^m, J^m) we obtain that $\theta_i + \sum_{j\in N^\beta \setminus (J^m \cup \{m\})} \theta_j - c_{\alpha\beta} \leq 0 \; \forall i \in N^\alpha \setminus I^m$. As $\theta_i \geq 0$, $\sum_{j\in N^\beta \setminus (J^m \cup \{m\})} \theta_j - c_{\alpha\beta} \leq 0$.

Now we can rewrite (3.9.3) as

$$\begin{aligned}
&\sum_{i\in N^\alpha} (\pi(l_N; e) - \pi(l_N^i; e^i)) + \sum_{j\in J^m \cup \{m\}} (z(I^m, J^m \cup \{m\}) - \pi(l_N; e) \\
&\geq |N^\beta| \sum_{i\in N^\alpha} \theta_i + |N^\alpha| \sum_{j\in (J^m \cup \{m\})} \theta_j + |J^m \cup \{m\}| \sum_{i\in N^\alpha \setminus I^m} \theta_i \\
&- |J^m \cup \{m\}| c_{\alpha\beta} + |I^m| \sum_{j\in N^\beta \setminus (J^m \cup \{m\})} \theta_j - |I^m| c_{\beta\alpha} \\
&= |N^\beta| \sum_{i\in N_\alpha \setminus I^m} \theta_i + z(I^m, J^m \cup \{m\}) \geq z(I^m, J^m \cup \{m\})
\end{aligned} \tag{3.9.4}$$

Therefore, we obtain that $\sum_{i\in N_{\alpha\beta}} (\pi(l_N; e) - \pi(l_N^i; e^i)) + |J^m| z(I^m, J^m \cup \{m\}) - \sum_{j\in J^m \cup \{m\}} \pi(l_N^j; e^j)) \geq 0$ implying that $(|N^\alpha| + |N^\beta| - 1)\pi(l_N; e) - \sum_{i\in N^\alpha} \pi(l_N^i; e^i) - \sum_{j\in N^\beta} \pi(l_N^j; e^j) \geq 0$. Similarly for $J^m = N^\beta$. ∎

Lemma 13 *Suppose that*

1. $1 \notin I^1$ *and* $m \notin J^m$,
2. $I^1 \cup \{1\} \supseteq I^m$.

Then the conjecture holds.

Proof. By Proposition 11,

$$\begin{aligned}
&\sum_{i\in I^1 \cup \{1\}} (z(I^1 \cup \{1\}, J^1) - \pi(l_N^i; e^i)) + \sum_{i\in J^1} (z(I^m, J^m \cup \{m\} - \pi^j) \\
&\geq |N^\beta| \sum_{i\in (I^1 \cup \{1\})} \theta_i + |I^1 \cup \{1\}| \sum_{j\in N^\beta \setminus J^1} \theta_j - |I^1 \cup \{1\}| c_{\beta\alpha} \\
&+ |N^\alpha| \sum_{j\in J^1} \theta_j + |J^1| \sum_{j\in N^\alpha \setminus I^m} \theta_i - |J^1| c_{\alpha\beta}
\end{aligned} \tag{3.9.5}$$

Since $I^1 \cup \{1\} \supseteq I^m$, we have that $N^\alpha \setminus I^m \supseteq N^\alpha \setminus (I^1 \cup \{1\})$. Thus $|J^1| \sum_{j \in N^\alpha \setminus I^m} \theta_i \geq |J^1| \sum_{N^\alpha \setminus (I^1 \cup \{1\})} \theta_i$. Therefore, we may rewrite 3.9.5 as,

$$\begin{aligned}
&\sum_{i \in I^1 \cup \{1\}} (z(I^1 \cup \{1\}, J^1) - \pi(l_N^i; e^i)) + \sum_{i \in J^1} (z(I^m, J^m \cup \{m\}) - \pi(l_N^j; e^j)) \\
&\geq |N^\beta| \sum_{i \in (I^1 \cup \{1\})} \theta_i + |I^1 \cup \{1\}| \sum_{j \in N^\beta \setminus J^1} \theta_j - |I^1 \cup \{1\}| c_{\beta\alpha} \\
&+ |N^\alpha| \sum_{j \in J^1} \theta_j + |J^1| \sum_{j \in N^\alpha \setminus I^m} \theta_i - |J^1| c_{\alpha\beta} \\
&\geq |N^\beta| \sum_{i \in (I^1 \cup \{1\})} \theta_i + |I^1 \cup \{1\}| \sum_{j \in N^\beta \setminus J^1} \theta_j - |I^1 \cup \{1\}| c_{\beta\alpha} \\
&+ |N^\alpha| \sum_{j \in J^1} \theta_j + |J^1| \sum_{j \in N^\alpha \setminus (I^1 \cup \{1\})} \theta_i - |J^1| c_{\alpha\beta} \\
&= z(I^1 \cup \{1\}, J^1) \qquad (3.9.6)
\end{aligned}$$

■

Putting Lemmas 12 and 13 together we obtain that following result.

Lemma 14 *If $I^1 \cup \{1\} \supseteq I^m$ or $J^m \cup \{m\} \supseteq J^1$ then the conjecture is true.*

Proof of Proposition 4: Let $\bar{N}^{\beta,T} = \zeta(T,e) + 2 + \frac{c_{\alpha\beta}}{T} + \frac{c_{\beta\alpha}}{T}$ for some $T > 0$. To prove the result it is sufficient to verify that $N^\beta \geq \bar{N}^{\beta,T}$.

Assume that for some $i \in N$, the optimal structure when $\theta_i = 0$ is of the type (I^i, J^i), with $|J^i| \geq 1$. By the F.O.C.s it must be $\sum_{k \in N^\alpha \setminus I^i} \theta_k \leq c_{\beta\alpha}$ and $\sum_{k \in N^\beta \setminus J^i} \theta_k \leq c_{\alpha\beta}$, or more succinctly

$$\sum_{k \in N^\alpha \setminus I^i} \theta_k + \sum_{k \in N^\beta \setminus J^i} \theta_k \leq c_{\alpha\beta} + c_{\beta\alpha}. \qquad (3.9.7)$$

We assume that $i \in N^\alpha$. The same proof is valid when $i \in N^\beta$. We have that $\sum_{k \in (N^\alpha \setminus I^i) \setminus (L(T,e) \cup \{i\})} \theta_k \geq T|(N^\alpha \setminus I^i) \setminus (L(T,e) \cup \{i\})|$ and $\sum_{k \in (N^\beta \setminus I^i) \setminus (L(T,e))} \theta_k \geq T|(N^\beta \setminus I^i) \setminus (L(T,e))|$. Therefore we obtain,

$$\frac{1}{T}\Big(\sum_{k \in (N^\alpha \setminus I^i) \setminus (L(T,e) \cup \{i\})} \theta_k + \sum_{k \in (N^\beta \setminus I^i) \setminus (L(T,e))} \theta_k\Big) \geq \qquad (3.9.8)$$

$$|(N^\alpha \setminus I^i) \setminus (L(T,e) \cup \{i\})| + |(N^\beta \setminus I^i) \setminus (L(T,e))| \qquad (3.9.9)$$

or by using 3.9.13 we obtain

$$\zeta(T,e) + 1 + \frac{c_{\alpha\beta}}{T} + \frac{c_{\beta\alpha}}{T} \geq |N^\alpha \setminus I^i| + |N^\beta \setminus J^i| \qquad (3.9.10)$$

or

$$|I^i| + |J^i| \geq |N^\alpha| + |N^\beta| - \zeta(T,e) - \frac{c_{\alpha\beta}}{T} - \frac{c_{\beta\alpha}}{T} - 1. \qquad (3.9.11)$$

Now if $N^\beta \geq \bar{N}^{\beta,T}$, then

$$|I^i| + |J^i| > |N^\alpha|, \qquad (3.9.12)$$

which is a contradiction to the fact that $|J^i| \geq 1$. By applying lemma 14 the statement follows. ■

Proof of Proposition 5: Let $\bar{N}^{\alpha,T} = \frac{c_{\beta\alpha}}{c_{\alpha\beta}}(\zeta(T,e)+1)+1+\frac{2c_{\beta\alpha}}{T}$ for some $T>0$. It is enough to verify the result if $N^\alpha \geq \bar{N}^{\alpha,T}$.

Assume that for some $i \in N$, the optimal structure when $\theta_i = 0$ is of the type (I^i, J^i), with $|I^i| \geq 1$. By the F.O.C.s it must be $\sum_{k\in N^\alpha\setminus I^i} \theta_k \leq c_{\beta\alpha}$ and $\sum_{k\in N^\beta\setminus J^i} \theta_k \leq c_{\alpha\beta}$, or more succinctly

$$c_{\alpha\beta}\sum_{k\in N^\alpha\setminus I^i}\theta_k + c_{\beta\alpha}\sum_{k\in N^\beta\setminus J^i}\theta_k \leq 2c_{\alpha\beta}c_{\beta\alpha}. \tag{3.9.13}$$

We assume that $i \in N^\alpha$. The same proof is valid when $i \in N^\beta$. We have

$$\sum_{k\in(N^\alpha\setminus I^i)\setminus(L(T,e)\cup\{i\})}\theta_k \geq T|(N^\alpha\setminus I^i)\setminus(L(T,e)\cup\{i\})|$$

and

$$\sum_{k\in(N^\beta\setminus I^i)\setminus(L(T,e))}\theta_k \geq T|(N^\beta\setminus I^i)\setminus(L(T,e))|.$$

Therefore we obtain,

$$\begin{aligned}&\frac{1}{T}(c_{\alpha\beta}\sum_{k\in(N^\alpha\setminus I^i)\setminus(L(T,e)\cup\{i\})}\theta_k + c_{\beta\alpha}\sum_{k\in(N^\beta\setminus I^i)\setminus(L(T,e))}\theta_k)\\ &\geq c_{\alpha\beta}|(N^\alpha\setminus I^i)\setminus(L(T,e)\cup\{i\})| + c_{\beta\alpha}|(N^\beta\setminus I^i)\setminus(L(T,e))|\end{aligned} \tag{3.9.14}$$

or using (3.9.13)

$$c_{\beta\alpha}\zeta(T,e)+c_{\beta\alpha}+\frac{2c_{\beta\alpha}c_{\alpha\beta}}{T} \geq c_{\alpha\beta}|N^\alpha\setminus I^i| + c_{\beta\alpha}|N^\beta\setminus J^i| \tag{3.9.15}$$

or

$$c_{\alpha\beta}|I^i| + c_{\beta\alpha}|J^i| \geq c_{\alpha\beta}|N^\alpha| + c_{\beta\alpha}|N^\beta| - c_{\beta\alpha}\zeta(T,e) - \frac{2c_{\alpha\beta}c_{\beta\alpha}}{T} - c_{\beta\alpha}. \tag{3.9.16}$$

Now if $N^\alpha \geq \bar{N}^{\alpha,T}$, then

$$c_{\alpha\beta}|I^i| + c_{\beta\alpha}|J^i| > c_{\beta\alpha}|N^\beta| \tag{3.9.17}$$

which is a contradiction to the fact that $|I^i| \geq 1$. By applying lemma 14 the statement follows. ■

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