Combinatorial Methods for Invariance and Safety of Hybrid Systems

Nikolaos Athanasopoulos [a], Raphaël M. Jungers [b],⋆

[a]*School of Electronics, Electrical Engineering and Computer Science, Queen's University Belfast, NI, UK*

[b]*ICTEAM Institute, Universite Catholiqué de Louvain, Louvain-la-Neuve, Belgium*

Abstract

Inspired by Switching Systems and Automata theory, we investigate how combinatorial analysis techniques can be performed on a hybrid automaton in order to enhance its safety or invariance analysis. We focus on the particular case of *Constrained Switching Systems*, that is, hybrid automata with linear dynamics and no guards. We follow two opposite approaches, each with unique benefits: First, we construct invariant sets via the *'Reduced'* system, induced by a smaller graph which consists of the essential nodes, called the *unavoidable nodes*. The computational amelioration of working with a smaller, and in certain cases the minimum necessary number of nodes, is significant. Second, we exploit graph liftings, in particular the *Iterated Dynamics Lift* (T-Lift) and the *Path-Dependent Lift* (P-Lift). For the former case, we show that invariant sets can be computed in a fraction of the iterations compared to the non-lifted case, while we show how the latter can be utilized to compute non-convex approximations of invariant sets of a controlled complexity.

We also revisit well studied problems, highlighting the potential benefits of the approach. In particular, we apply our framework to (i) invariant sets computations for systems with dwell-time restrictions, (ii) fast computations of the maximal invariant set for uncertain linear systems and (iii) non-convex approximations of the minimal invariant set for arbitrary switching linear systems.

1 Introduction

Discrete-time linear switching systems consist of a finite collection of dynamics, called modes, which are allowed to switch at each time instant, according to a set of rules (see Equations (1)–(6) below for a precise description). They constitute a particularly interesting and important family of hybrid systems Goebel et al. (2012); Jungers (2009); Liberzon (2003); Shorten et al. (2007). Apart from their simplicity, their ability to capture particular hybrid phenomena (Dehghan and Ong, 2012b; Donkers et al., 2011; Hernandez-Mejias et al., 2015; Zhang et al., 2016) and approximate arbitrarily well nonlinear dynamics (Girard and Pappas, 2011) makes them a central model in the class of hybrid systems. Thus, it is not surprising that switching systems have been the subject of huge research efforts with existing techniques arguably more powerful than the ones targeted to general hybrid systems. Our goal in this paper is to push further the boundary of application of these techniques, by com-

bining them with combinatorial techniques from graph and automata theory. As a first step, we tackle an intermediate family of systems, known as *constrained switching systems* (Dai (2012), Athanasopoulos and Lazar (2014); Philippe et al. (2015); Wang et al. (2016)). These systems are more general than classical switching systems in that they have their switching signals restricted by a labeled directed graph, namely the switching constraints graph. For example, in Figure 1, the system switches between the modes 1 and 2 and an admissible switching sequence is the one that can be realized by a path in the directed graph $\mathcal{G}_1$.
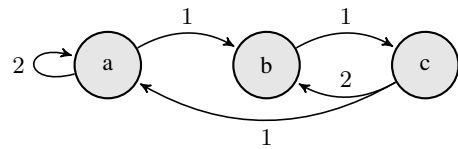

Fig. 1. A switching constraints graph $\mathcal{G}_1$ for a system with two modes. For example, the sequence 21121 is admissible whereas 212122 is not.

Recently (Athanasopoulos et al., 2017), we leveraged the results in (Philippe et al., 2015) to construct invariant sets (Blanchini and Miani, 2008; De Santis et al., 2004) for the constrained switching case. A *multi-set*, is a collection of sets, one per node of the graph that defines the switching

constraints. When a multi-set is invariant, the system trajectories that start from within this multi-set are always confined in one of its members. In this article we establish new, efficient, invariant (multi-)set constructions by exploiting the topological properties of the switching constraints graph. We highlight that the notion of multi-set is useful, beyond its proper physical meaning, for improving the state of the art in classical problems on simpler models, like LTI systems, or arbitrary switching systems. We adopt two opposite and complementary approaches, one reducing and the other increasing the size of the graph.

The first direction borrows the concept of *unavoidability* of a set of nodes, a notion used in Computer Science, e.g., (Lothaire, 2002, Proposition 1.6.7). Roughly, by keeping only a subset of 'important' nodes we are able to show that we can work with a reduced graph, and consequently a reduced system, and associate explicitly invariance properties of the reduced system with the original one, leading to efficient algorithmic constructions. See for example two possible reduced graphs of $\mathcal{G}_1$ in Figure 2.
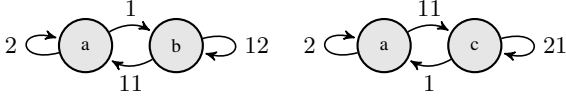

Fig. 2. Two possible reductions of $\mathcal{G}_1$ (Figure 1). We observe that the sequence of labels appearing in any infinite length path of $\mathcal{G}_1$ can be generated by either graph.

The second direction considers the lifting of the switching constraints graph, a classical idea in switching systems analysis, e.g., Bliman and Ferrari-Trecate (2003); Lee and Dullerud (2006); Philippe et al. (2015). Firstly, we consider the *Iterated Dynamics Lifted graph (abbr. T-lifted graph)*, which captures the switching constraints for the iterated dynamics of the systems, see, e.g., Figure 3 for the 2-lift of $\mathcal{G}_1$ of Figure 1. We exploit this construction to improve existing invariant multi-set computation algorithms by reducing the number of iterations required.

Secondly, we explore the *Path-Dependent Lifted graph (abbr. P-lifted graph)*, see, e.g., Figure 4 for the 1-lifted graph of $\mathcal{G}_1$, in forward reachability computations. This choice enables us to establish algorithms for non-convex approximations of invariant multi-sets described by a union of a *prespecified* number of convex sets.
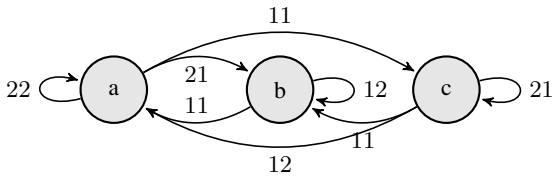

Fig. 3. The T-lifted graph of $\mathcal{G}_1$ of Figure 1, $T = 2$. There are as many edges as admissible switching sequences of length 2 in $\mathcal{G}_1$.
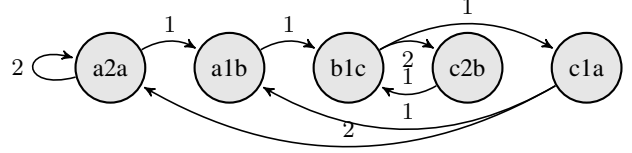

Fig. 4. The P-lifted graph of $\mathcal{G}_1$ of Figure 1 using the Path-Dependent Lift, $P = 1$. The graph has as many nodes as different walks of length 1 in $\mathcal{G}_1$.

Together with the theoretical contributions, we revisit three problems of set invariance in control. In particular, we consider systems under *dwell-time specifications* (Dehghan and Ong, 2012a,b; Liberzon, 2003; Zhang et al., 2016). We compute, to the best of our knowledge for the first time, the minimal invariant multi-set and its approximations, via a Reduced graph consisting of the minimum number of nodes. Moreover, we compute the maximal invariant set for *uncertain linear systems* faster compared to the standard backward reachability algorithm, see e.g., Blanchini and Miani (2008). Last, we propose a new method to compute *non-convex approximations* of the minimal invariant set for switching systems (Artstein and Rakovic, 2008; I. V. Kolmanovsky and E. G. Gilbert, 1998; Rakovic et al., 2005a,b).

Notation: The ball of radius α of an arbitrary norm is $\mathbb{B}(\alpha)$ and of the infinity norm is $\mathbb{B}_\infty(\alpha)$. The Minkowski sum of two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\mathcal{S}_1 \oplus \mathcal{S}_2$. A C-set $\mathcal{S} \subset \mathbb{R}^n$ is a convex compact set which contains the origin in its interior Blanchini (1999). The cardinality of a set $\mathcal{V}$ is denoted by $|\mathcal{V}|$. Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$, or $\mathcal{G}$, be a labeled directed graph with a set of nodes $\mathcal{V}$ and a set of edges $\mathcal{E}$. The set of sequences of labels appearing in a path from a node $s \in \mathcal{V}$ to a node $d \in \mathcal{V}$ is denoted by $\sigma(s, d)$. The set of sequences of nodes appearing in a walk from $s \in \mathcal{V}$ to $d \in \mathcal{V}$ is $m(s, d)$. We denote the 1-norm of a vector x with $\|x\|_1$, and the vector with elements equal to one with $\mathbf{1}$. The convex hull of a set $\mathcal{S} \subset \mathbb{R}^n$ is denoted by $\mathrm{conv}(\mathcal{S})$.

2 Preliminaries

We consider a set of matrices $\mathcal{A} := \{A_1, ..., A_N\} \subset \mathbb{R}^{n \times n}$ and disturbance sets $\mathbb{W} = \{\mathcal{W}_1, ..., \mathcal{W}_N\}$, $\mathcal{W}_i \subset \mathbb{R}^n$. We consider the sets of nodes and edges $\mathcal{V} := \{1, 2, ..., M\}$ and $\mathcal{E} = \{(s, d, \sigma) : s \in \mathcal{V}, d \in \mathcal{V}, \sigma \in \{1, ..., N\}\}$. We denote the corresponding graph by $\mathcal{G}(\mathcal{V}, \mathcal{E})$, or $\mathcal{G}$. The set of outgoing nodes of a node $s \in \mathcal{V}$ is $\mathrm{Outgoing}(s, \mathcal{G}) := \{d \in \mathcal{V} : (\exists \sigma \in \{1, ..., N\} : (s, d, \sigma) \in \mathcal{E})\}$. Finally, we consider constraint sets $\mathcal{X}_i \subset \mathbb{R}^n$, $i \in \{1, ..., M\}$.

Formally, the systems we study are described by the following set of relations

$$x(t + 1) = A_{\sigma(t)} x(t) + w(t), \qquad (1)$$
$$z(t + 1) \in \mathrm{Outgoing}(z(t), \mathcal{G}(\mathcal{V}, \mathcal{E})), \qquad (2)$$
$$w(t) \in \mathcal{W}_{\sigma(t)}, \qquad (3)$$
$$(x(0), z(0)) \in \mathbb{R}^n \times \mathcal{V}, \qquad (4)$$

subject to the constraints

$$(z(t), z(t+1), \sigma(t)) \in \mathcal{E}, \qquad (5)$$
$$x(t) \in \mathcal{X}_{z(t)}, \qquad (6)$$

for all $t \geq 0$. We underline that the switching signal $\sigma(t)$ depends on the discrete variable $z(t)$ at each time instant, however for notational convenience we make a slight abuse and write $\sigma(t)$ instead of $\sigma(z(t))$. We note the system (1)–(6) is defined in the hybrid state space [1] $[x^\top \quad z]^\top \in \mathbb{R}^n \times \mathcal{V}$. We call *nominal* the disturbance-free system, i.e., the system $x(t+1) = A_{\sigma(t)} x(t)$ together with (2), (4)–(6). The stability of the nominal system is characterized by the *constrained joint spectral radius* (Dai, 2012) $\check{\rho}(\mathcal{A}, \mathcal{G}) = \lim\limits_{t \to \infty} \check{\rho}_t(\mathcal{A}, \mathcal{G})$, where

$$\check{\rho}_t(\mathcal{A}, \mathcal{G}) := \max\{\|A_{\sigma(t-1)} \cdots A_{\sigma(0)}\|^{1/t} : z(0) \in \mathcal{V},$$
$$z(t) \text{ satisfies (2)}, \sigma(t) \text{ satisfies (5)}, t = 0, ..., t-1\}.$$

The nominal system is asymptotically stable if and only if $\check{\rho}(\mathcal{A}, \mathcal{G}) < 1$ (Dai, 2012, Corollary 2.8). We consider the following assumptions.

Assumption 1 *The constraint and disturbance sets* $\mathcal{X}_i \subset \mathbb{R}^n$, $i = 1, ..., M$ *and* $\mathcal{W}_i$, $i = 1, ..., N$, *are C-sets.*

Assumption 2 $\check{\rho}(\mathcal{A}, \mathcal{G}) < 1$.

Assumption 3 *The sets* $\text{Outgoing}(i, \mathcal{G}(\mathcal{V}, \mathcal{E}))$, $i \in \mathcal{V}$, *are nonempty.*

The assumptions are standard, see e.g., Blanchini and Miani (2008) for Assumption 1. Assumption 2 is not restrictive since $\check{\rho}(\mathcal{A}, \mathcal{G}) > 1$ excludes non-trivial invariant multi-sets or safe sets [2], while Assumption 3 guarantees the completeness of solutions. We highlight that Assumption 2 can be verified by computing arbitrarily close approximations of the constrained joint spectral radius, utilizing, e.g., the results in Philippe et al. (2015), implemented in software in Cambier et al. (2015).

Definition 1 (Multi-sets) *We call* multi-set *a collection of sets* $\{\mathcal{S}^i\}_{i \in \mathcal{V}}$, $\mathcal{S}^i \subset \mathbb{R}^n$, $i \in \mathcal{V}$.

Definition 2 [Multi–set invariance, Athanasopoulos et al. (2017)] *The multi-set* $\{\mathcal{S}^i\}_{i \in \mathcal{V}}$ *is an* invariant multi-set *with respect to the System (1)–(5) if* $x(0) \in \mathcal{S}^{z(0)}$ *implies* $x(t) \in \mathcal{S}^{z(t)}$, *for all* $t \geq 0$, $z(0) \in \mathcal{V}$ *and* $\sigma(t)$ *satisfying (5). If additionally* $\mathcal{S}^i \subseteq \mathcal{X}_i$, $i \in \mathcal{V}$, *then* $\{\mathcal{S}^i\}_{i \in \mathcal{V}}$ *is called an* admissible invariant multi-set *with respect to (1)–(6). The multi-set* $\{\mathcal{S}^i_M\}_{i \in \mathcal{V}}$ *is the maximal admissible invariant set*

if for any admissible invariant multi-set $\{\mathcal{S}^i\}_{i \in \mathcal{V}}$ it holds that $\mathcal{S}^i \subseteq \mathcal{S}^i_M$, $i \in \mathcal{V}$. The multi-set $\{\mathcal{S}^i_m\}_{i \in \mathcal{V}}$ is the minimal (convex) admissible invariant set if for any admissible (convex) invariant multi-set $\{\mathcal{S}^i\}_{i \in \mathcal{V}}$ it holds $\mathcal{S}^i_m \subseteq \mathcal{S}^i$, $i \in \mathcal{V}$.

Remark 1 (Related notions) *Multi-set invariance, as opposed to classical set invariance, is a necessary generalization for translating the invariance property to systems under constrained switching. Indeed, it is not difficult to construct a simple, scalar system for which an invariant multi-set exists, however no common invariant set can be found, see, e.g., (Athanasopoulos et al., 2017, footnote 3). In the literature, different notions of invariance have been introduced, motivated mostly by the need to establish efficient algorithmic characterizations. In Rakovic et al. (2010), Rakovic et al. (2011) the concept of an* invariant collection of sets *is introduced for decentralized and large scale systems respectively, with each member of the collection being defined in a subspace of the state space, allowing to construct linear comparison systems and analyze stability and stabilizability. In Lazar et al. (2013),* (k, λ)-contractiveness *is proposed as a generalization of invariance for homogeneous discrete-time systems. Roughly,* (k, λ)-contractiveness *(or* $(k, 1)$-invariance*) is related to periodic invariance of a single set with respect to the trajectories of the system, or, alternatively, to invariance of the iterated dynamics of the system. In our setting, this notion is related to the T-Lift developed in Section 4.1, when the considered dynamics is time-invariant. Another relevant contribution is in Artstein and Rakovic (2008), Artstein and Rakovic (2011): In Artstein and Rakovic (2008) the study of nonlinear autonomous systems is performed by an* infinite-dimensional representation *of the system in the space of sets. This makes the convergence proofs and approximation estimates possible, using contraction theory. In Artstein and Rakovic (2011), the same reasoning is utilized to propose a new theoretical framework of invariance under output feedback, which can be properly defined only on the space of sets. Contrary to this piece of work, our tools propose a finite dimensional and directly implementable combinatorial lifting, induced by the switching constraints graph. Last, it must be mentioned that De Santis et al. (2004) study invariance properties of general hybrid systems with inputs, capturing in some cases the multi-set invariance, referred to simply as invariance.*

Consider the System (1)–(4) and a switching sequence $\sigma_1 ... \sigma_p$, $\sigma_i \in \{1, ..., N\}$, $p \geq 1$. The *p-step forward reachability map* is $\mathcal{R}(\sigma_1 ... \sigma_p, \mathcal{S}) := (\prod_{i=1}^{p} A_{\sigma_{p+1-i}} \mathcal{S}) \oplus (\bigoplus_{j=1}^{p} \prod_{i=1}^{p-j} A_{\sigma_{p+1-i}} \mathcal{W}_{\sigma_j})$. The *p-step backward reachability map* is $\mathcal{C}(\sigma_1 ... \sigma_p, \mathcal{S}) := \{x : (\prod_{i=1}^{p} A_{\sigma_{p+1-i}}\{x\}) \oplus (\bigoplus_{j=1}^{p} \prod_{i=1}^{p-j} A_{\sigma_{p+1-i}} \mathcal{W}_{\sigma_j}) \in \mathcal{S}\}$. We write $\mathcal{R}_{\mathrm{N}}(\sigma_1 ... \sigma_p, \mathcal{S}) := \{\prod_{i=1}^{p} A_{\sigma_{p+1-i}} x : x \in \mathcal{S}\}$.

We consider the multi-set sequence $\{\mathcal{N}_l^j\}_{j \in \mathcal{V}}$, $l \geq 0$, gen-

---

[1] Indeed, from (2), (4) it follows that $z(t) \in \mathcal{V}$, for all $t \geq 0$.

[2] For the case when $\check{\rho}(\mathcal{A}, \mathcal{G}) = 1$ we still cannot guarantee boundedness of trajectories and a fortiori the existence of invariant sets. See Protasov and Jungers (2015) for the arbitrary switching case.

erated by [3]

$$\mathcal{N}_0^j := \cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{W}_\sigma, \quad j \in \mathcal{V}, \tag{7}$$

$$\mathcal{N}_{l+1}^j := \cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{R}_{\mathrm{N}}(\sigma,\mathcal{N}_l^s), \quad j \in \mathcal{V}. \tag{8}$$

The multi-set sequence (7), (8) has as elements the forward reachability multi-sets of the nominal system starting from the disturbance sets. Under Assumptions 1–2, the sets $\mathcal{N}_l^j$, $l \geq 0$, $j \in \mathcal{V}$ are compact, and for any $\check{\rho}(\mathcal{A},\mathcal{G}) \leq \rho < 1$, the relation

$$\mathcal{N}_t^j \subseteq \Gamma\rho^t\mathcal{N}_0^j, \quad \forall j \in \mathcal{V}, \quad \forall t \geq 0 \tag{9}$$

holds, for some $\Gamma \geq 1$. Several methods exist for computing the constants in (9), e.g., Athanasopoulos and Lazar (2014), Philippe et al. (2015), Cambier et al. (2015).

We consider the forward reachability multi-set sequence $\{\mathcal{F}_l^j\}_{j\in\mathcal{V}}$, $l \geq 0$, with

$$\mathcal{F}_0^j := \{0\}, \quad j \in \mathcal{V}, \tag{10}$$

$$\mathcal{F}_{l+1}^j := \cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{R}(\sigma,\mathcal{F}_l^s), \quad j \in \mathcal{V}, \tag{11}$$

and the backward reachability multi-set sequence $\{\mathcal{B}_l^j\}_{j\in\mathcal{V}}$, where

$$\mathcal{B}_0^j = \mathcal{X}_j, \quad j \in \mathcal{V}, \tag{12}$$

$$\mathcal{B}_{l+1}^j = (\cap_{(j,d,\sigma)\in\mathcal{E}}\mathcal{C}(\sigma,\mathcal{B}_l^d)) \cap \mathcal{B}_0^j, \quad j \in \mathcal{V}. \tag{13}$$

The sequence (10), (11) captures the propagation of all solutions starting from the zero singleton in time. The l-th term of the multi-set sequence (12), (13) contains the initial conditions $(x(0), z(0)) \in \mathcal{X}_{z(0)} \times \mathcal{V}$ which satisfy the state constraints for at least the first l time instants. Intuitively, each set $\mathcal{B}_{l+1}^j$, $l \geq 0$, $j \in \mathcal{V}$, contains the set of states in the constraint set $\mathcal{B}_0^j$ that can be transferred to each set $\mathcal{B}_l^d$ via the dynamics σ, where d is any outgoing node of j. We recall the main theoretical results from (Athanasopoulos et al., 2017, Theorems 1–3) that are utilized in this paper. To this purpose, given the System (1)–(6) and the minimal invariant multi-set $\{\mathcal{S}_m^j\}_{j\in\mathcal{V}}$, consider the scalars

$$\alpha_1 := \min\{\alpha > 0 : \cup_{\sigma\in\{1,...,N\}}\mathcal{W}_\sigma \subseteq \mathbb{B}(\alpha)\}, \tag{14}$$

$$\alpha_2 := \min\{\alpha > 0 : \cup_{i\in\{1,...,N\}}\mathcal{W}_i \subseteq \alpha \cap_{i\in\{1,...,N\}}\mathcal{W}_i\}, \tag{15}$$

$$\alpha_3 := \min\{\alpha > 0 : \mathbb{B}(1) \subseteq \alpha \cap_{i\in\{1,...,N\}}\mathcal{W}_i\}, \tag{16}$$

$$R_j := \max\{R : \mathbb{B}(R) \subseteq \mathcal{X}_j\}, \tag{17}$$

$$r_j := \min\{r : \mathcal{S}_m^j \subseteq \mathbb{B}(r)\}, \tag{18}$$

$$c := \min\{c : \mathcal{X}_j \subseteq c\mathcal{N}_0^j, j \in \mathcal{V}\}. \tag{19}$$

---

[3] We slightly abuse the notation and write $(s, j, \sigma) \in \mathcal{E}$ instead of $\{\sigma : (s, j, \sigma) \in \mathcal{E}\}$.

Theorem 1 (Athanasopoulos et al. (2017)) *The minimal invariant multi–set $\{\mathcal{S}_m^j\}_{j\in\mathcal{V}}$ with respect to the System (1)–(6) is unique and equal to $\mathcal{S}_m^j = \lim_{l\to\infty}\mathcal{F}_l^j$, $j \in \mathcal{V}$. Consider a pair (Γ, ρ) satisfying (9), the multi-set sequence (10), (11). Given an accuracy $\epsilon > 0$ the following hold.*

(i) *For any $l \geq \lceil\log_\rho(\frac{\epsilon(1-\rho)}{\alpha_1\Gamma})\rceil$, it holds that $\mathcal{F}_l^j \subseteq \mathcal{S}_m^j \subseteq \mathcal{F}_l^j \oplus \mathbb{B}(\epsilon)$, $j \in \mathcal{V}$.*

(ii) *For any pair (k, λ) that satisfies the inequalities $\alpha_2\Gamma\rho^k \leq \lambda$, $\frac{\Gamma(1-\rho^{k-1})}{1-\rho} \leq \frac{\epsilon(1-\lambda)}{\alpha_1\lambda}$, the multi-set $\{\frac{1}{1-\lambda}\mathcal{F}_{k-1}^j\}_{j\in\mathcal{V}}$ is invariant, and furthermore, $\mathcal{S}_m^j \subseteq \frac{1}{1-\lambda}\mathcal{F}_{k-1}^j \subseteq \mathcal{S}_m^j \oplus \mathbb{B}(\epsilon)$, $j \in \mathcal{V}$.*

(iii) *Consider the sequence (12), (13) and assume $\mathcal{S}_m^j \subseteq \mathrm{int}(\mathcal{X}_j)$, $j \in \mathcal{V}$ and let $\mathcal{N}_0^j$ given in (7). Then, there is $\overline{k}$ such that $\mathcal{B}_{\overline{k}+1}^j = \mathcal{B}_{\overline{k}}^j$, $j \in \mathcal{V}$, with $\overline{k} \leq \log_\rho(\frac{\min_{j\in\mathcal{V}}(R_j-r_j)}{\alpha_1\Gamma c})$. Moreover, $\{\mathcal{B}_{\overline{k}}^j\}_{j\in\mathcal{V}}$ is the maximal admissible invariant multi-set.*

In the absence of disturbances, i.e., when $\mathcal{W}_\sigma = 0$, $\sigma \in \{1,...,N\}$, we can adapt Theorem 1(iii), with the equivalent bound being $\overline{k} \leq \log_\rho(\frac{\min_{j\in\mathcal{V}}(R_j)}{\Gamma c})$, where Γ, ρ satisfy $\|x(t)\| \leq \Gamma\rho^t\|x(0)\|$ and $c := \min\{c : \mathcal{X}_j \subseteq \mathbb{B}(c)\}$, $R_j = \{R : \mathbb{B}(R) \subseteq \mathcal{X}_j\}$, $j \in \mathcal{V}$.

We note that under Assumptions 1–3 the elements of the maximal invariant multi-set $\{\mathcal{S}_M^i\}_{i\in\mathcal{V}}$ are convex while the elements of the minimal invariant multi-set $\{\mathcal{S}_m^i\}_{i\in\mathcal{V}}$ are in general non-convex (precisely they are radially convex). Following the same line of reasoning with the literature, e.g., Blanchini and Miani (2008), we can define the 'convexified' forward reachability multi-set sequence $\{\overline{\mathcal{F}}_l^j\}_{j\in\mathcal{V}}$, $l \geq 0$, with $\overline{\mathcal{F}}_0^j := \{0\}$, $\overline{\mathcal{F}}_{l+1}^j := \cup_{(s,j,\sigma)\in\mathcal{E}}\mathrm{conv}(\mathcal{R}(\sigma,\mathcal{F}_l^s))$, $j \in \mathcal{V}$. The convex hull of the fixed point of the above sequence which converges to the minimal *convex* invariant multi-set. This allows for more efficient computations, at the price of losing accuracy on the approximation of the minimal invariant set. For more details on the convex version of Theorem 1(i), (ii), see (Athanasopoulos et al., 2017, Section III.A).

3 The Reduced System

In this section, we compute invariant multi-sets efficiently by reducing the number of modes in the constraints graph. We focus on a subset $\mathcal{Y}$ of the nodes of $\mathcal{G}(\mathcal{V}, \mathcal{E})$, for which at least one element is visited in a walk of length $m \geq 1$.

Definition 3 (Lothaire (2002), Proposition 1.6.7.) *Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and an integer $m \geq 1$, a set of nodes $\mathcal{Y} \subseteq \mathcal{V}$ is called m-unavoidable if any walk of length m starting from any node passes through at least one node $v \in \mathcal{Y}$ at*

least once. We call a set of nodes unavoidable *if it is m-unavoidable for some $m \geq 1$.*

Unavoidable sets of nodes are classically known in graph-theory as *feedback vertex sets*, which are the sets of nodes whose removal leaves a graph acyclic. Finding the feedback vertex set of the minimum cardinality has been shown to be NP-complete Karp (1972). Nevertheless, efficient approximation schemes of the minimal feedback vertex sets exist, see for example the seminal paper Even et al. (1998). Moreover, our construction only requires to find a set *containing* such an unavoidable set, which allows one to operate a trade-off between algorithmic efficiency and size of the obtained set.

Example 1 *The graph $\mathcal{G}_1$ shown in Figure 1, has two minimal 2-unavoidable sets of nodes, namely $\{a, b\}$ and $\{a, c\}$.*

Letting $\mathcal{Y} \subseteq \mathcal{V}$ be a set of m-unavoidable nodes of $\mathcal{G}(\mathcal{V}, \mathcal{E})$, we define the *Reduced Graph* $\mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})$, where

$$\mathcal{E}_{\mathcal{Y}} := \{(s, d, \sigma^\star) : (s, d) \in \mathcal{Y} \times \mathcal{Y}, \sigma^\star \in \sigma(s, d),$$

with no unavoidable node in the path from s to $d\}$. (20)

The edges of $\mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})$ are labeled with the sequences of labels appearing in paths between unavoidable nodes in $\mathcal{G}(\mathcal{V}, \mathcal{E})$. We consider the set of matrices $\tilde{\mathcal{A}} \subset \mathbb{R}^{n \times n}$, where $\tilde{\mathcal{A}} := \{ \prod_{i=0}^{p} A_{\sigma_{p-i}} : (s, d, \sigma_0 ... \sigma_p) \in \mathcal{E}_{\mathcal{Y}} \}$ and denote each member of $\tilde{\mathcal{A}}$ by $\tilde{A}_i$, $i \in \{1, ..., \tilde{N}\}$, for some $\tilde{N} \geq 1$. We consider the corresponding set of disturbance sets $\tilde{\mathbb{W}} := \{ \bigoplus_{j=0}^{p-1} (\prod_{i=0}^{p-1-j} A_{\sigma_{p-1-i}(t)} \mathcal{W}_j) \oplus \mathcal{W}_p : (s, d, \sigma_0 ... \sigma_p) \in \mathcal{E}_{\mathcal{Y}} \}$ and use the notation $\tilde{\mathcal{W}}_i$ for each member of $\tilde{\mathbb{W}}$.

Definition 4 (Reduced System) *The* Reduced System *related to the System* (1)–(6) *via the unavoidable set of nodes $\mathcal{Y} \subset \mathcal{V}$ is a constrained switching system with constraints graph $\mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})$, matrix set $\tilde{\mathcal{A}}$, disturbance sets $\tilde{\mathbb{W}}$ and state constraints $\mathcal{X}_i$, $i \in \mathcal{Y}$.*

Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a set $\mathcal{Y} \subseteq \mathcal{V}$, we denote by θ_m and θ_M the smallest and largest number of edges in a path connecting two nodes $i \in \mathcal{Y}$ in $\mathcal{G}$, i.e.,

$$\theta_m := \min_{(i,j) \in \mathcal{Y} \times \mathcal{Y}} \{ |\sigma^\star| : \sigma^\star \in \sigma(i, j) \}, \quad (21)$$

$$\theta_M := \max_{(i,j) \in \mathcal{Y} \times \mathcal{Y}} \{ |\sigma^\star| : \sigma^\star \in \sigma(i, j) \}. \quad (22)$$

The stability properties of the nominal (i.e., its disturbance free version) System (1)–(6) and the nominal Reduced System coincide, as shown in the following result.

Lemma 1 *Consider the System* (1)-(4) *and the Reduced System associated to the System via a set of unavoidable nodes*

$\mathcal{Y} \subseteq \mathcal{V}$. *The following equivalence holds.*

$$\check{\rho}(\mathcal{A}, \mathcal{G}(\mathcal{V}, \mathcal{E})) < 1 \Leftrightarrow \check{\rho}(\tilde{\mathcal{A}}, \mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})) < 1. \quad (23)$$

Proof Direction $(\Rightarrow)$ can be shown from the fact that the trajectories of the Reduced System are generated by subsequences of the ones of the original one. To show $(\Leftarrow)$, let $c := \max_{i_1 ... i_l} \{ \| A_{i_l} ... A_{i_1} \|, 1 \}$, where $1 < l \leq \theta_M - 1$, θ_M is given in (22), and $i_1 ... i_l$ is an admissible switching sequence for the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. For any $t \geq \theta_M$, we can divide any switching signal in three parts, namely $\sigma(0) ... \sigma(t_1 - 1)$, $\sigma(t_1) .... \sigma(t_2)$, $\sigma(t_2 + 1) .... \sigma(t - 1)$, where $\sigma(t_1) .... \sigma(t_2)$ corresponds to a path in $\mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})$. Then, we have

$$\begin{aligned} \check{\rho}_t(\mathcal{A}, \mathcal{G}(\mathcal{V}, \mathcal{E})) &= \max_{\sigma(0) ... \sigma(t-1)} \{ \| A_{\sigma(t-1)} \cdots A_{\sigma(0)} \|^{1/t} \} \\ &\leq c^{1/t} \max_{\sigma(t_1) ... \sigma(t_2)} \| A_{\sigma(t_2)} ... A_{\sigma(t_1)} \|^{1/t} c^{1/t} \\ &\leq c^{2/t} (\check{\rho}_{t^\star}(\tilde{\mathcal{A}}, \mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}}))^{t^\star/t}, \end{aligned}$$

with $\theta_m t \leq t^\star \leq \theta_M t$. Taking the limit when $t \to \infty$, the result follows. ∎

The starting point for deriving the results is the observation that the forward reachability maps on the Reduced System provide inner and outer bounds on the multi-set $\{\mathcal{F}_l^j\}_{j \in \mathcal{V}}$ of the original System (1)–(6).

Lemma 2 *Let $\{\mathcal{F}_l^j\}_{j \in \mathcal{V}}$, $\{\tilde{\mathcal{F}}_l^j\}_{j \in \mathcal{Y}}$, $l \geq 0$, be the forward reachability multi-set sequences* (10), (11) *of the System* (1)–(4) *and the Reduced System associated to the System via the set of nodes $\mathcal{Y}$ respectively. Then,*

$$\mathcal{F}_{l\theta_m}^j \subseteq \tilde{\mathcal{F}}_l^j \subseteq \mathcal{F}_{l\theta_M}^j, \quad \forall j \in \mathcal{Y}, \quad \forall l \geq 0, \quad (24)$$

where θ_m, θ_M are in (21) *and* (22).

Proof To prove the left inclusion, we exploit that each path appearing in $\mathcal{F}_{l\theta_m}^j$ is a part of a path appearing in $\tilde{\mathcal{F}}_l^j$. Relation (24) holds with equality for $l = 0$. Given an integer $j \in \mathcal{Y}$, we define the sets of pairs $\mathcal{S}_1(j, \mathcal{G}(\mathcal{V}, \mathcal{E}))$, $\mathcal{S}_2(j, \mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}}))$,

$$\begin{aligned} \mathcal{S}_1(j, \mathcal{G}(\mathcal{V}, \mathcal{E})) = \{(i_1, \sigma_1) : \\ \sigma_1 \in \sigma(i_1, j), i_1 \in \mathcal{V}, |m(i_1, j)| = l\theta_m + 1\}, \\ \mathcal{S}_2(j, \mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})) = \{(i_2, \sigma_2) : \\ \sigma_2 \in \sigma(i_2, j), i_2 \in \mathcal{Y}, |m(i_2, j)| = l + 1\}. \end{aligned}$$

Let us consider an arbitrary pair $(i_1, \sigma_1) \in \mathcal{S}_1(j, \mathcal{G}(\mathcal{V}, \mathcal{E}))$. By construction of the Reduced graph, there exists an admissible subsequence [4] $\bar{\sigma}$, a pair $(i_2, \sigma_2) \in \mathcal{S}_2(j, \mathcal{G}(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}}))$

---

[4] $\bar{\sigma}$ might be possibly empty, i.e., $\bar{\sigma} = \varepsilon$ where ε denotes the empty word.

such that $\sigma_2 = \bar{\sigma}\sigma_1$. Moreover,

$$\mathcal{R}(\sigma_2, \mathcal{F}_0^{i_2}) = \mathcal{R}(\sigma_1, \mathcal{R}(\bar{\sigma}, \mathcal{F}_0^{i_2})) \supseteq \mathcal{R}(\sigma_1, \mathcal{F}_0^{i_1}),$$

because $\mathcal{F}_0^i = \mathcal{F}_0^j = \{0\}$ for any $i, j \in \mathcal{V}$. Consequently,

$$
\begin{aligned}
\mathcal{F}_{l\theta_m}^j &= \bigcup_{(i,\sigma)\in\mathcal{S}_1(j,\mathcal{G}(\mathcal{V},\mathcal{E}))} \mathcal{R}(\sigma, \mathcal{F}_0^i) \\
&\subseteq \bigcup_{(i,\sigma)\in\mathcal{S}_2(j,\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y}))} \mathcal{R}(\sigma, \mathcal{F}_0^i) = \tilde{\mathcal{F}}_l^j.
\end{aligned}
$$

We can show the right inclusion in (24) using a similar reasoning. ∎

Given the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and the related Reduced Graph $\mathcal{G}(\mathcal{Y}, \tilde{\mathcal{E}})$, we define the map $f(\cdot)$ from a multi-set $\{\mathcal{M}^j\}_{j\in\mathcal{Y}}$, $\mathcal{M}^j \subset \mathbb{R}^n$ to a multi-set $\{\mathcal{K}^j\}_{j\in\mathcal{V}}$, $\mathcal{K}^j \subset \mathbb{R}^n$ to be

$$
f\left(\{\mathcal{M}^j\}_{j\in\mathcal{Y}}\right) :=
\begin{cases}
\mathcal{M}^j, & j \in \mathcal{Y}, \\
\bigcup_{(\sigma,i)\in\mathcal{P}(j)} \mathcal{R}(\sigma, \mathcal{M}^i), & j \in \mathcal{V} \setminus \mathcal{Y},
\end{cases}
\tag{25}
$$

where

$$
\begin{aligned}
\mathcal{P}(j) := \{&(\sigma^\star, i) : i \in \mathcal{Y}, \sigma^\star \in \sigma(i,j) \text{ and} \\
&\text{no } k \in \mathcal{Y} \text{ appears in the path from } i \text{ to } j\}. \quad (26)
\end{aligned}
$$

We write the minimal and maximal invariant multi-set of the System (1)–(6) and the Reduced System with $\{\mathcal{S}_m^j\}_{j\in\mathcal{V}}$, $\{\mathcal{S}_M^j\}_{j\in\mathcal{V}}$ and $\{\tilde{\mathcal{S}}_m^j\}_{j\in\mathcal{Y}}$, $\{\tilde{\mathcal{S}}_M^j\}_{j\in\mathcal{Y}}$ respectively.

Theorem 2 *Consider the System* (1)–(6)*, a set of unavoidable nodes* $\mathcal{Y} \subseteq \mathcal{V}$ *and the corresponding Reduced System. The minimal invariant multi-set* $\{\mathcal{S}_m^j\}_{j\in\mathcal{V}}$ *with respect to the System* (1)–(6) *is*

$$\{\mathcal{S}_m^j\}_{j\in\mathcal{V}} = f\left(\{\tilde{\mathcal{S}}_m^j\}_{j\in\mathcal{Y}}\right). \tag{27}$$

Proof Taking the limit in (24) for $l \to \infty$, we have from Theorem 1 that $\mathcal{S}_m^j \subseteq \tilde{\mathcal{S}}_m^j \subseteq \mathcal{S}_m^j$, thus, $\mathcal{S}_m^j = \tilde{\mathcal{S}}_m^j$, for all $j \in \mathcal{Y}$. For all $v \in \mathcal{V} \setminus \mathcal{Y}$ we have for any $l \geq \theta_M$ $\mathcal{F}_l^v = \bigcup_{p=1}^{\theta_M} \bigcup_{\{(\sigma^\star,i)\in\mathcal{P}(j):|\sigma^\star|=p\}} \mathcal{R}(\sigma^\star, \mathcal{F}_{l-p}^i)$, and taking the limit as $l \to \infty$ relation (27) follows. ∎

Analogously to Theorem 2 we can establish similar results for the corresponding ϵ–approximations. In particular, we can obtain inner and outer approximations of the minimal invariant multi-set utilizing only the Reduced System. This is stated formally in the following Corollary.

Corollary 1 *Consider the System* (1)–(6)*, a set of unavoidable nodes* $\mathcal{Y} \subset \mathcal{V}$ *and the associated Reduced System. Let* α_1–α_3*,* $\tilde{\alpha}_1$–$\tilde{\alpha}_3$*,* (14)*,* (15)*,* (16) *correspond to the System and the Reduced System respectively. Let the pairs* (Γ, ρ)*,*

$(\tilde{\Gamma}, \tilde{\rho})$ *satisfy* (9) *for the System and the Reduced System respectively. Consider the multi-set sequence* $\{\tilde{\mathcal{F}}_l^j\}_{j\in\mathcal{V}}$*,* $l \geq 0$*, where* $\{\tilde{\mathcal{F}}_l^j\}_{j\in\mathcal{V}} = f(\{\tilde{\mathcal{F}}_l^j\}_{j\in\mathcal{Y}})$ (25) *and* $\{\tilde{\mathcal{F}}_l^j\}_{j\in\mathcal{Y}}$*,* $l \geq 0$ *is generated as in* (10)*,* (11) *for the Reduced System. For any accuracy* $\epsilon > 0$ *the following hold.*

(i) *For any integer* $l \geq \lceil \log_{\tilde{\rho}}(\frac{\epsilon(1-\tilde{\rho})}{\tilde{\alpha}_1\tilde{\Gamma}\max\{1,\Gamma\rho\alpha_3\alpha_1\}}) \rceil$*, it holds that* $\tilde{\mathcal{F}}_l^j \subseteq \mathcal{S}_m^j \subseteq \tilde{\mathcal{F}}_l^j \oplus \mathbb{B}(\epsilon)$*,* $\forall j \in \mathcal{V}$*.*

(ii) *For any pair* (k, λ)*,* $k \geq 1$*,* $\lambda \in (0,1)$ *that satisfy the inequalities* $\tilde{\alpha}_2\tilde{\Gamma}\tilde{\rho}^k \leq \lambda$*, and* $\max\{1, \alpha_1\alpha_3\Gamma\rho\}\tilde{\Gamma}\tilde{\alpha}_1\lambda(1 - \tilde{\rho}^k) leq \epsilon(1 - \tilde{\rho})(1 - \lambda)$*, the multi-set* $\{\frac{1}{1-\lambda}\tilde{\mathcal{F}}_l^j\}_{j\in\mathcal{V}}$ *is invariant with respect to the System* (1)–(6)*. Furthermore,* $\mathcal{S}_m^j \subseteq \frac{1}{1-\lambda}\tilde{\mathcal{F}}_{k-1}^j \subseteq \mathcal{S}_m^j \oplus \mathbb{B}(\epsilon)$*,* $\forall j \in \mathcal{V}$*.*

The proof is in Appendix A for completeness. We show next an analogous result for the maximal invariant multi-set.

Lemma 3 *Consider the maximal invariant multi-set* $\{\mathcal{S}_M^j\}_{j\in\mathcal{V}}$ *with respect to the System* (1)–(6)*. Consider the multi-set sequence* $\{\tilde{\mathcal{B}}_l^j\}_{j\in\mathcal{V}}$ *generated by* (13) *with* $\tilde{\mathcal{B}}_0^j = \hat{\mathcal{X}}_j$*, where* $\mathcal{S}_M^j \subseteq \hat{\mathcal{X}}_j \subseteq \mathcal{X}_j$*,* $j \in \mathcal{V}$ *and* $\tilde{\mathcal{X}}_j$ *are C-sets. Then, the multi-set sequence converges to the maximal invariant multi-set* $\{\mathcal{S}_M^j\}_{j\in\mathcal{V}}$*.*

Proof By Theorem 1(iii) the multi-set $\{\tilde{\mathcal{B}}_l^j\}_{j\in\mathcal{V}}$ converges to the maximal invariant multi-set $\{\tilde{\mathcal{S}}_M^j\}_{j\in\mathcal{V}}$ of (1)–(6) with constraints $\hat{\mathcal{X}}_j$, $j \in \mathcal{V}$. Since $\mathcal{S}_M^j \subseteq \hat{\mathcal{X}}_j$, $j \in \mathcal{V}$, then necessarily $\tilde{\mathcal{S}}_M^j \supseteq \mathcal{S}_M^j, j \in \mathcal{V}$. On the other hand, since $\hat{\mathcal{X}}_j \subseteq \mathcal{X}_j$, $j \in \mathcal{V}$, it holds that $\tilde{\mathcal{S}}_M^j \subseteq \mathcal{S}_M^j$ and the result follows. ∎

We define the sets of pairs

$$
\begin{aligned}
\mathcal{T}(j) := \{&(\sigma^\star, i) : \sigma^\star \in \sigma(j,i) \text{ and} \\
&\text{no } k \in \mathcal{Y} \text{ appears in the path from } j \text{ to } i\}. \quad (28)
\end{aligned}
$$

Theorem 3 *Consider the System* (1)–(6)*, a set of unavoidable nodes* $\mathcal{Y} \subseteq \mathcal{V}$ *and the corresponding Reduced System with a different constraint multi-set* $\{\tilde{\mathcal{X}}_j\}_{j\in\mathcal{Y}}$*, where*

$$\tilde{\mathcal{X}}_j := \mathcal{X}_j \bigcap_{(\sigma,d)\in\mathcal{T}(j)} \mathcal{C}(\sigma, \mathcal{X}_d),$$

$j \in \mathcal{Y}$ *and* $\mathcal{T}(j)$ *defined in* (28)*. Let* $\{\tilde{\mathcal{S}}_M^j\}_{j\in\mathcal{Y}}$ *be the maximal invariant multi-set of the Reduced System. Then, the maximal invariant multi-set* $\{\mathcal{S}_M^j\}_{j\in\mathcal{V}}$ *with respect to the System* (1)–(6) *is given by* (30)*, where*

$$
\mathcal{B}_0^j :=
\begin{cases}
\tilde{\mathcal{S}}_M^j, & j \in \mathcal{Y}, \\
\mathcal{X}_j, & j \in \mathcal{V} \setminus \mathcal{Y}.
\end{cases}
\tag{29}
$$

$$\mathcal{S}_M^j := \begin{cases} \tilde{\mathcal{S}}_M^j, & j \in \mathcal{Y}, \\ \displaystyle\bigcap_{\{d \in \mathcal{Y}: m(j,d) \cap \mathcal{Y} = \{d\}\}} \left(\left(\bigcap_{\{i \in m(j,d) \setminus \{j\}\}} \mathcal{C}(\sigma(j,i), \mathcal{B}_0^i) \right) \right) \cap \mathcal{X}_j, & j \in \mathcal{V} \setminus \mathcal{Y}. \end{cases} \tag{30}$$

Proof Consider the multi-set sequence $\{\mathcal{B}_l^j\}_{j \in \mathcal{V}}$, $l \geq 0$ generated by (13) with initial condition (29). By Lemma 3, $\{\mathcal{B}_l^j\}_{j \in \mathcal{V}}$ converges to the maximal invariant multi-set $\{\mathcal{S}_M^j\}_{j \in \mathcal{V}}$, since for each $j \in \mathcal{V}$, it holds that $\mathcal{S}_M^j \subseteq \mathcal{B}_0^j \subseteq \mathcal{X}_j$. We first show that the upper branch of (30) holds for all $l \geq 0$, $j \in \mathcal{Y}$, i.e.,

$$\mathcal{B}_l^j = \tilde{\mathcal{S}}_M^j, \quad j \in \mathcal{Y}, l \geq 0. \tag{31}$$

For $l = 0$, (31) holds by definition. Assuming that (31) holds for $l = k$, for $l = k+1$ we can write $\mathcal{B}_{k+1}^j$ as

$$\mathcal{B}_{k+1}^j = \tilde{\mathcal{S}}_M^j \cap \mathcal{H}_1 \cap \mathcal{H}_2,$$

where

$$\mathcal{H}_1 = \bigcap_{p=1}^{\max\{\theta_M, k\}} \bigcap_{\{(\sigma,d) \in \mathcal{T}(j): d \in \mathcal{Y}, |\sigma| = p\}} \mathcal{C}(\sigma, \mathcal{B}_{k-p}^d), \tag{32}$$

$$\mathcal{H}_2 = \bigcap_{\{(\sigma,d) \in \mathcal{T}(j): d \in \mathcal{V} \setminus \mathcal{Y}, |\sigma| = k\}} \mathcal{C}(\sigma, \mathcal{B}_0^d). \tag{33}$$

Since $\mathcal{B}_0^j = \mathcal{B}_k^j = \tilde{\mathcal{S}}_M^j$ and the set sequence $\{\mathcal{B}_l^j\}_{l \geq 0}$, for any index $j \in \mathcal{V}$, is monotonically non-increasing, it necessarily holds $\mathcal{B}_l^j = \tilde{\mathcal{S}}_M^j$ for all $l = 0, ..., k$. Consequently, in (32) we have $\mathcal{B}_{k-p}^d = \tilde{\mathcal{S}}_M^d$, and since $\{\tilde{\mathcal{S}}_M^j\}_{j \in \mathcal{V}}$ is the maximal invariant multi-set for the Reduced System, we have for each element of the intersection that $\mathcal{C}(\sigma, \tilde{\mathcal{S}}_M^d) \supseteq \tilde{\mathcal{S}}_M^j$, thus, $\tilde{\mathcal{S}}_M^j \cap \mathcal{H}_1 = \mathcal{B}_0^j$. Regarding the term $\mathcal{H}_2$, it is sufficient to observe that for each element of the intersection we have $\mathcal{C}(\sigma, \mathcal{B}_0^d) = \mathcal{C}(\sigma, \mathcal{X}_d) \supseteq \tilde{\mathcal{X}}_d$, thus, $\tilde{\mathcal{S}}_M^j \cap \mathcal{H}_2 = \tilde{\mathcal{S}}_M^j$ and $\mathcal{B}_{k+1}^j = \tilde{\mathcal{S}}_M^j$.

To show the lower branch of (30), it suffices to take any $l \geq \theta_M$ and calculate $\mathcal{B}_l^j$, for all $j \in \mathcal{V} \setminus \mathcal{Y}$. ∎

3.1 Application to minimum dwell time constraints

Systems under dwell time constraints can be modeled as constrained switching systems. These constraints impose a restriction on how fast the switching from one mode to another is allowed. In specific, given a set of N modes, $N > 1$, and a dwell time $\tau > 1$, the dynamics of the system may switch from a mode i to another mode j only if the system followed the dynamics of the mode i for at least τ consecutive time instants. This system can be described by (1)–(6) with a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N(N-1)(\tau-1) + N$, $|\mathcal{E}| = N(N-1)\tau + N$. For example, when $N = 2$, a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that captures these constraints is shown in Figure 5.
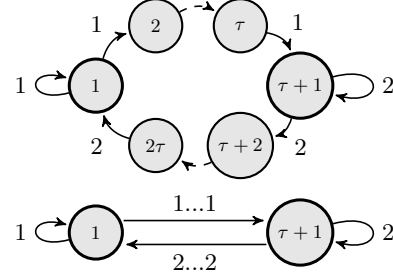

Fig. 5. The dwell-time constraints graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ (above) for a system consisting of two nodes, with a dwell-time $\tau > 1$ and the Reduced Graph taking $\mathcal{Y} = \{1, \tau+1\}$ (below).
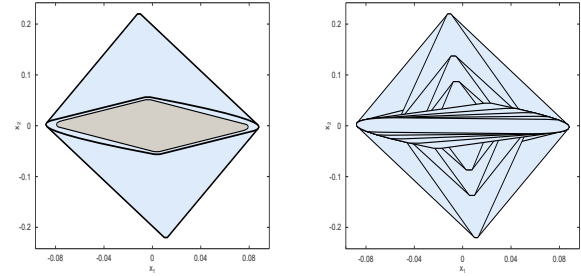

Fig. 6. Example 2: Left, the two elements of the multi-set corresponding to the 10^{-2}-approximation of the minimal invariant multi-set of the Reduced system (see Fig. 5, lower part, for $\tau = 6$) are shown in blue. The minimal DDT-invariant set computed in Dehghan and Ong (2012a) is depicted in grey. One can see that the minimal-DDT-invariant set fails to represent a minimal set of points in which the trajectories are confined. Right, all elements of the 10^{-2}-approximation of the minimal invariant multi-set of the system are shown.

The smallest set of unavoidable nodes is unique and consists of N nodes. The Reduced graph $\mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y})$ is a fully connected graph with $|\mathcal{Y}| = N$, $|\mathcal{E}_\mathcal{Y}| = N^2$, significantly less compared to the original graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$.

Example 2 *We consider the example in (Dehghan and Ong, 2012a, Section 6, Systems Ia, Ib). Therein, the notions of the minimal Disturbance Dwell-Time (DDT) invariance and constraint admissible maximal DDT invariance were introduced. DDT invariance is a type of invariance that relates to a truncated system trajectory corresponding to the so-called 'admissible sequence' (Dehghan and Ong, 2012a, Definitions 1, 2). In our framework, this switching sequence corresponds to a path $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that necessarily starts and ends in an unavoidable node (for example the node 1 or $\tau+1$ in Figure 5).*

We consider two modes $A_1 = \begin{bmatrix} 0.1321 & 0.2494 \\ -2.4940 & -0.1173 \end{bmatrix}$, $A_2 =$

dwell time τ	max. multi-set	max. DDT	10^{-2}–min. multi-set	min. DDT
6	0.036s	0.104s	7.03s	32.10s
10	0.044s	0.135s	6.65s	122.56s

Table 1

Example 2, computation times for the construction of the maximal invariant multi-set (second column, Theorem 3), the 10^{-2}–approximation of the minimal invariant multi-set (fourth column, Corollary 1(i)) and the maximal and minimal Disturbance Dwell Time Invariant set from Dehghan and Ong (2012a) (third and fifth column respectively).

$\begin{bmatrix} 0.9885 & 0.4406 \\ -0.0441 & 0.7682 \end{bmatrix}$, $\mathcal{W}_1 = \mathcal{W}_2 = \mathbb{B}_\infty(10^{-3})$, *constraint sets* $\mathcal{X}_i = \mathbb{B}_\infty(1)$, $i \in \mathcal{V}$ *and dwell times equal to* $\tau = 6$ *and* $\tau = 10$. *From Theorem 3 and by the convex version of Corollary 1(i), we compute the maximal invariant multi-set and a* 10^{-2}*–approximation of the convex minimal invariant multi-set. The computation times are shown in Table 1* [5] *. They are faster by at least an order of magnitude, compared to Dehghan and Ong (2012a), although additional information is generated about the behaviour of the system. The computation time for obtaining the reduced system is negligible compared to the time required for the reachability computations. The* 10^2*-approximation of the minimal invariant multi-set is shown in Figure 6 (left), while the union of the multi-set is compared to the computed minimal DDT set in Figure 6 (right). We observe that the concept of multi-set invariance offers an accurate characterization of where the system trajectories lie at all times, as opposed to the conservative approximation offered by the minimal DDT set.*

4 The Lifted System

In this section we take a somewhat opposite approach, and show that one can also benefit from increasing the size of the switching constraints graph. We apply two liftings on the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that defines the switching constraints on the System (1)–(6). The relation between the minimal and maximal invariant multi-sets between the System and the Lifted System can be exploited in several directions.

4.1 The Iterated Dynamics Lift (T-Lift)

We consider first the Iterated dynamics Lift, see e.g. the relevant works dealing with the stability analysis problem Lazar et al. (2013), Geiselhart et al. (2014), Philippe et al. (2015).

Definition 5 (Iterated Dynamics Lift (T-Lift) Philippe et al. (2015)) *Consider the System* (1)–(6) *and the switching constraints graph* $\mathcal{G}(\mathcal{V}, \mathcal{E})$. *Given an integer* $T \geq 1$, *the T-product lifted graph* $\mathcal{G}_T(\mathcal{V}, \mathcal{E}_T)$, *or,* $\mathcal{G}_T$, *is a graph having the same nodes as* $\mathcal{G}(\mathcal{V}, \mathcal{E})$ *and the set of edges* $\mathcal{E}_T := \{(i, j, \sigma^\star) : (i, j) \in \mathcal{V} \times \mathcal{V}, \sigma \in \sigma(i, j), |\sigma^\star| = T\}$.

Intuitively, there is an edge between a node $i \in \mathcal{V}$ and $j \in \mathcal{V}$ in $\mathcal{G}_T(\mathcal{V}, \mathcal{E}_T)$ whenever there is a walk between i and j in $\mathcal{G}(\mathcal{V}, \mathcal{E})$ of length T. We let $\mathcal{A}^T := \{\prod_{i=1}^{T} A_{\sigma_{T-i}} : (i, j, \sigma_1...\sigma_T) \in \mathcal{E}_T\}$ denote the set of matrices formed by the products corresponding to labels appearing in a walk of length T in the graph $(\mathcal{G}, \mathcal{V})$ and $\mathbb{W}^T := \{\bigoplus_{i=1}^{T} (\prod_{j=1}^{T-i} A_{\sigma_{T-j}} \mathcal{W}_{\sigma_i}) : (i, j, \sigma_1...\sigma_T) \in \mathcal{E}_T\}$ the corresponding set of disturbance sets of the iterated dynamics.

We note that in terms of memory storage space, the size of the lifted graph is in general an exponential function of the lifting parameter T. Nevertheless, the sets computed by the backward or forward reachability iterative methods have a description of an exponentially increasing complexity as well in general (for example half-space or vertex representation when dealing with polytopes). As a consequence, the price paid by increasing the problem size by the lifting may well be negligible in comparison to the cost of the brute-force application of a propagation algorithm. In Section 4.1.1, we explicitly describe the computational gain obtained due to the lifting method.

Definition 6 [Iterated Dynamics Lifted System (T-Lifted System)] *Given an integer* $T \geq 1$, *the* Iterated Dynamics Lifted System, *or,* T-Lifted System, *related to the System* (1)–(6) *is a constrained switching system with constraints graph* $\mathcal{G}_T(\mathcal{V}, \tilde{\mathcal{E}}_T)$, *matrix set* $\mathcal{A}_T$, *disturbance sets* $\mathbb{W}^T$ *and state constraints* $\mathcal{X}_i$, $i \in \mathcal{V}$.

The asymptotic stability properties of the System (1)–(6) and the T-Lifted System coincide (Philippe et al., 2015, Theorem 3.2), precisely it holds

$$\check{\rho}(\mathcal{A}_T, \mathcal{G}_T) = (\check{\rho}(\mathcal{A}, \mathcal{G}))^T. \tag{34}$$

We establish next the relationship between the minimal and maximal invariant multi-sets of the System and the Lifted System.

Theorem 4 *Consider the System* (1)–(6), *an integer* $T \geq 1$ *and the corresponding T-Lifted System. Let* $\{\check{\mathcal{S}}_m^j\}_{j \in \mathcal{V}}$, $\{\check{\mathcal{S}}_M^j\}_{j \in \mathcal{V}}$ *be the minimal and the maximal invariant multi-set of the T-Lifted System. The following hold.*

(i) The multi-set $\{\mathcal{S}^j\}_{j \in \mathcal{V}}$, *with*

$$\mathcal{S}^j = \check{\mathcal{S}}_m^j \bigcup_{\{(\sigma^\star, s) : \sigma^\star \in \sigma(s, j), |\sigma^\star| \leq T-1\}} \mathcal{R}(\sigma^\star, \check{\mathcal{S}}_m^s), \tag{35}$$

8

$j \in \mathcal{V}$, is the minimal invariant multi-set with respect to the System (1)–(6).

(ii) The multi-set $\{\mathcal{S}^j\}_{j\in\mathcal{V}}$, with

$$\mathcal{S}^j = \check{\mathcal{S}}_M^j \bigcap_{\{(\sigma^\star, d):\sigma^\star \in \sigma(j,d)|,|\sigma^\star|\leq T-1\}} \mathcal{C}(\sigma^\star, \check{\mathcal{S}}_M^d), \quad (36)$$

$j \in \mathcal{V}$, is the maximal invariant multi-set with respect to the System (1)–(6).

Proof (i) Given a node $j \in \mathcal{V}$ and a graph $\mathcal{G}$ we define the set

$$\mathcal{L}(\mathcal{G}, j) := \lim_{p\to\infty}\{\sigma^\star \in \sigma(i,j) : i \in \mathcal{V}, |\sigma^\star| = p\}.$$

By considering the sets $\mathcal{L}_1 = \mathcal{L}(\mathcal{G}, j)$, $\mathcal{L}_2 = \mathcal{L}(\mathcal{G}_T, j)$, we can express the elements $\mathcal{S}_m^j$ and $\check{\mathcal{S}}_m^j$ of the minimal invariant multi-set of the System and the T-Lifted System respectively by

$$\mathcal{S}_m^j = \bigcup_{\sigma\in\mathcal{L}_1}\mathcal{R}(\sigma, \{0\}), \quad \check{\mathcal{S}}_m^j = \bigcup_{\sigma\in\mathcal{L}_2}\mathcal{R}(\sigma, \{0\}).$$

For any sequence $\sigma_1 \in \mathcal{L}_1$, there is a (possibly empty) sequence $\overline{\sigma}$, $0 \leq \overline{\sigma} \leq T-1$ such that $\sigma_1 = \sigma_2\overline{\sigma}$. Taking into account that $\mathcal{R}(\sigma_2\overline{\sigma}, \{0\}) = \mathcal{R}(\overline{\sigma}, \mathcal{R}(\sigma_2, \{0\}))$ the result follows. The proof of **(ii)** follows a similar path. ∎

Similarly to Theorem 1(i), (iii), we establish upper bounds for computing invariant multi-sets for the T-Lifted System and show that they can be computed in a fraction of steps compared to System (1)–(6).

Proposition 1 *Consider the System (1)–(6), an integer $T \geq 1$, the corresponding T-Lifted System, the respective scalars (14)–(19) for the System and the T-Lifted System, an accuracy $\epsilon > 0$ and the quantities $l_1 = \lceil\log_\rho(\frac{\epsilon(1-\rho)}{\alpha_1\Gamma})\rceil$, $l_2 = \lceil\log_\rho\frac{\min_{j\in\mathcal{V}}R_j-r_j}{\alpha_1 c}\rceil$. An inner ϵ–approximation of the minimal invariant multi-set $\{\check{\mathcal{S}}_m^j\}_{j\in\mathcal{V}}$ and the maximal invariant multi-set $\{\check{\mathcal{S}}_M^j\}_{j\in\mathcal{V}}$ of the T-Lifted System can be computed at most after $\check{l}_1$ and $\check{l}_2$ iterations respectively, where*

$$\check{l}_1 = \lceil\tfrac{l_1+\delta_1}{T}\rceil, \quad \check{l}_2 = \lceil\tfrac{l_2+\delta_2}{T}\rceil, \quad (37)$$

and $\delta_1 = \log_\rho\frac{\alpha_1(1-\rho^T)}{\check{\alpha}_1(1-\rho)}$, $\delta_2 = \log_\rho(\frac{\alpha_1 c\min_{j\in\mathcal{V}}(R_j-\check{r}_j)}{\check{\alpha}_1\check{c}\min_{j\in\mathcal{V}}(R_j-r_j)})$.

Proof From (34), there exist pairs (Γ, ρ), $(\check{\Gamma}, \check{\rho})$ such that (9) holds for the System (1)–(6) and the T-product Lifted System respectively. Moreover, we have $\mathcal{N}_0^j = \cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{W}_\sigma \subseteq \cup_{\{\sigma^\star\in\sigma(i,j):|\sigma|=T\}}\mathcal{R}(\sigma^\star, \{0\}) = \check{\mathcal{N}}_0^j$, where $\mathcal{N}_0^j, \check{\mathcal{N}}_0^j, j \in \mathcal{V}$ are defined as in (7). We consider the multi-set sequences $\{\mathcal{N}_l^j\}_{j\in\mathcal{V}}$, $\{\check{\mathcal{N}}_l^j\}_{j\in\mathcal{V}}$, $l \geq 0$, generated by (8). Since $\mathcal{N}_0^j \subseteq$

$\check{\mathcal{N}}_0^j, j \in \mathcal{V}$, we have $\mathcal{N}_{tT}^j \subseteq \Gamma\rho^{tT}\mathcal{N}_0^j \subseteq \Gamma\rho^{tT}\check{\mathcal{N}}_0^j$. Thus, we can choose the pair $(\check{\Gamma}, \check{\rho})$ to be

$$\check{\rho} = \rho^T, \quad \check{\Gamma} = \Gamma. \quad (38)$$

Applying Theorem 1(i), (iii) to the T-Lifted System and taking into account (38), the upper bounds (37) are calculated directly. ∎

4.1.1 How to choose the lift degree T

Proposition 1 and Theorem 4 can be combined to compute ϵ–approximations of the minimal invariant multi-set, as well as the maximal invariant multi-set of the System (1)–(6) via the T-Lifted System. In this subsection, we focus on quantifying this potential computational gain by formulating a suitable cost function on the computational burden that depends on the lift degree T. From the backward reachability multi-set sequence (12), (13) and relation (36) we can distinguish between two main operations, the *basic iteration* and the *intersection operation*.

Definition 7 *Given a matrix $A \in \mathbb{R}^{n\times n}$ and the sets $\mathcal{S} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^n$, we call basic iteration the mapping $f_-(\mathcal{S}, \mathcal{W}) := \{x \in \mathbb{R}^n : \forall w \in \mathcal{W}, Ax + w \in \mathcal{S}\}$ and the mapping $f_+(\mathcal{S}, \mathcal{W}) = \{Ax + w : x \in \mathcal{S}, w \in \mathcal{W}\}$. Moreover, given an integer $q \geq 2$ and the sets $\mathcal{S}_i \subset \mathbb{R}^n$, $i = 1, ..., q$, the intersection operation is the mapping $f_\cap(\mathcal{S}_1, ..., \mathcal{S}_q) = \cap_{i=1}^q\mathcal{S}_i$.*

Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, we define its adjacency matrix $A \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ to have elements a_{ij} equal to the number of edges starting from node i and ending to node j, and equal to 0 otherwise. Letting $d_1(T)$ denote the number of basic iterations required for the computation of the maximal invariant multi-set and $d_2(T)$ denote the corresponding number of intersections of sets, we have

$$d_1(T) = \left(\lceil\tfrac{l_2+\delta_2}{T}\rceil\|A^T 1\|_1 + \sum_{i=0}^{T-1}\|A^i 1\|_1\right), \quad (39)$$

$$d_2(T) = \left(\lceil\tfrac{l_2+\delta_2}{T}\rceil + 1\right)|\mathcal{V}|, \quad (40)$$

where l_2, δ_2 are given in Proposition 1. One can formulate a cost function $J(d_1(T), d_2(T))$ that models the computational burden of the procedure and retrieve an optimal lift degree [6]. Similar formulations are possible for describing the computational cost of computing ϵ–approximations of the minimal invariant multi-set via Theorem 4(i).

---

[6] A more accurate expression of the computational burden depends strongly on the shape of the state and disturbance sets, the choice of algorithms implementing the set intersections, e.g., convex hull algorithms, etc.

Example 3 *We consider the arbitrary switching system in (Blanchini and Miani, 2008, Example 5.23), consisting of two modes with $A_1 = \begin{bmatrix} 1 & 2.5 \cdot 10^{-4} \\ -5 \cdot 10^{-4} & 0.99975 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2.5 \cdot 10^{-4} \\ -2.2425 \cdot 10^{-3} & 0.99975 \end{bmatrix}$. The switching constraints graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ has one node with two self-loops, labeled by 1 and 2. We consider $\mathcal{X} = \mathbb{B}_\infty(1)$ and $\mathcal{W}_1 = \mathcal{W}_2 = \{0\}$. The adjacency matrix of the graph is $A = 2$. To decide the lift degree T, we choose the cost $J(T) = d_1(T) + 40 d_2(T)$, where $d_1(T)$, $d_2(T)$ are given in (39) and (40) respectively. We choose the weight taking into account that in the specific example the intersection operation includes the convex hull operation. This is considerably more demanding than the basic iterations, which in this case it corresponds to computing and storing matrix-vector products. The minimizer of the cost function is $T^\star = 4$. Utilizing Proposition 1, we first compute the maximal invariant set of the 4-Lifted system $\check{\mathcal{S}}_M$. Next, from Theorem 4(ii) we compute the max-imal invariant set $\mathcal{S}_M$, with a corresponding computation time 23 seconds [7]. In comparison, the classical backward reachability algorithm corresponds to $T = 1$ and 90 seconds computation time. In Figure 7 the actual time spent*
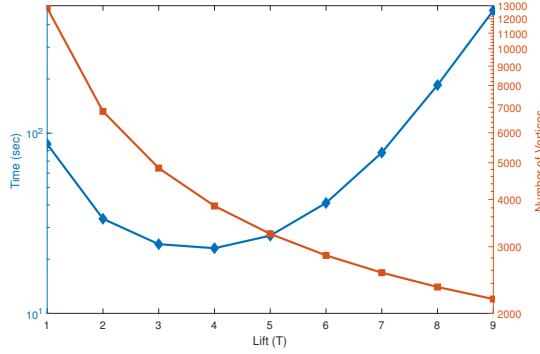

Fig. 7. Example 3, the time spent for computing $\mathcal{S}_M$ (blue) and the number of vertices of $\check{\mathcal{S}}_M$ (red), for different choices of the lift T.

for computing $\mathcal{S}_M$ for different choices of the T is shown in blue line in logarithmic scale. We observe that the actual optimal lift degree is equal to $T^\star = 4$. In the same Figure, the number of vertices of $\check{\mathcal{S}}_M$ is also shown in red.

4.2 The Path-Dependent Lift (P-Lift)

We exploit the Path-Dependent lifting, studied in Bliman and Ferrari-Trecate (2003), Lee and Dullerud (2006), and Philippe et al. (2015) where it provided asymptotically tight approximations to the constrained joint spectral radius.

Definition 8 [Path Dependent Lift (P-Lift) Lee and Dullerud (2006)] *Consider an integer $P \geq 1$ and a*

---

[7] The computation times include the time required to construct the T-lifts, which in this example is negligible compared to the reachability iterations.

System (1)–(6) corresponding to a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The Path-Dependent lifted graph $\mathcal{G}_P(\mathcal{V}_P, \mathcal{E}_P)$, or $\mathcal{G}_P$, is a graph with the set of nodes $\mathcal{V}_P := \{v_{i_1} \sigma_{i_1} v_{i_2} \cdots v_{i_{P+1}} : (v_{i_j}, v_{i_{j+1}}, \sigma_{i_j}) \in \mathcal{E}, j \in \{1, ..., P\}\}$, and the set of edges $\mathcal{E}_P := \{(v_a, v_b, \sigma) : v_a = v_{i_1} \sigma_{i_1} \cdots v_{i_{P+1}}, v_b = v_{i_2} \sigma_{i_2} \cdots v_{i_{P+2}}, \sigma = \sigma_{i_{P+1}}, \sigma_{i_j} \in \{1, ..., N\}, v_{i_j} \in \mathcal{V}, j \in \{1, ..., P+2\}\}$.

Roughly, the P-Lifted graph $\mathcal{G}_P(\mathcal{V}_P, \mathcal{E}_P)$ has as many nodes as different walks of length P in the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. For a node $j \in \mathcal{V}$, we define the subsets of nodes $\mathcal{J}(j) \subseteq \mathcal{V}_P$ to be

$$\mathcal{J}(j) := \{v \in \mathcal{V}_P : v = v_{i_1} \sigma_{i_1} \cdots j\}. \qquad (41)$$

Similarly to the T-lifted graph, the memory size required to store the obtained lifted graph is generally an exponential function of the lifting parameter P. Nevertheless, as we see in the sequel, it is a controlled price that is paid to establish non-convex approximations of the minimal invariant multi-set. Compared to the brute-force strategy of storing the non-convex forward reachability multi-sets (which are unions of an exponentially increasing number of convex sets), this additional degree of freedom allows us to establish non-convex approximations of forward reachability operations efficiently.

Definition 9 [Path-Dependent Lifted System (P-Lifted System)] *Given an integer $P \geq 1$, the Path-Dependent Lifted System, or P-Lifted System, related to the System (1)–(6) is a constrained switching system with constraints graph $\mathcal{G}_P(\mathcal{V}_P, \mathcal{E}_P)$, matrix set $\mathcal{A}$, disturbance sets $\mathbb{W}$ and state constraints $\mathcal{X}_a := \mathcal{X}_j$ for any node $a \in \mathcal{J}(j)$.*

Example 4 *The P-Lifted Graph, $P = 1$, of the Graph in Figure 1 is in Figure 4. Moreover, we have $\mathcal{X}_{a2a} := \mathcal{X}_a$, $\mathcal{X}_{a1b} = \mathcal{X}_b, \mathcal{X}_{b1c} = \mathcal{X}_c, \mathcal{X}_{c2b} = \mathcal{X}_b, \mathcal{X}_{c1a} = \mathcal{X}_a$ and $\mathcal{J}(a) = \{a1a, c1a\}, \mathcal{J}(b) = \{a1b, c2b\}, \mathcal{J}(c) = \{b1c\}$.*

Theorem 5 *Consider the System (1)–(6), an integer $P \geq 1$ and the corresponding P-Lifted System. Let $\{\mathcal{S}_m\}_{j \in \mathcal{V}}$, $\{\mathcal{S}_M^j\}_{j \in \mathcal{V}}$, $\{\check{\mathcal{S}}_m^j\}_{j \in \mathcal{V}_P}$ and $\{\check{\mathcal{S}}_M^j\}_{j \in \mathcal{V}_P}$ be the minimal and maximal invariant multi-set of the System (1)–(6) and the P-Lifted System respectively. The following hold.*

(i) $\mathcal{S}_m^j = \bigcup\limits_{i \in \mathcal{J}(j)} \check{\mathcal{S}}_m^i$, for all $j \in \mathcal{V}$.

(ii) $\mathcal{S}_M^j = \check{\mathcal{S}}_M^i$, for all $j \in \mathcal{V}$, for all i in $\mathcal{J}(j)$,

Proof (i) Let $\{\mathcal{F}_l^j\}_{j \in \mathcal{V}}$ and $\{\check{\mathcal{F}}_l^j\}_{j \in \mathcal{V}_P}$, $l \geq 0$ denote the members of the multi-set sequences generated by (10), (11), for the System (1)–(6) and the P-Path-dependent Lifted system respectively. We show that

$$\mathcal{F}_l^j = \bigcup\limits_{i \in \mathcal{J}(j)} \check{\mathcal{F}}_l^i, \qquad j \in \mathcal{V}. \qquad (42)$$

10

For $l = 0$ we have $\mathcal{F}_0^j = \cup_{i \in \mathcal{J}(j)} \check{\mathcal{F}}_0^i = \{0\}$. Assuming (42) holds for $l = k$, we have for $l = k + 1$ that $\mathcal{F}_{k+1}^j = \cup_{(s_j, j, \sigma) \in \mathcal{E}} \mathcal{R}(\sigma, \mathcal{F}_k^{s_j}) = \cup_{(s_j, j, \sigma) \in \mathcal{E}} \mathcal{R}(\sigma, \cup_{i \in \mathcal{I}(s_j)} \check{\mathcal{F}}_k^i) = \cup_{(s_j, j, \sigma) \in \mathcal{E}} \cup_{i \in \mathcal{J}(s_j)} \mathcal{R}(\sigma, \check{\mathcal{F}}_k^i) = \cup_{i \in \mathcal{J}(s_j)} \cup_{(s_j, j, \sigma) \in \mathcal{E}} \mathcal{R}(\sigma, \check{\mathcal{F}}_k^i) = \cup_{i \in \mathcal{J}(j)} \check{\mathcal{F}}_{k+1}^i$, thus, (42) holds for all $l \geq 0$. Taking the limit as $l \to \infty$, the result follows.

(ii) By construction of $\mathcal{G}_P$, for any $j \in \mathcal{V}$, for any $i \in \mathcal{I}(j)$ and for each edge $(j, d, \sigma) \in \mathcal{E}$, there exists a node $\check{d} \in \mathcal{V}_P$ such that $(i, \check{d}, \sigma) \in \mathcal{E}_P$. Taking into account that for all $j \in \mathcal{V}$, for all $i \in \mathcal{J}(j)$ it holds that $\mathcal{X}_i = \mathcal{X}_j$, the members of the multi-set sequences $\{\mathcal{B}_l^j\}_{j \in \mathcal{V}}$, $\{\check{\mathcal{B}}_l^j\}_{j \in \mathcal{V}_P}$, generated by (12), (13) for the System (1)–(6) and the P-Lifted System respectively satisfy $\mathcal{B}_l^j = \check{\mathcal{B}}_l^i$. The result follows from Theorem 1(iii) by taking a large enough integer $l^\star$ such that $\mathcal{B}_l^j = \check{\mathcal{B}}_l^i = \mathcal{S}_M^j$. ∎

By combining Theorem 5 and Theorem 1(i), (ii) one can adapt in a straightforward manner all results concerning the ϵ–approximations of the minimal invariant multi-set of the System (1)–(6). This is illustrated in the following subsection for the case of arbitrary switching systems.

4.2.1 Approximations of the minimal invariant set

Utilizing Theorem 5(i), we can compute non-convex approximations of the minimal invariant multi-set. In specific and especially for arbitrary switching systems, by utilizing the P-Lift and the convex version of Theorem 5(i) we can approximate the minimal invariant set with the union of a *finite, pre-specified number, of convex sets*. The P-Lifted graph of an arbitrary switching system is a *de Bruijn graph* (Lothaire, 2002, Section 1.3.4). For example, for $N = 2$ modes, the 1 and 2 Path-Dependent Lifted graphs are shown in the upper and lower part of Figure 8.
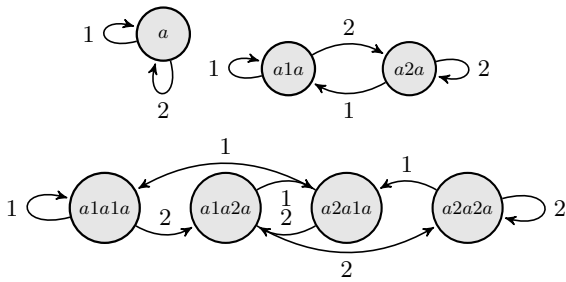

Fig. 8. Upper left, a graph representing an arbitrary switching system consisting of two modes. Upper right, its 1-Path-Dependent Lifted Graph. Lower part, its 2-Path-Dependent Lifted Graph.

Example 5 *We compute the minimal invariant set for an arbitrary switching linear system inspired by (Rakovic et al., 2005b, Example 1). In specific, we consider the system $x(t+1) = (A_{\sigma(t)} + BK)x(t) + w(t)$, $\sigma(t) : \mathbb{N} \to \{1, 2\}$, with $A_1 = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.8 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $K = \begin{bmatrix} -1.2 & -1 \end{bmatrix}$,*

$w(t) \in \mathbb{B}_\infty(10)$. *By considering the P-Lift, $P = 1$, we obtain the graph $\mathcal{G}_P(\mathcal{V}_P, \mathcal{E}_P)$, shown in the upper right part of Figure 8. Utilizing the convex version of Theorem 1(i) (Athanasopoulos et al., 2017, Corollary 1) we compute an inner ϵ–approximation of the minimal convex invariant multi-set of the Lifted System for $\epsilon = 10^{-3}$, reached in 40 iterations. In Figure 9, the sets $\{\mathcal{S}_i^{a1a}\}_{i \in \{1,...,40\}}$ and $\{\mathcal{S}_i^{a2a}\}_{i \in \{1,...,40\}}$ are shown in blue and red respectively. Consequently, by Theorem 5(i), the 10^{-3}–approximation of the minimal invariant set of the original system is $\mathcal{S}_{40}^{a1a} \cup \mathcal{S}_{40}^{a2a}$. We note that for the specific example this set is the exact non-convex inner approximation of the minimal invariant set.*
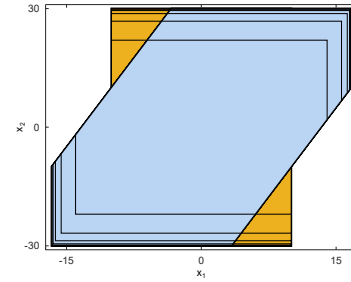

Fig. 9. Example 5, the sets $\{\mathcal{S}_i^{a1a}\}_{i \in \{1,...,40\}}$ and $\{\mathcal{S}_i^{a2a}\}_{i \in \{1,...,40\}}$ are depicted in blue and yellow respectively. The minimal invariant set for the system is $\check{\mathcal{S}} = \mathcal{S}_{40}^{a1a} \cup \mathcal{S}_{40}^{a2a}$. The arrows indicate the direction of expansion for the sequences $\{\mathcal{S}_i^{a1a}\}_{i \in \{1,..,40\}}$, $\{\mathcal{S}_i^{a2a}\}_{i \in \{1,..,40\}}$,

5 Conclusions

We demonstrated how combinatorial constructions can be used for general models of hybrid systems. As a starting point, we considered constrained switching systems and studied the computation of the minimal and maximal invariant (multi-)sets. We illustrated the usefulness of our constructions in applications dealing with minimum dwell-time specifications and arbitrary switching linear systems. We believe that the combinatorial and algebraic reductions and liftings established can be useful for more general classes of hybrid automata, both for qualitative and quantitative analysis. Further work should investigate the so-called *horizon-dependent lifting* from Essick et al. (2014), which seems well fit for parallelizing optimally the maximal invariant multi-set computations. Moreover, it is worth investigating the possibility to extend the results to state-dependent constrained switching. Last, it would be interesting to explore the possible gain in applying the techniques presented here to systems with inputs, whether by investigating classical problems, or by extending the notion of multi-set invariance to controlled multi-set invariance and compare with existing notions from the literature, e.g., De Santis et al. (2004).

References

Artstein, Z., Rakovic, S. V., 2008. Feedback and Invariance under Uncertainty via Set Iterates. Automatica 44 (2), 520–525.

Artstein, Z., Rakovic, S. V., 2011. Set Invariance Under Output Feedback : A Set–Dynamics Approach. International Journal of Systems Science 42, 539–555.

Athanasopoulos, N., Lazar, M., 2014. Stability analysis of switched systems defined by graphs. In: 53rd IEEE Conference on Decision and Control. Los Angeles, CA, USA, pp. 5451–5456.

Athanasopoulos, N., Smpoukis, K., Jungers, R. M., 2017. Invariant sets analysis for constrained switching systems. IEEE Control Systems Letters 1, 256–261.

Blanchini, F., 1999. Set Invariance in Control – A Survey. Automatica 35 (11), 1747–1767, Survey Paper.

Blanchini, F., Miani, S., 2008. Set–Theoretic Methods in Control. Systems & Control: Foundations & Applications. Birkhauser, Boston, Basel, Berlin.

Bliman, P.-A., Ferrari-Trecate, G., 2003. Stability analysis of discrete-time switched systems through Lyapunov functions with nonminimal state. In: IFAC Conference on the Analysis and Design of Hybrid Systems. St. Malo, France, pp. 325–330.

Cambier, L., Philippe, M., Jungers, R. M., 2015. CSS Toolbox for MATLAB. http://www.mathworks.com/matlabcentral/fileexchange/52723-the-cssystem-toolbox.

Dai, X., 2012. A Gel'fand-type spectral radius formula and stability of linear constrained switching systems. Linear Algebra and its Applications 436, 1099–1113.

De Santis, E., Di Benedetto, M. D., Berardi, L., 2004. Computation of maximal safe sets for switching systems. IEEE Transactions on Automatic Control 49, 184–195.

Dehghan, M., Ong, C.-J., 2012a. Characterization and computation of disturbance invariant sets for constrained switched linear systems with dwell time restriction. Automatica 48, 2175–2181.

Dehghan, M., Ong, C.-J., 2012b. Discrete-time switching linear systems with constraints: Characterization and computation of invariant sets under dwell-time consideration. Automatica 48, 964–969.

Donkers, M. C. F., Heemels, W. P. M., van den Wouw, N., Hetel, L., 2011. Stability Analysis of Networked Systems Using a Switched Linear Systems Approach. IEEE Transactions on Automatic Control 56, 2101–2115.

Essick, R., Lee, J.-W., Dullerud, G. E., 2014. Control of Linear Switched Systems with Receding Horizon Modal Information. IEEE Transactions on Automatic Control 59, 2340–2352.

Even, G., Naor, S. J., Schieber, B., Sudan, M., 1998. Approximating minimum feedback sets and multicuts in directed graphs. Algorithmica 20, 151–174.

Geiselhart, R., Gielen, R. H., Lazar, M., Wirth, F. R., 2014. An alternative converse Lyapunov theorem for discrete-time systems. Systems & Control Letters 70, 49–59.

Girard, A., Pappas, G. J., 2011. Approximate Bisimulation: A Bridge Between Computer Science and Control Theory. European Journal of Control 17, 568–578.

Goebel, R., Sanfelice, R. G., Teel, A. R., 2012. Hybrid Dynamical Systems: Modeling Stability, and Robustness. Princeton University Press.

Hernandez-Mejias, M. A., Sala, A., Arino, C., Querol, A., 2015. Reliable controllable sets for constrained Markov-Jump Linear Systems. International Journal of Robust and Nonlinear Control 26, 2075–2089.

I. V. Kolmanovsky and E. G. Gilbert, 1998. Theory and Computation of Disturbance Invariant Sets for Discrete–Time Linear Systems. Mathematical Problems in Egineering 4, 317–367.

Jungers, R. M., 2009. The joint spectral radius: theory and applications. Vol. 385 of Lecture Notes in Control and Information Sciences. Springer.

Karp, R. M., 1972. Reducibility among combinatorial problems. Complexity of computer computations, 85–103.

Lazar, M., Doban, A. I., Athanasopoulos, N., 2013. On stability analysis of discrete–time homogeneous dynamics. In: 17th International Conference on System Theory, Control and Computing. Sinaia, Romania, pp. 1–8.

Lee, J.-W., Dullerud, G. E., 2006. Uniform stabilization of discrete-time switched and Markovian jump linear systems. Automatica 42, 205–218.

Liberzon, D., 2003. Switching in systems and control. Birkhauser, Boston.

Lothaire, M., 2002. Algebraic Combinatorics on Words. Cambridge University Press, Vol. 90.

Philippe, M., Essick, R., Dullerud, R., Jungers, R. M., 2015. Stability of discrete-time switching systems with constrained switching sequences. Automatica 72, 242–250.

Protasov, V. Y., Jungers, R. M., 2015. Resonance and marginal instability of switching systems. Nonlinear Analysis: Hybrid Systems 17, 81–93.

Rakovic, S. V., Kern, B., Findeisen, R., 2010. Practical set invariance for decentralized discrete-time systems. In: 49th IEEE Conference on Decision and Control. pp. 3283–3288.

Rakovic, S. V., Kern, B., Findeisen, R., 2011. Practical robust positive invariance for large-scale discrete time systems. In: IFAC World Congress. pp. 6425–6430.

Rakovic, S. V., Kerrigan, E. C., Kouramas, K. I., Mayne, D. Q., 2005a. Invariant approximations of the minimal robustly positively invariant sets. IEEE Transactions on Automatic Control 50 (3), 406–410.

Rakovic, S. V., Kouramas, K. I., Kerrigan, E. C. Allwright, J. C., Mayne, D. Q., 2005b. The minimal robust positively invariant set for linear difference inclusions and its robust positively invariant approximations. Tech. Rep. EEE/C P

/SVR/9-d/2005, Imperial College, London, UK.

Shorten, R., Wirth, F., Mason, O., Wulff, K., King, C., 2007. Stabiliy criteria for switched and hybrid systems. SIAM Review 49, 545–592.

Wang, Y., Roohi, N., Dullerud, G. E., Viswanathan, M., 2016. Stability Analysis of Switched Linear Systems defined by Regular Languages. IEEE Transactions on Automatic Control 99, 1–1.

Zhang, L., Zhuang, S., Braatz, R. D., 2016. Switched model predictive control of switched linear systems: Feasibility, stability and robustness. Automatica 67, 8–21.

A Proof of Corollary 1

The following Lemma is required first.

Lemma 4 *Consider the System* (1)–(6)*, an admissible switching sequence* $\sigma_1...\sigma_l$, $l \geq 1$ *and two $\mathcal{C}$-sets* $\mathcal{S}_1 \subset \mathbb{R}^n$, $\mathcal{S}_2 \subset \mathbb{R}^n$. *It holds that* $\mathcal{R}(\sigma_1...\sigma_l, \mathcal{S}_1 \oplus \mathcal{S}_2) = \mathcal{R}(\sigma_1...\sigma_l, \mathcal{S}_1) \oplus \mathcal{R}_N(\sigma_1...\sigma_l, \mathcal{S}_2)$.

Proof We use induction: For $l = 1$ it holds that $\mathcal{R}(\sigma, \mathcal{S}_1 \oplus \mathcal{S}_2) = A_\sigma(\mathcal{S}_1 \oplus \mathcal{S}_2) \oplus \mathcal{W}_\sigma = (A_\sigma \mathcal{S}_1 \oplus \mathcal{W}_\sigma) \oplus A_\sigma \mathcal{S}_2 = \mathcal{R}(\sigma, \mathcal{S}_1) \oplus \mathcal{R}_N(\sigma, \mathcal{S}_2)$. Let us suppose that the relation holds for l. Then, $\mathcal{R}(\sigma_1...\sigma_{l+1}, \mathcal{S}_1 \oplus \mathcal{S}_2) = \mathcal{R}(\sigma_{l+1}, \mathcal{R}(\sigma_1...\sigma_l, \mathcal{S}_1 \oplus \mathcal{S}_2)) = \mathcal{R}(\sigma_{l+1}, \mathcal{R}(\sigma_1...\sigma_l, \mathcal{S}_1) \oplus \mathcal{R}_N(\sigma_1...\sigma_l, \mathcal{S}_2)) = \mathcal{R}(\sigma_1...\sigma_{l+1}, \mathcal{S}_1) \oplus \mathcal{R}_N(\sigma_1...\sigma_{l+1}, \mathcal{S}_2)$. Thus, the relation holds for $l + 1$. ∎

Similarly to the multi-set $\{\mathcal{N}_l^j\}_{j \in \mathcal{V}}$ generated by (7), (8), we denote forward reachability multi-set sequence of the nominal part of the Reduced System by $\{\tilde{\mathcal{N}}_l^j\}_{j \in \mathcal{Y}}$, $l \geq 0$.

Proof. (i) When $j \in \mathcal{Y}$, from Theorem 2 and Theorem 1(i), we have that $\tilde{\mathcal{F}}_l^j \subseteq \tilde{\mathcal{S}}_m^j = \mathcal{S}_m^j \subseteq \tilde{\mathcal{F}}_l^j \oplus \mathbb{B}(\epsilon)$, for any $l \geq \lceil \log_{\tilde{\rho}}(\frac{\epsilon(1-\tilde{\rho})}{\tilde{\alpha}_1 \tilde{\Gamma}}) \rceil$. When $j \in \mathcal{V} \setminus \mathcal{Y}$, for any integer $k \geq 0$, $l \geq \theta_M$, from (25), (26) and by setting $\mathcal{Q}(j,p) := \{(\sigma,i) \in \mathcal{P}(j) : |\sigma| = p\}$, where $\mathcal{P}(j)$ is defined in (26), we have

$$\tilde{\mathcal{F}}_{l+k}^j = \bigcup_{p=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} \mathcal{R}(\sigma, \tilde{\mathcal{F}}_{l+k}^i)$$
$$\subseteq \bigcup_{p=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} \mathcal{R}(\sigma, \tilde{\mathcal{F}}_{l+k-1}^i \oplus \tilde{\Gamma}\tilde{\rho}^{l+k-1}\tilde{\alpha}_1\mathbb{B}(1)) \subseteq ...$$
$$\subseteq \bigcup_{p=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} \mathcal{R}(\sigma, \tilde{\mathcal{F}}_l^i \oplus (\frac{\tilde{\Gamma}\tilde{\rho}^l\tilde{\alpha}_1(1-\tilde{\rho}^k)}{1-\tilde{\rho}})\mathbb{B}(1)).$$

Setting $\delta = \left(\frac{\tilde{\Gamma}\tilde{\rho}^l\tilde{\alpha}_1}{1-\tilde{\rho}}\right)$, taking the limit as $k \to \infty$ and from Lemma 4, it follows

$$\tilde{\mathcal{F}}_\infty^j \subseteq \bigcup_{p=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} [\mathcal{R}(\sigma(i,j), \tilde{\mathcal{F}}_l^i) \oplus \mathcal{R}_N(\sigma(i,j), \mathbb{B}(\delta))]$$
$$\subseteq \tilde{\mathcal{F}}_l^j \oplus \Gamma\rho\alpha_3\alpha_1\mathbb{B}(\delta),$$

which implies $\mathcal{S}_m^j \subseteq \tilde{\mathcal{F}}_l^j \oplus \mathbb{B}(\epsilon)$ for any $l \geq \theta_M$ satisfying $l \geq \lceil \log_{\tilde{\rho}}(\frac{\epsilon(1-\tilde{\rho})}{\tilde{\alpha}_1\tilde{\Gamma}\alpha_3\alpha_1\Gamma\rho}) \rceil$. Combining the two above inequalities on l the result follows.

(ii) By (23) and Theorem 1(ii) we have that the multi-set $\{\frac{1}{1-\lambda}\tilde{\mathcal{F}}_{k-1}^j\}_{j \in \mathcal{Y}}$ is invariant with respect to the Reduced System. Consequently, by construction of (25), $\{\frac{1}{1-\lambda}\tilde{\mathcal{F}}_{k-1}^j\}_{j \in \mathcal{V}}$ is also invariant with respect to the System (1)–(6), thus the left inclusion holds. By Theorem 1(ii) and the hypothesis, the right inclusion holds for all $j \in \mathcal{Y}$. For $j \in \mathcal{V} \setminus \mathcal{Y}$, we have

$$\frac{1}{1-\lambda}\tilde{\mathcal{F}}_{k-1}^j = \bigcup_{p=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} \mathcal{R}(\sigma, (1 + \frac{\lambda}{1-\lambda})\tilde{\mathcal{F}}_{k-1}^i)$$
$$= \bigcup_{p=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} (\mathcal{R}(\sigma, \tilde{\mathcal{F}}_{k-1}^i) \oplus \mathcal{R}_N(\sigma, \frac{\lambda}{1-\lambda}\tilde{\mathcal{F}}_{k-1}^i))$$
$$\subseteq \bigcup_{i=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} (\mathcal{R}(\sigma, \tilde{\mathcal{F}}_{k-1}^i) \oplus \mathcal{R}_N(\sigma, \delta\mathbb{B}(1)),$$

with $\delta = \frac{\lambda\tilde{\Gamma}\tilde{\alpha}_1(1-\tilde{\rho}^{k-1})}{(1-\lambda)(1-\tilde{\rho})}$. Consequently, we have that

$$\frac{1}{1-\lambda}\tilde{\mathcal{F}}_{k-1}^j \subseteq \bigcup_{i=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} (\mathcal{R}(\sigma, \tilde{\mathcal{F}}_{k-1}^i) \oplus$$
$$\mathcal{R}_N(\sigma, \delta\alpha_3\Gamma\rho^p\mathcal{N}_0^i))$$
$$\subseteq \left(\bigcup_{i=1}^{\theta_M} \bigcup_{(\sigma,i) \in \mathcal{Q}(j,p)} (\mathcal{R}(\sigma, \tilde{\mathcal{F}}_{k-1}^j)\right) \oplus \delta\Gamma\rho\alpha_1\alpha_3\mathbb{B}(1)$$
$$\subseteq \tilde{\mathcal{F}}_{k-1}^j \oplus \mathbb{B}(\epsilon) \subseteq \mathcal{S}_m^j \oplus \mathbb{B}(\epsilon).$$

Thus, $\frac{1}{1-\lambda}\tilde{\mathcal{F}}_{k-1}^j \subseteq \mathcal{S}_m^j \oplus \mathbb{B}(\epsilon)$ for all $j \in \mathcal{V}$. ∎