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Public versus Private Education in Leveraging Long-Run Growth*

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Abstract

There is strong evidence that school quality, complemented by the quality of institutional arrangements, is one of the most important determinants of a country's long-run economic growth. Thus, understanding the role that private and public schools play in this relationship is highly important for both scholars and policymakers. We develop an endogenous growth model to study the effects of different education financing modes on long-run economic growth. We distinguish between public and private education provision by explicitly modeling their differing levels of efficiency based on the distinct characteristics associated with each mode of financing, incorporating their intrinsic features as pointed out by the empirical literature and their predicted effects on quality. By allowing us to model a wide range of policy scenarios commonly observed in practice, our framework shows that, under a plausible characterization of the educational conditions prevailing in many Western countries, the interplay between institutional and socio-educational factors shaping the efficiency of both types of education gives public systems a greater advantage in promoting long-run economic growth.

Keywords: economic growth, human capital, education, education financing

JEL Codes: O41, I22, I25

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1 Introduction

Since the mid-1950s, many macroeconomists have been engaged in studying the long-term rate of economic growth. In addition, since the late 1980s, much attention has been given to knowledge and human capital accumulation as determinant factors. Although human capital involves education and health, the literature has focused on education, emphasizing the distinction between the quantity of education, measured by school attainment or years of schooling completed, as well as the quality of education (Barro, 2013; Aghion et al., 2025). Endogenous growth models have emphasized the role of the accumulation of knowledge and human capital as the basis of technological progress, facilitating the purposeful research, the adoption of technologies developed by others, and its application to new production methods and products (Benhabib and Spiegel, 2005). The productive value of human capital embodied in individuals comes from the ability of scientists for knowledge generation and solve problems, the dynamism of entrepreneurs to adopt new ideas and technologies, and from the workers' skill that facilitates the diffusion (Nelson and Phelps, 1966). Although there are different types of human capital and there are different ways to acquire it, Romer (1986) and Lucas (1988) focused on learning-by-doing and schooling, respectively. Their models generate persistent endogenous growth from the actions of individuals in the economy but, even if governments have strong direct involvement in the financing and provision of schooling, they do not account for the large participation of the public sector in human capital investment.

After these initial contributions to the study of the relationship between education and long-run growth, it has also been examined in conjunction with fertility choice, income inequality and differences in wealth related to disparities in human capital endowments. However, Romer and Lucas's models are representative agent models, in which a household-dynasty lives forever, and are not well-suited to addressing issues associated with distribution and inequality. For this reason, overlapping generations (OLG) models inspired by Diamond (1965) ultimately prevailed. Over the years, the models have progressively evolved to address a broader range of issues related to public and private education. These include matters such as funding, school choice, segregation, challenges posed by imperfect credit markets, and even considerations of sustainability. This expansion reflects a deeper understanding of the multifaceted nature of educational systems and their broader societal implications. Notable contributions include the influential works by Glomm and Ravikumar (1992) and De la Croix and Doepke (2004), which take the schooling regime as given and analyze the effects of public and private school systems on growth and inequality. The first assumes exogenous fertility and a fixed population, while the second treats fertility as an endogenous variable. On the other hand, Glomm and Ravikumar (1998), Bearse et al. (2005) and De la Croix and Doepke (2009) address the issue of the choice of schooling regime, emphasizing how this decision is influenced by income distribution and the specific characteristics of the voting process.

Be as it may, the accumulation of human capital in modern societies must necessarily occur through educational systems, which may be organized under a wide variety of institutional arrangements (Stiglitz, 1974). This means that the financial structure of schools depends on the nature of their governance, the type of control and the sources of their funding. Usually, the literature distinguishes between the following types: private household expenditure with private control; public government expenditure at different administrative levels (state, region, and municipality) with public control; subsidized private expenditure; and the market with loans and repayments from students. Toma (1996) and Awan and Zia (2015) describe how different countries have implemented different institutional arrangements to organize their education systems. However, researchers most commonly focus on the first two. Indeed, the OLG papers mentioned above go beyond confirming the interdependence of education and fertility choices, as well as the quantity-quality trade-off between the number of children and the resources spent on each child's education, and compare the implications of public and private schooling regimes for economic growth. In the pure private education regime, individuals allocate their income to school expenses and consumption. The quality of private education increasingly depends on school expenses by households. In the pure public education regime, a government levies taxes on the income and uses tax revenues to provide free public education. The quality of public education is an increasing function of the tax revenues.

A widely held assumption exists that private schools offer superior educational quality relative to public schools.[1] Many families choose private schools for their children because they believe they

[1] In this regard, see the words by Stiglitz (1974) regarding the two predominant school systems in the provision of education: "while all individuals can obtain free education at public expense, a significant proportion of the population send their children to private schools. This is partly because the 'commodity' they wish to purchase is somewhat different from that provided publicly. But it is also partly because some parents want a high quality education than that provided publicly".

will receive a better education there. A private school is selected with the expectation of getting a higher quality education than what most students receive in the public system. However, this doesn't necessarily have to be the case. Some nuances to such judgment have been provided by Eckstein and Zilcha (1994) showing that compulsory public schooling induces more economic growth than private education, Zhang (1996) and Gradstein and Justman (1997) showing that subsidies for private education can outperform public education in terms of growth and welfare, and Bräuninger and Vidal (2000) showing that pure public education maximizes the long-run growth rate and a partially subsidized education can lead to lower growth than pure private education.

At this point, however, it is important to emphasize a crucial methodological distinction. The main result concerning comparative growth—which has often been presented as the central conclusion of the literature on education and growth—is invariably derived from some version of an overlapping generations (OLG) model. This places the analysis at a considerable methodological distance from the seminal theoretical contributions on the relationship between human capital accumulation and long-run growth, which were developed within the framework of representative-agent models.[2] In the OLG model there is no Pareto efficiency because of an externality implying a too short education investment. In the Lucas model it is not necessary the presence of an education externality for sustained growth, linearity with respect to the stock of human capital is sufficient, while increasing returns to scale associated with an externality in educational technologies are related to multiplicity of balanced growth trajectories. Consequently, in the benchmark Lucas model without externality the outcome is Pareto efficient. On the other hand, the literature based on OLG models studies many topics simultaneously, combining inequality, fertility, and choice of education system. Although endogenous growth is often driven by the accumulation of human capital in general, these models predict that, when inequality is low, private education leads to higher growth. Parents with low human capital have many children and provide little education to each child. Consequently, when inequality is high, average education is low. When the fertility gap between the rich and the poor disappears or decreases, the average level of education rises. Conversely, public education leads to higher economic growth only when inequality is high. Therefore, along

[2] According to De la Croix and Michel (2002), when altruism is sufficiently strong and people leave bequests to the next generation, households behave considering all its infinite dynasty. In this case, the infinite-lived representative agent model can be viewed as a special case of the micro founded overlapping generations model.

a balanced growth path in which there is no inequality, the growth rate of income per capita is higher with private education.

In this paper, we therefore return to the basic Lucas model, with its assumption of identical households and the absence of income and wealth disparities, and extend it to accommodate alternative schooling regimes. This allows us to assess the implications of different educational systems for long-run growth and, in particular, to determine under which conditions each regime is growth-enhancing. In doing so, we are able to evaluate the theoretical prediction that, in an economy populated by homogeneous agents, a private education system generates faster per capita income growth than a public one, thereby establishing the conditions under which this prediction is confirmed or overturned. Thus, building on the works of Lucas (1988) and Boucekkine and Ruiz-Tamarit (2008), we propose a model that incorporates two distinct educational systems, private and public, which can be considered isolated or combined for the production of education and the accumulation of human capital. Our analysis will focus exclusively on the pure private system, funded by household expenditures, and the pure public system, funded by government expenditures, while deliberately setting aside the influence of subsidies and loans in education financing. This approach allows us to isolate and compare the effects of these two systems on human capital accumulation and economic growth.

In studying the relationship between education and economic growth, researchers suggest to distinguish the quantity of education, such as years of schooling, educational attainment or enrollment rates, from the quality of education, often assessed through cognitive skills measured in international tests of mathematics, science, and reading (Hanushek, 2010, 2020). This distinction emphasizes the difference between knowledge and skills, on the one hand, and mere schooling, on the other. According to De la Croix (2012), the mechanism by which the quality of education, measured by scores on international tests in mathematics and science, affects growth is through the cognitive capacities of the population that allow it to better adapt and innovate. Cognitive skills are then related to economic growth. Empirical studies suggest that improvements in cognitive skills generated in the school system through institutional features affecting school quality lead to higher long-run growth of economies (Hanushek and Woessmann, 2012).

When examining the literature, we find several papers that raise interesting judgments concerning school quality (Hanushek and Rivkin, 2006; Woessmann and West, 2006; Fredriksson et al., 2013; Bosworth, 2014; Hanushek, 2015; Leuven and Løkken, 2020; Bucci et al., 2025). According to them, the quality of education is determined by factors such as teachers' qualifications, teaching methods, and the characteristics of schools, including facilities, supplementary activities, learning materials, class size, and assessment methods. Based on this literature, the set of factors influencing educational outcomes also include the attributes of families and classroom peers, which provide social support, values, and encouragement, as well as the learning capacities of the students. Family background typically refers to parental education, income, and family size. Educated women often face higher opportunity costs when it comes to dedicating time to child-rearing. Consequently, they tend to prioritize investing in the education or quality of a smaller number of children. Peer characteristics in turn refer to sociodemographic status and cultural traits. Beyond primary and secondary education, all these factors with their corresponding effects extend to higher education. On the one hand, teachers' quality is crucial in educational production because without good and qualified teachers the education system cannot guarantee high standards of quality. On the other hand, external exit examinations are a device to increase accountability in the school system that is associated with cognitive skills. These exams are important because in the private school system, parental pressure can often influence achievement measures without having any real effect on learning.

Unlike studies that analyze the relationship between different types of education and growth, based on OLG models where it is directly assumed that the private education system exhibits higher quality than the public one and is more efficient in generating human capital, we do not assume at all that one of them shows higher quality than the other. In our model, the higher or lower quality of the educational system is the result of a trade-off between factors that influence it positively and those that do so negatively. The outcome is not predetermined, and as a result, the impact on human capital accumulation, as well as on the economy's long-run rate of growth, remains uncertain *a priori*. In any case, given the representative agent approach of our model, the different growth rates cannot be attributed to differences in the time allocated to human capital accumulation in the public and private education sectors. It is necessary to inspect the role of factors that positively and negatively influence the quality of each educational system in order to conclude which one contributes more to the long-term economic growth of the economy.

In the following pages, we shed some new light on the important issue of which educational system is best for the long-run growth of modern economies. The rest of the paper is structured as follows. Section 2 explains the elements of the model, providing a characterization of agents and sectors, and emphasizing the features of the production functions for private and public education. Section 3.1 outlines and solves the dynamic optimization problem by invoking the Pareto optimal solution of the central planner, while Section 3.2 explicitly outlines the Balanced Growth Path trajectories of the model's key variables, building on the analytical developments presented in the Appendix. Then, in Section 4 we provide specific functional forms for the efficiency of each of the educational systems, in order to be able to draw implications regarding the behavior of the long-run growth rate. Section 5 illustrates this by focusing on two extreme cases of fully public or fully private education. Lastly, in Section 6 we discuss our main findings and conclude.

2 The model

Consider an economy in which activity takes place in two sectors producing, respectively, a final good and education. The economy is closed with competitive markets and populated by many rational agents, faced with the problem of choosing, over infinite continuous time, the values of two controls: per capita consumption c, and the fraction of non-leisure time devoted to goods production u. The structure of this model is similar to that of the Uzawa-Lucas two-sector endogenous growth model (Lucas, 1988), extended to include two different technologies for human capital accumulation. These technologies will be used, respectively, in the private and public education systems. There are no externalities, asymmetric information or market power. Hence, the competitive equilibrium is Pareto optimal. Instead of focusing on the behavior of representative agents, firms, households and government with perfect foresight, we can address the underlying dynamic optimization problem by invoking the central planner's solution, applying the Pontryagin's Principle from optimal control theory.

2.1 Population and preferences

The population is composed of identical individuals, with size at time t denoted by $N(t)$ and initial value N_0. Every person can choose to attend a tax-funded public school or opting out of the public system and go to a school of the private system. We assume this decision has already been made, and people are enrolled in one or the other educational system. There are many sociological factors that influence the choice of one system over another, and people are not limited to selecting the system that provides the highest quality education. The study of how people make the choice between public and private schooling is beyond the scope of this paper,[3] but it is known that total population, N, is the sum of people enrolled in private schools, N_1, and people enrolled in public schools, N_2,

$$N(t) = N_1(t) + N_2(t)\,. \tag{1}$$

Individual preferences are represented by the instantaneous utility function

$$U(c(t)) = \frac{c(t)^{1-\sigma} - 1}{1-\sigma}, \tag{2}$$

where $0 < \sigma^{-1} \leqslant 1$ represents the constant intertemporal elasticity of substitution (CIES).

At the aggregate level, the intertemporal social welfare is the discounted value of the sum of all utilities over an infinite horizon,

$$W = \int_0^\infty \frac{c(t)^{1-\sigma} - 1}{1-\sigma} N(t)\, e^{-\rho t} dt, \tag{3}$$

where parameter ρ is the rate of time preference or social discount rate. We assume $\rho > 0$.

2.2 The final-good production sector

In each period, the economy produces a single homogeneous good by using as inputs the best available technology, physical capital K, and the effective work force uNh (labor in efficiency

[3] In this regard, see Glomm and Ravikumar (1998), Bearse et al. (2005) and De la Croix and Doepke (2009).

units) according to the production function

$$Y(t) = AK(t)^{\beta}(u(t)N(t)h(t))^{1-\beta} =$$

$$= \underbrace{\{N_1(t)c(t) + pN_1(t)c(t) + N_2(t)c(t)\}}_{\text{Private Consumption Expenditure}} + \underbrace{\{G(t)\}}_{\text{Public Expenditure}} + \underbrace{\left\{\dot{K}(t) + \pi K(t)\right\}}_{\text{Gross Investment}}. \tag{4}$$

The variable $h(t)$ is predetermined and represents the average human capital or skill level of a representative worker, and $u(t)$ is a flow variable representing the fraction of (non-leisure time) human capital devoted to goods production. Consequently, $1-u(t)$ represents the individual effort or amount of human capital committed to education. Technology in goods sector shows constant returns to scale over private internal factors. In the production function parameter $\beta \in (0,1)$ is the elasticity of output with respect to physical capital, and coefficient $A > 0$ represents the technological level in the goods sector of the economy.

Equation (4) also shows the dynamic resource constraint of the economy. Produced output $Y(t)$ is allocated to private consumption, public expenditure and gross investment, where π represents the constant positive rate of physical capital depreciation. Private consumption includes the expenditure on goods and services by the entire population, as well as the spending on education by families that choose the private education system. The latter is specified as a percentage p, meaning that families using the private education system must spend an extra amount above their expenditure on goods and services $(1+p)N_1(t)c(t)$. The value of p can be indirectly interpreted as the price that each family pays for private school services. The population attending private schools could have chosen public schools, avoiding to pay the extra amounts for private schooling. Population choosing public schools do not supplement it with private schooling services. On the other hand, all agents are taxed, with the government collecting taxes at a proportional constant rate τ on income. Tax rates are determined by citizens through majority voting, and therefore, public school funding is the result of a political process (Glomm and Ravikumar, 1992, 1998; Bearse et al., 2005; De la Croix and Doepke, 2009). It is assumed that public expenditure is fully devoted to finance public education, which is available at zero price to everyone, and that government ensures a strict budget balance from period to period

$$G(t) = \tau Y(t) = \tau A K(t)^{\beta} (u(t) N(t) h(t))^{1-\beta}. \tag{5}$$

Putting all of the above together, we can rewrite the dynamic resource constraint of the economy as follows

$$\overset{\bullet}{K}(t) = (1-\tau) A K(t)^{\beta} (u(t) N(t) h(t))^{1-\beta} - \pi K(t) - N(t) c(t) - p N_1(t) c(t). \tag{6}$$

2.3 The private and public production of education

The labor factor measured in efficiency units is determined by the decisions of households and the government regarding the quantity and quality of their children's human capital. Here we assume that a country can accumulate human capital according to two educational systems: the private school system, hereafter referred to as subindex 1, and the public school system, hereafter referred to as subindex 2. Each system produces human capital H_1 and H_2 that are perfect substitutes for each other

$$H(t) = H_1(t) + H_2(t). \tag{7}$$

This means that H_1 and H_2 do not differ in their nature, but in their quantities, which depend on the intensity and quality of the corresponding educational investments. For the sake of simplicity, and considering that the effects of demographic change are beyond the scope of this paper, we assume that total population remains constant over time and has been normalized to unity

$$N(t) = 1 = N_1 + N_2 \tag{8}$$

Although the fixed population is divided into two groups, we ignore intergroup mobility. Consequently, N_1 and N_2 are also exogenously given. Moreover, there is no heterogeneity inside the groups. We assume that the private education system provides the entire group N_1 with the same quantity and quality of education. In parallel, the public education system provides the entire group N_2 with the same quantity and quality of education. Given that now N_1 and N_2 represent the percentages of population enrolled in private and public education, respectively, the general per capita human capital is the weighted average of the two per capita human capital, each produced in an educational sector

$$h(t) = \frac{H(t)}{N(t)} = \frac{H_1(t)}{N_1} N_1 + \frac{H_2(t)}{N_2} N_2 = h_1(t) N_1 + h_2(t) N_2. \tag{9}$$

The private school system and the public school system contribute to the accumulation of human capital according to the following educational dynamics:

$$\overset{\bullet}{h}_1(t) = \delta_1 (1 - u(t)) h(t) - \theta h_1(t), \tag{10}$$

$$\overset{\bullet}{h}_2(t) = \delta_2 (1 - u(t)) h(t) - \theta h_2(t). \tag{11}$$

Technology in both educational sectors is linear in the stocks of human capital. These equations show how much the human capital of a person who decides to attend private school increases and that of another person who attends public school. In each period, the gross increase in the human capital stock depends on the stock of human capital already achieved, the time spent in school, and the quality of the schools. It is assumed that the increase in human capital does not depend on the stock of physical capital, and that there is no externality in the form of a spillover effect associated with the accumulation of human capital. In other words, it depends on the own effort devoted to the accumulation of human capital. Each person in the economy devotes to education the proportion $1 - u$ of his/her time and, consequently, of his/her embodied human capital (here represented by the average h because there is no clear separation implying that only the human capital produced in an educational system is used in that system). The school contributes to the increase in the per capita human capital through a productivity factor. However, efficiency parameters in each type of school, δ_1 and δ_2, which represent the corresponding technological levels, could be different from each other. Each of these parameters also represents the quality of each school system, private or public. Regardless of this, the positive depreciation rate, θ, is common to all human capital coming from any of the educational systems. We define the efficiency coefficients as follows:

$$\delta_1 = z_1 - x_1(p) \geqslant 0, \tag{12}$$

$$\delta_2 = z_2(\tau) - x_2 \geqslant 0. \tag{13}$$

Equations (12) and (13) may be read as technologies that transform expenditures on education into quality of education. We can see that the quality of private and public schools is not fixed or predetermined. The quality of public schools is greatly determined by the funding coming from taxpayers. The quality of private schools is largely determined by the tuition and other payments made by households. However, the whole efficiency of these investment technologies is affected by many different factors. In these equations $z_i \ \forall i = \{1, 2\}$ represents the combined positive contribution of a number of elements to the efficiency coefficient associated with the production of education in private and public schools respectively. By contrast, $x_i \ \forall i = \{1, 2\}$ represents the combined negative contribution of different factors to the efficiency coefficient in the production of education in private and public schools respectively.

Among the elements that contribute positively to the productivity of the educational system mentioned in the literature, and that could be found unbalanced between the two systems, we would like to highlight the following as the most relevant ones: *i*) the quality of the teaching staff (experience, training and the degree of involvement); *ii*) incentives for teachers (salaries and academic freedom); *iii*) classroom pedagogical support resources (both material and human); and *iv*) the accuracy and reliability of the examination procedures.[4] Furthermore, the size (also recognized by the student-teacher ratio) and the heterogeneous cultural and socioeconomic composition of the class are considered the two main candidates among the factors that reduce the productivity of the educative system. They can affect students' cognitive abilities and the rhythm of skill acquisition if larger and more diversified classes lead to poorer academic performance. In such a case, smaller size and uniform class are associated with higher quality and a faster growth rate of human capital

[4]It has been empirically proved that family background and characteristics such as parental education, income and family size, are important for students' academic performance. However, these are not relevant in our model of two educational systems because we consider that both private and public schools use uniformly the average per capita human capital of the whole economy. This assumption neutralizes in our model any influence of families on the respective levels of educational productivity.

per capita.

Based on the above arguments, we take a step forward and assume the following constraints between the determinants of the efficiency levels in each type of school

$$z_1 \leqslant z_2(\tau) < \Omega; \qquad \frac{\partial z_2}{\partial \tau} > 0, \quad \frac{\partial^2 z_2}{\partial \tau^2} \lesseqgtr 0, \tag{14}$$

$$x_2 \geqslant x_1(p) > 0; \qquad \frac{\partial x_1}{\partial p} < 0, \quad \frac{\partial^2 x_1}{\partial p^2} > 0. \tag{15}$$

According to (14) the joint positive contribution of teacher's quality, resources, and incentives to the efficiency in the private school system is taken as a benchmark for the economy and is pegged to the value z_1. This value also represents a lower bound for the corresponding value in the public school system, which is increasingly determined by the proportional income tax rate. Tax collection is the country's way of generating the public revenues that make it possible to finance educational expenditures in the public system. These revenues collected with τ allow z_2 to move away from z_1. According to (15) the joint negative contribution of class size and heterogeneity to the efficiency in the public school system is now the benchmark for the economy as a whole, which is set to the value x_2.[5] This one also represents an upper bound for the corresponding value in private school system. The private school is funded directly by population N_1 who individually pay the price p, and paying this price they attend a more homogeneous school with smaller class size. This is the meaning of the inverse relationship between x_1 and p.

Given the relationship between the quality of education, the income tax rate and the price paid for attending the private school, we observe that the quality of the publicly provided schooling is an increasing function of the uniform tax rate and the quality of the privately provided schooling is an increasing function of the uniform private price. This implies that both public and private schools can vary in their levels of quality.

At this point, we are interested in the comparison between the levels of efficiency with which

[5]Behind the constant value x_2 we should think of a maximum of diversity and density in the classroom. In many countries there are administrative regulations fixing maximum class sizes in their public schools and imposing quotas to promote cultural and socioeconomic integration. We assume that this ceiling applies to public schools, but private schools can reduce class size and increase homogeneity of students.

the private and public systems provide education

$$\delta_2 - \delta_1 = z_2(\tau) - z_1 + x_1(p) - x_2. \tag{16}$$

The result of this subtraction will lie in the interval $[-x_2, z_2(\tau) - z_1]$, where the infimum corresponds to the extreme case in which we observe the smallest $z_2(\tau) = z_1$ and $x_1(p)$ asymptotically approaching zero, while the supremum occurs when we observe the highest $z_2(\tau)$ value bounded by Ω and the largest $x_1(p) = x_2$.

By paying a higher unit price p for education the population N_1 increases the efficiency of the private education system, since it decreases x_1 and, other things being equal, increases δ_1. By spending a higher percentage of income τ on education, the government increases the efficiency of the public education system, as it increases z_2 and, other things being equal, increases δ_2. In this regard, it is important to note that empirical research in the field of educational economics suggests that differences in the quality of teaching across schools might be the most significant determinant of differences in student achievement, i.e. the efficiency level of the educational system. Moreover, this literature also suggests that class size and degree of homogeneity only represent a second-order factor as a possible determinant of productivity differences.

Let us now consider the aggregation of both processes of human capital accumulation net of depreciation. From Equations (10) and (11) we get

$$N_1 \dot{h}_1(t) = \delta_1 (1 - u(t)) N_1 h(t) - \theta N_1 h_1(t), \tag{17}$$

$$N_2 \dot{h}_2(t) = \delta_2 (1 - u(t)) N_2 h(t) - \theta N_2 h_2(t). \tag{18}$$

By differentiating (9) and adding the two previous terms we obtain

$$\dot{h}(t) = \dot{h}_1(t) N_1 + \dot{h}_2(t) N_2 = (\delta_1 N_1 + \delta_2 N_2)(1 - u(t)) h(t) - \theta h(t) =$$

$$= ((z_1 - x_1(p)) N_1 + (z_2(\tau) - x_2) N_2)(1 - u(t)) h(t) - \theta h(t). \tag{19}$$

Finally, we can express the dynamic constraint for the per capita human capital in a compact form

$$\overset{\bullet}{h}(t) = \delta (1 - u(t)) h(t) - \theta h(t), \tag{20}$$

where

$$\delta = (z_1 - x_1(p)) N_1 + (z_2(\tau) - x_2)(1 - N_1). \tag{21}$$

and

$$\frac{\partial \delta}{\partial p} = -N_1 \frac{\partial x_1}{\partial p} > 0, \tag{22}$$

$$\frac{\partial \delta}{\partial \tau} = (1 - N_1) \frac{\partial z_2}{\partial \tau} > 0, \tag{23}$$

$$\frac{\partial \delta}{\partial N_1} = z_1 - x_1(p) + x_2 - z_2(\tau) = \delta_1 - \delta_2 \gtreqless 0. \tag{24}$$

The efficiency parameter δ can also be read, roughly speaking because of the presence of the depreciation rate, as the maximal rate of growth for the average per capita human capital stock attainable when all effort is devoted to education.

3 Solving the model

3.1 Optimization

Given that it is assumed $N(t) = 1$ $\forall t$, we get $Y(t) = y(t)$, $K(t) = k(t)$, $G(t) = g(t)$. Then, we can rewrite the aggregate resource constraint in the following way

$$y(t) = Ak(t)^{\beta} (u(t) h(t))^{1-\beta} = \overset{\bullet}{k}(t) + \pi k(t) + N_1 c(t) + pN_1 c(t) + N_2 c(t) + g(t) \tag{25}$$

where

$$g(t) = \tau y(t) = \tau A k(t)^{\beta} (u(t) N(t) h(t))^{1-\beta} \tag{26}$$

Because of the context and structure introduced in the previous section, we realize that our model is an extension of the Uzawa-Lucas two-sector endogenous growth model. The economy is closed with competitive markets and populated with identical rational agents. There are two controls to be chosen by the agents: consumption per capita $c(t)$ and the fraction of their average human capital devoted to goods production $u(t)$, aiming at solving the following dynamic optimization problem

$$\max_{\{c(t),u(t)\}} \qquad W = \int_0^\infty \frac{c(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt \tag{P}$$

subject to

$$\overset{\bullet}{k}(t) = (1-\tau) A k(t)^\beta (u(t) h(t))^{1-\beta} - \pi k(t) - (1+pN_1) c(t),$$

$$\overset{\bullet}{h}(t) = ((z_1 - x_1(p)) N_1 + (z_2(\tau) - x_2)(1-N_1))(1-u(t)) h(t) - \theta h(t),$$

$$k(0) = k_0, \quad h(0) = h_0, \quad c(t) \geqslant 0, \quad u(t) \in [0,1], \quad k(t) \geqslant 0, \quad h(t) \geqslant 0.$$

We assume that $(z_1 - x_1(p)) N_1 + (z_2(\tau) - x_2)(1-N_1) > \theta + \rho$ for positive growth to arise in the long-run. Note that this parameter constraint implies that

$$z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau)) N_1 + \pi - \theta > 0. \tag{27}$$

The current value Hamiltonian associated with the previous intertemporal optimization problem is

$$\begin{aligned} H^c(k, h, \vartheta_1, \vartheta_2, c, u; \sigma, A, \beta, \pi, \theta, z_1, x_1, z_2, x_2, \tau, p, N_1 \{N(t) = 1 = N_1 + N_2 : t \geq 0\}) = \\ = \frac{c^{1-\sigma}-1}{1-\sigma} + \vartheta_1 \left[(1-\tau) A k^\beta (uh)^{1-\beta} - \pi k - (1+pN_1) c\right] \\ + \vartheta_2 \left[(z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau)) N_1)(1-u) h - \theta h\right], \end{aligned} \tag{28}$$

where ϑ_1 and ϑ_2 are the co-state variables for k and h, respectively. For the sake of simplicity, we have dropped the time index from the variables and will keep it removed unless strictly necessary.

On the margin, efficiency requires that goods and time be equally valuable in their different uses:

on the one hand, consumption and physical capital accumulation, and on the other, production and human capital accumulation. The first order necessary conditions are

$$c^{-\sigma} = (1 + pN_1)\,\vartheta_1, \tag{29}$$

$$\vartheta_1 (1-\beta)(1-\tau) Ak^{\beta} (uh)^{-\beta} = \vartheta_2 \left(z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau)) N_1\right), \tag{30}$$

the Euler equations

$$\overset{\bullet}{\vartheta}_1 = (\rho + \pi)\,\vartheta_1 - \vartheta_1 \beta (1-\tau) Ak^{\beta-1} (uh)^{1-\beta}, \tag{31}$$

$$\overset{\bullet}{\vartheta}_2 = (\rho + \theta)\,\vartheta_2 - \vartheta_1 (1-\beta)(1-\tau) Ak^{\beta} u^{1-\beta} h^{-\beta}$$

$$-\vartheta_2 \left(z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau)) N_1\right)(1-u), \tag{32}$$

the dynamic constraints

$$\overset{\bullet}{k} = (1-\tau) Ak^{\beta}(uh)^{1-\beta} - \pi k - (1 + pN_1)\,c, \tag{33}$$

$$\overset{\bullet}{h} = \left(z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau)) N_1\right)(1-u)\,h - \theta h, \tag{34}$$

the boundary initial conditions k_0, h_0, and transversality conditions

$$\lim_{t\to\infty} \vartheta_1 k \exp\{-\rho t\} = 0, \tag{35}$$

$$\lim_{t\to\infty} \vartheta_2 h \exp\{-\rho t\} = 0. \tag{36}$$

Notice that by (29), $\vartheta_1(t)$ cannot be equal to 0 at any finite date t because this would require that consumption is infinite at a finite date, which violates the resource constraint of the economy. Then, according to (30), $\vartheta_2(t) \neq 0$ at a finite t, provided the economy starts with finite and strictly positive endowments of physical and human capital, implying also finite and strictly positive output levels at any finite date.

3.2 The balanced growth path

In the Appendix below, the reader will find the methodological details that allow us to arrive at the analytical solution of the dynamic system just reported in the previous section. Guided by the results displayed there, we can now specify in closed-form the long-run trajectories of the variables as a function of the parameters of the model, and in particular those characterizing the private and public educational systems. The fraction of human capital devoted to goods production is constant

$$0 < \bar{u} = \frac{\rho - \left(z_2\left(\tau\right) - x_2 + \left(z_1 - x_1\left(p\right) + x_2 - z_2\left(\tau\right)\right)N_1 - \theta\right)\left(1-\sigma\right)}{\sigma\left(z_2\left(\tau\right) - x_2 + \left(z_1 - x_1\left(p\right) + x_2 - z_2\left(\tau\right)\right)N_1\right)} < 1, \tag{37}$$

as well as the ratio between the two shadow prices

$$\overline{\left(\frac{\vartheta_1}{\vartheta_2}\right)} = \frac{z_2\left(\tau\right) - x_2 + \left(z_1 - x_1\left(p\right) + x_2 - z_2\left(\tau\right)\right)N_1}{\left(1-\beta\right)\left(1-\tau\right)A}$$

$$\left(\frac{z_2\left(\tau\right) - x_2 + \left(z_1 - x_1\left(p\right) + x_2 - z_2\left(\tau\right)\right)N_1 + \pi - \theta}{\beta\left(1-\tau\right)A}\right)^{\frac{\beta}{1-\beta}} > 0, \tag{38}$$

and the ratio between the stocks of physical and human capital

$$\overline{\left(\frac{k}{h}\right)} = \frac{\rho - \left(z_2\left(\tau\right) - x_2 + \left(z_1 - x_1\left(p\right) + x_2 - z_2\left(\tau\right)\right)N_1 - \theta\right)\left(1-\sigma\right)}{\sigma\left(z_2\left(\tau\right) - x_2 + \left(z_1 - x_1\left(p\right) + x_2 - z_2\left(\tau\right)\right)N_1\right)}\left(\frac{\beta\left(1-\tau\right)A}{\delta + \pi - \theta}\right)^{\frac{1}{1-\beta}} > 0. \tag{39}$$

On the other hand, the levels of consumption, physical capital stock, human capital stock and output evolve according to the following exponential paths

$$\bar{c} = \frac{\left(\frac{z_2(\tau)-x_2+(z_1-x_1(p)+x_2-z_2(\tau))N_1+\pi-\theta}{\beta(1-\tau)A}\right)^{\frac{\beta}{\sigma(1-\beta)}}}{\left(1+pN_1\right)^{\frac{1}{\sigma}}\vartheta_2^{\frac{1}{\sigma}}\left(0\right)\left(\frac{(1-\beta)(1-\tau)A}{z_2(\tau)-x_2+(z_1-x_1(p)+x_2-z_2(\tau))N_1}\right)^{\frac{1}{\sigma}}}\exp\left\{\bar{g}_c t\right\}, \tag{40}$$

$$\bar{k} = \frac{\sigma\beta\left(\frac{(1-\beta)(1-\tau)A}{(z_2(\tau)-x_2+(z_1-x_1(p)+x_2-z_2(\tau))N_1)\vartheta_2(0)}\right)^{\frac{1}{\sigma}}\left(\frac{\beta(1-\tau)A}{z_2(\tau)-x_2+(z_1-x_1(p)+x_2-z_2(\tau))N_1+\pi-\theta}\right)^{\frac{\beta}{\sigma(1-\beta)}}}{\beta(\rho+\pi-\pi\sigma)-(z_2(\tau)-x_2+(z_1-x_1(p)+x_2-z_2(\tau))N_1+\pi-\theta)(\beta-\sigma)}\exp\left\{\bar{g}_k t\right\}, \tag{41}$$

$$\bar{h} = \frac{h_0}{{}_2\widetilde{F}_1(0)} \exp\left\{\bar{g}_h t\right\}, \tag{42}$$

$$\bar{y} = \frac{(1-\tau)\, A h_0}{{}_2\widetilde{F}_1(0)} \left(\frac{\beta\,(1-\tau)\, A}{z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau))\, N_1 + \pi - \theta}\right)^{\frac{\beta}{1-\beta}}$$
$$\left(\frac{\rho - (z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau))\, N_1 - \theta)(1-\sigma)}{\sigma\,(z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau))\, N_1)}\right) \exp\left\{\bar{g}_y t\right\}. \tag{43}$$

The above expression corresponds to the per capita narrow (market) output. However, we can define a new measure of output which includes not only the production of goods, but also education services. In per capita terms, the broad (aggregate) output is

$$q(t) = y(t) + \left(\frac{\vartheta_2(t)}{\vartheta_1(t)}\right) \dot{h}(t), \tag{44}$$

which moves along the balanced growth path

$$\bar{q} = \frac{(1-\tau)\, A h_0}{{}_2\widetilde{F}_1(0)} \left(\frac{\beta\,(1-\tau)\, A}{z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau))\, N_1 + \pi - \theta}\right)^{\frac{\beta}{1-\beta}}$$
$$\left(1 - \frac{\beta\,(z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau))\, N_1 - \theta - \rho + \sigma\theta)}{\sigma\,(z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau))\, N_1)}\right) \exp\left\{\bar{g}_q t\right\}. \tag{45}$$

Finally, the common long-run rate of growth is constant

$$\bar{g}_q = \bar{g}_y = \bar{g}_h = \bar{g}_k = \bar{g}_c = \bar{\gamma} = \frac{z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau))\, N_1 - \theta - \rho}{\sigma}. \tag{46}$$

It is easy to verify that the growth rate of the economy will be higher the higher the intertemporal elasticity of substitution and the average efficiency of the country's education system, and the lower the rate of depreciation of human capital and the discount rate. In the next section we will discuss how the parameters that characterize the private and public education systems influence qualitatively on the growth of the economy and how they quantitatively determine the intensity of growth.

4 Education and long-run growth

Education is considered one of the key determinants of economic growth (Hanushek and Woessmann, 2008, 2010). First, because education enhances the innovative potential of the economy by underpinning R&D, and the new production processes and products boost the growth of technology leaders who see their theoretical production frontiers shifting outward (Aghion and Howitt, 1998). Second, because education facilitates adoption and diffusion of technology, and this spurs the growth of the followers who see their effective production frontiers moving outward (Nelson and Phelps, 1966; Benhabib and Spiegel, 2005).[6] Finally, and closer to the approach adopted in this paper, because education increases the human capital embodied in the labor force and, consequently, the productivity of the economy which endogenously drives growth (Lucas, 1988).

Although much of the empirical literature that has studied the relationship between education of the working-age population and economic growth has focused on the quantity of schooling, studies currently consider that the relevant variable, the one that best explains growth, is the quality of education. In this paper we evaluate the quality of education by separating between two different educational systems, and attributing to them differentiated features that are closely related to the quality of education. These non-shared characteristics of the public and private school systems contribute decisively to determine the country's long-run levels of per capita consumption, physical capital stock, human capital stock and output. They are also the determinants of the stationary values of the fraction of human capital devoted to goods production, of the ratio between physical and human capitals, of the ratio between their shadow prices and, with a prominent role in our study, of the rate of growth.

According to equation (46), the economy's long-run rate of growth will be higher the higher the quality of teachers and the stronger the incentives structure, but also the higher the amount of pedagogical resources that students find in their classrooms and the more demanding the schools are in the evaluation exams. In short, the rate of growth will be higher the higher the compensations, monetary and non-monetary, to teachers, which we represent here by the terms z_i $\forall i = \{1, 2\}$ included in the efficiency coefficients of each education production function. At the same time,

[6] More specifically, adoption depends on the education of entrepreneurs, while diffusion depends more on the education of workers (Ruiz-Tamarit et al., 2021).

the long-run rate of growth will be higher the lower the student-teacher ratio and the lower the socioeconomic and cognitive skill mixture in the classroom. In other words, the growth rate will be higher the stronger the population's incentives to pay, prices or taxes, for less populated and more uniform pupils in classrooms, which effect on the efficiency coefficients is represented with the opposite of terms x_i $\forall i = \{1, 2\}$. As mentioned above, the compensations to teachers in the private school system that are behind the value z_1, are taken as a benchmark and floor value in the public school system. Similarly, the administrative regulations establishing quotas on the size and composition of classrooms in the public school system, which are behind the value x_2, are taken as a benchmark and ceiling value in the private school system. Differences in the efficiency levels between private and public school have a big impact on both the aggregate productivity and the growth rate of the economy.

At this point, we assume particular specifications for functions $x_1(p)$ (hyperbolic) and $z_2(\tau)$ (logistic), which meet all of the requirements stated along the previous sections

$$x_2 \geqslant x_1(p) = \frac{x_2}{1+p} > 0, \tag{47}$$

$$0 < z_1 \leqslant z_2(\tau) = \frac{z_1 \Omega \exp\{\phi(\tau - \tau_m)\}}{\Omega + z_1(-1 + \exp\{\phi(\tau - \tau_m)\})} < \Omega, \tag{48}$$

where $p \geqslant 0$, $\phi > 0$ and $0 \leqslant \tau_m \leqslant \tau < 1$. Parameters Ω and ϕ represent, respectively, the logistic *carrying capacity* (maximum) and rate of growth for $z_2(\tau)$. Substituting into (12) and (13) we obtain the following functions that determine the quality of private and public schools, respectively,

$$\delta_1(p) = \frac{z_1 - x_2 + z_1 p}{1+p}, \tag{49}$$

which is increasing and strictly concave in p because $\frac{\partial \delta_1}{\partial p} = \frac{x_2}{(1+p)^2} > 0$ and $\frac{\partial^2 \delta_1}{\partial p^2} = \frac{-2x_2}{(1+p)^3} < 0$, and

$$\delta_2(\tau) = \frac{1 - \frac{x_2}{\Omega} - x_2\left(\frac{1}{z_1} - \frac{1}{\Omega}\right)\exp\{-\phi(\tau - \tau_m)\}}{\frac{1}{\Omega} + \left(\frac{1}{z_1} - \frac{1}{\Omega}\right)\exp\{-\phi(\tau - \tau_m)\}}, \tag{50}$$

which is increasing and strictly concave in τ because $\frac{\partial \delta_2}{\partial \tau} = \phi\left(1 - \frac{z_2(\tau)}{\Omega}\right) z_2(\tau) = \frac{\partial z_2}{\partial \tau} > 0$ and

$\frac{\partial^2 \delta_2}{\partial \tau^2} = \phi \left(1 - \frac{2z_2(\tau)}{\Omega}\right) \frac{\partial z_2(\tau)}{\partial \tau} < 0$, given $z_2(\tau) \geqslant z_1$ and the assumption $z_1 > \frac{\Omega}{2}$. Moreover, we will assume that $z_1 - x_2 > 0$, which corresponds to the value of $\delta_1(p = 0)$ and $\delta_2(\tau = \tau_m)$, as a necessary but no sufficient condition for a positive long-run rate of growth whatever the combination of private and public schools in the economy, that is, for any value of N_1 between 1 and 0 (See Figures 1 and 2, respectively).

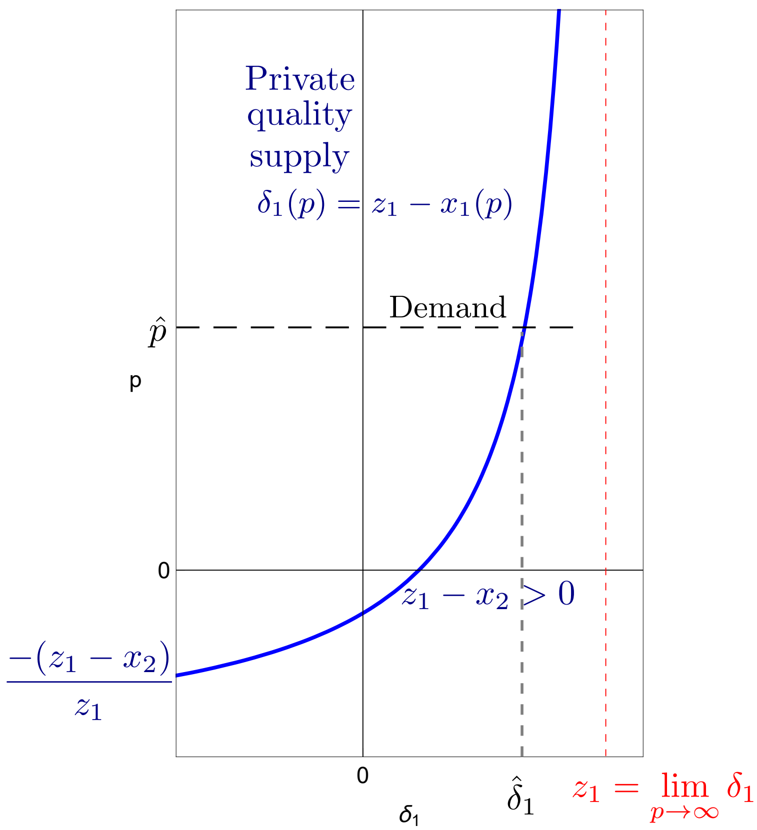

Figure 1: Graphical representation of $\delta_1(p)$ and equilibrium.

Functions (49) and (50) represent the respective scales of quality supply in the private and public schooling systems. The private school offers quality δ_1 depending on the consumption price p paid by students. Population N_1 makes a double choice when they choose to attend a private school and pay a price p associated with the student-teacher ratio and the degree of socioeconomic homogeneity they wish to find in the classroom. According to this, p is determined at the same time that people decide to go to private school. If population N_1 is willing to pay a higher price, the private school will offer higher quality by reducing class size and decreasing student heterogeneity, factors that reduce the component x_1 subtracted from δ_1. In our model only one price p and one quality level δ_1 are relevant at a time for all N_1 due to the representative agent hypothesis.

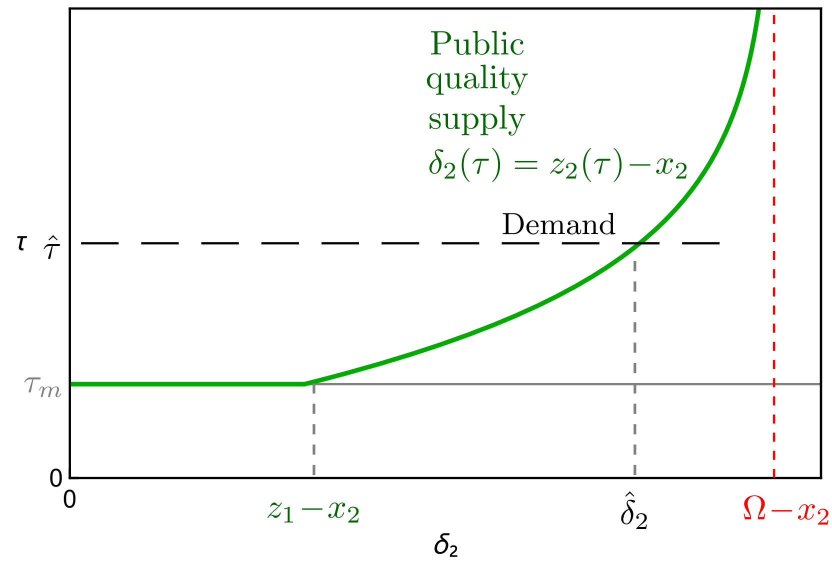

Figure 2: Graphical representation of $\delta_2(\tau)$ and equilibrium.

Furthermore, the quality of public school δ_2 supplied by the government depends on the tax revenue collected, which we assume is entirely spent on public education. Population assesses the performance of public school by the tax revenue available. If population pays more taxes, the government can provide a higher-quality public education service by selecting better teachers, improving their incentives, strengthening pedagogical resources, and implementing more reliable methods for learning assessment, factors that enhance the component z_2 that pushes δ_2. However, these resources depend on the income tax rate τ that taxpayers N are willing to pay. In our model all citizens pay the same proportional income tax rate and have access to the same quality of public education.

We assume that the choice of τ emerges from a voting process (Glomm and Ravikumar, 1992; De la Croix and Doepke, 2009; De la Croix, 2011). This process consists in the democratic rule of the majority vote in a context of universal suffrage (Gradstein and Justman, 1997; Glomm and Ravikumar, 1998; Bearse et al., 2005). Majority vote results in the socially preferred tax rate.[7] At this point, it is important to clarify how the expansion of private education can affect the quality,

[7] If we define social preferences in terms of individual preferences, the choices made by the N members of society can be combined to identify the socially preferred τ. According to Friedman (1986), "that is the idea of democracy: Let each individual vote for what he prefers and hope that the outcome will be good for the society. This is even more clear when we assume the representative agent model".

but also the quantity, of public education. In this political context, the N_1 citizens who attend private schools will prefer and probably vote for $\tau = 0$, while the remaining N_2 will vote for $\tau > 0$. Initially, the expansion of population N_1 could have a negative impact on the quality of public education by pushing for lower taxes and reduced spending on public schools. However, given the majority rule, the progressive increase until reaching $N_1 > 0.5$ could jeopardize the very existence of public education with majority votes for $\tau = 0$, an outcome that can only be avoided through the constitutional establishment of a minimum $\tau_m > 0$.

Finally, we come back to the economy's long-run rate of growth (46) and, after substituting (47) and (48), we get

$$\bar{\gamma} = \frac{\frac{z_1 \Omega \exp\{\phi(\tau-\tau_m)\}(1-N_1)}{\Omega + z_1(-1+\exp\{\phi(\tau-\tau_m)\})} - x_2 + \left(z_1 + \frac{p x_2}{1+p}\right) N_1 - \theta - \rho}{\sigma}. \tag{51}$$

Then, we find the following comparative results:

$$\frac{\partial \bar{\gamma}}{\partial \theta} = \frac{\partial \bar{\gamma}}{\partial \rho} = -\frac{1}{\sigma} < 0 \text{ and } \frac{\partial \bar{\gamma}}{\partial \sigma} = -\frac{\bar{\gamma}}{\sigma} < 0, \tag{52}$$

higher human capital depreciation and agent's impatience are associated with a lower rate of growth;

$$\frac{\partial \bar{\gamma}}{\partial x_2} = -\frac{1 + p\,(1 - N_1)}{\sigma\,(1+p)} < 0, \tag{53}$$

higher size and composition in the public school that implies a lower public school quality is associated with a lower rate of growth;

$$\frac{\partial \bar{\gamma}}{\partial z_1} = \frac{N_1}{\sigma} + \frac{1}{\sigma} \frac{\Omega^2 \exp\{\phi\,(\tau - \tau_m)\}\,(1 - N_1)}{\left(\Omega + z_1\,(-1 + \exp\{\phi\,(\tau - \tau_m)\})\right)^2} > 0, \tag{54}$$

higher teacher's quality in the private school that implies a higher private school quality is associated with a higher rate of growth;

$$\frac{\partial \bar{\gamma}}{\partial p} = \frac{x_2 N_1}{\sigma\,(1+p)^2} > 0, \tag{55}$$

higher education price in the private school that implies a lower size and composition and a higher private school quality is associated with a higher rate of growth;

$$\frac{\partial\bar{\gamma}}{\partial\tau} = -\frac{\partial\bar{\gamma}}{\partial\tau_m} = \frac{1}{\sigma}\left(\frac{\phi\left(\Omega - z_1\right) z_1 \Omega \exp\left\{\phi\left(\tau - \tau_m\right)\right\}\left(1 - N_1\right)}{\left(\Omega + z_1\left(-1 + \exp\left\{\phi\left(\tau - \tau_m\right)\right\}\right)\right)^2}\right) > 0, \tag{56}$$

higher income tax that implies higher teacher's and public school quality is associated with a higher rate of growth[8];

$$\frac{\partial\bar{\gamma}}{\partial N_1} = \frac{1}{\sigma}\left(-\frac{\left(\Omega - z_1\right) z_1\left(-1 + \exp\left\{\phi\left(\tau - \tau_m\right)\right\}\right)}{\Omega - z_1 + z_1 \exp\left\{\phi\left(\tau - \tau_m\right)\right\}} + \frac{p x_2}{1 + p}\right) \gtrless 0, \tag{57}$$

the composition of enrollment with a higher proportion of people attending private schools cannot be a priori associated with a higher or lower economic growth rate.

5 Long-run growth in the extremes of the financing distribution

From the previous section we know almost everything about the relationship between the growth rate of the economy and the parameters that characterize the two educational systems. And we know all that even for the particular forms adopted for the functions $x_1(p)$ and $z_2(\tau)$. However, if we wish to shed some light on the sign of the previous equation (57), and offer something practical and concrete about which school system leads to a higher level of per capita income and a higher growth rate, we have to take some steps forward and confront educational regimes by analyzing two polar cases: one in which the entire population attends private schools and another in which the entire population attends public schools.

5.1 The rates of growth

In the economy in which education is a completely private activity, except for the minimal public counterpart financed with $\tau = \tau_m$, from (37) and (46) with $N_1 = 1$ we get

[8] Tax revenues fund not only the public school system but also economic infrastructure and other essential socioeconomic institutions. In this context, Aghion et al. (2016) find an inverted U-shaped relationship between taxation and growth. They introduce corruption into the equation and show that the nonlinear relationship and the value of the resulting optimal tax rate depend on government inefficiency in managing tax revenues.

$$0 < 1 - \bar{u}_1 = \frac{z_1 - x_1(p) - (1-\sigma)\theta - \rho}{\sigma(z_1 - x_1(p))} = \frac{z_1 - \frac{x_2}{1+p} - (1-\sigma)\theta - \rho}{\sigma\left(z_1 - \frac{x_2}{1+p}\right)} < 1, \tag{58}$$

$$\bar{\gamma}_1 = \frac{z_1 - x_1(p) - \theta - \rho}{\sigma} = \frac{z_1 - \frac{x_2}{1+p} - \theta - \rho}{\sigma}. \tag{59}$$

In the economy where education is only provided or fully funded by the government, from (37) and (46) with $N_1 = 0$ we get

$$0 < 1 - \bar{u}_2 = \frac{z_2(\tau) - x_2 - (1-\sigma)\theta - \rho}{\sigma(z_2(\tau) - x_2)} = \frac{z_2(\tau) - x_2 - (1-\sigma)\theta - \rho}{\sigma(z_2(\tau) - x_2)} < 1, \tag{60}$$

$$\bar{\gamma}_2 = \frac{z_2(\tau) - x_2 - \theta - \rho}{\sigma} = \frac{\frac{z_1 \Omega \exp\{\phi(\tau-\tau_m)\}}{\Omega + z_1(-1+\exp\{\phi(\tau-\tau_m)\})} - x_2 - \theta - \rho}{\sigma}. \tag{61}$$

According to (59) and (61), the minimum growth rate attainable in the economy with a pure public school system is the same minimum growth rate attainable in the economy with a pure private school system, $\underline{\gamma}_1^{\inf}(p=0) = \underline{\gamma}_2^{\inf}(\tau = \tau_m) = \frac{z_1 - x_2 - \theta - \rho}{\sigma}$. On other hand, the maximum growth rate attainable in the economy with a pure public school system $\underline{\gamma}_2^{\sup} = \frac{\Omega - x_2 - \theta - \rho}{\sigma}$ will be greater (smaller) than the maximum growth rate attainable in the economy with a pure private school system $\underline{\gamma}_1^{\sup} = \frac{z_1 - \theta - \rho}{\sigma}$ if and only if $Range\{z_2(\tau)\} = \Omega - z_1$ is greater (smaller) than $Max\{x_1(p)\} = x_2$.[9,10]

Beyond this characterization of the rates of growth in terms of intervals, we will consider the consumption price $\hat{p}$ that population N_1 pays to attend a private school, and the income tax rate

[9] This relationship between the corresponding maximum growth rates is also supported by the following relationship between the percentages of human capital devoted to education: $\left(1 - \bar{u}_2\right)^{\sup} = \frac{\Omega - x_2 - (1-\sigma)\theta - \rho}{\sigma(\Omega - x_2)}$ greater (smaller) than $\frac{z_1 - (1-\sigma)\theta - \rho}{\sigma z_1} = \left(1 - \bar{u}_1\right)^{\sup}$.

[10] Actually, since there is a maximum value for the tax rate less than infinity, i.e. $\tau = 1$, the maximum growth rate attainable in the economy with a pure public school system is $\underline{\gamma}_2^{\sup} = \frac{\omega - x_2 - \theta - \rho}{\sigma}$, where $z_1 < \omega = \frac{z_1 \Omega \exp\{\phi(1-\tau_m)\}}{\Omega - z_1 + z_1 \exp\{\phi(1-\tau_m)\}} < \Omega$ because $z_2(\tau)$ is monotonically increasing in τ.

$\hat{\tau}$ that arises from the rule of the majority vote and is applied to total population N. If we define

$$\Pi\left(\hat{p}\right) = z_1 + \tfrac{\hat{p}}{1+\hat{p}} x_2 \qquad \text{and} \qquad \Gamma\left(\hat{\tau}\right) = \tfrac{z_1 \Omega \exp\{\phi(\hat{\tau}-\tau_m)\}}{\Omega - z_1 + z_1 \exp\{\phi(\hat{\tau}-\tau_m)\}} \ , \tag{62}$$

we can establish the following comparative result:

$$\bar{\gamma}_1 \gtreqless \bar{\gamma}_2 \qquad \text{iff} \quad \Pi\left(\hat{p}\right) \gtreqless \Gamma\left(\hat{\tau}\right). \tag{63}$$

Then, going back to equation (57), this result implies that when the conditions for $\bar{\gamma}_2 > \bar{\gamma}_1$ are met, we also have that $\frac{\partial \bar{\gamma}}{\partial N_1} < 0$, and vice versa.

Given the monotonicity of both functions in (62) and the following limit values

$$\left\{ \begin{array}{cc} \lim\limits_{\hat{p}\to 0} \Pi\left(\hat{p}\right) = z_1 & \lim\limits_{\hat{\tau}\to \tau_m} \Gamma\left(\hat{\tau}\right) = z_1 \\ & \\ \lim\limits_{\hat{p}\to +\infty} \Pi\left(\hat{p}\right) = z_1 + x_2 & \lim\limits_{\hat{\tau}\to +\infty} \Gamma\left(\hat{\tau}\right) = \Omega > z_1 \end{array} \right\}, \tag{64}$$

we can conclude that when $\hat{p}$ tends to be low and $\hat{\tau}$ high, an economy where education is only provided or fully funded by the government will enjoy a stronger growth rate than an economy where education is a completely private activity. Conversely, whenever $\hat{p}$ tends to be high and $\hat{\tau}$ low, an economy with a pure public school system will have a weaker growth rate than an economy with a pure private school system. When both $\hat{p}$ and $\hat{\tau}$ are low, it does not appear to be a clear hierarchy between growth rates and, consequently, not between educational systems either.

The most interesting case arises when population N_1 pays a high $\hat{p}$ and N votes for a high $\hat{\tau}$. This represents a situation in which all members of society greatly appreciate education as an engine of growth and a safeguard for their prosperity. In this case $\bar{\gamma}_2 > \bar{\gamma}_1$ if and only if $z_2\left(\tau\right) > z_1 + x_2$, that is

$$\frac{\hat{p}}{1+\hat{p}} x_2 < x_2 < z_2\left(\tau\right) - z_1 = \frac{\left(\Omega - z_1\right) z_1 \left(-1 + \exp\left\{\phi\left(\hat{\tau} - \tau_m\right)\right\}\right)}{\Omega - z_1 + z_1 \exp\left\{\phi\left(\hat{\tau} - \tau_m\right)\right\}} < \omega - z_1 < \Omega - z_1. \tag{65}$$

This means that the difference between the corresponding positive impacts of teacher's quality, resources, and other incentives on the quality of public and private schools, $z_2(\tau) - z_1$, must be greater than the negative impact of class size and composition on public school quality, x_2. From equations (47) and (48) we know that x_2 is an upper bound while z_1 is a lower bound, but $z_1 > x_2$ is a necessary condition for the minimum achievable growth rates in both the economy with a pure public school system and the economy with a pure private school system to be positive. Furthermore, the multiple empirical studies carried out by Hanushek and his different collaborators show that the factors underlying $z_i \; \forall i = \{1, 2\}$ have a strong and statistically significant effect on the efficiency-quality of school systems, while the factors underlying $x_i \; \forall i = \{1, 2\}$ have a weak and statistically insignificant effect. Then, given that society as a whole can democratically choose a value of $z_2(\tau)$ substantially higher than z_1, we conclude that public schools are potentially better positioned than private schools to serve as engines of long-term economic growth.

However, there is a limiting factor that could prevent this outcome: the value of the income tax rate τ that the population N is willing to choose in order for the government to fund the desired quality of public education. We have assumed that the level of public spending on education is determined by the revenue collected with the income tax rate τ, which has been previously decided by a majority vote simultaneously with the decision of individuals to enroll in the public or private education system.[11] In our model, there is no endogenous rule for selecting an optimal τ. It is predetermined outside the model, and since it represents a percentage of the economy's total income, it is reasonable to expect it to be far from unity and closer to τ_m. Once the decision has been made, taxes must be accepted and paid, and it is well-known that governments face significant challenges in collecting them.

5.2 The levels' pace

Finally, we will examine the long-run trajectories for the levels of per capita human capital, output, and broad output that emerge in the economy under each of the two polar educational systems. From equations (42), (43) and (45), in the economy in which education is almost exclusively a private activity, the values of human capital, output and broad output corresponding to the balanced

[11] De la Croix and Doepke (2009) analyze an alternative timing assumption: voters elect a government that commits in advance to a certain level of total spending on education. Then, once the tax is set, they choose between public and private education conditional on this spending.

growth path are

$$\bar{h}_{(N_1=1)} = \frac{h_0}{{}_2\widetilde{F}_1(0)_{\tau=\tau_m}} \exp\left\{\bar{\gamma}_1 t\right\}, \tag{66}$$

$$\bar{y}_{(N_1=1)} = \frac{(1-\tau_m)\, A h_0}{{}_2\widetilde{F}_1\,(0)_{\tau=\tau_m}} \left(\frac{\beta\,(1-\tau_m)\,A}{z_1 - x_1\,(p) + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \left(\frac{\rho - (z_1 - x_1\,(p) - \theta)\,(1-\sigma)}{\sigma\,(z_1 - x_1\,(p))}\right) \exp\left\{\bar{\gamma}_1 t\right\}, \tag{67}$$

$$\bar{q}_{(N_1=1)} = \frac{(1-\tau_m)\, A h_0}{{}_2\widetilde{F}_1(0)_{\tau=\tau_m}} \left(\frac{\beta\,(1-\tau_m)\,A}{z_1 - x_1\,(p) + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \left(1 - \frac{\beta\,(z_1 - x_1\,(p) - \theta - \rho + \sigma\theta)}{\sigma\,(z_1 - x_1\,(p))}\right) \exp\left\{\bar{\gamma}_1 t\right\}. \tag{68}$$

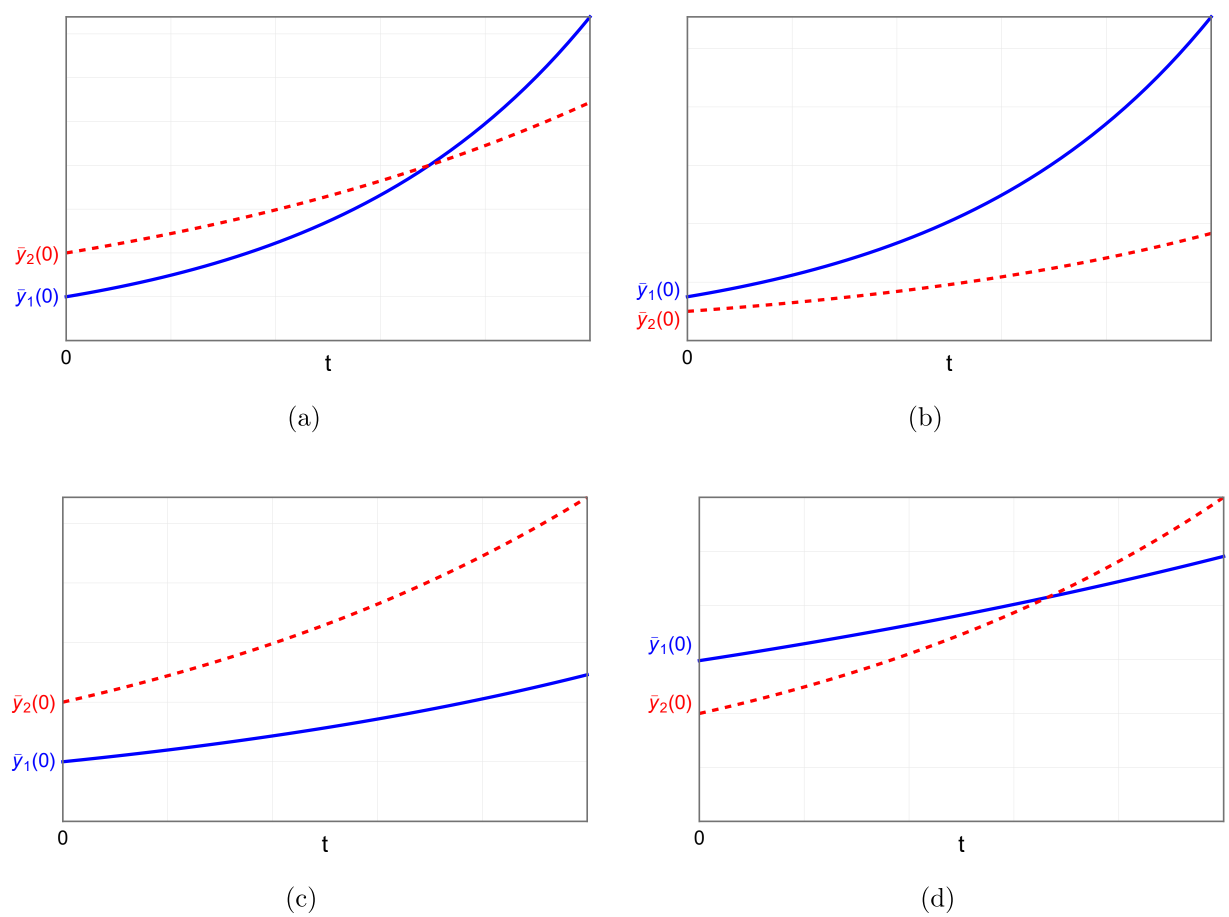

Figure 3: Long-run trajectories of output $\bar{y}(t)$ under pure private education (blue-solid line) and pure public education (red-dashed line).

In the economy where education is only provided or fully funded by the government, the values of human capital, output and broad output corresponding to the balanced growth path are

$$\bar{h}_{(N_1=0)} = \frac{h_0}{{}_2\tilde{F}_1(0)} \exp\left\{\bar{\gamma}_2 t\right\}, \tag{69}$$

$$\bar{y}_{(N_1=0)} = \frac{(1-\tau) A h_0}{{}_2\tilde{F}_1(0)} \left(\frac{\beta(1-\tau) A}{z_2(\tau) - x_2 + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \left(\frac{\rho - (z_2(\tau) - x_2 - \theta)(1-\sigma)}{\sigma(z_2(\tau) - x_2)}\right) \exp\left\{\bar{\gamma}_2 t\right\}, \tag{70}$$

$$\bar{q}_{(N_1=0)} = \frac{(1-\tau) A h_0}{{}_2\tilde{F}_1(0)} \left(\frac{\beta(1-\tau) A}{z_2(\tau) - x_2 + \pi - \theta}\right)^{\frac{\beta}{1-\beta}} \left(1 - \frac{\beta(z_2(\tau) - x_2 - \theta - \rho + \sigma\theta)}{\sigma(z_2(\tau) - x_2)}\right) \exp\left\{\bar{\gamma}_2 t\right\}. \tag{71}$$

From (67) and (68), it can be shown that in the economy in which education is a completely private activity, the result $\bar{q}_{(N_1=1)} > \bar{y}_{(N_1=1)}$ holds even for negative values of p because the parameter restriction associated with the positivity of $\underline{\gamma}_1^{\inf}$ implies that $\frac{x_2}{z_1-\rho-(1-\sigma)\theta} < 1$. From (70) and (71), it can be shown that in the economy where education is only provided or fully funded by the government, the result $\bar{q}_{(N_1=0)} > \bar{y}_{(N_1=0)}$ holds even for values of τ below τ_m because the parameter restrictions associated with the definition (48), and the positivity of $\underline{\gamma}_2^{\inf}$ and $\underline{\gamma}_2^{\sup}$ imply that $\ln\left(\frac{\frac{\Omega}{z_1}-1}{\frac{\Omega}{x_2+\rho+(1-\sigma)\theta}-1}\right) < 0$.

6 Conclusion

The pivotal role of human capital in driving economic growth is beyond dispute. Yet, the question of how best to finance its accumulation through public provision, private investment, or a combination of both remains a subject of vigorous debate in both academic and policy circles. This article contributes to that debate by addressing a fundamental question: Which type of education system, public or private, is better suited to fostering long-term economic growth?

To answer this, we extend the current state-of-the-art of endogenous growth models applied to human capital by incorporating the institutional and socio-educational characteristics that distinguish public and private school systems. We build an endogenous growth model *à la Lucas* to analyze how the presence of a private or public educational system affects the relationship between human capital accumulation and growth. In particular, we examine the impact of the composition of human capital formation on the economy's long-term evolution. Our model offers an alternative to the OLG framework where to study the trade-off between public and private education. Drawing on key findings from empirical research in the economics of education, our model captures how the distribution of students across the two educational systems influences the long-run rate of growth.

Our findings underscore an important insight: the impact of education systems on growth is not determined solely by their public or private nature, but by the institutional and socio-educational attributes that shape their respective levels of efficiency. When these characteristics are taken into account, our model suggests that a public education system is, in most empirically plausible scenarios, better positioned than a private one to promote economic growth in the long-run by providing higher-quality education. In this regard, our results go in line with recent empirical research that points out to the socioeconomic background of students attending private schools as the reason for their perceived success, or the role of cultural aspects (Vandenberghe and Robin, 2004), rather than the quality they provide (Rodríguez-Pose and Henry de Frahan, 2024). By considering a model that explicitly models these characteristics, we deem our results to provide a theoretical basis to guide such findings.

The accumulation of skills as a standard production factor, emphasized by augmented neoclassical growth models, is probably best captured by the basic-literacy term, which has positive effects that are similar in size across all countries. However, one important issue is whether to concentrate attention at the lowest or at the highest levels of education. It is well known that the larger growth effect of high-level skills in countries farther from the technological frontier is most consistent with technological diffusion models. From this perspective, countries need high-skilled human capital for an imitation strategy. Some argue in favor of elitist school systems which focus on the top performers as potential future managers of the economy and drivers of innovation. Others favor more egalitarian school systems to ensure well-educated masses that will be capable of implementing established and new technologies. In other words, should education policy focus on forming a small group of "rocket scientists," or are encompassing approaches more promising in spurring growth?

While highly educated individuals contribute to invention and technological advancement, the lack of a sufficiently educated workforce can hinder and delay technology adoption. More exactly, a large group of educated individuals in the labor force facilitates and drives technology diffusion but technology adoption is more dependent on the human capital of entrepreneurs. All together, invention, adoption and diffusion explain the economy's technological progress that enhance its economic growth. With regard to the implications for the economy's long-run growth, we conclude that developing basic skills and highly talented people reinforce each other. Achieving basic literacy for all may well be a precondition for identifying those who can reach engineer and scientist status. In other words, tournaments among a large pool of students with basic skills may be an efficient way to obtain a large share of high performers.

Finally, we can draw some new insights and predictions about the potential overall effects of educational reforms. From a public policy perspective, and particularly in light of the specific efficiency characteristics of human capital formation that should guide public intervention, government action should be oriented toward several key objectives: first, appropriate checks and balances should be implemented to enhance the performance and accountability of public schools; second, the operation of private schools should be closely monitored, and a clear and coherent regulatory framework should be established; third, the government should prevent the entry of private providers whose primary motivation is purely commercial, rather than educational; and fourth, this approach requires the design of policies that do not favor private initiative at the expense of public provision, especially in ways that may exacerbate negative factors within public schools, thereby widening existing disparities in educational outcomes.

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Appendix: the analytical solution of the dynamic system

From (29) and (30) we get the control functions

$$c = (1 + pN_1)^{-\frac{1}{\sigma}}\, \vartheta_1^{-\frac{1}{\sigma}}, \tag{A.1}$$

$$u = \left(\frac{(1-\beta)(1-\tau)A}{\delta}\right)^{\frac{1}{\beta}} \left(\frac{\vartheta_1}{\vartheta_2}\right)^{\frac{1}{\beta}} \frac{k}{h}, \tag{A.2}$$

where

$$\delta = z_2(\tau) - x_2 + (z_1 - x_1(p) + x_2 - z_2(\tau))\, N_1. \tag{A.3}$$

After substituting the above expressions into equations (31)-(34), we obtain

$$\overset{\bullet}{\vartheta}_2 = -(\delta - \rho - \theta)\, \vartheta_2, \tag{A.4}$$

$$\overset{\bullet}{\vartheta}_1 = (\rho + \pi)\, \vartheta_1 - \psi_1(t)\, \vartheta_1^{\frac{1}{\beta}}, \tag{A.5}$$

$$\overset{\bullet}{k} = \psi_2(t)\, k - \psi_3(t), \tag{A.6}$$

$$\overset{\bullet}{h} = (\delta - \theta)\, h - \psi_4(t), \tag{A.7}$$

where

$$\psi_1(t) = \beta(1-\tau)A\left(\frac{(1-\beta)(1-\tau)A}{\delta}\right)^{\frac{1-\beta}{\beta}} \vartheta_2^{-\frac{1-\beta}{\beta}}, \tag{A.8}$$

$$\psi_2(t) = (1-\tau)A\left(\frac{(1-\beta)(1-\tau)A}{\delta}\right)^{\frac{1-\beta}{\beta}} \left(\frac{\vartheta_1}{\vartheta_2}\right)^{\frac{1-\beta}{\beta}} - \pi, \tag{A.9}$$

$$\psi_3(t) = (1 + pN_1)^{\frac{\sigma-1}{\sigma}}\, \vartheta_1^{-\frac{1}{\sigma}}, \tag{A.10}$$

$$\psi_4(t) = \delta\left(\frac{(1-\beta)(1-\tau)A}{\delta}\right)^{\frac{1}{\beta}} \left(\frac{\vartheta_1}{\vartheta_2}\right)^{\frac{1}{\beta}} k. \tag{A.11}$$

These equations, together with the initial conditions, k_0 and h_0, and the transversality conditions (35) and (36) constitute the dynamic system which drives the economy over time. This dynamic system can be recursively solved in closed form. Boucekkine and Ruiz-Tamarit (2008) show that such a system can be solved explicitly without resorting to any dimension reduction. According to the procedure developed there, we can obtain the exact solution for the long-run trajectories. Any particular non-explosive solution to the dynamic system (A.4)-(A.7) has to satisfy the initial conditions k_0 and h_0, as well as the transversality conditions (35) and (36). These ones impose the constraints

$$(\delta+\pi-\theta)(\beta-\sigma)-\beta(\rho+\pi-\pi\sigma)<-\sigma(1-\beta)(\delta+\pi-\theta)<0, \tag{A.12}$$

$$(\delta-\theta)(1-\sigma)-\rho<0, \tag{A.13}$$

$$\frac{k_0}{{}_2F_1(0)}\left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{\frac{1}{\beta}}=-\frac{\sigma\beta\vartheta_2(0)^{-\frac{1}{\sigma}}\left(\frac{\delta+\pi-\theta}{\epsilon}\right)^{\frac{\sigma-\beta}{\sigma(1-\beta)}}}{(\delta+\pi-\theta)(\beta-\sigma)-\beta(\rho+\pi-\pi\sigma)}, \tag{A.14}$$

$$\frac{{}_2F_1(0)}{{}_2\widetilde{F}_1(0)}=\frac{(1-\beta)\,\epsilon\sigma}{-((\delta-\theta)(1-\sigma)-\rho)\,\beta}\frac{k_0}{h_0}\left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{\frac{1}{\beta}}, \tag{A.15}$$

where

$$\epsilon=\beta(1-\tau)A\left(\frac{(1-\beta)(1-\tau)A}{z_2(\tau)-x_2+(z_1-x_1(p)+x_2-z_2(\tau))N_1}\right)^{\frac{1-\beta}{\beta}}>0. \tag{A.16}$$

Conditions (A.14) and (A.15) make up a system of two equations with two unknowns, $\vartheta_1(0)$ and $\vartheta_2(0)$. Their values may be determined in the following way: (A.15) determines a unique value for the ratio $\frac{\vartheta_1(0)}{\vartheta_2(0)}$, then (A.14) determines the value of $\vartheta_2(0)$, which after multiplying by the value of the ratio itself gives the value of $\vartheta_1(0)$. In these two conditions we use the following Gaussian hypergeometric functions written under the Euler representation form[12]

$${}_2F_1(0)\equiv\ {}_2F_1(a,b;c;z_0)=$$

[12]Recall that $\vartheta_1(0)$ and $\vartheta_2(0)$ are both finite and different from zero, ${}_2F_1(0)={}_2F_1(a,b;c;z_0)$ and ${}_2\widetilde{F}_1(0)={}_2F_1\left(\widetilde{a},b;c;z_0\right)$ are constant, ${}_2F_1(\infty)={}_2F_1(a,b;c;0)=1$ and ${}_2\widetilde{F}_1(\infty)={}_2F_1\left(\widetilde{a},b;c;0\right)=1$.

$$= {}_2F_1\left(a, b; 1+a; z_0\right) = {}_2F_1\left(b, a; 1+a; z_0\right) =$$

$$= \frac{\Gamma(1+a)}{\Gamma(a)\Gamma(1)} \int_0^1 t^{a-1}(1-tz_0)^{-b} dt = a \int_0^1 t^{a-1}(1-tz_0)^{-b} dt =$$

$$= \left(1+\widetilde{a}\right) \int_0^1 t^{\widetilde{a}}(1-tz_0)^{-b} dt \tag{A.17}$$

and

$${}_2\widetilde{F}_1(0) \equiv {}_2F_1\left(\widetilde{a}, b; c; z_0\right) =$$

$$= {}_2F_1\left(\widetilde{a}, b; 2+\widetilde{a}; z_0\right) = {}_2F_1\left(b, \widetilde{a}; 2+\widetilde{a}; z_0\right) =$$

$$= \frac{\Gamma(2+\widetilde{a})}{\Gamma(\widetilde{a})\Gamma(2)} \int_0^1 t^{\widetilde{a}-1}(1-t)(1-tz_0)^{-b} dt =$$

$$= a(a-1) \int_0^1 t^{a-2}(1-t)(1-tz_0)^{-b} dt =$$

$$= \left(1+\widetilde{a}\right) \widetilde{a} \int_0^1 t^{\widetilde{a}-1}(1-t)(1-tz_0)^{-b} dt \tag{A.18}$$

where

$$a = -\frac{(\delta+\pi-\theta)(\beta-\sigma) - \beta(\rho+\pi-\pi\sigma)}{\sigma(\delta+\pi-\theta)(1-\beta)} > 1, \tag{A.19}$$

$$\widetilde{a} = a - 1 = -\frac{\beta\left((\delta-\theta)(1-\sigma)-\rho\right)}{\sigma(\delta+\pi-\theta)(1-\beta)} > 0, \tag{A.20}$$

$$b = -\frac{\beta-\sigma}{\sigma(1-\beta)}, \quad c = 1 + a = 2 + \widetilde{a}, \tag{A.21}$$

$$z_0 = 1 - \frac{\delta+\pi-\theta}{\epsilon}\left(\frac{\vartheta_1(0)}{\vartheta_2(0)}\right)^{-\frac{1-\beta}{\beta}} \in \left]-\infty, 1\right[. \tag{A.22}$$

The long-run closed-form trajectories for the variables in our model are as follows[13]

$$\bar{\vartheta}_1 = \left(\frac{\delta+\pi-\theta}{\epsilon}\right)^{\frac{\beta}{1-\beta}} \vartheta_2(0)\exp\left\{-\left(\delta-\rho-\theta\right)t\right\}, \tag{A.23}$$

$$\bar{\vartheta}_2 = \vartheta_2(0)\exp\left\{-\left(\delta-\rho-\theta\right)t\right\}, \tag{A.24}$$

$$\overline{\left(\frac{\vartheta_1}{\vartheta_2}\right)} = \frac{\delta}{\left(1-\beta\right)\left(1-\tau\right)A}\left(\frac{\delta+\pi-\theta}{\beta\left(1-\tau\right)A}\right)^{\frac{\beta}{1-\beta}} > 0, \tag{A.25}$$

$$0 < \bar{u} = -\frac{\left(\delta-\theta\right)\left(1-\sigma\right)-\rho}{\sigma\delta} < 1, \tag{A.26}$$

$$\bar{c} = \left(1+pN_1\right)^{-\frac{1}{\sigma}}\left(\frac{\delta+\pi-\theta}{\epsilon}\right)^{\frac{\beta}{\sigma(1-\beta)}} \vartheta_2^{-\frac{1}{\sigma}}(0)\exp\left\{\frac{\delta-\theta-\rho}{\sigma}t\right\}, \tag{A.27}$$

$$\overline{\left(\frac{k}{h}\right)} = -\frac{\left(\delta-\theta\right)\left(1-\sigma\right)-\rho}{\sigma\delta}\left(\frac{\beta\left(1-\tau\right)A}{\delta+\pi-\theta}\right)^{\frac{1}{1-\beta}} > 0, \tag{A.28}$$

$$\bar{k} = -\frac{\sigma\beta\left(\frac{(1-\beta)(1-\tau)A}{\delta\vartheta_2(0)}\right)^{\frac{1}{\sigma}}\left(\frac{\beta(1-\tau)A}{\delta+\pi-\theta}\right)^{\frac{\beta}{\sigma(1-\beta)}}}{(\delta+\pi-\theta)(\beta-\sigma)-\beta(\rho+\pi-\pi\sigma)}\exp\left\{\frac{\delta-\theta-\rho}{\sigma}t\right\}, \tag{A.29}$$

$$\bar{h} = \frac{h_0}{{}_2\widetilde{F}_1(0)}\exp\left\{\frac{\delta-\theta-\rho}{\sigma}t\right\}, \tag{A.30}$$

$$\bar{y} = \left(1-\tau\right)A\overline{\left(\frac{k}{h}\right)}^{\beta}\bar{u}^{1-\beta}\bar{h} = \left(1-\tau\right)A\left(\frac{\beta\left(1-\tau\right)A}{\delta+\pi-\theta}\right)^{\frac{\beta}{1-\beta}}$$

[13] The results encompass three different particular cases arising from the relationship between the inverse of the intertemporal elasticity of substitution, σ, and the physical capital share, β. They correspond to $b \gtreqless 0$, which depend on $\sigma \gtreqless \beta$.

$$\left(\frac{\rho-(\delta-\theta)(1-\sigma)}{\sigma\delta}\right)\frac{h_0}{{}_2\widetilde{F}_1(0)}\exp\left\{\frac{\delta-\theta-\rho}{\sigma}t\right\}. \tag{A.31}$$

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