CHAPTER I

Semilinear elliptic equation on $\mathbf{R}^N$ with unbounded coefficients

1. Introduction

We study the existence of localized solutions of the semilinear elliptic equation

$$-\Delta u + a(x)u = f(x,u) \qquad \text{on } \mathbf{R}^N.$$

Many papers deal with this problem when $a(x)$ is "large" at infinity and f is subcritical, i.e. when there exist $a_0 \in \mathbf{R}_0^+$, $c \in \mathbf{R}^+$ and $p < 2^* := \frac{2N}{N-2}$ such that

$$\begin{aligned} a(x) &\geqslant a_0 \qquad \forall x \in \mathbf{R}^N, \\ \lim_{|x|\to+\infty} a(x) &= +\infty, \\ |f(x,u)| &\leqslant c\left(1+|u|^{p-1}\right). \end{aligned}$$

See e.g. Bartsch-Wang [4], Costa [18], Omana-Willem [39] and Rabinowitz [43].

Sirakov [52] observed that the case when f is unbounded in x is rather delicate. Consider the model problem

$$\left\{ \begin{array}{l} -\Delta u + |x|^a u = |x|^b u^{p-1}, \\ u \in H^1(\mathbf{R}^N), \quad u > 0, \end{array} \right. \tag{1.1}$$

where $a > b \geqslant 0$ and $N \geqslant 3$. Sirakov proves the existence of a solution for

$$2 < p < p^{\#} := 2^* - \tfrac{4b}{a(N-2)},$$

On the other hand, using the Derrick-Pohozaev identity, it is shown that there is no solution for

$$p > \tilde{p} := 2^* + \tfrac{2b}{N-2}.$$

Hence 2^* lies in a gap between $p^{\#}$ and $\tilde{p}$. Ding-Ni [21] obtain the existence of a solution of (1.1) with

$$a = 0, \quad p < 2^* \quad \text{and} \quad 2b < (N-1)(p-2).$$

Our aim is to solve problem (1.1) for $a, b \geqslant 0$. Our approach is applicable in a more general setting, but for simplicity we consider only the model problem.

In Section 2, we prove the existence of a radial least energy solution of (1.1) when

$$2 < p < \tilde{p} \quad \text{and} \quad 2b - \left(1 + \tfrac{p}{2}\right) a < (N-1)(p-2). \tag{1.2}$$

In Section 3, we establish the existence of nonradial solutions. In Section 4, we derive necessary conditions for the existence of a solution of (1.1).

2. Radial solution

We introduce some spaces adapted to the problem. We denote by $H^1_r(\mathbf{R}^N)$, $H^1_{r,a}(\mathbf{R}^N)$ and $\mathcal{D}^{1,2}_r(\mathbf{R}^N)$ the spaces of radially symmetric functions respectively in

$$\begin{aligned}
H^1(\mathbf{R}^N) &= \left\{u \in L^2(\mathbf{R}^N) : \nabla u \in L^2(\mathbf{R}^N)\right\}, \\
H^1_a(\mathbf{R}^N) &= \left\{u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} |x|^a u^2 dx < \infty\right\}, \\
\mathcal{D}^{1,2}(\mathbf{R}^N) &= \left\{u \in L^{2^*}(\mathbf{R}^N) : \nabla u \in L^2(\mathbf{R}^N)\right\}.
\end{aligned}$$

Note that $H^1(\mathbf{R}^N) \subset \mathcal{D}^{1,2}(\mathbf{R}^N)$.

We define the space $H^1_0(\Omega)$ (resp. $\mathcal{D}^{1,2}_0(\Omega)$) as the closure of $\mathcal{D}(\Omega)$ with respect to the norm $(||\nabla u||_2^2 + ||u||_2^2)^{\frac{1}{2}}$ (resp. $||\nabla u||_2$). If $|\Omega| < \infty$ we have $H^1_0(\Omega) = \mathcal{D}^{1,2}_0(\Omega)$, see [59, Remarks 1.11].

The following radial lemma is an improvement of the Strauss radial Lemma (see [55, Lemma 1]).

Lemma 2.1. *If $N \geqslant 2$, there exists $A_N > 0$ such that, for every $u \in H^1_{r,a}(\mathbf{R}^N)$, we have $u \in \mathcal{C}(\mathbf{R}^N \backslash \{0\})$ and*

$$|x|^{\frac{N-1}{2} + \frac{a}{4}} |u(x)| \leqslant A_N \left(\int_{\mathbf{R}^N} |x|^a |u|^2 dx\right)^{\frac{1}{4}} ||\nabla u||_2^{\frac{1}{2}}.$$

Proof. By density, it suffices to consider $u \in H^1_{r,a}(\mathbf{R}^N) \cap \mathcal{D}(\mathbf{R}^N)$. Since

$$2\, u \frac{du}{dr}\, r^{\frac{a}{2}}\, r^{N-1} \leqslant \frac{d}{dr}\left(u^2\, r^{\frac{a}{2}}\, r^{N-1}\right),$$

we obtain

$$\begin{aligned} r^{N-1} r^{\frac{a}{2}} u^2(r) &\leqslant 2 \int_r^{\infty} |u| \left|\tfrac{du}{dr}\right| s^{N-1} s^{\frac{a}{2}} ds, \\ &\leqslant A_N \left(\int_{\mathbf{R}^N} |x|^a u^2 \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^N} |\nabla u|^2 \right)^{\frac{1}{2}} \end{aligned}$$

and the lemma is proved. □

The following inequality is due to Rother (see [44, Lemma 2]).

Lemma 2.2 (Rother). *If $N \geqslant 3$, $1 \leqslant p < \infty$ and $p = 2^* + \frac{2c}{N-2}$, then there exists $B_{N,c} > 0$ such that, for every $u \in \mathcal{D}_r^{1,2}(\mathbf{R}^N)$,*

$$\left(\int_{\mathbf{R}^N} |x|^c |u|^p dx \right)^{\frac{2}{p}} \leqslant B_{N,c} \int_{\mathbf{R}^N} |\nabla u|^2 dx.$$

We shall prove that, under some conditions,

$$m = m(a,b,p) := \inf_{\substack{u \in H^1_{r,a}(\mathbf{R}^N) \\ \int_{\mathbf{R}^N} |x|^b |u|^p dx = 1}} \int_{\mathbf{R}^N} |\nabla u|^2 + |x|^a u^2 dx$$

is achieved. Then by the Lagrange Multiplier Rule, the Symmetric Criticality Principle and the Maximum Principle, we obtain a solution of

$$\begin{cases} -\Delta v + |x|^a v = \lambda |x|^b v^{p-1}, \\ v \in H^1_{r,a}(\mathbf{R}^N), \quad v > 0. \end{cases}$$

Hence $u = \lambda^{\frac{1}{p-2}} v$ is a solution of (1.1).

Theorem 2.3. *If $a, b \geqslant 0$, $N \geqslant 3$,*

$$2 < p < \tilde{p} = 2^* + \tfrac{2b}{N-2} \quad \textit{and} \quad 2b - \left(1 + \tfrac{p}{2}\right) a < (N-1)(p-2)$$

then $m(a,b,p)$ is achieved and problem (1.1) has a radial solution.

Proof. Let $(u_n) \subset H^1_{r,a}(\mathbf{R}^N)$ be a minimizing sequence for $m = m(a,b,p)$, i.e.

$$\begin{aligned} \int_{\mathbf{R}^N} |x|^b |u_n|^p dx &= 1, \\ \int_{\mathbf{R}^N} |\nabla u_n|^2 + |x|^a u_n^2 dx &\to m. \end{aligned}$$

Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H^1_{r,a}(\mathbf{R}^N)$. Hence, by weak lower semicontinuity, we have

$$\begin{aligned} \int_{\mathbf{R}^N} |\nabla u|^2 + |x|^a u^2 dx &\leqslant m, \\ \int_{\mathbf{R}^N} |x|^b |u|^p dx &\leqslant 1. \end{aligned}$$

If c is defined by $p = 2^* + \frac{2c}{N-2}$, then $c < b$ and it follows from Lemma 2.2 that

$$\begin{aligned} \int_{|x|\leqslant\varepsilon} |x|^b |u_n|^p dx &\leqslant \varepsilon^{b-c} \int_{\mathbf{R}^N} |x|^c |u_n|^p dx, \\ &\leqslant C\varepsilon^{b-c}, \end{aligned}$$

since (u_n) is bounded in $H^1_{r,a}(\mathbf{R}^N)$. We deduce from Lemma 2.1 that

$$\begin{aligned} \int_{|x|\geqslant\frac{1}{\varepsilon}} |x|^b |u_n|^p dx &= \int_{|x|\geqslant\frac{1}{\varepsilon}} |x|^{b-a} \, |u_n|^{p-2} \, |x|^a \, u_n^2 \, dx, \\ &\leqslant \left(\tfrac{1}{\varepsilon}\right)^{b-a-(p-2)\left(\frac{(N-1)}{2}+\frac{a}{4}\right)} C \int_{\mathbf{R}^N} |x|^a \, u_n^2 \, dx, \\ &\leqslant C \, \varepsilon^{a\left(\frac{1}{2}+\frac{p}{4}\right)-b+\frac{(p-2)(N-1)}{2}}. \end{aligned}$$

It follows from the two preceding inequalities that, for every $t < 1$, there exists $\varepsilon > 0$ such that, for every n,

$$\int_{\varepsilon\leqslant|x|\leqslant\frac{1}{\varepsilon}} |x|^b |u_n|^p dx \geqslant t.$$

By the Rellich Theorem and Lemma 2.1,

$$1 \geqslant \int_{\mathbf{R}^N} |x|^b |u|^p dx \geqslant \int_{\varepsilon\leqslant|x|\leqslant\frac{1}{\varepsilon}} |x|^b |u|^p dx \geqslant t.$$

Finally $\int_{\mathbf{R}^N} |x|^b |u|^p dx = 1$ and $m = m(a, b, p)$ is achieved by u. $\square$

We now consider

$$M = M(a, b, p) := \inf_{\substack{u\in H^1_a(\mathbf{R}^N) \\ \int_{\mathbf{R}^N} |x|^b |u|^p dx = 1}} \int_{\mathbf{R}^N} |\nabla u|^2 + |x|^a u^2 dx.$$

Clearly $M \leqslant m$. We shall prove that M is achieved under the condition used by Sirakov.

Theorem 2.4. *If $a > 0$, $b \geqslant 0$, $N \geqslant 3$ and*

$$2 < p < p^{\#} = 2^* - \tfrac{4b}{a(N-2)},$$

then $M(a,b,p)$ is achieved.

Proof. Let $(u_n) \subset H_a^1(\mathbf{R}^N)$ be a minimizing sequence for $M = M(a,b,p)$, i.e.

$$\begin{aligned}\int_{\mathbf{R}^N} |x|^b |u_n|^p dx &= 1,\\ \int_{\mathbf{R}^N} |\nabla u_n|^2 + |x|^a u_n^2 dx &\to M.\end{aligned}$$

By going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H_a^1(\mathbf{R}^N)$. Hence, by weak lower semi-continuity we have

$$\begin{aligned}\int_{\mathbf{R}^N} |\nabla u|^2 + |x|^a u^2 dx &\leqslant M,\\ \int_{\mathbf{R}^N} |x|^b |u|^p dx &\leqslant 1.\end{aligned}$$

If c is defined by $p = 2^* - \frac{4c}{a(N-2)}$, then $c > b$ and the exponents

$$r = \tfrac{a}{c}, \quad s = \tfrac{a\,2^*}{ap-2c}$$

are conjugate. It follows from Hölder and Sobolev inequalities that

$$\begin{aligned}\int_{|x|\geqslant\frac{1}{\varepsilon}} |x|^b |u_n|^p dx &\leqslant \left(\tfrac{1}{\varepsilon}\right)^{b-c} \int_{\mathbf{R}^N} |x|^c |u_n|^p dx,\\ &\leqslant \varepsilon^{c-b} \left(\int_{\mathbf{R}^N} |x|^a u_n^2 dx\right)^{\frac{1}{r}} \left(\int_{\mathbf{R}^N} |u_n|^{2^*} dx\right)^{\frac{1}{s}},\\ &\leqslant C\varepsilon^{c-b}.\end{aligned}$$

As in Theorem 2.3, for every $t < 1$, there exists $\varepsilon > 0$ such that, for every n,

$$\int_{|x|\leqslant\frac{1}{\varepsilon}} |x|^b |u_n|^p dx \geqslant t.$$

By the Rellich Theorem, since $p < 2^*$,

$$1 \geqslant \int_{\mathbf{R}^N} |x|^b |u|^p dx \geqslant \int_{|x|\leqslant\frac{1}{\varepsilon}} |x|^b |u|^p \, dx \geqslant t.$$

Hence $\int_{\mathbf{R}^N} |x|^b |u|^p dx = 1$ and $M = M(a,b,p)$ is achieved at u. □

3. Nonradial solutions

We use the preceding results in order to construct nonradial solutions of

$$\left\{\begin{array}{rcll} -\Delta u + |x|^a u & = & |x|^b u^{p-1} & \text{in } B(0,R), \\ u & > & 0 & \text{in } B(0,R), \\ u & = & 0 & \text{on } \partial B(0,R) \end{array}\right. \tag{3.1}$$

when R is "large".

Theorem 3.1. *If $a, b \geqslant 0$, $N \geqslant 3$, $2 < p < 2^*$, $ap < 2b$ and*

$$2b - \left(1 + \tfrac{p}{2}\right) a < (N-1)(p-2),$$

then, for every R large enough, problem (3.1) *has a radial solution and a nonradial one.*

Proof. By Theorem 2.3, $m(a,b,p)$ is positive. Since $2b > ap$, it is easy to verify that $M(a,b,p) = 0$. Let us define

$$\begin{aligned} M(a,b,p,R) &:= \inf_{\substack{u \in H_0^1(B(0,R)) \\ \int_{B(0,R)} |x|^b |u|^p dx = 1}} \int_{B(0,R)} |\nabla u|^2 + |x|^a u^2 dx \\ m(a,b,p,R) &:= \inf_{\substack{u \in H_{0,rad}^1(B(0,R)) \\ \int_{B(0,R)} |x|^b |u|^p dx = 1}} \int_{B(0,R)} |\nabla u|^2 + |x|^a u^2 dx \end{aligned}$$

Clearly, $M(a,b,p,R)$ and $m(a,b,p,R)$ are achieved for every $R > 0$, and

$$\lim_{R \to \infty} m(a,b,p,R) = m(a,b,p) > 0.$$

Moreover, we have

$$\lim_{R \to \infty} M(a,b,p,R) = M(a,b,p) = 0, \tag{3.2}$$

which is equivalent to

$$\inf_{\substack{u \in H_0^1(B(0,R)) \\ u \neq 0}} \frac{\int_{\mathbf{R}^N} |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_{\mathbf{R}^N} |x|^b |u|^p dx\right)^{\frac{2}{p}}} = 0.$$

To obtain the latter equality, consider a test function $u \in \mathcal{D}(B(0,1))$ and its translations $u_\lambda(x) := u(x - (\lambda, 0, \ldots, 0))$. Simple computations lead to

$$\begin{aligned}
\int_{\mathbf{R}^N} |\nabla u_\lambda|^2 dx &= \int_{\mathbf{R}^N} |\nabla u|^2 dx, \\
\int_{\mathbf{R}^N} |x|^a u_\lambda^2 dx &\leqslant (\lambda+1)^a \int_{\mathbf{R}^N} u^2 dx, \\
\int_{\mathbf{R}^N} |x|^b |u_\lambda|^p dx &\geqslant (\lambda-1)^b \int_{\mathbf{R}^N} |u|^p dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\inf_{\substack{u\in H_0^1(B(0,R))\\ u\neq 0}} \frac{\int_{\mathbf{R}^N} |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_{\mathbf{R}^N} |x|^b |u|^p dx\right)^{\frac{2}{p}}} &\leqslant \frac{\int_{\mathbf{R}^N} |\nabla u_\lambda|^2 + |x|^a u_\lambda^2 dx}{\left(\int_{\mathbf{R}^N} |x|^b |u_\lambda|^p dx\right)^{\frac{2}{p}}}, \\
&\leqslant \frac{(\lambda+1)^a \int_{\mathbf{R}^N} |\nabla u|^2 + u^2 dx}{(\lambda-1)^{b\frac{2}{p}} \left(\int_{\mathbf{R}^N} |u|^p dx\right)^{\frac{2}{p}}},
\end{aligned}$$

which tends to 0 when λ tends to $+\infty$ provided $ap < 2b$, implying (3.2).

Using the Lagrange Multiplier Rule, the Symmetric Criticality Principle and the Maximum Principle as in Section 2, we obtain, for every R large enough, a radial and a non radial solution of (3.1). □

4. Necessary conditions

We derive from the Derrick-Pohozaev identity some necessary conditions for the existence of solutions of problem (1.1).

Theorem 4.1. *Let $a, b \geqslant 0$, if*

$$p \geqslant \tilde{p} := 2^* + \tfrac{2b}{N-2} \tag{4.1}$$

or if

$$p \leqslant 2 + 2\tfrac{b-a}{N+a} \tag{4.2}$$

then there is no non-trivial solution for problem (1.1).

Proof. By contradiction, assume $u \neq 0$ is a solution of problem (1.1). Then u satisfies the following Derrick-Pohozaev identity:

$$\tfrac{N-2}{2} \int_{\mathbf{R}^N} |\nabla u|^2 dx + \tfrac{N+a}{2} \int_{\mathbf{R}^N} |x|^a u^2 dx - \tfrac{N+b}{p} \int_{\mathbf{R}^N} |x|^b u^p dx = 0.$$

On the other hand, multiplying (1.1) by u and integrating, we see that

$$\int_{\mathbf{R}^N} |x|^b u^p dx = \int_{\mathbf{R}^N} |\nabla u|^2 + |x|^a u^2 dx.$$

Hence we obtain

$$\left(\tfrac{N-2}{2}-\tfrac{N+b}{p}\right)\int_{\mathbf{R}^N}|\nabla u|^2dx+\left(\tfrac{N+a}{2}-\tfrac{N+b}{p}\right)\int_{\mathbf{R}^N}|x|^a u^2dx=0.$$

So, we must have

$$\tfrac{N-2}{2}-\tfrac{N+b}{p}<0 \quad\text{and}\quad \tfrac{N+a}{2}-\tfrac{N+b}{p}>0.$$

This is equivalent to

$$p\ <\ \tfrac{2N}{N-2}+\tfrac{2b}{N-2}\quad\text{and}\quad p\ >\ 2+2\tfrac{b-a}{N+a},$$

which is a contradiction. □

Remark 4.2.

(a) The first assumption of Theorem 2.3, $2<p<\tilde{p}$ is optimal.

(b) The second assumption of Theorem 2.3,

$$2b-\left(1+\tfrac{p}{2}\right)a<(N-1)(p-2),$$

implies that $p>2+2\frac{b-a}{N+a}$.

(c) The following example shows that $m(a,b,p)>0$ implies

$$2b-ap\leqslant(N-1)(p-2).$$

This example is inspired by [21, Example 5.9].

Example 4.3. We assume that

$$2b-ap>(N-1)(p-2).$$

For each integer $n>0$, we define $r=|x|$ and

$$u_n(r)=\begin{cases}\frac{1}{n^{\frac{N-1}{2}+\frac{a}{2}}}e^{\frac{r-n}{2}}, & 0\leqslant r<n,\\ \frac{1}{r^{\frac{N-1}{2}+\frac{a}{2}}}, & n\leqslant r\leqslant n+1,\\ \frac{1}{(n+1)^{\frac{N-1}{2}+\frac{a}{2}}}e^{\frac{n+1-r}{2}}, & n+1<r.\end{cases}$$

These (u_n) belong to $H^1_{r,a}(\mathbf{R}^N)$. Moreover, we verify that

$$\begin{aligned}\int_{\mathbf{R}^N}|x|^b|u_n|^pdx\ &\geqslant\ \int_n^{n+1}r^{N-1}r^br^{-(\frac{N-1}{2}+\frac{a}{2})p}dr\\ &\geqslant\ n^{\frac{1}{2}(2b-ap-(N-1)(p-2))}\\ &\to\ +\infty\quad\text{as } n\to\infty.\end{aligned}$$

On the other hand, it is easy to verify that there exists a constant C such that, for every n,

$$\int_{\mathbf{R}^N}|\nabla u_n|^2+|x|^a u_n^2dx\leqslant C.$$

Hence

$$\frac{\int_{\mathbf{R}^N} |\nabla u_n|^2 + |x|^a u_n^2 dx}{\left(\int_{\mathbf{R}^N} |x|^b |u_n|^p dx\right)^{\frac{2}{p}}} \to 0 \text{ as } n \to \infty,$$

and thus $m(a,b,p) = 0$ when $2b - ap > (N-1)(p-2)$.

CHAPTER II

Symmetry of solutions of a semilinear elliptic equation with coefficients

1. Introduction

Consider the problem

$$(1.1) \qquad \begin{cases} -\Delta u + g(x)|u|^{q-2}u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω denotes $\mathbf{R}^N$ or the unit ball $B(0,1)$. We are interested in the existence of radial solutions of (1.1). As explained in Chapter I, many papers deal with the case $\Omega = \mathbf{R}^N$, $q = 2$, g large at infinity and f superlinear, subcritical and bounded in x (see e.g. [4, 39, 43]). The case f unbounded in x was studied by Sirakov [51] and the case $\Omega = B(0,1)$, $g(x) = 0$ and $f(x,u) = |x|^b u^{p-1}$ was studied by Smets-Su-Willem [54]. In the latter work, it is proved that under some hypothesis the ground states are not radially symmetric. For an introduction to symmetry-breaking results, see the survey of Brezis [10].

Here below, we consider the model problem

$$(1.2) \qquad \begin{cases} -\Delta u + |x|^a|u|^{q-2}u &= |x|^b|u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

First, we study the general case (i.e. $q > 1$) with $\Omega = \mathbf{R}^N$. In Section 2.1, using the Mountain-Pass Theorem, we prove the existence of a positive radial solution of (1.2) when

$$\max(2,q) < p < 2^* + \tfrac{2b}{N-2}, \qquad (b-a) < \left(\tfrac{a+2(N-1)}{q+2}\right)(p-q).$$

In Section 2.2, using a symmetric Mountain-Pass Theorem, we prove that there exist infinitely many radial solutions of (1.2). In Section 2.3, we obtain a necessary condition for the existence of solutions of (1.2), using a Pohozaev-type equality.

Sections 3 and 4 are devoted to ground states for p close to the limit values 2 and 2^* in the quadratic case $q = 2$ and with $\Omega = B(0,1)$.

2. The general problem

2.1. *Existence of a radial solution*

We consider the problem

$$(2.1)\qquad \begin{cases} -\Delta u + |x|^a|u|^{q-2}u &= |x|^b|u|^{p-2}u \qquad \text{on } \mathbf{R}^N, \\ u &> 0, \end{cases}$$

with $q > 1$, $a, b > -N$ and $u(x)$ radially symmetric (i.e. $u(x) = u(|x|)$) such that

$$(2.2)\qquad \int_{\mathbf{R}^N} \frac{|\nabla u|^2}{2} + |x|^a \frac{|u|^q}{q} dx < \infty.$$

The functional associated with this problem is

$$(2.3)\qquad \Phi(u) := \int_{\mathbf{R}^N} \left(\frac{|\nabla u|^2}{2} + |x|^a \frac{|u|^q}{q} - |x|^b \frac{|u|^p}{p} \right) dx.$$

We use the following norm:

$$||u||_X := ||\nabla u||_2 + \left\| |x|^{\frac{a}{q}} u \right\|_q,$$

and we define X as the completion of $\mathcal{D}(\mathbf{R}^N)$ for this norm.

We also define the space

$$Y := \left\{ u \,:\, u \text{ is measurable on } \mathbf{R}^N \text{ and } \int_{\mathbf{R}^N} |x|^b |u|^p dx < \infty \right\}.$$

with the norm $||u||_Y := \left\| |x|^{\frac{b}{p}} u \right\|_p$. We denote by X_r and Y_r the spaces of radially symmetric functions in X and Y respectively, and we prove that $X_r \subset Y_r$ and that the embedding is compact.

First, we recall an inequality due to Rother [44, Lemma 2].

Lemma 2.1 (Rother). *If $N \geqslant 3$, $1 \leqslant p < \infty$ with $p = 2^* + \frac{2c}{N-2}$, there exists $B_{N,c} > 0$ such that, for every $u \in \mathcal{D}^{1,2}_r(\mathbf{R}^N)$,*

$$\left(\int_{\mathbf{R}^N} |x|^c |u|^p dx \right)^{\frac{2}{p}} \leqslant B_{N,c} \int_{\mathbf{R}^N} |\nabla u|^2 dx.$$

The following radial lemma is an improvement of the Strauss radial Lemma (see [55, Lemma 1]). We obtain it in a similar way as Lemma I.2.1.

Lemma 2.2. *If $N \geqslant 3$, there exists $C_{N,q} > 0$ such that, for every $u \in X_r$,*

$$|u(x)| \leqslant C_{N,q} \; |||x|^{\frac{a}{q}} u||_q^{\frac{q}{q+2}} \; ||\nabla u||_2^{\frac{2}{q+2}} \; |x|^{-\frac{a+2(N-1)}{q+2}} \qquad a.e. \text{ on } \mathbf{R}^N.$$

Proof. By density, it suffices to consider $u \in X_r \cap \mathcal{D}(\mathbf{R}^N)$. Since

$$(\tfrac{q}{2}+1)\ |u|^{\frac{q}{2}}\ \tfrac{d|u|}{dr}\ r^{\frac{a}{2}}\ r^{N-1}\ \leqslant\ \frac{d}{dr}\left(|u|^{\frac{q}{2}+1}\ r^{\frac{a}{2}}\ r^{N-1}\right),$$

we obtain

$$\begin{aligned} r^{N-1}\ r^{\frac{a}{2}}\ |u(r)|^{\frac{q}{2}+1} &\leqslant (\tfrac{q}{2}+1)\int_r^{\infty} s^{\frac{a}{2}}\,|u|^{\frac{q}{2}}\ \left|\tfrac{du}{dr}\right|\ s^{N-1}ds,\\ &\leqslant C_{N,q}\left(\int_{\mathbf{R}^N}|x|^a|u|^q\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}^N}|\nabla u|^2\right)^{\frac{1}{2}} \end{aligned}$$

and the lemma is proved. □

We can now prove the compactness of the embedding $X_r \subset Y_r$.

Lemma 2.3. *If* $2 < p < 2^* + \frac{2b}{N-2}$ *and if* $(b-a) < (\frac{a+2(N-1)}{q+2})(p-q)$, *then* $X_r \subset Y_r$ *and the embedding is compact.*

Proof. Assume that $u_n \rightharpoonup 0$ in X_r. We must prove $||u_n||_Y \to 0$. Note first that the sequence $||u_n||_X$ is bounded. We divide the domain $\mathbf{R}^N$ in three parts: $|x| \leqslant \varepsilon$, $\varepsilon \leqslant |x| \leqslant \frac{1}{\varepsilon}$ and $|x| \geqslant \frac{1}{\varepsilon}$. Accordingly,

$$||u_n||_Y^p =$$

$$\int_{|x|\leqslant\varepsilon}|x|^b|u_n|^p dx+\int_{\varepsilon\leqslant|x|\leqslant\frac{1}{\varepsilon}}|x|^b|u_n|^p dx+\int_{|x|\geqslant\frac{1}{\varepsilon}}|x|^b|u_n|^p dx. \tag{2.4}$$

We prove that each summand tends to 0 when n tends to $+\infty$. To do that, we tackle them separately using appropriate tools.

First summand. We define c by $p = 2^* + \frac{2c}{N-2}$. Lemma 2.1 gives us the following result, knowing that $||\nabla u_n||_2$ is bounded (see Chapter I, page 30):

$$\int_{|x|\leqslant\varepsilon}|x|^b|u_n|^p dx \leqslant C\,\varepsilon^{b-c}.$$

The hypothesis $p < 2^* + \frac{2b}{N-2}$ entails that the first summand of (2.4) tends to 0 uniformly in n as $\varepsilon \to 0$.

Third summand. We clearly have

$$\int_{|x|\geqslant\frac{1}{\varepsilon}}|x|^b|u_n|^p dx = \int_{|x|\geqslant\frac{1}{\varepsilon}}|x|^{b-a}\,|u_n|^{p-q}\,|x|^a\,|u_n|^q\,dx.$$

Using Lemma 2.2, we obtain

$$\int_{|x|\geqslant\frac{1}{\varepsilon}} |x|^b|u_n|^p dx \leqslant C\int_{|x|\geqslant\frac{1}{\varepsilon}} |x|^{b-a-(\frac{a+2(N-1)}{q+2})(p-q)}|x|^a|u_n|^q dx,$$
$$\leqslant C\left(\tfrac{1}{\varepsilon}\right)^{b-a-(\frac{a+2(N-1)}{q+2})(p-q)}.$$

As $(b-a) < (\frac{a+2(N-1)}{q+2})(p-q)$, this term tends to 0 uniformly in n as $\varepsilon \to 0$.

Second summand. We must prove that $\int_{\Omega_\varepsilon} |x|^b|u_n|^p dx \to 0$ as $n\to\infty$ with $\Omega_\varepsilon := \left\{x\in \mathbf{R}^N : \varepsilon \leqslant |x| \leqslant \frac{1}{\varepsilon}\right\}$. As Ω_ε is bounded, there exists $C\in\mathbf{R}$ such that

$$||u_n||_{H^1(\Omega_\varepsilon)} \leqslant C||u_n||_X.$$

As $u_n \rightharpoonup 0$ in X, we obtain $u_n \rightharpoonup 0$ in $H^1(\Omega_\varepsilon)$. Using the fact that the embedding $H^1(\Omega_\varepsilon)\subset L^2(\Omega_\varepsilon)$ is compact, we obtain

$$u_n \to 0 \text{ in } L^2(\Omega_\varepsilon). \tag{2.5}$$

We now compute, using Lemma 2.2,

$$\int_{\Omega_\varepsilon} |x|^b|u_n|^p dx \leqslant C||u_n||_\infty^{p-2}\int_{\Omega_\varepsilon}|u_n|^2dx \leqslant C\int_{\Omega_\varepsilon}|u_n|^2dx,$$

which tends to 0 by (2.5). □

We will use the following variant of the Mountain-Pass Theorem (see [5, Theorem 10]).

Lemma 2.4 (Brezis-Nirenberg). *Let E a Banach space, $\Phi\in\mathcal{C}^1(E,\mathbf{R})$ which verify $\Phi(0)=0$ and the following conditions:*

- [i] *there exists $r>0$ such that $\inf_{||u||=r}\Phi(u)>0$;*
- [ii] *there exists v with $||v||>r$ such that $\Phi(v)<0$;*
- [iii] *the Palais-Smale condition: if $\Phi'(u_n)\to 0$ and if $\Phi(u_n)\to c$ then (u_n) has a subsequence converging to $u\in X$;*
- [iv] *there exists a continuous mapping $P:E\to E$ such that $P(0)=0$, $P(v)=v$ and $\Phi(P(u))\leqslant\Phi(u)$ for every $u\in E$.*

Then there exists a critical point $u^*\in\overline{P(E)}$ such that

$$\Phi(u^*)\geqslant \inf_{||u||=r}\Phi(u).$$

We now prove the main result of this section.

Theorem 2.5. *If $N\geqslant 3$, $q>1$, $\max(2,q)<p<2^*+\frac{2b}{N-2}$ and $(b-a)<(\frac{a+2(N-1)}{q+2})(p-q)$, then problem (2.1) has a radial solution.*

Proof. Consider the functional Φ restricted to X_r. We shall verify the assumptions of Lemma 2.4. First, we verify the geometry of the Mountain-Pass Theorem.

Condition [i]. We must find $r > 0$ such that $\inf_{||u||=r} \Phi(u) > 0$.

Suppose that $p > q \geqslant 2$ and take $||u||_X \ll 1$. By Lemma 2.3,

$$\begin{aligned}\Phi(u) &= \tfrac{1}{2} \|\nabla u\|_2^2 + \tfrac{1}{q} \left\| |x|^{\frac{a}{q}} u \right\|_q^q - \tfrac{1}{p} \|u\|_Y^p , \\ &\geqslant \tfrac{1}{2} \|\nabla u\|_2^q + \tfrac{1}{q} \left\| |x|^{\frac{a}{q}} u \right\|_q^q - C \|u\|_X^p .\end{aligned}$$

As $||\nabla u||_2 < 1$, we obtain

$$\Phi(u) \geqslant \tfrac{1}{q} \left(\|\nabla u\|_2^q + \left\| |x|^{\frac{a}{q}} u \right\|_q^q \right) - C \|u\|_X^p .$$

However $\left(\frac{a+b}{2}\right)^q \leqslant \max(a^q, b^q) \leqslant a^q + b^q$, so

$$\begin{aligned}\Phi(u) &\geqslant \frac{1}{q\, 2^q} \left(\|\nabla u\|_2 + \left\| |x|^{\frac{a}{q}} u \right\|_q \right)^q - C||u||_X^p, \\ &= C'||u||_X^q - C||u||_X^p.\end{aligned}$$

As $p > q$, we have $\Phi(u) > 0$ for $||u||_X = r$ small enough.

On the other hand, if $p > 2 \geqslant q$ a similar argument is valid. In this case we obtain

$$\Phi(u) \geqslant C'||u||_X^2 - C||u||_X^p.$$

Condition [ii]. Let $w > 0$ be fixed, we consider $v := tw$, with $t \in \mathbf{R}$, so

$$\begin{aligned}\Phi(v) &= \tfrac{t^2}{2} \|\nabla w\|_2^2 + \tfrac{t^q}{q} \left\| |x|^{\frac{a}{q}} w \right\|_q^q - \tfrac{t^p}{p} \left\| |x|^{\frac{b}{p}} w \right\|_p^p , \\ &\leqslant C\, t^2 + C'\, t^q - C''\, t^p.\end{aligned}$$

Hence, there exists t such that $\Phi(v) < 0$ and $||v|| > r$.

Condition [iii]*, the Palais-Smale condition.* Let $(u_n) \subset X_r$ such that $\Phi'(u_n) \to 0$ and $\Phi(u_n) \to c$; we must prove that $u_n \to u \in X_r$. First, we prove that the sequence (u_n) is bounded.

For n large enough, $\Phi(u_n) \leqslant c+1$ and $||\Phi'(u_n)|| \leqslant p$. So we obtain

$$\begin{aligned} c+1+||u_n||_X &\geqslant \Phi(u_n) - \frac{1}{p}\langle \Phi'(u_n), u_n\rangle, \\ &= \left(\tfrac{1}{2}-\tfrac{1}{p}\right)\|\nabla u_n\|_2^2 + \left(\tfrac{1}{q}-\tfrac{1}{p}\right)\left\||x|^{\frac{a}{q}}u_n\right\|_q^q, \\ &\geqslant m\left(\left(||\nabla u_n||_2 + \left\||x|^{\frac{a}{q}}u_n\right\|_q\right)^{\min(2,q)} - 1\right), \end{aligned}$$

where $m = \min(\frac{1}{2}-\frac{1}{p}, \frac{1}{q}-\frac{1}{p})$. Since $q>1$, the sequence (u_n) is bounded. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in X_r.

We now prove that $u_n \to u \in X_r$.

As $(u_n - u) \rightharpoonup 0$ in X_r, we have $\langle \Phi'(u), u_n - u\rangle \to 0$. On the other hand, as $\Phi'(u_n) \to 0$ in X_r', we also have $\langle \Phi'(u_n), u_n - u\rangle \to 0$. In consequence, we obtain

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0.$$

Expanding $\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle$, we get

$$\begin{aligned} &||\nabla(u_n-u)||_2^2 + \int |x|^a(u_n^{q-1} - u^{q-1})(u_n-u)dx \\ &\qquad = \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle + \int |x|^b(u_n^{p-1} - u^{p-1})(u_n-u)dx, \\ &\qquad \leqslant o(1) + o(1). \end{aligned}$$

So,

$$||\nabla(u_n-u)||_2^2 \to 0 \qquad \text{and} \qquad \int |x|^a(u_n^{q-1} - u^{q-1})(u_n-u)dx \to 0.$$

A computation similar to [57, Lemma 4.3] shows that this implies $\int |x|^a u_n^q \to \int |x|^a u^q$. Hence $u_n \to u$ in X_r.

Condition [iv]. To obtain a positive solution, we choose the mapping $u \mapsto |u|$ as P in Lemma 2.4. This mapping is continuous, $\Phi(P(u)) = \Phi(u)$ for all $u \in X_r$, $P(0)=0$ and, taking $v>0$ in [ii], it is clear that $P(v)=v$.

All the conditions [i]–[iv] are fulfilled. So, using Lemma 2.4, we obtain a nontrivial critical point $u \geqslant 0$ of Φ restricted to X_r. By the Maximum Principle we obtain $u>0$ on $\mathbf{R}^N$, and the Symmetric Criticality Principle (see e.g. [59, Theorem 1.28]) ends the proof. □

2.2. *Infinity of radial solutions*

Consider the analogous problem

$$
\left\{
\begin{array}{rcll}
-\Delta u + |x|^a |u|^{q-2} u & = & |x|^b |u|^{p-2} u & \text{on } \mathbf{R}^N, \\
\int_{\mathbf{R}^N} \frac{|\nabla u|^2}{2} + |x|^a \frac{|u|^q}{q} dx & < & \infty, & \\
u(x) & = & u(|x|). &
\end{array}
\right.
\tag{2.6}
$$

Here we take radial functions, so the embedding $X_r \subset Y_r$ is compact.

We use the same norm $||.||_X$ and the same functional Φ, restricted to X_r, as in Section 2.1. We will apply the following result, a symmetric version of the Mountain-Pass Theorem (see Rabinowitz [42, Theorem 9.12]).

Lemma 2.6 (Rabinowitz). *Let E be an infinite-dimensional Banach space and let $\Phi \in \mathcal{C}^1(E, \mathbf{R})$ be even, satisfy the Palais-Smale condition and $\Phi(0) = 0$. If $E = V \oplus W$, where V is finite dimensional and Φ satisfies*

(A_1) *there exists $r > 0$ such that*

$$\inf_{\substack{u \in W, \\ ||u||_E = r.}} \Phi(u) > 0;$$

(A_2) *for each finite dimensional subspace $F \subset E$, there exists ρ_F such that*

$$\max_{\substack{u \in F, \\ ||u||_E = \rho_F.}} \Phi(u) < 0;$$

then Φ possesses an unbounded sequence of critical values.

Theorem 2.7. *If $N \geqslant 3$, $q > 1$, $\max(2, q) < p < 2^* + \frac{2b}{N-2}$ and if $(b - a) < (\frac{a+2(N-1)}{q+2})(p - q)$, then problem (2.6) has a sequence of radial solutions (u_k) such that $\Phi(u_k) \to \infty$ as $k \to \infty$.*

Proof. We verify the hypotheses of Lemma 2.6 with $E = X_r$, $V = 0$ and $W = X_r$. First, note that X_r is an infinite-dimensional Banach space and that $\Phi \in \mathcal{C}^1(E, \mathbf{R})$ is even, vanishes for $u = 0$ and satisfies the Palais-Smale condition.

Condition (A_1) was already verified in the proof of Theorem 2.5. In order to verify condition (A_2), we find an upper bound of Φ:

$$
\begin{array}{rcl}
\Phi(u) & = & \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{q} \left\| |x|^{\frac{a}{q}} u \right\|_q^q - \frac{1}{p} \left\| |x|^{\frac{b}{p}} u \right\|_p^p, \\
& \leqslant & ||u||_X^2 + ||u||_X^q - \frac{1}{p} ||u||_Y^p.
\end{array}
$$

Since all norms are equivalent on the finite-dimensional space F, relation (A_2) is satisfied for every $\rho_F > 0$ large enough.

So, by Lemma 2.6, the functional Φ has an unbounded sequence of critical values on X_r. The Symmetric Criticality Principle concludes the proof. $\square$

2.3. *Non-existence of solutions*

We deduce a non-existence result of solutions for problem (2.1) thanks to a Pohozaev-type identity. The difficulty is that the domain is unbounded, so we use a truncation argument (see e.g. [28, Chapter 6]).

Lemma 2.8. *Let $u \in X$ be a solution of* (2.1). *Then u satisfies*

$$\tfrac{N-2}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx + \tfrac{N+a}{q}\int_{\mathbf{R}^N} |x|^a|u|^q\,dx - \tfrac{N+b}{p}\int_{\mathbf{R}^N} |x|^b|u|^p\,dx = 0.$$

Proof. Let $\psi \in \mathcal{D}(\mathbf{R})$ such that $0 \leqslant \psi \leqslant 1$, $\psi(r) = 1$ for $r \leqslant 1$ and $\psi(r) = 0$ for $r \geqslant 2$. We define on $\mathbf{R}^N$ the functions $\psi_n(x) := \psi(\frac{|x|^2}{n^2})$. There exists $c \geqslant 0$ such that, for every $n \in \mathbf{N}$:

$$\psi_n \leqslant c \qquad \text{and} \qquad |x|\,|\nabla\psi_n(x)| \leqslant c.$$

It follows from (2.1) that, for every n

$$0 = \big(\Delta u - |x|^a|u|^{q-1} + |x|^b|u|^{p-1}\big)\psi_n x\cdot\nabla u. \tag{2.7}$$

In order to simplify notations, let

$$\begin{aligned} f(x,u) &:= -|x|^a|u|^{q-1} + |x|^b|u|^{p-1},\\ F(x,u) &:= -|x|^a\tfrac{|u|^q}{q} + |x|^b\tfrac{|u|^p}{p}. \end{aligned}$$

We have

$$\begin{aligned} f(x,u)\psi_n x\cdot\nabla u &= \operatorname{div}\left(F(x,u)\psi_n x\right) - N\psi F(x,u) - F(x,u)x\cdot\nabla\psi_n\\ &\quad - \left(-\tfrac{a}{q}|x|^a|u|^q + \tfrac{b}{p}|x|^b|u|^p\right)\psi_n,\\ \psi_n\,\Delta u\,x\cdot\nabla u &= \operatorname{div}\left(\psi_n\left(\nabla u x\cdot\nabla u - x\tfrac{|\nabla u|^2}{2}\right)\right) + \tfrac{N-2}{2}\psi_n|\nabla u|^2\\ &\quad + \tfrac{|\nabla u|^2}{2}x\cdot\nabla\psi_n - x\cdot\nabla u\nabla\psi_n\cdot\nabla u. \end{aligned}$$

Using these relations in (2.7) and integrating, we obtain, for every $n \in \mathbf{N}$,

$$\begin{aligned} \int_{\mathbf{R}^N} \Big(\Big[NF(x,u) - \tfrac{a}{q}|x|^a|u|^q + \tfrac{b}{p}|x|^b|u|^p - \tfrac{N-2}{2}|\nabla u|^2\Big]\psi_n&\\ + F(x,u)x\cdot\nabla\psi_n - \tfrac{|\nabla u|^2}{2}x\cdot\nabla\psi_n + x\cdot\nabla u\nabla\psi_n\cdot\nabla u\Big)\,dx &= 0. \end{aligned}$$

The Lebesgue Dominated-Convergence Theorem implies that

$$\int_{\mathbf{R}^N} \left(NF(x,u) - \tfrac{a}{q}|x|^a|u|^q + \tfrac{b}{p}|x|^b|u|^p - \tfrac{N-2}{2}|\nabla u|^2 \right) dx = 0,$$

which concludes the proof. □

We can now prove the non existence of non trivial solution result.

Theorem 2.9. *If*

$$p < 2^* + \tfrac{2b}{N-2} \qquad \textit{and} \qquad p(N+a) \leqslant q(N+b)$$

or if

$$p > 2^* + \tfrac{2b}{N-2} \qquad \textit{and} \qquad p(N+a) \geqslant q(N+b)$$

then the trivial solution $u = 0$ is the unique solution of problem (2.1)*.*

Proof. By contradiction, assume $u \neq 0$ is a nontrivial solution of (2.1). We have then

$$\int_{\mathbf{R}^N} |\nabla u|^2 dx + \int_{\mathbf{R}^N} |x|^a|u|^q dx = \int_{\mathbf{R}^N} |x|^b|u|^p dx.$$

Using Lemma 2.8, we obtain

$$\left(\tfrac{N-2}{2} - \tfrac{N+b}{p}\right) \int_{\mathbf{R}^N} |\nabla u|^2 dx + \left(\tfrac{N+a}{q} - \tfrac{N+b}{p}\right) \int_{\mathbf{R}^N} |x|^a|u|^q dx = 0.$$

As the two integrals are positive, their coefficients must be of opposite sign. Since there is no predefined order relation between these coefficients, we consider two cases.

Case 1 when $\frac{N+a}{q} - \frac{N+b}{p} > 0 > \frac{N-2}{2} - \frac{N+b}{p}$: we obtain

$$p < 2^* + \tfrac{2b}{N-2} \qquad \text{and} \qquad p(N+a) > q(N+b),$$

which is a contradiction.

Case 2 when $\frac{N-2}{2} - \frac{N+b}{p} > 0 > \frac{N+a}{q} - \frac{N+b}{p}$: we obtain

$$p > 2^* + \tfrac{2b}{N-2} \qquad \text{and} \qquad p(N+a) < q(N+b),$$

which is a contradiction.

□

3. Quadratic problem on $B(0,1)$ when p is close to 2

Consider

$$\begin{cases} -\Delta u + |x|^a u = |x|^b u^{p-1}, \\ u \in H_0^1(B(0,1)). \end{cases} \tag{3.1}$$

Here below we observe there exists a limiting exponent b^* above which the corresponding infimum cannot be radial. Our goal is to study the asymptotic comportment of b^* when $p \to 2$.

We first define the global and the radial infima corresponding to the problem:

$$S_{a,b,p} := \inf_{\substack{u\in X\\ u\neq 0}} R(u) \qquad \text{and} \qquad S^R_{a,b,p} := \inf_{\substack{u\in X_r\\ u\neq 0}} R(u),$$

where $R(u)$ is the associated Rayleigh quotient

$$R(u) := \frac{\int_{B(0,1)} |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_{B(0,1)} |x|^b u^p dx\right)^{\frac{2}{p}}}.$$

Let $(-\Delta)^{-1}$ be the inverse of the Laplace operator with Dirichlet boundary conditions. Equation (3.1) can be written, with $\lambda = 1$,

$$\begin{cases} (-\Delta)^{-1}\big(\lambda |x|^b u^{p-1} - |x|^a u\big) - u = 0, \\ u \in H_0^1(B(0,1)). \end{cases} \tag{3.2}$$

We define the operator

$$P : [2, 2^*[\times \mathbf{R}^+ \times \mathbf{R}^+ \times H_0^1(B(0,1)) \times \mathbf{R} \quad \longrightarrow \quad H_0^1(B(0,1)) \times \mathbf{R},$$

with

$$P(p,a,b,u,\lambda) = \left((-\Delta)^{-1}\left(\lambda |x|^b u^{p-1} - |x|^a u\right) - u \,,\, ||u||^2 - 1\right),$$

where

$$||u||^2 := \int_{B(0,1)} |\nabla u|^2 dx.$$

In this section, S_0 denotes $S_{a_0,b_0,2}$ and u_0 is a minimizer of S_0 such that $\int_{B(0,1)} |x|^b u_0^p dx = 1$. We first prove the unicity of u_0.

Lemma 3.1. *Let $a_0, b_0 > 0$ fixed. The kernel of the first component of $P(2, a_0, b_0, \cdot, S_0)$, namely*

$$(-\Delta)^{-1}\big(S_0 |x|^{b_0} I - |x|^{a_0} I\big) - I,$$

is the set of multipliers of u_0. Moreover, u_0 is radial.

Proof. We begin proving the simplicity of the first eigenvalue of problem

$$-\Delta u + |x|^a u = \lambda |x|^b u.$$

Consider u a corresponding eigenfunction. So $|u|$ is also an eigenfunction, as well as $u^+ := \frac{u+|u|}{2}$. Hence we have

$$-\Delta u^+ + |x|^a u^+ = \lambda_1 |x|^b u^+ \geqslant 0.$$

By the Maximum Principle, we have $u^+ > 0$ a.e., so $u = u^+$. Accordingly, every eigenfunction associated to the first eigenvalue is positive or negative, which implies simplicity. Hence the kernel of the first component of $P(2, a_0, b_0, \cdot, S_0)$ is the set of multipliers of u_0.

If u is a solution, the compositions $u \circ g_\theta$ of u by every rotation g_θ are obviously also solutions. So u has to be radial. This implies that u_0 is radial. □

Let a_0, b_0 fixed and p close to 2. Using the Implicit Function Theorem, we prove that the (u, λ) for which P vanishes are functions of (a, b, p). This means that the non-trivial solutions of (3.1) are function of (a, b, p).

Lemma 3.2. *Let $a_0, b_0 > 0$ fixed, then there exist $\varepsilon > 0$ and functions*

$$\Lambda : [2, 2+\varepsilon[\times]a_0-\varepsilon, a_0+\varepsilon[\times]b_0-\varepsilon, b_0+\varepsilon[\;\to\;]S_0-\varepsilon, S_0+\varepsilon[,$$
$$U : [2, 2+\varepsilon[\times]a_0-\varepsilon, a_0+\varepsilon[\times]b_0-\varepsilon, b_0+\varepsilon[\;\to\; B(u_0, \varepsilon) \subset H_0^1(B(0,1)),$$

such that in
$[2, 2+\varepsilon[\times]a_0-\varepsilon, a_0+\varepsilon[\times]b_0-\varepsilon, b_0+\varepsilon[\times B(u_0,\varepsilon)\times]S_0-\varepsilon, S_0+\varepsilon[$
we have

$$P(p, a, b, u, \lambda) = 0 \qquad \Leftrightarrow \qquad \begin{cases} u = U(p, a, b) \\ \lambda = \Lambda(p, a, b). \end{cases}$$

Proof. It suffices to prove that the partial derivative ∂P with respect to (u, λ) at $(2, a_0, b_0, u_0, S_0)$ is a homeomorphism of $H_0^1(B(0,1)) \times \mathbf{R}$.

We have

$$\begin{aligned} &\langle \partial P(2, a_0, b_0, u_0, S_0), (v, t)\rangle \\ &\quad = \left((-\Delta)^{-1}\left(S_0|x|^{b_0}v - |x|^{a_0}v\right) - v + t(-\Delta)^{-1}\left(|x|^{b_0}u_0\right), 2\langle u_0, v\rangle\right). \end{aligned}$$

So, if $\langle \partial P(2, a_0, b_0, u_0, S_0), (v, t)\rangle = 0$, then

$$\begin{cases} t &= 0, \\ (-\Delta)^{-1}\left(S_0|x|^{b_0}v - |x|^{a_0}v\right) - v &= 0, \\ \langle u_0, v\rangle &= 0. \end{cases}$$

But, by Lemma 3.1, the kernel of $(-\Delta)^{-1}\big(S_0|x|^{b_0}I - |x|^{a_0}I\big) - I$ is the set of multipliers of u_0. Hence

$$\langle \partial P(2, a_0, b_0, u_0, S_0), (v,t)\rangle = 0 \quad \Leftrightarrow \quad (v,t) = (0,0).$$

Let $(w,s) \in H_0^1(B(0,1)) \times \mathbf{R}$. We must prove that in this case there exists $(v,t) \in H_0^1(B(0,1)) \times \mathbf{R}$ such that

$$\langle \partial P(2, a_0, b_0, u_0, S_0), (v,t)\rangle = (w,s). \tag{3.3}$$

We define a new scalar product on $H_0^1(B(0,1))$:

$$((u,v)) := \int_{B(0,1)} \nabla u \nabla v + |x|^{a_0} uv\, dx$$

and we consider the functional

$$\begin{aligned} H_0^1(B(0,1)) &\to \mathbf{R}, \\ v &\mapsto \int_{B(0,1)} |x|^{b_0} uv\, dx. \end{aligned}$$

As this functional is continuous and by virtue of the Riesz Representation Theorem, it can be represented by a scalar product:

$$((Au,v)) = \int_{B(0,1)} |x|^{b_0} uv\, dx.$$

So the weak form of problem (3.2) can be written

$$((u,v)) = \lambda((Au,v)).$$

The operator A is self-adjoint and compact. So, by the elementary spectral theorem (see e.g. [9, Théorème VI.11]), there exists a sequence of eigenvalues $\lambda_1, \lambda_2, \ldots$ and an orthonormal basis $e_1, e_2, \ldots$ of the space $Z := \big(H_0^1(B(0,1)), ((.,.))\big)$ such that

$$e_n - \lambda_n\, A e_n = 0$$

for every $n \in \mathbf{N}$. As u_0 is the solution of (3.2) with $\lambda = S_0$ and $p = 2$, we can suppose without loss of generality that $e_1 = u_0$ and $\lambda_1 = S_0$. Let

$$\begin{aligned} L : Z &\to Z, \\ u &\mapsto u - \lambda_1\, Au, \end{aligned}$$

we have $KerL = e_1 = u_0$. Since the operator L is self-adjoint, we have $Im(L) = (KerL)^\perp = \langle u_0\rangle^\perp$ $\big($orthogonality with respect to the scalar product $((.,.))\big)$.

Let t such that $\Delta w + t|x|^{b_0}u_0$ is orthogonal to u_0 with respect to the scalar product $((.,.))$. In this case, there exists $\tilde{v}$ satisfying $L\tilde{v} =$

$\Delta w + t|x|^{b_0}u_0$. In order to verify relation (3.3), let $v := \tilde{v} + v_1$ with $v_1 \in KerL$. So (v,t) verifies the first part of relation (3.3) and, taking the suitable v_1, also the second. □

We now prove the following result.

Theorem 3.3. *Let a_0 fixed, then for every $n \in \mathbf{N}$ there exists $\delta_n > 0$ such that the unique minimizer of $S_{a_0,b,p}$ is radial for $b \leqslant n$ and $p \leqslant 2+\delta_n$.*

This means that for p close to 2, there exists a limiting exponent b^* such that, for every $b < b^*$, we have $S_{a_0,b,p} = S^R_{a_0,b,p}$. Moreover, b^* tends to $+\infty$ when p tends to 2. Note it is possible that b^* is infinite for $p > 2$, which would imply the radiality of solutions for *every b*.

Proof. By contradiction, assume that there exists $n \in \mathbf{N}$, a sequence $(\delta_k) \to 0_+$ and a sequence $(b_k) \subset [0,n]$ such that a minimizer of $S_{a_0,b_k,2+\delta_k}$ is non radial for every k, i.e. such that $S_{a_0,b_k,2+\delta_k} < S^R_{a_0,b_k,2+\delta_k}$.

Note that, without loss of generality, $b_k \to b_\infty$ as $b_\infty \in [0,n]$. Moreover, if we denote by u_k the minimizer of $S_{a_0,b_k,2+\delta_k}$, we have $u_k \rightharpoonup u_\infty$ in $H^1_0(B(0,1))$. Also without loss of generality, we suppose that $\int_{B(0,1)} |x|^{b_k} u_k^{2+\delta_k} dx = 1$.

We now prove that $S_{a_0,b_\infty,2} = \lim_{k\to\infty} S_{a_0,b_k,2+\delta_k}$. To this end, we prove the following relations

$$S_{a_0,b_\infty,2} \leqslant \varliminf_{k\to\infty} S_{a_0,b_k,2+\delta_k}, \tag{3.4}$$

$$S_{a_0,b_\infty,2} \geqslant \varlimsup_{k\to\infty} S_{a_0,b_k,2+\delta_k}. \tag{3.5}$$

To obtain inequality (3.4), we begin by proving $\int_{B(0,1)} |x|^{b_\infty} u_\infty^2 \geqslant 1$.

With $\sigma := N\left(\frac{1}{2+\delta_k} - \frac{1}{2^*}\right)$, we obtain the inequality

$$\int_{B(0,1)} |x|^{b_k} u_k^{2+\delta_k} dx \leqslant \left(\int_{B(0,1)} |x|^{b_k} u_k^2 dx\right)^{\frac{2+\delta_k}{2}\sigma} \left(\int_{B(0,1)} u_k^{2^*} dx\right)^{\frac{2+\delta_k}{2^*}(1-\sigma)}.$$

The left member equals to 1 for every k, so

$$1 \leqslant \left(\int_{B(0,1)} |x|^{b_k} u_k^2 dx\right)^{\frac{2+\delta_k}{2}\sigma} \left(\int_{B(0,1)} u_k^{2^*} dx\right)^{\frac{2+\delta_k}{2^*}(1-\sigma)}. \tag{3.6}$$

When $k \to \infty$, the second term converges to 1: the integral is uniformly bounded and the exponent tends to 0. On the other hand, the first term converges to $\int_{B(0,1)} |x|^{b_\infty} u_\infty^2 dx$. Indeed,

$$\begin{aligned}
\int_{B(0,1)} &\left(|x|^{b_k} u_k^2 - |x|^{b_\infty} u_\infty^2\right) dx \\
&\leqslant \int_{B(0,1)} \left(|x|^{b_k} u_k^2 - |x|^{b_k} u_\infty^2\right) dx + \int_{B(0,1)} \left||x|^{b_k} - |x|^{b_\infty}\right| u_\infty^2 dx, \\
&\leqslant \int_{B(0,1)} |x|^{b_k} \left(u_k^2 - u_\infty^2\right) dx + \int_{B(0,1)} \left||x|^{b_k} - |x|^{b_\infty}\right| u_\infty^2 dx,
\end{aligned}$$

the latter two terms converge to 0: the first by Rellich Theorem and the second by the Lebesgue Dominated Convergence Theorem. So equation (3.6) boils down to

$$1 \leqslant \int_{B(0,1)} |x|^{b_\infty} u_\infty^2 dx.$$

This entails

$$\begin{aligned}
S_{a_0,b_\infty,2} &\leqslant \frac{\int_{B(0,1)} |\nabla u_\infty|^2 + |x|^a u_\infty^2 dx}{\int_{B(0,1)} |x|^{b_\infty} u_\infty^2 dx}, \\
&\leqslant \int_{B(0,1)} |\nabla u_\infty|^2 + |x|^a u_\infty^2 dx.
\end{aligned}$$

So, by weak lower semi-continuity, we obtain inequality (3.4):

$$S_{a_0,b_\infty,2} \leqslant \varliminf_{k\to\infty} S_{a_0,b_k,2+\delta_k}.$$

To obtain relation (3.5), we prove the upper semi-continuity of $S_{a,b,p}$ with respect to (a,b,p):

If $(a_n, b_n, p_n) \to (a,b,p)$, then $\varlimsup_{n\to\infty} S_{a_n,b_n,p_n} \leqslant S_{a,b,p}$.

Let us recall that $S_{a,b,p} := \inf_{\substack{u\in X \\ u\neq 0}} R(u)$. As the Rayleigh quotient $R(u)$ is continuous (hence upper semi-continuous) with respect to (a,b,p), the infimum $S_{a,b,p}$ will also be upper semi-continuous, implying (3.5). This is dual to the corresponding result for lower semi-continuity, which is more popular:

> If $(R(u))_{u\in X}$ is a set of lower semi-continuous functions, then $\sup_{u\in X} R(u)$ is lower semi-continuous.

So it is proved that

$$S_{a_0,b_\infty,2} = \lim_{k\to\infty} S_{a_0,b_k,2+\delta_k}.$$

By uniform convexity, $u_k \to u_\infty \in H_0^1(B(0,1))$. This limit function u_∞ correspond to u_0 in Lemma 3.2. Hence, when $2+\delta_k$ is sufficiently close to 2 the solution u_k is unique. So u_k is radial, which is a contradiction. □

4. Quadratic problem on $B(0,1)$ when p is close to 2^*

Here, we study the limiting exponent introduced in Section 3 when p tends to the critical Sobolev exponent 2^*.

The following inequality is due to Ni [38].

Lemma 4.1 (Ni). *If $N \geqslant 3$, there exists $C > 0$ such that for every $u \in \mathcal{D}_r^{1,2}(\mathbf{R}^N)$*

$$|x|^{\frac{N-2}{2}}|u(x)| \leqslant C||\nabla u||_2 \qquad a.e.\ in\ \mathbf{R}^N.$$

This inequality gives us the following upper bound:

$$\begin{aligned} \int_{B(0,1)} |x|^b|u(r)|^p dr &\leqslant C||\nabla u||_2^p \int_0^1 r^{b+N-1+p\frac{2-N}{2}}dr, \\ &= C||\nabla u||_2^p \left(\tfrac{2-N}{2}p + b + N\right)^{-1}. \end{aligned}$$

On the other hand, it is obvious that

$$||\nabla u||_2^2 \leqslant \int |\nabla u|^2 + |x|^a u^2 dx.$$

So we get

$$\frac{\int_{B(0,1)} |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_{B(0,1)} |x|^b u^p dx\right)^{2/p}} \geqslant C\left(\tfrac{2-N}{2}p + b + N\right)^{2/p} \geqslant Cb^{2/p}$$

as $\frac{2-N}{2}p + N > 0$. And we obtain the following lemma.

Lemma 4.2. *If $N \geqslant 3$, then there exists $C > 0$ such that for every $p \in [2, 2^*]$ and $b \geqslant 0$, we have*

$$Cb^{2/p} \leqslant S_{a,b,p}^R.$$

Now, we obtain a strict inequality between $S_{a,b,2^*}$ and $S_{a,b,2^*}^R$.

Lemma 4.3. *If $N \geqslant 3$, $a \geqslant 0$ and $b > 0$, then*

$$S_{a,b,2^*} < S_{a,b,2^*}^R.$$

Proof. We begin proving that the infimum $S_{a,b,2^*}$ equals

$$S := \inf_{u \in H_0^1(B(0,1))} Q(u) \qquad \text{where} \qquad Q(u) := \frac{\int_{B(0,1)} |\nabla u|^2}{\left(\int_{B(0,1)} u^{2^*}\right)^{2/2^*}}.$$

The first step is to show that $S_{a,b,2^*} \leqslant S$. Let $(v_n) \subset H_0^1(B(0,1))$ a sequence such that $\frac{||v_n||_2^2}{||v_n||_{2^*}^2} \to S$ when $n \to \infty$ and $\operatorname{supp} v_n \subset B(0, \frac{1}{n})$. For the existence of such a sequence, see e.g. [56, Section I.4.5].

For $y \in B(0,1)$, we consider

$$u_n(x) := \begin{cases} v_n(x) & \text{if } \operatorname{dist}(y, \partial B(0,1)) \leqslant \frac{1}{n}, \\ v_n(x-y) & \text{if } \operatorname{dist}(y, \partial B(0,1)) > \frac{1}{n}. \end{cases}$$

For n sufficiently large, namely $\frac{1}{n} < \operatorname{dist}(y, \partial B(0,1))$, we have:

$$\begin{aligned} \int_{B(0,1)} |x|^b |u_n|^{2^*} dx &= \int_{B(y,\frac{1}{n})} |x|^b |u_n|^{2^*} dx, \\ &\geqslant \int_{B(y,\frac{1}{n})} (|y| - \tfrac{1}{n})^b \, |u_n|^{2^*} dx, \\ &\geqslant (|y| - \tfrac{1}{n})^b \, ||u_n||_{2^*}^{2^*}. \end{aligned}$$

On the other hand, it is obvious that

$$\int_{B(0,1)} |x|^a |u_n|^2 dx \leqslant ||u_n||_2^2.$$

and $||u_n||_2 \to 0$ as supp $u_n \subset B(y, \frac{1}{n})$ for n sufficiently large, and $||\nabla u||_2$ is bounded.

Hence

$$\begin{aligned} S_{a,b,2^*} &\leqslant \lim_{n\to\infty} \frac{\int_{B(0,1)} |\nabla u_n|^2 + |x|^a |u_n|^2 dx}{\left(\int_{B(0,1)} |x|^b |u_n(x)|^{2^*} dx\right)^{2/2^*}}, \\ &\leqslant \lim_{n\to\infty} |y|^{-b\frac{2}{2^*}} \frac{\int_{B(0,1)} |\nabla u_n|^2 dx}{\left(\int_{B(0,1)} u_n^{2^*} dx\right)^{2/2^*}}, \\ &= S. \end{aligned}$$

The converse inequality $S \leqslant S_{a,b,2^*}$ easily follows from the fact that, for every u, we have the two inequalities:

$$\begin{aligned} \int_{B(0,1)} |\nabla u|^2 dx &\leqslant \int_{B(0,1)} |\nabla u|^2 + |x|^a |u|^2 dx, \\ \int_{B(0,1)} |u|^{2^*} dx &\geqslant \int_{B(0,1)} |x|^b |u|^{2^*} dx. \end{aligned}$$

On the other hand, it is easy to see that $S^R_{a,b,2^*}$ is achieved: it suffices to prove that there is no loss of compactness at the origin, as in Chapter I.

So we conclude $S_{a,b,2^*} = S < S^R_{a,b,2^*}$. $\square$

Using Lemmas 4.2 and 4.3, we prove the main result of this section.

Theorem 4.4. *Let $N \geqslant 3$, $a \geqslant 0$ and $b > 0$, for every $n \in \mathbf{N}$ there exists $\delta > 0$ such that if $b \geqslant \frac{1}{n}$ and $2^* - \delta_n < p < 2^*$, then*

$$S_{a,b,p} < S^R_{a,b,p}.$$

Proof. For the sake of contradiction, we suppose there exists $n \in \mathbf{N}$, $(b_k) \geqslant \frac{1}{n}$ and $(\delta_k) \to 0$ such that

$$S_{a,b_k,2^*-\delta_k} = S^R_{a,b_k,2^*-\delta_k}. \tag{4.1}$$

First, we prove there exists $C > 0$ such that for every $p \in [2, 2^*]$ and $b \to +\infty$ we have $S_{a,b,p} \leqslant C b^{2-N+\frac{2N}{p}}$. Consider test function $u \in \mathcal{D}(B(0,1))$ that we *crash* on the boundary on the following way:

$$u_\lambda(x) := u(\lambda(x - x_\lambda))$$

with $x_\lambda = (1 - \frac{1}{\lambda}, 0, \dots, 0)$ and $\lambda \to \infty$. So

$$\begin{aligned}
\int_{B(0,1)} |\nabla u_\lambda|^2 dx &= \lambda^{2-N} \int_{B(0,1)} |\nabla u|^2 dx, \\
\int_{B(0,1)} |x|^a u_\lambda^2(x) dx &\leqslant \lambda^{-N} \int_{B(0,1)} |u(x)|^2 dx, \\
\int_{B(0,1)} |x|^b u_\lambda^p(x) dx &\geqslant \left(1 - \tfrac{2}{\lambda}\right)^b \lambda^{-N} \int_{B(0,1)} |u|^p dx.
\end{aligned}$$

Hence,

$$S_{a,b,p} \leqslant \frac{\lambda^{2-N} \int_{B(0,1)} |\nabla u|^2 dx + \lambda^{-N} \int_{B(0,1)} |u|^2 dx}{\left(\left(1 - \frac{2}{\lambda}\right)^b \lambda^{-N} \int_{B(0,1)} |u|^p dx\right)^{\frac{2}{p}}}.$$

Letting $\lambda := b$, this implies

$$S_{a,b,p} \leqslant C\, b^{2-N+\frac{2N}{p}} \qquad \text{when} \qquad b \to +\infty. \tag{4.2}$$

On the other hand, Lemma 4.2 implies there exists $C > 0$ such that

$$C\, b^{\frac{2}{p}} \leqslant S^R_{a,b,p} \qquad \text{when} \qquad b \to +\infty. \tag{4.3}$$

Relations (4.2) and (4.3) imply that the sequence (b_k) is bounded. Without loss of generality, we can suppose that this sequence converges: $b_k \to b_\infty \geqslant \frac{1}{n}$. In this case, as in Section 3, we have

$$S^R_{a,b_\infty,2^*} = \lim_{k\to\infty} S^R_{a,b_k,2^*-\delta_k}. \tag{4.4}$$

On the other hand, by upper semi-continuity, we have

$$S_{a,b_\infty,2^*} \geqslant \varlimsup_{k\to\infty} S_{a,b_k,2^*-\delta_k}. \tag{4.5}$$

Using (4.5), (4.1) and (4.4) we obtain

$$S_{a,b_\infty,2^*} \;\geqslant\; \varlimsup_{k\to\infty} S_{a,b_k,2^*-\delta_k} \;=\; \varlimsup_{k\to\infty} S^R_{a,b_k,2^*-\delta_k} \;=\; S^R_{a,b_\infty,2^*}.$$

So $S_{a,b_\infty,2^*} \geqslant S^R_{a,b_\infty,2^*}$, which contradicts Lemma 4.3. $\square$

CHAPTER III

Existence of many solutions of a semilinear elliptic equation with coefficients

1. Introduction

We investigate the existence of many nonradial solutions of a semilinear elliptic equation with coefficients in the unit disk or in an annulus in $\mathbf{R}^2$.

We consider the following problem:

$$\tag{1.1} \left\{ \begin{array}{rcll} -\Delta u + |x|^a u & = & |x|^b u^{p-1} & \text{in } \Omega, \\ u & > & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

with $a > 0$, $b > 0$, $p > 2$ and $\Omega \subset \mathbf{R}^2$.

The case $a = b = 0$ was already studied: it is the classical problem

$$\tag{1.2} \left\{ \begin{array}{rcll} -\Delta u + u & = & u^{p-1} & \text{in } \Omega, \\ u & > & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

with $\Omega \subset \mathbf{R}^N$ and $2 < p < 2^* = \frac{2N}{N-2}$. Gidas-Ni-Nirenberg [26] proved that with $\Omega = B(0,1)$, any smooth solution of (1.2) must be radially symmetric.

On the annulus, i.e. on

$$\tag{1.3} \Omega = D(R,d) = \left\{ x \in \mathbf{R}^N : R < |x| < R + d \right\},$$

this problem was also studied. If $N \geqslant 3$, Brezis-Nirenberg [11] proved that there exists a nonradial solution if p is closed to 2^* from below.

Coffman [16] obtained a better result under restrictive hypotheses: if $N = 2$ and $p > 2$ or if $N \geqslant 4$ even and $2 < p < \frac{2N-2}{N-2}$, then the number of rotationally nonequivalent nonradial solutions of (1.2) tends to $+\infty$ as R tends to $+\infty$. This result was extended by Li [32] to arbitrary $N \geqslant 4$ and by Byeon [13] to $N = 3$.

We study problem (1.1) using the ideas developed in the above works. It was first studied by Sirakov [51] for $\Omega = \mathbf{R}^N$. He obtained results about the existence and non-existence of solutions. These results are improved in Chapter I; where it is also proved that, on $B(0,R)$ under some conditions and for R large enough, problem (1.1) has a radial and a nonradial solution. In Chapter II, we proved that when $\Omega = \mathbf{R}^N$, problem (1.1) has a radial and a nonradial solution if b is sufficiently large.

Those results give us the following intuition: on the ball, the coefficient $|x|^b$ concentrates the energy of the solution near the boundary. So, if b is large, we can expect that the solution u of problem (1.1) with $\Omega = B(0,1)$ is small near the origin and large near the boundary. The situation is similar to the case of an annulus. For the ball, we obtain a result similar to Coffman's one: the number of rotationally nonequivalent nonradial solutions of (1.1) tends to $+\infty$ as b tends to $+\infty$. The proof of this result is given in Section 2. This proof only holds for the two-dimensional case: the general case seems more delicate and remains open.

In Section 3, we study this problem on a two-dimensional annulus and we obtain a similar result: the number of rotationally nonequivalent nonradial solutions of (1.1) tends to $+\infty$ under asymptotic condition on a, b, R and d.

2. Multibumps on the unit disk

We consider the problem

$$(2.1) \qquad \begin{cases} -\Delta u + |x|^a u &= |x|^b |u|^{p-2} u \quad \text{in } \Omega, \\ \qquad\qquad u &> 0 \qquad\qquad\quad \text{in } \Omega, \\ \qquad\qquad u &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases}$$

and we assume

$$(2.2) \qquad a, b > 0,\ p > 2 \text{ and } \Omega = B(0,1) \subset \mathbf{R}^2.$$

Since we work in $\mathbf{R}^2$, the embedding $H_0^1(B(0,1)) \subset L^p(B(0,1))$ is compact for every $p \geqslant 1$.

We use the following notations:

$$J(u) = \frac{\int_\Omega |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_\Omega |x|^b |u|^p dx\right)^{\frac{2}{p}}}$$

$$G_k = \Big\{ g \in \mathbf{O}(2) : g(x_1, x_2) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$(x_1, x_2) \in \mathbf{R}^2,\ \theta = \tfrac{2\pi l}{k},\ 0 \leqslant l \leqslant k-1 \Big\},$$

$$\begin{aligned}
\Sigma_k &= \left\{v \in H_0^1(\Omega) : v(x) = v(gx), g \in G_k\right\},\\
\Sigma_\infty &= \left\{v \in H_0^1(\Omega) : v(x) \text{ is a radial function}\right\},\\
j_k &= \inf\left\{J(v) : v \in \Sigma_k \backslash \{0\}\right\},\\
j_\infty &= \inf\left\{J(v) : v \in \Sigma_\infty \backslash \{0\}\right\}.
\end{aligned}$$

To prove our main result, we need three lemmas.

Lemma 2.1. *If $k \geqslant 2$ and $p > 2$ are fixed and if* (2.2) *is satisfied. Then there exists $C \in \mathbf{R}$ such that, for every $b > 0$,*

$$j_k \leqslant C\ k^{\frac{p}{2}+1}\left(1 - \tfrac{2\delta}{k}\right)^{-\frac{2b}{p}},$$

where $\delta = \min\left\{\frac{1}{10}, \frac{1}{2k}, \frac{1}{b}\right\}$.

Proof. We construct $v \in \Sigma_k \backslash \{0\}$ and estimate his Rayleigh coefficient $J(v)$. Let $\psi \in \mathcal{D}(B(0,\delta))$ and $v_0(x_1, x_2) = \psi\left(kx_1, k\left(x_2 - 1 + \frac{\delta}{k}\right)\right)$, so $v_0 \in \mathcal{D}\left(B\left(\left(0, 1 - \frac{\delta}{k}\right), \frac{\delta}{k}\right)\right)$.

Now, let $v(x) := \sum_{g \in G_k} v_0(gx)$. As δ is sufficiently small, we have

$$J(v) = \frac{k \int_\Omega |\nabla v_0|^2 + |x|^a v_0^2 dx}{k^{\frac{2}{p}}\left(\int_\Omega |x|^b |v_0|^p dx\right)^{\frac{2}{p}}}.$$

We look for an upper bound of this quantity. Using the fact that $|x|^a \leqslant 1$ on Ω, a translation, a change of variables and $\frac{1}{k^2} \leqslant 1$, we obtain

$$\begin{aligned}
\int_\Omega |\nabla v_0|^2 + |x|^a v_0^2 dx &\leqslant \int_\Omega |\nabla v_0|^2 + v_0^2 dx,\\
&= \int_\Omega k^2 |\nabla \psi(kx)|^2 + |\psi(kx)|^2 dx,\\
&\leqslant \int_\Omega \left(k^2 |\nabla \psi(y)|^2 + |\psi(y)|^2\right) \tfrac{1}{k^2}\, dy,\\
&\leqslant \int_\Omega |\nabla \psi|^2 + |\psi|^2 dx.
\end{aligned}$$

On the other hand, as $|x| \geqslant 1 - \frac{2\delta}{k}$ on $B\left(\left(0, 1 - \frac{\delta}{k}\right), \frac{\delta}{k}\right)$ and making a translation and a dilation:

$$\begin{aligned} \int_\Omega |x|^b v_0^p dx &= \int_{B\left(\left(0,1-\frac{\delta}{k}\right),\frac{\delta}{k}\right)} |x|^b |\psi(kx_1, k(x_2 - 1 + \tfrac{\delta}{k}))|^p dx, \\ &\geqslant \left(1 - \tfrac{2\delta}{k}\right)^b \int_{B(0,\frac{\delta}{k})} |\psi(kx)|^p dx, \\ &\geqslant \left(1 - \tfrac{2\delta}{k}\right)^b \tfrac{1}{k^2} \int_\Omega |\psi|^p dx. \end{aligned}$$

So we deduce the following inequality:

$$\begin{aligned} J(v) &\leqslant \frac{k \int_\Omega |\nabla\psi|^2 + |\psi|^2 dx}{k^{\frac{2}{p}} \frac{1}{k^{\frac{4}{p}}} \left(1 - \frac{2\delta}{k}\right)^{\frac{2b}{p}} \left(\int_\Omega |\psi|^p dx\right)^{\frac{2}{p}}}, \\ &= k^{1+\frac{2}{p}} \left(1 - \tfrac{2\delta}{k}\right)^{-\frac{2b}{p}} C. \end{aligned}$$

Hence the thesis, since $j_k \leqslant J(v)$ for all $v \in \Sigma_k \setminus \{0\}$. □

Lemma 2.2. *If p is fixed and (2.2) is satisfied, then there exists a constant $C \in \mathbf{R}$ such that, for every $b > 0$,*

$$j_\infty \geqslant \left(\frac{b+2}{2}\right)^{1+\frac{2}{p}} C.$$

Proof. First, note that j_∞ is achieved as Σ_∞ is closed in $H_0^1(\Omega)$, so there exists $u \in \Sigma_\infty \setminus \{0\}$ such that $J(u) = j_\infty$. We find a lower bound of the Rayleigh quotient $J(u)$ using the change of variables $v(|x|) = u(|x|^\beta)$, where $\beta = \frac{2}{b+2}$. So we obtain

$$\begin{aligned} \int_\Omega |x|^b |u(x)|^p dx &= \beta \int_\Omega |v(x)|^p dx, \\ \int_\Omega |\nabla u|^2 dx &= \frac{1}{\beta} \int_\Omega |\nabla v|^2 dx, \\ \int_\Omega |x|^a u^2 dx &= \beta \int_\Omega |x|^{\beta(a+2)-2} |v|^2 dx. \end{aligned}$$

Thus we have

$$\begin{aligned} J(u) &= \frac{\beta^{-1}\int_\Omega |\nabla v|^2 dx + \beta \int_\Omega |x|^{\beta(a+2)-2}|v|^2 dx}{\left(\beta \int_\Omega |v|^p dx\right)^{\frac{2}{p}}}, \\ &\geqslant \frac{\beta^{-1}\int_\Omega |\nabla v|^2 dx}{\beta^{\frac{2}{p}} \left(\int_\Omega |v|^p dx\right)^{\frac{2}{p}}}, \end{aligned}$$

implying the result. □

Lemma 2.3. *Let $k \in \mathbf{N}_0$, $m \in \mathbf{N}\backslash\{0,1\}$ and (2.2) satisfied. If $j_{km} < j_\infty$, then $j_k < j_{km}$.*

Proof. First, note that j_{km} is achieved by a function $u \in \Sigma_{km}\backslash\{0\}$. The action of G_{km} on $H_0^1(\Omega)$ is isometric and the functional J is invariant under the action of G_{km}. By the Symmetric Criticality Principle (see e.g. [59, Theorem 1.28]), we obtain u is a critical point of J on $H_0^1(\Omega)$. This implies that u is a solution of problem (2.1).

To prove the lemma, we construct a function $v \in \Sigma_k\backslash\{0\}$ such that $J(v) < J(u)$. First, we describe u in polar coordinates (r,θ), so

$$\int_\Omega |\nabla u|^2 + |x|^a u^2 dx = \iint \left(u_r^2 + \frac{1}{r^2}u_\theta^2 + r^a u^2\right) r\, dr\, d\theta,$$
$$\int_\Omega |x|^b |u|^p dx = \iint r^b |u|^p r\, dr\, d\theta.$$

Let $v(r,\theta) = u(r, \frac{\theta}{m})$; obviously $v \in \Sigma_k\backslash\{0\}$ and

$$\begin{aligned} v_r(r,\theta) &= u_r(r,\theta), \\ v_\theta(r,\theta) &= \tfrac{1}{m}\, u_\theta(r, \tfrac{\theta}{m}). \end{aligned}$$

Now, we compare the integrals of u and v:

$$\int_\Omega |\nabla v|^2 + |x|^a v^2 dx = \iint \left(u_r^2 + \frac{1}{m^2 r^2}u_\theta^2 + r^a u^2\right) r\, dr\, d\theta,$$
$$\int_\Omega |x|^b |v|^p dx = \int_\Omega |x|^b |u|^p dx.$$

As $j_{km} < j_\infty$, it is obvious that $u \notin \Sigma_\infty$, hence

$$\int u_\theta^2\, d\theta > 0,$$

and we obtain

$$\int_\Omega |\nabla v|^2 + |x|^a v^2 dx < \int_\Omega |\nabla u|^2 + |x|^a u^2 dx.$$

Using this relation, we complete the proof:

$$j_k \leqslant J(v) < J(u) = j_{km}.$$

□

Using these lemmas, we now prove the main result.

Theorem 2.4. *If a and p are fixed and if* (2.2) *is satisfied, then the number of rotationally nonequivalent nonradial solutions of the unit disk problem* (2.1) *tends to $+\infty$ as b tends to $+\infty$.*

Proof. Let $k \in \mathbf{N}_0$ fixed. By Lemma 2.1, we have

$$j_{2^k} \leqslant C\big(2^k\big)^{1+\frac{2}{p}}\left(1-\tfrac{2\delta}{2^k}\right)^{-\frac{2b}{p}}.$$

As $\lim_{b\to+\infty}\left(1-\frac{2\delta}{2^k}\right)^{-\frac{2b}{p}} = \exp\left(\frac{1}{2^{k-2}p}\right)$, there exists a constant $C \in \mathbf{R}$ such that

$$j_{2^k} \to C\,2^{k(1+\frac{2}{p})}\exp\left(\frac{1}{2^{k-2}p}\right) \qquad \text{as} \qquad b \to +\infty.$$

On the other hand, Lemma 2.2 implies that

$$j_\infty \to +\infty \qquad \text{as} \qquad b \to +\infty,$$

so, there exists some $b_0 > 0$ such that, for every $b \geqslant b_0$,

$$j_{2^k} < j_\infty.$$

Using Lemma 2.3, we get

$$j_2 < j_{2^2} < j_{2^3} < \ldots < j_{2^k} < j_\infty.$$

So, there exists a family of functions $v_i \in \Sigma_i$, $v_i \geqslant 0$ $(i = 1, \ldots, k)$ such that $J(v_i) = j_{2^i}$ and each v_i is a critical point of $J|_{\Sigma_i}$. By the Symmetric Criticality Principle (as in the proof of Lemma 2.3), each function v_i is also a critical point of J. Rescaling these functions, we obtain k nonradial and rotationally nonequivalent solutions of problem (2.1). As this is true for every $k \in \mathbf{N}_0$, the proof is complete. □

3. Multibumps on an annulus in $\mathbf{R}^2$

Consider now the same problem on the annulus in $\mathbf{R}^2$:

$$\left\{\begin{array}{rcll} -\Delta u + |x|^a u &=& |x|^b|u|^{p-2}u & \text{in } D, \\ u &=& 0 & \text{on } \partial D, \\ u &>& 0 & \text{in } D, \end{array}\right. \tag{3.1}$$

and assume

$$
\begin{aligned}
&p > 2,\ R, d > 0 \\
&D := D(R,d) \subset \mathbf{R}^2 \text{ is the annulus defined as in (1.3).}
\end{aligned}
\tag{3.2}
$$

We prove that the number of rotationally nonequivalent nonradial solutions of (3.1) tends to $+\infty$ under some conditions. The idea of the proof is similar to the one developed in Section 2.

We begin giving some technical lemmas.

Lemma 3.1. *Let k, p, b, d fixed and (3.2) satisfied, then there exists $C \in \mathbf{R}$ such that*

$$
j_k \leqslant C\, k^{1+\frac{2}{p}} \frac{\max\{R^a, (R+d)^a\}}{\min\{R^{\frac{2b}{p}}, (R+d)^{\frac{2b}{p}}\}}.
$$

Proof. We construct $v \in \Sigma_k$ and estimate $J(v)$. Let $\psi \in \mathcal{D}(B(0,\delta))$ where $\delta = \min\{\frac{d}{10}, \frac{R}{10}, 1\}$, let $v_0(x_1, x_2) := \psi\left(kx_1, k\left(x_2 - 1 - R + \frac{d}{2}\right)\right)$.

We define $v(x) := \sum_{g \in G_k} v_0(gx)$. Using similar arguments as in the proof of Lemma 2.1 and the fact that on the annulus Ω we have $|x|^a \leqslant \max\{R^a, (R+d)^a\}$ and $|x|^b \geqslant \min\{R^{\frac{2b}{p}}, (R+d)^{\frac{2b}{p}}\}$, we get

$$
J(v) \leqslant \frac{\max\{R^a, (R+d)^a\}}{\min\{R^{\frac{2b}{p}}, (R+d)^{\frac{2b}{p}}\}} k^{1+\frac{2}{p}} \frac{\int_{B(0,\delta)} |\nabla\psi|^2 + |\psi|^2}{\left(\int_{B(0,\delta)} |\psi|^p\right)^{\frac{2}{p}}},
$$

which ends the proof. □

Remark 3.2. In the proofs, we use particular cases of this lemma; the following elementary facts are important:

- if $a > 0$ then $\max\{R^a, (R+d)^a\} = (R+d)^a$;
- if $a < 0$ then $\max\{R^a, (R+d)^a\} = R^a$;
- if $b > 0$ then $\min\{R^{\frac{2b}{p}}, (R+d)^{\frac{2b}{p}}\} = R^{\frac{2b}{p}}$;
- if $b < 0$ then $\min\{R^{\frac{2b}{p}}, (R+d)^{\frac{2b}{p}}\} = (R+d)^{\frac{2b}{p}}$.

Lemma 3.3. *Suppose (3.2) is satisfied. There exists a constant $C \in \mathbf{R}$ such that if*

$$
2b < a\left(1 + \frac{p}{2}\right) + (p-2),
\tag{3.3a}
$$

then

$$
j_\infty \geqslant C\, R^{-\frac{1}{p}\left(2b - a(1+\frac{p}{2}) - (p-2)\right)}
$$

and if

$$2b > a\left(1+\frac{p}{2}\right)+(p-2), \tag{3.3b}$$

then

$$j_\infty \geqslant C\,(R+d)^{-\frac{1}{p}\left(2b-a(1+\frac{p}{2})-(p-2)\right)}.$$

Proof. Under our hypotheses, j_∞ is achieved by some $u \in \Sigma_\infty$. We find a lower bound of $J(u)$. Using hypothesis (3.3a) and Lemma 2.1, we obtain

$$\left(\int_D |x|^b u^p dx\right)^{\frac{2}{p}} \leqslant R^{\frac{1}{p}\left(2b-a(1+\frac{p}{2})-(p-2)\right)} ||\nabla u||^{\frac{p-2}{p}} \left(\int_D |x|^a u^2\right)^{\frac{p+2}{2p}}.$$

The case (3.3a) follows.

The case (3.3b) is similar. □

Lemma 3.4. *If* (3.2) *is satisfied and* $j_{km} < j_\infty$*, then* $j_k < j_{km}$.

Proof. It suffices to adapt the proof of Lemma 2.3. □

Now we can prove the main results of this section. The upper and lower bounds of j_k and j_∞ given by Lemmas 3.1 and 3.3 respectively depend on the sign of a, b and on the relations (3.3a) and (3.3b). As a consequence, we consider 6 different cases:

Case A: $a > 0$ and $b < 0$ (so (3.3a) is satisfied);
Case B: $a > 0$, $b > 0$ and (3.3a) is satisfied;
Case C: $a > 0$, $b > 0$ and (3.3b) is satisfied;
Case D: $a < 0$ and $b > 0$ (so (3.3b) is satisfied);
Case E: $a < 0$, $b < 0$ and (3.3b) is satisfied;
Case F: $a < 0$, $b < 0$ and (3.3a) is satisfied.

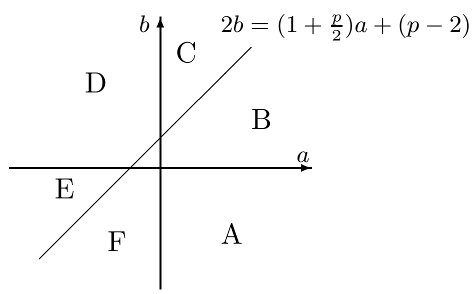

FIGURE 1. The six different cases in function of a and b.

The cases A, B and C are proven in Theorems 3.5, 3.6 and 3.7 respectively. We state the result for cases D, E and F in Theorem 3.8 without explicit proof as the arguments are similar.

Theorem 3.5. *Assume* (3.2)*,* $a > 0$ *and* $b < 0$ *with*

$$b = \frac{p}{2}\, h(a),$$

where $h \in \mathcal{F}(\mathbf{R}^+, \mathbf{R}^-)$*. If* R, d, p *are such that*

$$R^{\frac{1}{p}+\frac{1}{2}-\frac{h(a)}{a}} > (R+d)^{1-\frac{h(a)}{a}} \tag{3.4}$$

for a sufficiently large, then the number of rotationally nonequivalent nonradial solutions of problem (3.1) *tends to* $+\infty$ *as* a *tends to* $+\infty$*.*

Proof. The idea of the proof is similar to the one of Theorem 2.4. It suffices to show that, for k fixed, $j_{2^k} < j_\infty$, or equivalently that $\frac{j_{2^k}}{j_\infty} < 1$. As $a > 0$ and $b < 0$, it is obvious that inequality (3.3a) is satisfied. Using Lemmas 3.1 and 3.3, we obtain

$$\begin{aligned} j_{2^k} &\leqslant C(R+d)^{a-\frac{2b}{p}}, \\ j_\infty &\geqslant C R^{-\frac{2b}{p}+a(\frac{1}{p}+\frac{1}{2})+(1-\frac{2}{p})}. \end{aligned}$$

So we have

$$\begin{aligned} \frac{j_{2^k}}{j_\infty} &\leqslant C\,(R+d)^{a-\frac{2b}{p}}\, R^{\frac{2b}{p}-a(\frac{1}{p}+\frac{1}{2})}, \\ &\leqslant C\left(\frac{(R+d)^{1-\frac{h(a)}{a}}}{R^{\frac{1}{p}+\frac{1}{2}-\frac{h(a)}{a}}}\right)^{a}. \end{aligned}$$

Using inequality (3.4), this quantity tends to 0 as a tends to $+\infty$. We conclude the proof using Lemma 3.4, as in Theorem 2.4. ☐

Theorem 3.6. *Let* (3.2) *satisfied and* $a, b > 0$ *satisfying* (3.3a)*. If* R, d, p *are such that*

$$R^{\frac{1}{p}+\frac{1}{2}} > R+d, \tag{3.5}$$

then the number of rotationally nonequivalent nonradial solutions of problem (3.1) *tends to* $+\infty$ *as* a *tends to* $+\infty$*.*

Proof. As before, it suffices to prove that, for k fixed, $\frac{j_{2^k}}{j_\infty} < 1$. Using Lemmas 3.1 and 3.3, we obtain

$$\begin{aligned} j_{2^k} &\leqslant C(R+d)^a\, R^{-\frac{2b}{p}}, \\ j_\infty &\geqslant C R^{-\frac{2b}{p}+a(\frac{1}{p}+\frac{1}{2})+(1-\frac{2}{p})}. \end{aligned}$$

So we have

$$\frac{j_{2^k}}{j_\infty} \leqslant C\left(\frac{R+d}{R^{\frac{1}{p}+\frac{1}{2}}}\right)^a.$$

By (3.5) we see that $\frac{j_{2^k}}{j_\infty} < 1$ for a sufficiently large. The end of the proof is similar to that of Theorem 2.4. $\square$

Theorem 3.7. *Let* (3.2) *satisfied and* $a, b > 0$ *satisfying* (3.3b) *and with*

$$b = \frac{p}{2}\,h(a)$$

where $h \in \mathcal{F}(\mathbf{R}^+, \mathbf{R}^+)$. *If* R, d, p *are such that*

$$R^{\frac{h(a)}{a}} > (R+d)^{\frac{h(a)}{a}+(\frac{1}{2}-\frac{1}{p})} \tag{3.6}$$

for a *sufficiently large, then the number of rotationally nonequivalent nonradial solutions of problem* (3.1) *tends to* $+\infty$ *as* a *tends to* $+\infty$.

Proof. It suffices to prove that, for k fixed, $\frac{j_{2^k}}{j_\infty} < 1$. Using Lemmas 3.1 and 3.3, we obtain

$$\begin{aligned}
\frac{j_{2^k}}{j_\infty} &\leqslant C\,\frac{(R+d)^a R^{-\frac{2b}{p}}}{(R+d)^{-\frac{2b}{p}+a(\frac{1}{p}+\frac{1}{2})+(1-\frac{2}{p})}},\\
&= C\,\frac{(R+d)^{\frac{2b}{p}+a(\frac{1}{2}-\frac{1}{p})}}{R^{\frac{2b}{p}}},\\
&= C\left(\frac{(R+d)^{\frac{h(a)}{a}+(\frac{1}{2}-\frac{1}{p})}}{R^{\frac{h(a)}{a}}}\right)^a.
\end{aligned}$$

Now, by (3.6) we see that $\frac{j_{2^k}}{j_\infty} < 1$ for a sufficiently large. The end of the proof is similar to that of Theorem 2.4. $\square$

In the following theorem, we list similar results for $a < 0$, proved using the same kind of argument.

Theorem 3.8. *Suppose that* (3.2) *is satisfied.*

Let $a < 0$, $b > 0$ *(so* (3.3b) *is verified) with* $b = \frac{p}{2}\,h(a)$ *where* $h \in \mathcal{F}(\mathbf{R}^-, \mathbf{R}^+)$. *If* R, d, p *are such that*

$$R^{1-\frac{h(a)}{a}} > (R+d)^{\frac{1}{p}+\frac{1}{2}-\frac{h(a)}{a}} \tag{3.7}$$

for $|a|$ *sufficiently large, then the number of rotationally nonequivalent nonradial solutions of the annulus problem* (3.1) *tends to* $+\infty$ *as* a *tends to* $-\infty$.

Let $a, b < 0$ satisfying (3.3a) *and $b = \frac{p}{2}\, h(a)$ where $h \in \mathcal{F}(\mathbf{R}^-, \mathbf{R}^-)$. If R, d, p are such that*

$$R^{\frac{h(a)}{a}+(\frac{1}{2}-\frac{1}{p})} > (R+d)^{\frac{h(a)}{a}} \tag{3.8}$$

for $|a|$ sufficiently large, then the number of rotationally nonequivalent nonradial solutions of problem (3.1) *tends to $+\infty$ as a tends to $-\infty$.*

Let $a, b < 0$ satisfying (3.3b)*. If R, d, p are such that*

$$R > (R+d)^{\frac{1}{p}+\frac{1}{2}}, \tag{3.9}$$

then the number of rotationally nonequivalent nonradial solutions of problem (3.1) *tends to $+\infty$ as a tends to $-\infty$.*

Remark 3.9. If $a > 0$, conditions (3.4), (3.5) or (3.6) implies that $R < 1$ and d tends to 0 as p tends to 2.

If $a < 0$, conditions (3.7), (3.8) or (3.9) implies that $R > 1$ and d tends to 0 as p tends to 2.

Remark 3.10*.* Using the same kind of technique, we obtain a result similar to the Coffman's one: let a, b, p, d fixed, if $2b > ap$ and condition (3.3a) is verified, then the number of rotationally nonequivalent nonradial solutions of problem (3.1) tends to $+\infty$ as R tends to $+\infty$.