Exponential stability of nonlinear infinite-dimensional systems: application to nonisothermal axial dispersion tubular reactors

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Abstract

Exponential stability of equilibria of nonlinear distributed parameter systems is considered. A general framework is set with related assumptions. In particular it is shown how to get local exponential stability of an equilibrium profile for the corresponding nonlinear system based on stability results for the linearized one. For this purpose a weakend concept of Fréchet differentiability is required for the nonlinear semigroup generated by the nonlinear model, with links to "Al Jamal, R., Morris, K., 2018, Linearized stability of partial differential equations with application to stabilization of the Kuramoto-Sivashinsky equation, SIAM Journal on Control and Optimization 56 (1), 120–147". The theoretical results are applied to a nonisothermal axial dispersion tubular reactor model and are illustrated with numerical simulations.

Keywords: Nonlinear distributed parameter systems - Fréchet/Gâteaux derivatives - Equilibrium profiles - Nonisothermal axial dispersion tubular reactor - Bistability

1. Introduction

Analyzing the stability of equilibrium profiles for infinite dimensional systems is paramount in order to stabilize a system. Indeed this could be done either by stabilizing an unstable equilibrium or even by improving the stability margin of a stable one.

When one deals with nonlinear distributed parameter systems, conclusions on the stability can be hard to achieve. For finite dimensional systems it is well-known that the conclusions on the local stability of an equilibrium profile of a nonlinear system are the same as for its linearized tangent version. The approach used here is quite similar but in the infinite-dimensional setting.

The concept of linearized stability is discussed in (Henry, 1981) wherein assumptions on the nonlinear operator describing the dynamics are stated. The nonlinear operator has to be continuously Fréchet differentiable, locally Hölder continuous in time and locally Lipschitz continuous. As long as the linearized system around a chosen equilibrium is exponentially stable, (Henry, 1981, Theorem 5.1.1) guarantees that the same conclusion holds for the original system, locally around that equilibrium. In (Smoller, 1983), the same conclusion on the nonlinear system is shown to hold provided that the nonlinear semigroup generated by the nonlinear dynamics is Fréchet differentiable, see (Smoller, 1983, Theorem 11.22). Sufficient conditions on the nonlinear operator are needed to get this assumption, see e.g. (Smoller, 1983, Theorems 11.17, 11.18). Two years later, Webb showed the same result with other assumptions and applied it to the age-dependent population problem, see (Webb, 1985, Theorem 4.12). Fréchet differentiability of nonlinear semigroups is also studied in (Temam, 1997). Conditions are required on both linear and nonlinear parts of the semilinear dynamics. For instance, the linear operator has to be closed, negative and self-adjoint while the nonlinear one has to satisfy some appropriate decomposition, see e.g. (Temam, 1997, Section 8). This result has been presented again in (Al Jamal et al., 2014, Theorem 3.8). In (Kato, 1995), it is shown that exponential stability of an equilibrium of a nonlinear system is guaranteed as long as the same behavior holds for a Gâteaux linearized version of the nonlinear operator describing the dynamics, as long as the nonlinear operator is Fréchet differentiable. Other assumptions like the Lipschitz continuity of the Fréchet derivative in the operator norm are also needed.

More recently in (Al Jamal et al., 2014), similar results are established and applied to the Kuramoto-Sivashinski nonlinear partial differential equations (PDEs). In that paper the limitations of the theory are also discussed. Indeed the assumptions that are required on both the linearized system and the nonlinearity could be hard to verify for many models of nonlinear PDEs. The main difficulty is due to the Fréchet differentiability of nonlinear operators which is in general not guaranteed since the considered operators

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are unbounded. This is the reason why one often works on the semigroup instead of the generator, see for instance (Al Jamal, 2013; Al Jamal and Morris, 2018).
Our approach is inspired by previous works but is quite different in the sense that the assumptions we need are weaker and generally easier to apply. For instance, the results presented in (Al Jamal et al., 2014) to get Fréchet differentiability of nonlinear semigroups can not be applied on the example that is considered therein and a case-by-case study has generally to be done. These results are also not applicable to our case study, mainly due to the fact that Fréchet differentiability depends strongly on the chosen state space. It is therefore challenging to obtain since the norms are not equivalent in infinite dimension. In this perspective, we adapted the concept of Fréchet differentiability by incorporating different norms in the definition. This allows more flexibility in the manipulation of norm inequalities, providing Fréchet differentiability conditions that can be verified more easily. The considered systems are semilinear. Some standard assumptions on the linearized model are formulated. Other assumptions on the nonlinear operator are also needed. It has notably to verify some relaxed Fréchet differentiability condition. Under these assumptions, it is shown how to get Fréchet differentiability of the nonlinear semigroup in a weaker sense in order to provide appropriate local exponential stability of the nonlinear system. Note that the Lipschitz continuity of the Fréchet derivative of the nonlinear operator in the operator topology, often hard to obtain, is not requested here. Note that the exponential stability of the linearized system is paramount in the analysis. As it is highlighted in (Al Jamal et al., 2014), the results do not apply if one assumes asymptotic stability for instance. It is meant here that if the linearized system is asymptotically stable, there is no guarantee that the nonlinear system preserves that property, even if the nonlinear semigroup is Fréchet differentiable, see e.g. (Al Jamal et al., 2014, Example 3.4).
Our new results are also applied on a specific nonlinear distributed parameter system consisting in a nonisothermal axial dispersion tubular reactor with Arrhenius type reaction kinetics, in order to deduce exponential stability or instability of the equilibria for the nonlinear model. Such systems have been studied in (Aksikas et al., 2007b; Laabissi et al., 2001) among others. Asymptotic stability of the equilibria of the nonlinear model has been studied in (Varma and Aris, 1977; Aksikas et al., 2007a) among others. In (Varma and Aris, 1977), this is studied for a Gâteaux linearized version of the nonlinear dynamics while (Aksikas et al., 2007a) solves the question of asymptotic stability for the nonlinear system without taking diffusion phenomena into account. Numerical methods have also been introduced in (McGowin and Perlmutter, 1970; Amundson, 1965; Lefèvre et al., 2000). Here the novelty consists in analyzing the exponential stability of the equilibria for the nonlinear PDE system by using exponential stability of a corresponding linearized model and by using a specific concept of Fréchet differentiability of the nonlinear semigroup that is generated by the nonlinear dynamics. The paper is organized as follows. In Section 2 the general framework we are working with is introduced and the corresponding assumptions are stated. Moreover, the concepts of generalized Fréchet differentiability and of local and global exponential stability are described. Section 3 is dedicated to the statements and the proofs of some important results in order to make the link between exponential stability of a linearized system and exponential stability of the nonlinear nominal one around an equilibrium profile. These results are shown to be extensions of theorems in (Al Jamal and Morris, 2018). In Section 4, a class of nonlinear distributed parameter systems related to chemical engineering is shown to fit the assumptions required in Section 2. Then a linearized model around any equilibrium is built and it is shown how to get exponential stability of the equilibrium. The last part of the section is devoted to the application of the main results in order to establish local exponential stability of the equilibrium for the nonlinear model. In Section 5 the results are illustrated by some numerical simulations.

2. Background and problem statement

Let X be a real Hilbert space. Let us consider the following semi-linear infinite-dimensional system

$$\begin{cases} \dot{\xi} = A\xi + f(\xi) \\ \xi(0) = \xi_0, \end{cases} \tag{1}$$

where the linear operator $A : D(A) \subset X \to X$ is supposed to be the infinitesimal generator of a quasicontractive[1] C_0-semigroup on X, $f : \mathcal{D} \subset X \to X$ is a static nonlinear operator defined on a closed convex subset $\mathcal{D}$ and the initial condition ξ_0 is chosen in $D(A) \cap \mathcal{D}$. Such a class of systems is studied in (Curtain and Zwart, 1995; Engel and Nagel, 2006) among others.

Remark 2.1. *Since the operator A is the infinitesimal generator of a quasicontractive C_0-semigroup, there exists some nonnegative constant l_A such that $A - l_A I$ is dissipative on X.*

The following assumptions are quite standard in order to study systems like (1), see for instance (Aksikas et al., 2007a).

Assumption 2.1. *The closed subset $\mathcal{D}$ is $T(t)-$invariant, i.e. $T(t)\mathcal{D} \subset \mathcal{D}$, for all $t \geq 0$, where $(T(t))_{t\geq 0}$ is the linear C_0-semigroup generated by the operator A. In addition, assume that for all $\xi \in \mathcal{D}, \lim_{h\to 0^+} \frac{1}{h} d(\xi + hf(\xi); \mathcal{D}) = 0$ where the distance function d is defined as $d(\xi; \mathcal{D}) := \inf \{\|\xi - y\|, y \in \mathcal{D}\}$*

[1] A linear C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X satisfying $\|T(t)\|_X \leq e^{\omega t}, t \geq 0$ is called quasicontractive, see e.g. (Engel and Nagel, 2006, Chapter 2).

for any $\xi \in X$. Moreover, suppose that the operator f is Lipschitz continuous on $\mathcal{D}$ and that there exists $l_f \in \mathbb{R}^+$ such that the operator $f - l_f I$ is dissipative on $\mathcal{D}$.

The following definition is dedicated to the dissipativity of nonlinear operators, see e.g. (Augner, 2019, Definition 2.4).

Definition 2.1. *The operator $f : \mathcal{D} \subset X \to X$ is called dissipative on $\mathcal{D}$ if for all $\xi_1, \xi_2 \in \mathcal{D}$,*

$$\langle f(\xi_1) - f(\xi_2), \xi_1 - \xi_2 \rangle_X \leq 0.$$

By (Laabissi et al., 2001, Theorem 2.1), Assumption 2.1 implies that equation (1) has a unique mild solution on $[0, +\infty[$, for all $\xi_0 \in \mathcal{D}$. By defining $\tilde{S}(t)\xi_0 = \xi(t)$ for all $t \geq 0$, $(\tilde{S}(t))_{t\geq 0}$ is a nonlinear semigroup on $\mathcal{D}$ whose infinitesimal generator is $A + f$. Note that the condition $\lim_{h\to 0^+} \frac{1}{h} d(\xi + hf(\xi); \mathcal{D}) = 0$ implies that $\mathcal{D}$ is $\tilde{S}(t)-$invariant.
Suppose that system (1) has some equilibrium $\xi^e \in D(A) \cap \mathcal{D}$, i.e. $A\xi^e + f(\xi^e) = 0$ and let us consider the following assumption.

Assumption 2.2. *The nonlinear operator f is Gâteaux differentiable at ξ^e, i.e. there exists a linear operator $df(\xi^e) : X \to X$ such that*

$$\lim_{\epsilon\to 0} \frac{f(\xi^e + \epsilon\xi) - f(\xi^e)}{\epsilon} = df(\xi^e)\xi,$$

where $\xi^e, \xi^e + \epsilon\xi \in \mathcal{D}$. Note that, since $\mathcal{D}$ is a convex subset, any convex combination of ξ^e and $\xi^e + \epsilon\xi$ lies in $\mathcal{D}$, that is, $a\xi^e + (1-a)(\xi^e + \epsilon\xi) \in \mathcal{D}$ for all $a \in [0, 1]$. In other words we have that $\xi^e + (1-a)\epsilon\xi \in \mathcal{D}$, meaning that if $\xi^e, \xi^e + \epsilon_0\xi \in \mathcal{D}$ for some nonnegative ϵ_0 then $\xi^e + \epsilon\xi \in \mathcal{D}$ for any nonnegative $\epsilon < \epsilon_0$. Moreover, suppose that the operator $df(\xi^e)$, which will be called the Gâteaux derivative of f (at ξ^e), is bounded on X.

Let us define $\hat{\xi} = \xi - \xi^e$ and $\mathcal{D}^e = \mathcal{D} - \xi^e$. Hence, system (1) becomes

$$\begin{cases} \dot{\hat{\xi}} = A\hat{\xi} + f(\hat{\xi} + \xi^e) - f(\xi^e) \\ \hat{\xi}(0) = \xi_0 - \xi^e := \hat{\xi}_0. \end{cases} \tag{2}$$

The null state is obviously an equilibrium of (2). Since, by Assumption 2.1, system (1) is well-posed, system (2) is still well-posed in the sense that the operator $A + f(\cdot + \xi^e) - f(\xi^e)$ generates a nonlinear C_0-semigroup on $D(A) \cap \mathcal{D}^e$ denoted by $(S(t))_{t\geq 0}$. That is $\hat{\xi}(t) = S(t)\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$ for every $t \geq 0$ and $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$.

Lemma 2.1. *Under the assumption that $\mathcal{D}$ is $\tilde{S}(t)-$invariant, the shifted subset $\mathcal{D}^e$ is $S(t)-$invariant.*

Proof. In terms of semigroups, the change of variable $\hat{\xi} = \xi - \xi^e$ is expressed as

$$S(t)\hat{\xi}_0 = \tilde{S}(t)\xi_0 - \xi^e, \tag{3}$$

where $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e, \xi_0 \in D(A) \cap \mathcal{D}$ and $t \geq 0$. By assumption, $\tilde{S}(t)\xi_0 \in \mathcal{D}$ which entails that $S(t)\hat{\xi}_0 \in \mathcal{D}^e$. ■

The following lemma characterizes the state trajectory corresponding to (2) on a finite time interval $[0, t_0]$ where $t_0 > 0$.

Lemma 2.2. *Suppose that $f \in L^p([0, t_0]; X)$ for some $p \geq 1$, $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$ and that Assumptions 2.1 and 2.2 are satisfied. Then $\hat{\xi}$ and $f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}$ lie in $L^\infty([0, t_0]; X)$ and are continuous.*

Proof. Let us consider $t \in [0, t_0]$. By (Curtain and Zwart, 1995, Lemma 3.1.5) and (Temam, 1997, Theorem II.3.4), the state trajectory $\hat{\xi}$ lies in $L^\infty([0, t_0]; X)$ and is continuous since $f \in L^p([0, t_0]; X)$ for some $p \geq 1$. Moreover

$$\begin{aligned} &\|f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)} \\ &= \sup_{t\in[0,t_0]} \|f(\hat{\xi}(t) + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}(t)\|_X \\ &\leq \sup_{t\in[0,t_0]} (\|f(\hat{\xi}(t) + \xi^e) - f(\xi^e)\|_X + \|df(\xi^e)\|_{op}\|\hat{\xi}(t)\|_X), \end{aligned}$$

where $\|\cdot\|_{op}$ denotes the appropriate operator norm and where the boundedness of $df(\xi^e)$ has been used. Since f is Lipschitz continuous, one gets that

$$\begin{aligned} &\|f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)} \\ &\leq (l + \|df(\xi^e)\|_{op}) \sup_{t\in[0,t_0]} \|\hat{\xi}(t)\|_X \\ &= (l + \|df(\xi^e)\|_{op})\|\hat{\xi}\|_{L^\infty([0,t_0];X)} < \infty. \blacksquare \end{aligned}$$

The following definition generalizes the concept of Fréchet derivative for nonlinear operators that are defined on a subspace of X in a weaker sense than the usual concept. Note that, in the following, the nonlinear operator is defined on $D(A) \cap \mathcal{D}^e$.

Definition 2.2. *Let us consider the nonlinear operator $f : D(A) \cap \mathcal{D} \subset X \to X$. Let $(Y, \|\cdot\|_Y)$ be an infinite-dimensional space (possibly Banach) such that $D(A) \cap \mathcal{D}^e \subset Y \subseteq X$ and $\|h\|_X \leq \sigma\|h\|_Y$ for all $h \in D(A) \cap \mathcal{D}^e$ and $\sigma > 0$. The operator f is called $(Y, X)-$Fréchet differentiable at ξ^e if there exists a bounded linear operator $df(\xi^e) : X \to X$ such that for all $h \in D(A) \cap \mathcal{D}^e$, $f(\xi^e + h) - f(\xi^e) = df(\xi^e)h + R(\xi^e, h)$ where*

$$\lim_{\|h\|_Y\to 0} \frac{\|R(\xi^e, h)\|_X}{\|h\|_X} = 0,$$

or equivalently,

$$\lim_{\|h\|_Y\to 0} \frac{\|f(\xi^e + h) - f(\xi^e) - df(\xi^e)h\|_X}{\|h\|_X} = 0,$$

that is for all $\epsilon > 0$, there exists $\delta > 0$ such that for all

$h \in D(A) \cap \mathcal{D}^e, \|h\|_Y < \delta$ *implies that*

$$\frac{\|f(\xi^e + h) - f(\xi^e) - df(\xi^e)h\|_X}{\|h\|_X} < \epsilon.$$

Note that convergence in Y *implies convergence in* X *since* $\|h\|_X \leq \sigma \|h\|_Y$ *for all* $h \in D(A) \cap \mathcal{D}^e$.

Observe that if Y coincides with X, and if the nonlinear operator f is defined on the whole space X, Definition 2.2 corresponds to the classical definition of Fréchet differentiably, see for instance (Al Jamal and Morris, 2018), which will be called $X-$Fréchet differentiability here.
Let us recall that our objective is to link stability and instability, especially exponential stability properties of a linearized model of (2), with the original nonlinear one. For this purpose, let us introduce the following definitions.

Definition 2.3. *(See (Al Jamal, 2013)) The equilibrium* ξ^e *of (1) is said to be globally exponentially stable if there exist* $\alpha, \beta > 0$ *such that for all* $\xi_0 \in D(A) \cap \mathcal{D}$, *it holds* $\|\xi(t) - \xi^e\|_X \leq \alpha e^{-\beta t} \|\xi_0 - \xi^e\|_X, t \geq 0$, *or equivalently,* $\|\hat{\xi}(t)\|_X \leq \alpha e^{-\beta t} \|\hat{\xi}_0\|_X$ *for all* $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$.

Definition 2.4. *The equilibrium* ξ^e *of (1) is said to be* $(Y, X)-$*locally exponentially stable if there exist* $\delta, \alpha, \beta > 0$ *such that, for all* $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$ *with* $\|\hat{\xi}_0\|_Y < \delta$, *there holds* $\|\hat{\xi}(t)\|_X \leq \alpha e^{-\beta t} \|\hat{\xi}_0\|_X, t \geq 0$.

Definition 2.5. *The equilibrium* ξ^e *of (1) is said to be* $(Y, X)-$*locally stable if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e, \|\hat{\xi}_0\|_Y < \delta$ *implies* $\|\hat{\xi}(t)\|_X < \epsilon, t \geq 0$. *The equilibrium* ξ^e *is* $(Y, X)-$*(locally) unstable if it is not stable.*

Let us state the following assumption that is related to the $(Y, X)-$Fréchet differentiability of the nonlinear operator f and the continuous dependence of the solution $\hat{\xi}$ of equation (2) with respect to the initial condition $\hat{\xi}_0$.

Assumption 2.3. *Let us consider* $\hat{\xi}(t)$, *the solution to (2) for the initial condition* $\hat{\xi}_0$, *where* $0 \leq t \leq t_0$. *Let us consider a space* $(Y, \|\cdot\|_Y)$ *that satisfies* $D(A) \cap \mathcal{D}^e \subset Y \subseteq X$. *It is assumed that the nonlinear operator* f *is* $(Y, X)-$*Fréchet differentiable at* ξ^e *and that* $\hat{\xi}$ *is continuously dependent of the initial condition* $\hat{\xi}_0$ *on* X *and on* Y *at zero in the sense that the inequalities*

$$\|\hat{\xi}(t)\|_X \leq \gamma_t \|\hat{\xi}_0\|_X, \tag{4}$$

$$\|\hat{\xi}(t)\|_Y \leq \tilde{\gamma}_t \|\hat{\xi}_0\|_Y \tag{5}$$

hold for some positive $\gamma_t, \tilde{\gamma}_t$ *that may depend on time.*

Note that, if $f \in L^p([0, t_0]; X)$ for some $p \geq 1$, it follows that $\hat{\xi}$ and $f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}$ are in $L^\infty([0, t_0]; X)$ and are continuous, see Lemma 2.2. In that case, let us consider the following lemma that is a consequence of Assumption 2.3.

Lemma 2.3. *Let us consider* $\hat{\xi}(t)$, *the solution of the abstract differential equation (2), where* $t \in [0, t_0]$ *for some positive* t_0 *and where* f *is assumed to be in* $L^p([0, t_0]; X)$ *for some* $p \geq 1$. *Then, under Assumptions 2.1, 2.2 and 2.3,*

$$\lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} = 0.$$

Proof. Observe that the function $f(\hat{\xi}(\cdot) + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}(\cdot)$ is time–continuous on the interval $[0, t_0]$. Hence there exists $t^* \in [0, t_0]$ such that

$$\begin{aligned} &\sup_{t \in [0,t_0]} \|f(\hat{\xi}(t) + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}(t)\|_X \\ &= \|f(\hat{\xi}(t^*) + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}(t^*)\|_X. \end{aligned} \tag{6}$$

Moreover, according to (4), one has

$$\frac{1}{\|\hat{\xi}_0\|_X} \leq \gamma_{t^*} \frac{1}{\|\hat{\xi}(t^*)\|_X}. \tag{7}$$

Combining (6) and (7) yields

$$\begin{aligned} &\frac{\|f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} \\ &= \frac{\|f(\hat{\xi}(t^*) + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}(t^*)\|_X}{\|\hat{\xi}_0\|_X} \\ &\leq \gamma_{t^*} \frac{\|f(\hat{\xi}(t^*) + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}(t^*)\|_X}{\|\hat{\xi}(t^*)\|_X}. \end{aligned}$$

Thanks to (4), imposing that $\|\hat{\xi}_0\|_Y$ converges to 0 implies that so does $\|\hat{\xi}(t^*)\|_Y$. By the $(Y, X)-$Fréchet differentiability of f, we have

$$\lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\gamma_{t^*} \|f(\hat{\xi}(t^*) + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}(t^*)\|_X}{\|\hat{\xi}(t^*)\|_X} = 0.$$

In view of the inequality above, it follows that

$$\lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} = 0.$$

■

The following assumption is needed in order to guarantee that the problem is well-posed on the auxiliary space Y.

Assumption 2.4. *The nonlinear abstract Cauchy problem (2) is well-posed on* Y. *Moreover, it is assumed that the Gâteaux derivative* $df(\xi^e)$ *of* f *is bounded on* Y, *hence the linearized dynamics corresponding to (2) is well-posed on* Y. *The nonlinear* C_0-*semigroup* $(S(t))_{t \geq 0}$ *is also assumed to be* $Y-$*Fréchet differentiable.*

3. Stability analysis

In this section the link between exponential stability of a linearized model corresponding to (2) and (local) exponential stability of the original nonlinear model (2) is presented. First let us linearize (2) around its equilibrium $\hat{\xi} = 0$ where the linearization is computed via the Gâteaux derivative of f. This yields

$$\begin{cases} \dot{\overline{\xi}} = A\overline{\xi} + df(\xi^e)\overline{\xi} \\ \overline{\xi}(0) = \hat{\xi}_0 \in D(A) \cap \mathcal{D}^e, \end{cases} \tag{8}$$

where ξ^e still denotes the chosen equilibrium to (1) and the initial condition $\overline{\xi}(0)$ is willingly the same as the one chosen for (2). Note that, since the operator A is the infinitesimal generator of a C_0-semigroup on X and $df(\xi^e)$ is bounded by Assumption 2.2, the operator $A + df(\xi^e)$ is still the generator of a C_0-semigroup on X, see (Engel and Nagel, 2006, Bounded Perturbation Theorem). We denote this C_0-semigroup by $(T(t))_{t\geq 0}$. That is, $\overline{\xi}(t) = T(t)\hat{\xi}_0, t \geq 0, \hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$.

The following lemma shows the $(Y, X)-$Fréchet differentiability of the nonlinear semigroup $S(t)$ with $T(t)$ as its Fréchet derivative, where $T(t)$ is the semigroup generated by the Gâteaux derivative of $A + f(\cdot + \xi^e) - f(\xi^e)$ at 0.

The last assumption one needs concerns the linearized C_0-semigroup $(T(t))_{t\geq 0}$ on Y, around an exponentially stable equilibrium ξ^e.

Assumption 3.1. *For the case where $(T(t))_{t\geq 0}$ is exponentially stable on X, it is assumed that $(T(t))_{t\geq 0}$ satisfies*

$$\|T(t)\hat{\xi}_0\|_Y \leq \eta\|\hat{\xi}_0\|_Y, t \geq 0, \forall \hat{\xi}_0 \in Y,$$

for some $\eta > 0$.

Remark 3.1. *1) Assumption 3.1 implies Lyapunov stability of the equilibrium ξ^e of system (1) on the space Y.*

2) Under Assumptions 2.4 and 3.1, the estimate

$$\|S(t)\hat{\xi}_0\|_Y \leq M\|\hat{\xi}_0\|_Y, t \geq 0, \text{ for } \|\hat{\xi}_0\|_Y < \delta \tag{9}$$

holds for some $M > 0$ and $\delta > 0$ that may depend on M. Indeed, the $Y-$Fréchet differentiability of $(S(t))_{t\geq 0}$ in Assumption 2.4 yields the identity

$$S(t)\hat{\xi}_0 = T(t)\hat{\xi}_0 + r(\xi^e, \hat{\xi}_0),$$

where

$$\lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|r(\xi^e, \hat{\xi}_0)\|_Y}{\|\hat{\xi}_0\|_Y} = 0,$$

that is, for all $\epsilon > 0$, there exists $\tilde{\delta} > 0$ such that $\|r(\xi^e, \hat{\xi}_0)\|_Y < \epsilon\|\hat{\xi}_0\|_Y$ for $\|\hat{\xi}_0\|_Y < \tilde{\delta}$. Let us pick any $\epsilon > 0$. Then $\delta := \tilde{\delta} > 0$ is such that

$$\begin{aligned} \|S(t)\hat{\xi}_0\|_Y &\leq \|T(t)\hat{\xi}_0\|_Y + \|r(\xi^e, \hat{\xi}_0)\|_Y \\ &\leq \eta\|\hat{\xi}_0\|_Y + \epsilon\|\hat{\xi}_0\|_Y =: M\|\hat{\xi}_0\|_Y, \end{aligned}$$

for $\|\hat{\xi}_0\|_Y \leq \delta$ and $M := \eta + \epsilon$.

3) Let us consider the function $|||x||| := \sup_{t\geq 0} \|S(t)x\|_Y$ for $x \in Y, \|x\|_Y < \delta$. The quantities $|||\cdot|||$ and $\|\cdot\|_Y$ are locally equivalent around the equilibrium ξ^e, that is

$$\|x\|_Y \leq |||x||| \leq M\|x\|_Y,$$

for some $M > 0$ and $\|x\|_Y < \delta, \delta > 0$. This is valid since (9) is satisfied. Moreover, $(S(t))_{t\geq 0}$ satisfies a contraction property when evaluated with $|||\cdot|||$ on Y: see the proof of Theorem 3.1 below.

Lemma 3.1. *Let us consider a space $(Y, \|\cdot\|_Y)$ satisfying $D(A)\cap\mathcal{D}^e \subset Y \subseteq X$. Under Assumptions 2.1, 2.2 and 2.3, the nonlinear C_0-semigroup $(S(t))_{t\geq 0}$ is $(Y, X)-$Fréchet differentiable at 0 and its Fréchet derivative is given by the linear C_0-semigroup $(T(t))_{t\geq 0}$ whose infinitesimal generator is $A + df(\xi^e)$, that corresponds to the Gâteaux derivative of $A + f(\cdot + \xi^e) - f(\xi^e)$ at 0.*

Proof. Take $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$ and $t \in [0, t_0]$. Let us define $\phi(t) = \hat{\xi}(t) - \overline{\xi}(t)$, where $\hat{\xi}(t)$ and $\overline{\xi}(t)$ are the solutions to (2) and (8) at time t, respectively, with the same initial conditions $\hat{\xi}_0$. It follows that

$$\begin{aligned} \dot{\phi}(t) &= \dot{\hat{\xi}}(t) - \dot{\overline{\xi}}(t) \\ &= A\hat{\xi}(t) + f(\hat{\xi}(t) + \xi^e) - f(\xi^e) - A\overline{\xi}(t) - df(\xi^e)\overline{\xi}(t) \\ &= A\phi(t) + df(\xi^e)\hat{\xi}(t) - df(\xi^e)\hat{\xi}(t) + f(\hat{\xi}(t) + \xi^e) \\ &\quad - f(\xi^e) - df(\xi^e)\overline{\xi}(t) = A\phi(t) + df(\xi^e)\phi(t) + R(\xi^e, \hat{\xi}(t)), \end{aligned}$$

where $R(\xi^e, \hat{\xi}(t)) = f(\hat{\xi}(t) + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}(t)$. Obviously, $\phi(0) = 0$. By taking the inner product of $\phi(t)$ and $\dot{\phi}(t)$, one gets that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\phi(t)\|_X^2 &= \langle\dot{\phi}(t), \phi(t)\rangle_X \\ &= \langle A\phi(t) + df(\xi^e)\phi(t) + R(\xi^e, \hat{\xi}(t)), \phi(t)\rangle_X \\ &= \langle(A - l_A I)\phi(t), \phi(t)\rangle_X + \langle l_A\phi(t), \phi(t)\rangle_X \\ &\quad + \langle df(\xi^e)\phi(t), \phi(t)\rangle_X + \langle R(\xi^e, \hat{\xi}(t)), \phi(t)\rangle_X \\ &\leq (l_A + \|df(\xi^e)\|_{op})\,\|\phi(t)\|_X^2 + \|R(\xi^e, \hat{\xi}(t))\|_X\|\phi(t)\|_X, \end{aligned}$$

where Assumption 2.2 and the Cauchy-Schwarz inequality have been used. The notation $l_A + \|df(\xi^e)\|_{op} =: k$ is adopted in what follows. By applying Young's inequality to the previous inequality, see e.g. (Krstic and Smyshlyaev, 2008), it follows that

$$\frac{1}{2}\frac{d}{dt}\|\phi(t)\|_X^2 \leq (k + \frac{1}{2})\|\phi(t)\|_X^2 + \frac{1}{2}\|R(\xi^e, \hat{\xi}(t))\|_X^2.$$

Applying Gronwall's inequality with $\phi(0) = 0$, see for instance (Curtain and Zwart, 1995, Lemma A.6.7), gives

$$\begin{aligned}\|\phi(t)\|_X^2 &\le e^{(2k+1)t} \int_0^t e^{-(2k+1)s} \|R(\xi^e, \hat{\xi}(s))\|_X^2 ds \\ &\le e^{(2k+1)t_0} \int_0^{t_0} \|R(\xi^e, \hat{\xi}(s))\|_X^2 ds.\end{aligned}$$

Equivalently,

$$\begin{aligned}\|\phi(t)\|_X &\le e^{\frac{(2k+1)t_0}{2}} \left(\int_0^{t_0} \|R(\xi^e, \hat{\xi}(s))\|_X^2 ds \right)^{\frac{1}{2}} \\ &= e^{\frac{(2k+1)t_0}{2}} \|R(\xi^e, \hat{\xi})\|_{L^2([0,t_0];X)} \\ &\le t_0^{\frac{1}{2}} e^{\frac{(2k+1)t_0}{2}} \|R(\xi^e, \hat{\xi})\|_{L^\infty([0,t_0];X)}.\end{aligned}$$

The positive constant $t_0^{1/2} e^{\frac{(2k+1)t_0}{2}}$ is denoted by λ_{t_0} in the following. Assumption 2.3 implies that

$$\begin{aligned}&\lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|\phi(t)\|_X}{\|\hat{\xi}_0\|_X} = \lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|S(t)\hat{\xi}_0 - T(t)\hat{\xi}_0\|_X}{\|\hat{\xi}_0\|_X} \\ &\le \lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\lambda_{t_0} \|R(\xi^e, \hat{\xi})\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} \\ &= \lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\lambda_{t_0} \|f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} = 0.\end{aligned}$$

Hence, $S(t)$ is $(Y, X)-$Fréchet differentiable at 0 with $T(t)$ as Fréchet derivative. ■

Remark 3.2. *Lemma 3.1 can be extended to a Banach space setting by straightforward adaptations of its proof. Similarly, it can be shown that the $Y-$Fréchet differentiability of f at ξ^e and the continuous dependence of $\hat{\xi}$ with respect to the initial condition $\hat{\xi}_0$ on Y are sufficient conditions for $(S(t))_{t\ge 0}$ to be $Y-$Fréchet differentiable.*

Combining a specific stability assumption of an equilibrium profile of (8) with Lemma 3.1 allows to deduce some stability of this equilibrium profile for (2). Note that the proof of the following result goes along the lines of (Al Jamal et al., 2014, Theorem 3.3), wherein the concepts have been adapted to our framework.

Theorem 3.1. *Let us consider Assumptions 2.1 to 2.4 and 3.1. If 0 is a globally exponentially stable equilibrium of the linearized model (8), then it is a $(Y, X)-$locally exponentially stable equilibrium of (2). Conversely, if 0 is a $(Y, X)-$unstable equilibrium of (8), it is $(Y, X)-$locally unstable for the nonlinear system (2).*

Proof. Let us choose $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$. First, remark that by Lemma 3.1, $S(t)$ is $(Y, X)-$Fréchet differentiable at 0, i.e. $S(t)\hat{\xi}_0 = T(t)\hat{\xi}_0 + r(\xi^e, \hat{\xi}_0)$, where

$$\lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_X} = 0. \tag{10}$$

As the functions $|||\cdot|||$ and $\|\cdot\|_Y$ are locally equivalent, see Remark 3.1 3), it holds that

$$\lim_{|||\hat{\xi}_0||| \to 0} \frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_X} = 0.$$

In other words, for any $t > 0$ and $\epsilon > 0$, there exists $\delta(t, \epsilon) > 0$ such that, if $|||\hat{\xi}_0||| < \delta(t, \epsilon)$,

$$\frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_X} < \epsilon.$$

By the strong continuity in t of the semigroups $S(t)$ and $T(t)$, the function $r(\xi^e, \hat{\xi}_0)$ is also continuous in t. Since 0 is a globally exponentially stable equilibrium of (8), there exist $\alpha \ge 1$ and $\beta > 0$ such that for all $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$

$$\|T(t)\hat{\xi}_0\|_X \le \alpha e^{-\beta t} \|\hat{\xi}_0\|_X, \, t \ge 0. \tag{11}$$

Hence there exist $\epsilon > 0$ and $t_0 < +\infty$ such that for $\tau \in [0, t_0]$ it holds that

$$\begin{aligned}\|S(\tau)\hat{\xi}_0\|_X &\le \|T(t)\hat{\xi}_0\|_X + \|r(\xi^e, \hat{\xi}_0)\|_X \\ &\le \alpha e^{-\beta\tau} \|\hat{\xi}_0\|_X + \epsilon \|\hat{\xi}_0\|_X \le C \|\hat{\xi}_0\|_X,\end{aligned} \tag{12}$$

where $C = \alpha + \epsilon$. Let us choose $t_0 = \frac{\ln(4\alpha)}{\beta} > 0$. Writing (11) with t replaced by t_0 gives

$$\|T(t_0)\hat{\xi}_0\|_X \le \frac{1}{4} \|\hat{\xi}_0\|_X.$$

In addition,

$$\lim_{|||\hat{\xi}_0||| \to 0} \frac{\|S(t_0)\hat{\xi}_0 - T(t_0)\hat{\xi}_0\|_X}{\|\hat{\xi}_0\|_X} = 0,$$

that is, there exists $\delta > 0$ such that, if $|||\hat{\xi}_0||| < \delta$, then

$$\|S(t_0)\hat{\xi}_0 - T(t_0)\hat{\xi}_0\|_X \le \frac{1}{4} \|\hat{\xi}_0\|_X.$$

Hence,

$$\begin{aligned}\|S(t_0)\hat{\xi}_0\|_X &= \|S(t_0)\hat{\xi}_0 - T(t_0)\hat{\xi}_0 + T(t_0)\hat{\xi}_0\|_X \\ &\le \|S(t_0)\hat{\xi}_0 - T(t_0)\hat{\xi}_0\|_X + \|T(t_0)\hat{\xi}_0\|_X \\ &\le \frac{1}{2} \|\hat{\xi}_0\|_X = e^{-\ln 2} \|\hat{\xi}_0\|_X.\end{aligned}$$

Let $k > 0$ be an integer. By using the semigroup property and the fact that $S^k(t_0)$ maps $D(A) \cap \mathcal{D}^e$ into $D(A) \cap \mathcal{D}^e$ for every $k \in \mathbb{N}$, for every $t_0 \ge 0$, one gets

$$\begin{aligned}\|S(kt_0)\hat{\xi}_0\|_X =& \|S^k(t_0)\hat{\xi}_0\|_X = \|S(t_0)S^{k-1}(t_0)\hat{\xi}_0\|_X \\ &\le e^{-\ln 2} \|S^{k-1}(t_0)\hat{\xi}_0\|_X \le e^{-(\ln 2)k} \|\hat{\xi}_0\|_X,\end{aligned} \tag{13}$$

where we have been using recursively the fact that if

$|||\hat{\xi}_0||| < \delta$, then $|||S(t_0)\hat{\xi}_0||| < \delta$. Indeed,

$$\begin{aligned}|||S(t_0)\hat{\xi}_0||| &= \sup_{t>0} \|S(t)S(t_0)\hat{\xi}_0\|_Y = \sup_{t>0} \|S(t+t_0)\hat{\xi}_0\|_Y \\ &= \sup_{t>t_0} \|S(t)\hat{\xi}_0\|_Y \le \sup_{t\ge 0} \|S(t)\hat{\xi}_0\|_Y = |||\hat{\xi}_0||| < \delta.\end{aligned}$$

For $t > 0$, let[2] $k = \lfloor \frac{t}{t_0} \rfloor$ and $\tau = t - kt_0 \in [0, t_0]$. By using the semigroup property, (12) and (13),

$$\begin{aligned}\|S(t)\hat{\xi}_0\|_X &= \|S(\tau + kt_0)\hat{\xi}_0\|_X = \|S(\tau)S(kt_0)\hat{\xi}_0\|_X \\ &\le C\|S(kt_0)\hat{\xi}_0\|_X \le Ce^{-(\ln 2)k}\|\hat{\xi}_0\|_X \le Ce^{-\gamma t}\|\hat{\xi}_0\|_X\end{aligned}$$

for $\gamma \le \frac{\ln 2}{t_0}$. This implies that 0 is a $(Y, X)-$locally exponentially stable equilibrium for (2).
In order to prove the second part of the theorem, let 0 be a $(Y, X)-$locally stable equilibrium to the nonlinear system (2). One has

$$S(t)\hat{\xi}_0 = T(t)\hat{\xi}_0 + r(\xi^e, \hat{\xi}_0). \tag{14}$$

Since 0 is $(Y, X)-$locally stable, it follows that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\hat{\xi}_0\|_Y < \delta$, then

$$\|S(t)\hat{\xi}_0\|_X \le \frac{\epsilon}{2},$$

for all $t \ge 0$. From (10) and since $\|\hat{\xi}_0\|_X \le \sigma\|\hat{\xi}_0\|_Y$, it follows that

$$\lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_Y} \le \lim_{\|\hat{\xi}_0\|_Y \to 0} \sigma \frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_X} = 0.$$

Hence, $\|r(\xi^e, \hat{\xi}_0)\|_X$ has to converge to 0 when so does $\|\hat{\xi}_0\|_Y$. Consequently, there exists δ^* with $0 < \delta^* < \delta$ such that, if $\|\hat{\xi}_0\|_Y < \delta^*$, then $\|r(\xi^e, \hat{\xi}_0)\|_X \le \frac{\epsilon}{2}$. Since $\|\hat{\xi}_0\|_Y < \delta^* < \delta$, it follows from (14) and from the last inequality that

$$\|T(t)\hat{\xi}_0\|_X \le \|r(\xi^e, \hat{\xi}_0)\|_X + \|S(t)\hat{\xi}_0\|_X \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \blacksquare$$

Remark 3.3. *Note that the main difference in our proof with respect to the one of (Al Jamal et al., 2014, Theorem 3.3) lies in the fact that* $\|\hat{\xi}_0\|_Y$ *is assumed to converge to* 0 *instead of* $\|\hat{\xi}_0\|_X$*, because of the* $(Y, X)-$*Fréchet differentiability of the nonlinear semigroup. This leads to additional technical difficulties, notably when we need to apply successively the proprety that* $\|S(t_0)\hat{\xi}_0\|_X \le e^{-ln2}\|\hat{\xi}_0\|_X$ *whenever* $\|\hat{\xi}_0\|_Y < \delta$*, on higher composition orders of the nonlinear semigroup, i.e. on* $S^k(t_0)\hat{\xi}_0, k \in \mathbb{N}, k > 1$. *This is possible due to Assumptions 2.4 and 3.1 that allow to use a locally equivalent norm to* $\|\cdot\|_Y$*, namely* $|||\cdot|||$. *This technical detail is not needed in (Al Jamal et al., 2014) because* $\|S(t_0)\hat{\xi}_0\|_X \le e^{-ln2}\|\hat{\xi}_0\|_X$ *implies automatically that* $\|S(t_0)\hat{\xi}_0\|_X \le \delta$ *whenever* $\|\hat{\xi}_0\|_X \le \delta$.

[2]The notation $\lfloor \cdot \rfloor$ is used as the integer part.

To the best of our knowledge, the notion of $(Y, X)-$Fréchet differentiability is proposed here for the first time and is the basis of all the analysis and considerations. The alternative space Y has to be chosen e.g. in order to avoid limitations in the manipulations of norm-inequalities. Good choices are in general L^∞ or Sobolev spaces $(H^p, p \in \mathbb{N}_0)$ which are all multiplicative algebras[3]. Hence, they allow for example to split the norm of the product of two functions into the product of the norms, which is not permitted in L^p-spaces, $1 \le p < \infty$, in which Hölder inequality has to be applied. To give intuition about the alternative space Y it may be noted that the condition $D(A) \cap \mathcal{D}^e \subset Y \subseteq X$ is quite natural depending on the application. Indeed, the choice of Y is induced by the domain of the linear operator A which takes more regularity properties into account.
In order to guide the reader in the application of the method, the following scheme is proposed, see Figure 1.

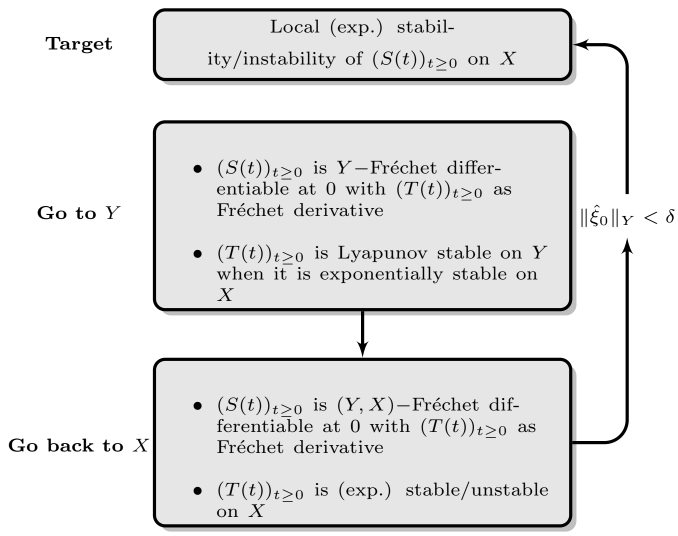

Figure 1: Guidelines in order to apply the new method.

It can be summarized as follows: the objective is to deduce exponential stability or instability of equilibrium profiles for nonlinear distributed parameter systems, where the state space is called X. First, a Gâteaux linearized version of the nonlinear system is built and its exponential stability is studied. Then, after the choice of the alternative space Y, the nonlinear semigroup is proved to be $Y-$Fréchet differentiable. In addition, its linearization has to be "Lyapunov stable" on Y when it is exponentially stable on X. Next, the new concept of $(Y, X)-$Fréchet differentiability plays its role to make the connection between Y and X to deduce exponential stability or instability of the equilibria for the nonlinear system (by using $X-$norms). What is specific here is that local means that the $Y-$norm of the initial condition is small instead of its $X-$norm.

[3]Note that there are no canonical choices for a given problem.

4. Application to nonisothermal axial dispersion tubular reactors

This section is devoted to the application of the previous results to a specific semilinear distributed parameter system. After introducing the model, the analysis of the exponential stability of its linearized version is investigated. Then the link with the nominal nonlinear system is made by using Theorem 3.1. Note that the assumptions needed in the theoretical framework of Section 3 will be successively checked.

Let us consider the following nonlinear PDE system

$$\begin{cases} \frac{\partial x_1}{\partial t} = \frac{1}{Pe_h}\frac{\partial^2 x_1}{\partial z^2} - \frac{\partial x_1}{\partial z} + \gamma(x_w - x_1) + \delta h(x_1, x_2), \\ \frac{\partial x_2}{\partial t} = \frac{1}{Pe_m}\frac{\partial^2 x_2}{\partial z^2} - \frac{\partial x_2}{\partial z} + h(x_1, x_2), \\ \frac{\partial x_1}{\partial z}(0) = Pe_h x_1(0), \frac{\partial x_2}{\partial z}(0) = Pe_m x_2(0), \\ \frac{\partial x_1}{\partial z}(1) = 0, \frac{\partial x_2}{\partial z}(1) = 0, \end{cases} \tag{15}$$

that describes the time evolution of the temperature and the reactant concentration in a nonisothermal axial dispersion tubular reactor, where x_1 and x_2 denote the dimensionless temperature and reactant concentration, respectively. The nonlinear function $h(x_1, x_2) := \alpha(1-x_2)e^{\frac{-\mu}{1+x_1}}$ is deduced from the Arrhenius law that models the evolution of the reaction rate as a function of the temperature. The variable x_w is the dimensionless coolant temperature in the heat exchanger that acts as a distributed control along the reactor. The dimensionless numbers Pe_h and Pe_m are the thermal and mass Peclet numbers, respectively, and α, δ and γ are constants that depend on the parameters of the system, see (Dochain, 2016; Hastir et al., 2020) for instance.

By (Laabissi et al., 2001; Winkin et al., 2000; Laabissi et al., 2005), it is well-known that (15) possesses a unique solution: the nonlinear operator describing the dynamics is the infinitesimal generator of a nonlinear C_0-semigroup on $D \cap \mathcal{K}$ where

$$D := \left\{x \in H^2(0,1) \times H^2(0,1) | \frac{dx_1}{d\zeta}(0) = Pe_h x_1(0), \frac{dx_2}{d\zeta}(0) = Pe_m x_2(0), \frac{dx_1}{d\zeta}(1) = 0, \frac{dx_2}{d\zeta}(1) = 0\right\} \tag{16}$$

and

$$\mathcal{K} := \{x \in H | -1 \leq x_1(z), 0 \leq x_2(z) \leq 1, \text{ a.e. on } [0,1]\}, \tag{17}$$

where $H = L^2(0,1) \times L^2(0,1)$ and a.c. stands for "absolutely continuous". Hence Assumption 2.1 is satisfied.

The existence and the multiplicity of equilibria of (15) have been studied in (Hastir et al., 2020) where it is shown that such reactors can exhibit different numbers of equilibria, one or three, depending on the parameters of the system, especially on the Peclet numbers. The analysis of existence and multiplicity of the equilibria of (15) is built by using perturbation theory, which assumes to take small parameters into account. Here these parameters are $1/D_{ma}$ and $1/\lambda_{ea}$ where D_{ma} and λ_{ea} are the axial mass dispersion and axial energy dispersion coefficients, respectively. This entails that diffusion is supposed to be dominant in all the following considerations, meaning also that $Pe_m \propto 1/D_{ma}$ and $Pe_h \propto 1/\lambda_{ea}$ are considered small.

Note that, for the analysis of the stability of these equilibria, we shall assume adiabatic conditions and equal Peclet numbers, that is $\gamma = 0$ and $Pe_h = Pe_m =: Pe$. Let us denote (x_1^e, x_2^e) an equilibrium of (15). Consider the following change of variables $\xi_1 = x_1 - x_1^e, \xi_2 = x_2 - x_2^e$, $\hat{\xi}_1 = e^{-\frac{Pe_h}{2}z}\xi_1, \hat{\xi}_2 = e^{-\frac{Pe_m}{2}z}\xi_2$. Then (15) becomes

$$\begin{cases} \frac{\partial \hat{\xi}_1}{\partial t} = \frac{1}{Pe_h}\frac{\partial^2 \hat{\xi}_1}{\partial z^2} - \frac{Pe_h}{4}\hat{\xi}_1 + \delta e^{\frac{-Pe_h}{2}z} f(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e), \\ \frac{\partial \hat{\xi}_2}{\partial t} = \frac{1}{Pe_m}\frac{\partial^2 \hat{\xi}_2}{\partial z^2} - \frac{Pe_m}{4}\hat{\xi}_2 + e^{\frac{-Pe_m}{2}z} f(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e), \\ \frac{\partial \hat{\xi}_1}{\partial z}(0) = \frac{Pe_h}{2}\hat{\xi}_1(0), \frac{\partial \hat{\xi}_2}{\partial z}(0) = \frac{Pe_m}{2}\hat{\xi}_2(0), \\ \frac{\partial \hat{\xi}_1}{\partial z}(1) = -\frac{Pe_h}{2}\hat{\xi}_1(1), \frac{\partial \hat{\xi}_2}{\partial z}(1) = -\frac{Pe_m}{2}\hat{\xi}_2(1), \end{cases} \tag{18}$$

where $f(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e) = g(e^{\frac{Pe_h}{2}z}\hat{\xi}_1 + x_1^e, e^{\frac{Pe_m}{2}z}\hat{\xi}_2 + x_2^e) - g(x_1^e, x_2^e)$ with $g(x, y) = \alpha(1-y)e^{\frac{-\mu}{1+x}}$ for $(x, y) \in \mathcal{K}$ and $g(-\mathbb{1}, y) = 0$. First, let us define the linear operator $A := \frac{1}{Pe}\frac{d^2 \cdot}{dz^2} - \frac{Pe}{4}\cdot$ on the domain

$$D(A) = \left\{\xi \in H^2(0,1), \frac{d\xi}{dz}(0) = \frac{Pe}{2}\xi(0), \frac{d\xi}{dz}(1) = -\frac{Pe}{2}\xi(1)\right\}.$$

We consider also the two linear and bounded functionals $\mathcal{B}_0 := \frac{d\cdot}{dz}(0) - \frac{Pe}{2}\cdot(0)$ and $\mathcal{B}_1 := \frac{d\cdot}{dz}(1) + \frac{Pe}{2}\cdot(1)$, both defined on the domain $D(A)$. Moreover, let us introduce the nonlinear operator

$$\mathcal{N}\begin{pmatrix}\hat{\xi}_1 \\ \hat{\xi}_2\end{pmatrix} := \begin{pmatrix}\delta e^{-\frac{Pe}{2}z} f(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e) \\ e^{-\frac{Pe}{2}z} f(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e),\end{pmatrix}$$

on the domain $e^{-\frac{Pe}{2}z}(\mathcal{K} - (x_1^e \quad x_2^e)^T)$. We shall now define the linear operators $\mathcal{A}\begin{pmatrix}\hat{\xi}_1 & \hat{\xi}_2\end{pmatrix}^T := \begin{pmatrix}A\hat{\xi}_1 & A\hat{\xi}_2\end{pmatrix}^T, \mathfrak{B}_0\begin{pmatrix}\hat{\xi}_1 & \hat{\xi}_2\end{pmatrix}^T := \begin{pmatrix}\mathcal{B}_0\hat{\xi}_1 & \mathcal{B}_0\hat{\xi}_2\end{pmatrix}^T$ and $\mathfrak{B}_1\begin{pmatrix}\hat{\xi}_1 & \hat{\xi}_2\end{pmatrix}^T := \begin{pmatrix}\mathcal{B}_1\hat{\xi}_1 & \mathcal{B}_1\hat{\xi}_2\end{pmatrix}^T$ for $\begin{pmatrix}\hat{\xi}_1 & \hat{\xi}_2\end{pmatrix}^T \in D(A) \times D(A)$. With these notations, (18) takes the form

$$\begin{cases} \dot{\hat{\xi}} = \mathcal{A}\hat{\xi} + \mathcal{N}(\hat{\xi}) \\ \mathfrak{B}_0\hat{\xi} = 0, \mathfrak{B}_1\hat{\xi} = 0, \end{cases} \tag{19}$$

where $\hat{\xi} = \begin{pmatrix}\hat{\xi}_1 & \hat{\xi}_2\end{pmatrix}^T$. Let us consider the linear operator matrix $U = \begin{pmatrix} I & 0 \\ I & -\delta I\end{pmatrix}$. The change of variables $\begin{pmatrix}\hat{\xi}_1 \\ \chi\end{pmatrix} = U\begin{pmatrix}\hat{\xi}_1 \\ \hat{\xi}_2\end{pmatrix}$ allows to triangularize (18). Note that U is bounded on H and its inverse, given by $U^{-1} = \begin{pmatrix} I & 0 \\ \frac{1}{\delta}I & -\frac{1}{\delta}I\end{pmatrix}$ is also bounded on H. The equivalent system to (19) in the variables $\tilde{\xi} = \begin{pmatrix}\hat{\xi}_1 & \chi\end{pmatrix}^T$ is given by

$$\begin{cases} \dot{\tilde{\xi}} = U\mathcal{A}U^{-1}\tilde{\xi} + U\mathcal{N}(U^{-1}\tilde{\xi}) \\ U\mathfrak{B}_0 U^{-1}\tilde{\xi} = 0, U\mathfrak{B}_1 U^{-1}\tilde{\xi} = 0, \end{cases} \tag{20}$$

or equivalently

$$\left\{\begin{array}{l} \frac{\partial\hat{\xi}_1}{\partial t} = \frac{1}{Pe}\frac{\partial^2\hat{\xi}_1}{\partial z^2} - \frac{Pe}{4}\hat{\xi}_1 + \delta e^{\frac{-Pe}{2}z} f(\hat{\xi}_1, \frac{\hat{\xi}_1-\chi}{\delta}, x_1^e, \frac{1}{\delta}x_1^e),\\ \frac{\partial\chi}{\partial t} = \frac{1}{Pe}\frac{\partial^2\chi}{\partial z^2} - \frac{Pe}{4}\chi\\ \frac{\partial\hat{\xi}_1}{\partial z}(0) = \frac{Pe}{2}\hat{\xi}_1(0), \frac{\partial\hat{\xi}_1}{\partial z}(1) = -\frac{Pe}{2}\hat{\xi}_1(1),\\ \frac{\partial\chi}{\partial z}(0) = \frac{Pe}{2}\chi(0), \frac{\partial\chi}{\partial z}(1) = -\frac{Pe}{2}\chi(1). \end{array}\right. \tag{21}$$

The following proposition allows to rewrite the problem of exponential stability for (18).

Proposition 4.1. *The component χ in (21) is expressed as*

$$\chi(t,z) = \sum_{n=1}^{+\infty} \psi_n\,\phi_n(z)\,e^{-(\beta_n^2+\frac{Pe}{4})t}, \tag{22}$$

for $z \in [0,1], t \geq 0$ where $\psi_n = \int_0^1 f(z)\phi_n(z)dz$ where $f(z)$ denotes the initial condition to $\hat{\xi}_1 - \delta\hat{\xi}_2$. The set $\{\phi_n\}_{n\geq 1}$ contains the eigenfunctions of the linear operator A, defined by $\phi_n(z) = K_n[\beta_n\sqrt{Pe}\cos(\beta_n\sqrt{Pe}z) + \frac{Pe}{2}\sin(\beta_n\sqrt{Pe}z)]$. Note that $\{K_n\}_{n\geq 1}$ is a set of normalization constants expressed as $K_n = (\frac{2}{\beta_n^2 Pe+Pe+Pe^2/4})^{\frac{1}{2}}$ and $\{\beta_n\}_{n\geq 1}$ are solutions of the resolvent equation $\tan(\beta\sqrt{Pe}) = \frac{4\beta\sqrt{Pe}}{4\beta^2-Pe}$, see e.g. (Delattre et al., 2003).

Proof. See e.g. (Varma and Aris, 1977, Section 2.5.2) and (Hastir et al., 2019). ■

Note that U defines a similarity transformation. Then (18) is equivalent to (21). Consequently, exponential stability can either be studied in the first set of variables (18) or in the second one (21), the same conclusions hold for both systems. In addition, $\chi(t,z)$, given by (22), converges exponentially fast to 0 as t tends to $+\infty$. Moreover, by noting the reaction invariant[4] $x_1^e - \delta x_2^e$, the stability analysis is based on the following nonlinear PDE

$$\left\{\begin{array}{l} \frac{\partial\hat{\xi}_1}{\partial t} = \frac{1}{Pe}\frac{\partial^2\hat{\xi}_1}{\partial z^2} - \frac{Pe}{4}\hat{\xi}_1 + \delta e^{\frac{-Pe}{2}z} f(\hat{\xi}_1, \frac{1}{\delta}\hat{\xi}_1, x_1^e, \frac{1}{\delta}x_1^e),\\ \frac{\partial\hat{\xi}_1}{\partial z}(0) = \frac{Pe}{2}\hat{\xi}_1(0), \frac{\partial\hat{\xi}_1}{\partial z}(1) = -\frac{Pe}{2}\hat{\xi}_1(1). \end{array}\right. \tag{23}$$

For the ease of notation we introduce $\xi := \hat{\xi}_1, x^e := x_1^e$ and $\delta g(x, \frac{1}{\delta}x) = \alpha(\delta - x)e^{\frac{-\mu}{1+x}} =: \tilde{g}(x)$. In this way, (23) is rewritten as follows:

$$\left\{\begin{array}{l} \frac{\partial\xi}{\partial t} = \frac{1}{Pe}\frac{\partial^2\xi}{\partial z^2} - \frac{Pe}{4}\xi + e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\xi + x^e) - \tilde{g}(x^e)],\\ \frac{\partial\xi}{\partial z}(0) = \frac{Pe}{2}\xi(0), \frac{\partial\xi}{\partial z}(1) = -\frac{Pe}{2}\xi(1). \end{array}\right. \tag{24}$$

4.1. Linearized bistability of equilibrium profiles

First we recall results for exponential bistability of the equilibrium profiles of (24) for a linearized model of (24) see (Hastir et al., 2019). Linearizing (24) via a Gâteaux derivative yields the following system:

$$\left\{\begin{array}{l} \frac{\partial\overline{\xi}}{\partial t} = \frac{1}{Pe}\frac{\partial^2\overline{\xi}}{\partial z^2} - \frac{Pe}{4}\overline{\xi} + \left(\alpha\frac{\mu(\delta-x^e)}{(1+x^e)^2}e^{\frac{-\mu}{1+x^e}} - \alpha e^{\frac{-\mu}{1+x^e}}\right)\overline{\xi},\\ \frac{\partial\overline{\xi}}{\partial z}(0) = \frac{Pe}{2}\overline{\xi}(0), \frac{\partial\overline{\xi}}{\partial z}(1) = -\frac{Pe}{2}\overline{\xi}(1). \end{array}\right. \tag{25}$$

First note that system (25) is well-posed in the sense that the linear operator describing the dynamics is the infinitesimal generator of a C_0-semigroup on $L^2(0,1)$, see (Hastir et al., 2019, Lemma 2.3). In the same reference, it is proved that the Gâteaux derivative of the nonlinear operator is a bounded linear operator, showing that Assumption 2.2 is satisfied. Furthermore, in (Hastir et al., 2019), it is shown that the state trajectory of (25) satisfies

$$\|\overline{\xi}(t,\cdot)\|_{L^2(0,1)} \leq e^{-\gamma t}\|\overline{\xi}(0,\cdot)\|_{L^2(0,1)},$$

where $\gamma = \frac{\pi^2}{\pi^2+4Pe} + q(c)$ and $q(z) = \frac{Pe}{4} + \alpha e^{\frac{-\mu}{1+x^e}} - \alpha\frac{\mu(\delta-x^e)}{(1+x^e)^2}e^{\frac{-\mu}{1+x^e}}$. Moreover, in the case where the reactor can exhibit only one equilibrium profile, the latter is exponentially stable (Hastir et al., 2019, Corollary 3.1). When three equilibria are present, bistability holds: the pattern "exponentially stable - unstable - exponentially stable" is exhibited for the three equilibria in (Hastir et al., 2019, Corollary 3.2).

4.2. Exponential stability of the nonlinear model

The first step of this section is concerned with the time boundedness of the state trajectory of (24) on a finite time interval.

Let us define a constraint domain $\mathcal{D}$ built from $\mathcal{K}$ and expressed as $\mathcal{D} =$

$$\left\{\xi \in L^2(0,1) | \xi(z) \geq e^{-\frac{Pe}{2}z}(-1-x^e), \text{ for a.e. } z \in [0,1]\right\}.$$

Note that the operator $A + e^{-\frac{Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\cdot + x^e) - \tilde{g}(x^e)]$ is the infinitesimal generator of a nonlinear C_0-semigroup on $D(A) \cap \mathcal{D}$ denoted by $(S(t))_{t\geq 0}$ and that satisfies $S(t)\hat{\xi}_0 = \xi(t,z)$, see e.g. (Laabissi et al., 2001). Moreover, it can be shown easily that the operator A is dissipative on $D(A)\cap\mathcal{D}$ see e.g. (Aksikas et al., 2007a) among others. In the same reference, it is shown that the nonlinear operator $\tilde{f}(\hat{\xi}_1, \hat{\xi}_2, x_1^e, x_2^e)$ is Lipschitz continuous in the variables $\hat{\xi}_1$ and $\hat{\xi}_2$ for $(e^{\frac{Pe_h}{2}z}\hat{\xi}_1(t,z) + x_1^e \quad e^{\frac{Pe_m}{2}z}\hat{\xi}_2(t,z) + x_2^e) \in \mathcal{K}$. In (Hastir et al., 2019), a well-posedness analysis of (25) is performed and it is notably shown that the nonlinear operator $e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\xi + x^e) - \tilde{g}\,(x^e)]$ is Gâteaux differentiable at 0 and its Gâteaux derivative, given by the linear operator $d\tilde{g}(\xi^e) : L^2(0,1) \to L^2(0,1)$ for $\xi \in L^2(0,1)$ by $d\tilde{g}(\xi^e)\xi = (\alpha\frac{\mu(\delta-x^e)}{(1+x^e)^2}e^{\frac{-\mu}{1+x^e}} - \alpha e^{\frac{\mu}{1+x^e}})\xi$, is bounded on $L^2(0,1)$. This means that Assumptions 2.1 and 2.2 are satisfied.

The following lemma shows the uniform (in time and space) boundedness of the function $e^{\frac{Pe}{2}z}\xi(t,z) + x^e(z)$, where ξ is the solution to (24).

[4] Since $x_1^e - \delta x_2^e$ is a reaction invariant, the relation $x_1^e(z) = \delta x_2^e(z)$ holds a.e. on $[0,1]$.

Lemma 4.1. *Suppose that there exists a positive and sufficiently large constant κ such that the condition*

$$\sum_{n=1}^{+\infty} |\psi_n| \leq \frac{1}{\sqrt{2}\gamma^*} e^{-\frac{Pe}{2}} \left(1 - \frac{1}{\kappa}\right) \tag{26}$$

holds, where $\gamma^* = (\frac{(Pe+2)(2Pe+4)}{6Pe+8+Pe^2})^{\frac{1}{2}} \to 1$ as $Pe \to 0$. Then

$$\frac{1}{\kappa} - 1 \leq e^{\frac{Pe}{2}z}\xi(t,z) + x^e(z) \leq \delta + 1 - \frac{1}{\kappa},$$

for all $t \geq 0$ and a.e. $z \in [0,1]$.

Proof. Fix $t \geq 0$ and $z \in [0,1]$. Observe that

$$|\phi_n(z)| \leq \left[\frac{2}{\beta_n^2 Pe + Pe + \frac{Pe^2}{4}}\right]^{\frac{1}{2}} \left(\beta_n\sqrt{Pe} + \frac{Pe}{2}\right)$$
$$= \sqrt{2}\left[\frac{\beta_n^2 Pe + \beta_n Pe\sqrt{Pe} + \frac{Pe^2}{4}}{\beta_n^2 Pe + Pe + \frac{Pe^2}{4}}\right]^{\frac{1}{2}}.$$

Replacing β_n by β in the right hand side of the inequality and interpreting it as a function of β, this function reaches its maximum at $\beta = \frac{Pe+4}{2\sqrt{Pe}}$. For that value of β, it is equal to $\sqrt{2}(\frac{(Pe+2)(2Pe+4)}{6Pe+8+Pe^2})^{\frac{1}{2}} =: \sqrt{2}\gamma^*$. It follows that

$$|\hat{\xi}_1(t,z) - \delta\hat{\xi}_2(t,z)| \leq \sum_{n=1}^{+\infty} |\psi_n \phi_n(z)| e^{-(\beta_n^2 + \frac{Pe}{4})t}$$
$$\leq \sqrt{2}\gamma^* \sum_{n=1}^{+\infty} |\psi_n| \leq e^{-\frac{Pe}{2}} \left(1 - \frac{1}{\kappa}\right).$$

Furthermore, remark that $|\hat{\xi}_1(t,z) - \delta\hat{\xi}_2(t,z)| =$

$$\left|e^{-\frac{Pe}{2}z}\left(x_1(t,z) - \delta x_2(t,z)\right) - e^{-\frac{Pe}{2}z}\left(x_1^e(z) - \delta x_2^e(z)\right)\right|$$
$$= \left|e^{-\frac{Pe}{2}z}\left(x_1(t,z) - \delta x_2(t,z)\right)\right|,$$

where $x_1, x_2 \in D \cap \mathcal{K}$. Consequently,

$$e^{\frac{-Pe}{2}}\left(\frac{1}{\kappa} - 1\right) \leq e^{\frac{-Pe}{2}z}(x_1(t,z) - \delta x_2(t,z)) \leq e^{\frac{-Pe}{2}}\left(1 - \frac{1}{\kappa}\right),$$

which can be written as

$$e^{\frac{-Pe}{2}} e^{\frac{Pe}{2}z}\left(\frac{1}{\kappa} - 1\right) \leq x_1(t,z) - \delta x_2(t,z) \leq e^{\frac{-Pe}{2}} e^{\frac{Pe}{2}z}\left(1 - \frac{1}{\kappa}\right).$$

Since $\frac{1}{\kappa} - 1 < 0$, it follows that

$$\frac{1}{\kappa} - 1 + \delta x_2(t,z) \leq x_1(t,z) \leq 1 - \frac{1}{\kappa} + \delta x_2(t,z).$$

Since x_2 lies in $\mathcal{K}, 0 \leq x_2(t,z) \leq 1$. Hence,

$$\frac{1}{\kappa} - 1 \leq x_1(t,z) \leq 1 - \frac{1}{\kappa} + \delta. \tag{27}$$

The conclusion follows by noting that $x_1(t,z) = e^{\frac{Pe}{2}z}\xi(t,z) + x^e(z)$. ■

Remark 4.1. *The assumption that is needed in Lemma 4.1 means that the sequence of the projections of the initial condition of the asymptotic reaction invariant on the basis of eigenfunctions of the operator A has to be summable, i.e. $\{\psi_n\}_{n\in\mathbb{N}} \subset l_1$. Moreover, its l_1-norm can not exceed the constant $\frac{1}{\sqrt{2}\gamma^*}e^{-\frac{Pe}{2}}(1 - \frac{1}{\kappa})$.*

As an illustration, we shall take as initial conditions[5] $\hat{\xi}_{i,0}(z) := \omega_i\left(\sin(\pi z) + \frac{2\pi}{Pe}\right), i = 1,2$, where ω_i are constant numbers that have to be chosen and that represent degrees of freedom in the analysis. Note that the subscript i denotes the initial condition given to the $i-$th variable. It is straightforward to see that this kind of initial conditions is in $D(A)$. Here after, we first show that the series $\sum_{n=1}^{+\infty}|\psi_n|$ is convergent in this case. Then we shall give the value above which the constant κ has to be chosen in such a way that condition (26) holds. Since the series is convergent it is always possible to adapt the constants $\omega_i, i = 1,2$ and then to chose κ such that (26) holds. With the choice of $\hat{\xi}_{i,0}$, the coefficient $|\psi_n|$ satisfies

$$|\psi_n| \leq |\omega_1 - \delta\omega_2| \cdot \left(\left|\int_0^1 \sin(\pi z)\phi_n(z)dz\right| + \frac{2\pi}{Pe}\left|\int_0^1 \phi_n(z)dz\right|\right).$$

Computing first $|\int_0^1 \phi_n(z)dz|$ yields

$$\left|\int_0^1 \phi_n(z)dz\right| = \left|K_n\left[\sin(\beta_n\sqrt{Pe})\right.\right.$$
$$\left.\left.-\frac{Pe}{2\beta_n\sqrt{Pe}}\cos(\beta_n\sqrt{Pe}) + \frac{Pe}{2\beta_n\sqrt{Pe}}\right]\right|.$$

By (Dehaye, 2015, Section 4.3.2), the approximation $\beta_n \approx \pi(n-1)/\sqrt{Pe}$ holds for n large. Moreover by considering Pe small and by plugging the asymptotic form of K_n in the previous equality implies that $\left|\int_0^1 \phi_n(z)dz\right| \approx$

$$\left|\frac{\sqrt{2}}{\pi(n-1)}\left(\frac{Pe}{2\pi(n-1)}(1-(-1)^{n+1})\right)\right| \leq \frac{\sqrt{2}Pe}{\pi^2(n-1)^2}.$$

Moreover, one has

$$\left|\int_0^1 \sin(\pi z)\phi_n(z)dz\right|$$
$$= \left|\frac{K_n\beta_n\sqrt{Pe}}{2}\int_0^1 \sin((\pi + \beta_n\sqrt{Pe})z)dz\right.$$
$$+\frac{K_n\beta_n\sqrt{Pe}}{2}\int_0^1 \sin((\pi - \beta_n\sqrt{Pe})z)dz$$
$$+\frac{K_n Pe}{4}\int_0^1 \cos((\pi - \beta_n\sqrt{Pe})z)dz$$
$$\left.-\frac{K_n Pe}{4}\int_0^1 \cos((\pi + \beta_n\sqrt{Pe})z)dz\right|.$$

[5] These initial conditions are considered in the numerical simulations, see Section 5

Considering n large enough yields that $K_n \approx \frac{\sqrt{2}}{\pi(n-1)}$, which results in

$$\begin{aligned}
&\left|\int_0^1 \sin(\pi z)\phi_n(z)dz\right| \\
&\approx \left|\frac{\sqrt{2}}{2}\int_0^1 \sin(\pi n z)dz + \frac{\sqrt{2}}{2}\int_0^1 \sin((2-n)\pi z)dz \right.\\
&\left. + \frac{\sqrt{2}Pe}{4\pi(n-1)}\int_0^1 \cos((2-n)\pi z)dz - \frac{\sqrt{2}Pe}{4\pi(n-1)}\int_0^1 \cos(n\pi z)dz\right| \\
&= \sqrt{2}\left|\frac{((-1)^n-1)}{\pi n^2 - 2\pi n}\right| \leq \frac{2\sqrt{2}}{|\pi n^2 - 2\pi n|} = \frac{2\sqrt{2}}{\pi n^2 - 2\pi n}.
\end{aligned}$$

The last equality holds since n is sufficiently large[6]. Combining the previous computations gives that

$$\begin{aligned}
&\left|\int_0^1 f(z)\phi_n(z)dz\right| \\
&= \mathcal{O}\left(|\omega_1 - \delta\omega_2| \cdot \left(\frac{2\sqrt{2}}{\pi n^2 - 2\pi n} + \frac{2\pi}{Pe}\frac{\sqrt{2}Pe}{\pi^2(n-1)^2}\right)\right) \\
&= \mathcal{O}\left(|\omega_1 - \delta\omega_2| \frac{4\sqrt{2}}{\pi n^2 - 2\pi n}\right),
\end{aligned}$$

which is the general term of a convergent series. This proves that $\{\psi_n\}_{n\in\mathbb{N}_0}$ is l_1-summable. In other words, there exists a positive and finite real number Ω such that

$$\Omega := \sum_{n=1}^{\infty}\left|\int_0^1 \left(\sin(\pi z) + \frac{2\pi}{Pe}\right)\phi_n(z)dz\right|.$$

Let us denote $\omega^* := |\omega_1 - \delta\omega_2|$. The inequality (26) becomes

$$\omega^*\Omega \leq \frac{1}{\sqrt{2}\gamma^*}e^{-\frac{Pe}{2}}(1 - \frac{1}{\kappa}).$$

Equivalently, we get the inequality $\frac{1}{\kappa} \leq 1 - \sqrt{2}\gamma^* e^{\frac{Pe}{2}}\omega^*\Omega$, whose right hand-side is supposed to be positive[7]. Hence, the constant κ has to be taken such that

$$\kappa \geq \frac{1}{1 - \sqrt{2}\gamma^* e^{\frac{Pe}{2}}\omega^*\Omega}. \tag{28}$$

Since Ω is a finite positive real number and ω^* is a degree of freedom in the analysis, there will always exist κ such that (28) is satisfied, i.e. such that assumption (26) of Lemma 4.1 holds.

The following remark concerns the stability of the linearized model by considering $L^\infty(0,1)-$norms.

Remark 4.2. *It can be shown that the solution of*

$$\begin{cases} \frac{\partial\overline{\xi}}{\partial t} = \frac{1}{Pe}\frac{\partial^2\overline{\xi}}{\partial z^2} - q(c)\overline{\xi}, \\ \frac{\partial\overline{\xi}}{\partial z}(0) = \frac{Pe}{2}\overline{\xi}(0), \frac{\partial\overline{\xi}}{\partial z}(1) = -\frac{Pe}{2}\overline{\xi}(1), \end{cases} \tag{29}$$

is given by $\overline{\xi}(t,z) = \sum_{n=1}^{+\infty}\psi_n\,\phi_n(z)\,e^{-\left(\beta_n^2 + q(c)\right)t}$, *for all* $c \in [0,1]$. *Let us either consider the case where the reactor can exhibit one equilibrium or consider the first and the third equilibrium in the case of three equilibria. Hence the inequality* $\sup_{n\geq 1}\left\{-\beta_n^2 - q(c)\right\} < 0$ *holds, see (Hastir et al., 2019, Section III). Moreover, by the proof of Lemma 4.1, one has that*

$$\begin{aligned}
|\overline{\xi}(t,z)| &\leq \sum_{n=1}^{+\infty} e^{-(\beta_n^2 + q(c))t}|\psi_n|\,|\phi_n(z)| \\
&\leq \sqrt{2}\gamma^*\sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t}\int_0^1 |\xi_0(z)\phi_n(z)|dz \\
&\leq 2{\gamma^*}^2\sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t}\|\xi_0\|_{L^1(0,1)} \\
&\leq 2{\gamma^*}^2\|\xi_0\|_{L^\infty(0,1)}\sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t},
\end{aligned}$$

where ξ_0 *denotes the initial condition that is given to (29). Consequently, it holds that*

$$\|\overline{\xi}(t,\cdot)\|_{L^\infty(0,1)} \leq 2{\gamma^*}^2\|\xi_0\|_{L^\infty(0,1)}\sum_{n=1}^{+\infty} e^{-(\beta_n^2+q(c))t}.$$

A straightforward computation leads to

$$\int_0^{+\infty}\|\overline{\xi}(t,z)\|_{L^\infty(0,1)}dt \leq 2{\gamma^*}^2\|\xi_0\|_{L^\infty(0,1)}\sum_{n=1}^{+\infty}\frac{1}{\beta_n^2 + q(c)} \tag{30}$$

According to (Dehaye, 2015, Section 4.3.2), for n sufficiently large, $\beta_n \simeq \pi(n-1)/\sqrt{Pe}$. *Hence (30) becomes*

$$\begin{aligned}
&\int_0^{+\infty}\|\overline{\xi}(t,z)\|_{L^\infty(0,1)}dt \\
&\quad\leq 2{\gamma^*}^2\nu\|\xi_0\|_{L^\infty(0,1)}\left(\sum_{n=1}^{+\infty}\frac{Pe}{\pi^2(n-1)^2 + Pe\,q(c)}\right) \\
&\quad= 2{\gamma^*}^2\nu\|\xi_0\|_{L^\infty(0,1)}\eta < +\infty,
\end{aligned}$$

for some $\nu > 0$ *and* $\eta > 0$. *From Dakto's lemma, see e.g. (Buse et al., 2006, Section 1), it follows that* $\overline{\xi}(t,z)$ *is exponentially stable on* $L^\infty(0,1)$, *i.e. there exists* $M \geq 1$ *and* $\alpha > 0$ *such that*

$$\|\overline{\xi}(t,\cdot)\|_{L^\infty(0,1)} \leq Me^{-\alpha t}\|\xi_0\|_{L^\infty(0,1)}, t \geq 0. \tag{31}$$

Note that considering the $L^\infty(0,1)-$*norm of* $\overline{\xi}$ *makes sense since* $\overline{\xi}$ *lies in* $D(A)$, *which is a subset of* $L^\infty(0,1)$. *It holds that* $-\alpha = \sup_{n\geq 1}\left\{-\beta_n^2 + q(c)\right\} < 0$. *Consider now the system (25), which can also be expressed as*

$$\begin{cases} \frac{\partial\overline{\xi}}{\partial t} = \frac{1}{Pe}\frac{\partial^2\overline{\xi}}{\partial z^2} - q(c)\overline{\xi} + [q(c) - q(z)]\,\overline{\xi}, \\ \frac{\partial\overline{\xi}}{\partial z}(0) = \frac{Pe}{2}\overline{\xi}(0), \frac{\partial\overline{\xi}}{\partial z}(1) = -\frac{Pe}{2}\overline{\xi}(1). \end{cases} \tag{32}$$

[6] Without loss of generality, since n is large, one can assume that $n > 2$ which entails that $\pi n^2 - 2\pi n > 0$.

[7] This is always possible to render this quantity positive because ω^* can be chosen arbitrarily.

Since the estimate (31) holds, by (Engel and Nagel, 2006, Bounded Perturbation Theorem), one has that

$$\|\overline{\xi}(t,\cdot)\|_{L^\infty(0,1)} \le Me^{\left(-\alpha+M\|q(c)-q(\cdot)\|_{L^\infty(0,1)}\right)t}\|\xi_0\|_{L^\infty(0,1)},$$

$t \ge 0$, where $\overline{\xi}$ is the solution to (25) (or equivalently (32)) and ξ_0 is the initial condition given to (25). By considering large diffusion phenomenon (meaning that Pe is sufficiently small), the equilibrium $x^e \to a$ uniformly, where $a \in (0,\delta)$, see (Hastir et al., 2020, Section IV). It follows that $\|q(c)-q(\cdot)\|_{L^\infty(0,1)} \to 0$ as $Pe \to 0$. Fix $\epsilon > 0$ such that $M\epsilon < \alpha$. There exists $\delta > 0$ such that $Pe < \delta$ implies that $\|q(c)-q(\cdot)\|_{L^\infty(0,1)} < \epsilon$. Hence

$$-\alpha + M\|q(c)-q(\cdot)\|_{L^\infty(0,1)} < -\alpha + M\epsilon := -\zeta < 0$$

by construction. It follows that the solution to (25) satisfies

$$\|\overline{\xi}(t,\cdot)\|_{L^\infty(0,1)} \le Me^{-\zeta t}\|\xi_0\|_{L^\infty(0,1)}, t \ge 0,$$

which ensures that Assumption 3.1 is satisfied.

As a consequence of Lemma 4.1 we have the next result, where $\xi(t)$ denotes the function $\xi(t,\cdot)$. The function η is defined as $\eta(\xi) =: e^{\frac{Pe}{2}z}\xi + x^e$ in what follows.

Corollary 4.1. *The following inequalities hold:*

$$\left\|\alpha(\delta - \eta(\xi(t)))e^{\frac{-\mu}{1+\eta(\xi(t))}}\right\|_{L^\infty(0,1)} \le \alpha\left(\delta + 1 - \frac{1}{\kappa}\right) \quad (33)$$

and

$$\left\|\frac{1}{(1+\eta(\xi(t)))(1+x^e)}\right\|_{L^\infty(0,1)} \le \kappa \quad (34)$$

for all $t \ge 0$.

Proof. By Lemma 4.1,

$$\frac{1}{\kappa} \le 1 + e^{\frac{Pe}{2}z}\xi(t,z) + x^e(z) \le \delta + 2 - \frac{1}{\kappa},$$

for all $t \ge 0$ and a.e. $z \in [0,1]$. Moreover the equilibria x_1^e and x_2^e form the reaction invariant $x_1^e - \delta x_2^e$ and $(x_1^e \quad x_2^e) \in \mathcal{K}$. Hence, $1 \le 1 + x_1^e(z) \le 1 + \delta$ for a.e. $z \in [0,1]$. These two double inequalities imply (34). Remark also that, by Lemma 4.1,

$$\frac{1}{\kappa} - 1 \le \delta - e^{\frac{Pe}{2}z}\xi(t,z) + x^e(z) \le \delta + 1 - \frac{1}{\kappa}.$$

This completes the proof of (33) since $1 + e^{\frac{Pe}{2}z}\xi(t,z) + x^e(z) \ge \frac{1}{\kappa} > 0$. ■

Note that the inequalities of Corollary 4.1 are valid for all $t \ge 0$, where the bound does not depend on time. This is a statement of uniform boundedness in time, and so it constitutes a quite strong result for further considerations.

Let us modify the domain $\mathcal{D}$ as follows:

$$\tilde{\mathcal{D}} := \left\{\xi \in L^2(0,1) | -1 + \frac{1}{\kappa} \le e^{\frac{Pe}{2}z}\xi(z) + x^e(z) \right.$$
$$\left. \le \delta + 1 - \frac{1}{\kappa}, \text{ for a.e. } z \in [0,1]\right\} \subset \mathcal{D}.$$

In the following, we restrict ourselves to functions in $D(A) \cap \tilde{\mathcal{D}}$. In order to get the time boundedness of the state trajectory, we state the following lemma.

Lemma 4.2. *The function $\tilde{h}(\cdot) = e^{-\frac{Pe}{2}\cdot}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e))$ is in $L^2([0,t_0];L^2(0,1))$ for every positive t_0, for every $\xi \in D(A) \cap \tilde{\mathcal{D}}$.*

Proof. Fix $t_0 > 0$ and $\xi \in D(A) \cap \tilde{\mathcal{D}}$. By computing the $L^2([0,t_0];L^2(0,1))-$norm of $\tilde{h}$, one gets

$$\begin{aligned}
&\|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e))\|^2_{L^2([0,t_0];L^2(0,1))}\\
&= \int_0^{t_0} \|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi(s))) - \tilde{g}(x^e))\|^2_{L^2(0,1)}ds\\
&\le \int_0^{t_0} (\|\tilde{g}(\eta(\xi(s)))\|_{L^2(0,1)} + \|\tilde{g}(x^e)\|_{L^2(0,1)})^2 ds\\
&= \int_0^{t_0} (\|\alpha(\delta - \eta(\xi(s)))e^{\frac{-\mu}{1+\eta(\xi(s))}}\|_{L^2(0,1)}\\
&\qquad + \|\alpha(\delta - x^e)e^{\frac{-\mu}{1+x^e}}\|_{L^2(0,1)})^2 ds
\end{aligned}$$

Since $e^{\frac{Pe}{2}z}\xi + x^e \in D(A) \cap \tilde{\mathcal{D}}$ and $(x^e \quad \frac{1}{\delta}x^e) \in \mathcal{K}$, by Lemma 4.1, $-1 + \frac{1}{\kappa} \le e^{\frac{Pe}{2}z}\xi(t,z) + x^e \le \delta + 1 - \frac{1}{\kappa}$ for a.e. $z \in [0,1]$, for all $t \in [0,t_0]$ and $0 \le x^e(z) \le \delta$, for a.e. $z \in [0,1]$. Consequently,

$$\left\|\alpha(\delta - \eta(\xi(t))e^{\frac{-\mu}{1+\eta(\xi(t))}}\right\|_{L^2(0,1)} \le \alpha(\delta + 1 - \frac{1}{\kappa})$$

and $\|\alpha(\delta - x^e)e^{-\mu/(1+x^e)}\|_{L^2(0,1)} \le \alpha\delta$. It follows that

$$\begin{aligned}
&\|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e))\|_{L^2([0,t_0];L^2(0,1))}\\
&\le \left(\int_0^{t_0} (\alpha(\delta + 1 - \frac{1}{\kappa}) + \alpha\delta)^2 dt\right)^{\frac{1}{2}} \le \alpha(2\delta + 1 - \frac{1}{\kappa})t_0^{\frac{1}{2}} < +\infty. \blacksquare
\end{aligned}$$

By Lemma 2.2, the state trajectory $\xi(t)$ of (24) and the function $e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)$ are in $L^\infty([0,t_0];L^2(0,1))$. Let us consider the $L^\infty([0,t_0];L^2(0,1))-$norm of ξ. The use of such norm clearly makes sense in the present context. We shall now give an estimate of that norm.

Lemma 4.3. *The state trajectory $\xi(t)$ of system (24) with the initial condition $\xi_0 \in D(A) \cap \tilde{\mathcal{D}}$ satisfies the estimate*

$$\|\xi\|_{L^\infty([0,t_0];L^2(0,1))} \le e^{kt_0}\|\xi_0\|_{L^2(0,1)}$$

for some positive constant k.

Proof. Consider the same $t_0 > 0$ as before and $0 \le t \le t_0$. Define $V(\xi(t)) = \frac{1}{2}\int_0^1 \xi^2(t)dz$ as a Lyapunov functional candidate for system (24). Differentiating V w.r.t. t along the state trajectory (24) yields

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2_{L^2(0,1)} \leq \left(\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4}\right)\|\xi(t)\|^2_{L^2(0,1)} + \int_0^1 e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi(t))) - \tilde{g}(x^e))\xi(t)dz, \quad (35)$$

see (Hastir et al., 2019, Section III) for further calculation details. The Cauchy-Schwarz inequality implies that

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2_{L^2(0,1)} \leq \left(\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4}\right)\|\xi(t)\|^2_{L^2(0,1)} + \|e^{\frac{-Pe}{2}z}(\tilde{g}(\eta(\xi(t))) - \tilde{g}(x^e))\|_{L^2(0,1)} \cdot \|\xi(t)\|_{L^2(0,1)}.$$

Since the operator $\tilde{g}$ is Lipschitz continuous, one obtains

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2_{L^2(0,1)} \leq \left(\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4}\right)\|\xi(t)\|^2_{L^2(0,1)} + l_{\tilde{g}}\|e^{\frac{Pe}{2}z}\xi(t)\|_{L^2(0,1)} \cdot \|\xi(t)\|_{L^2(0,1)},$$

where $l_{\tilde{g}} > 0$ denotes a Lipschitz constant associated to $\tilde{g}$. It follows that

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2_{L^2(0,1)} \leq \left(\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4} + l_{\tilde{g}}e^{\frac{Pe}{2}}\right)\|\xi(t)\|^2_{L^2(0,1)}.$$

By using Gronwall's Lemma and by denoting the constant $|\frac{-\pi^2}{\pi^2+4Pe} - \frac{Pe}{4} + l_{\tilde{g}}e^{\frac{Pe}{2}}|$ by k, one gets the inequality:

$$\|\xi(t)\|_{L^2(0,1)} \leq e^{kt}\,\|\xi_0\|_{L^2(0,1)}\,, \quad (36)$$

for all $t \in [0, t_0]$, which in turn leads to

$$\|\xi\|_{L^\infty([0,t_0];L^2(0,1))} \leq \sup_{t\in[0,t_0]} e^{kt}\|\xi_0\|_{L^2(0,1)} = e^{kt_0}\|\xi_0\|_{L^2(0,1)}.\blacksquare$$

The dense subspace Y of $L^2(0,1)$ that will be considered in the following is the Banach space $C(0,1)$ equipped with the uniform norm $\|\xi\|_{C(0,1)} := \|\xi\|_\infty = \sup_{z\in[0,1]}|\xi(z)|$.

By (Deimling, 1985, Chapter 6, Example 20.4.) the PDE (24) admits a mild solution $\xi(t,z) := S(t)\hat{\xi}_0$ on $Y = C(0,1)$, for $t \geq 0$, for every $\hat{\xi}_0 \in D(A)\cap\tilde{\mathcal{D}}$, wherein $L^2(0,1)$ is replaced by $C(0,1)$. That is,

$$S(t)\hat{\xi}_0 = \tilde{T}(t)\hat{\xi}_0 + \int_0^t \tilde{T}(t-s)(e^{\frac{-Pe}{2}z}[\tilde{g}(\eta(S(s)\hat{\xi}_0)) - \tilde{g}(x^e)])ds \quad (37)$$

for every $\hat{\xi}_0 \in D(A)\cap\tilde{\mathcal{D}}$, where $(\tilde{T}(t))_{t\geq0}$ is the C_0-semigroup whose operator A is the infinitesimal generator.

Moreover by (Martin, 1986, Lemma 5.1.), since for fixed $t_0 > 0$ the restricted functions $S(\cdot)\hat{\xi}_0$ and $S(\cdot)\tilde{\xi}_0 : [0,t_0) \to D(A)\cap\tilde{\mathcal{D}}$ are continuous and satisfy identity (37) with initial state $\hat{\xi}_0$ and $\tilde{\xi}_0$, respectively,

$$\|S(t)\hat{\xi}_0 - S(t)\tilde{\xi}_0\|_Y \leq \gamma_t\|\hat{\xi}_0 - \tilde{\xi}_0\|_Y, \quad (38)$$

for all $t \in [0,t_0)$, for some positive γ_t that depends on t and which is bounded on $[0,t_0)$. Inequality (38) is commonly called the continuous dependence property of the well-posed system (24) in the Banach space $Y = C(0,1)$. In particular for $\tilde{\xi}_0 = 0$,

$$\|S(t)\hat{\xi}_0\|_Y \leq \gamma_t\|\hat{\xi}_0\|_Y, \quad (39)$$

for all $t \in [0,t_0)$, since $S(t)$ maps the origin to itself. Looking at (39), one observes that

$$\|S(\cdot)\hat{\xi}_0\|_{L^\infty([0,t_0];C(0,1))} := \sup_{t\in[0,t_0]}\|S(t)\hat{\xi}_0\|_{C(0,1)} \leq \gamma_{t_0}\|\hat{\xi}_0\|_{C(0,1)} < \infty, \quad (40)$$

for all $\hat{\xi}_0 \in D(A)\cap\tilde{\mathcal{D}}$, where $\gamma_{t_0} := \sup_{t\in[0,t_0]}\gamma_t < \infty$.

Remark 4.3. *Observe that* $\|\cdot\|_{C(0,1)} = \|\cdot\|_{L^\infty(0,1)}$ *and* $\|\cdot\|_{L^\infty([0,t_0];C(0,1))} = \|\cdot\|_{L^\infty([0,t_0];L^\infty(0,1))}$.

Let us a consider the following result that will ensure equality in Lemma 2.3 to be true.

Lemma 4.4. *By denoting $\xi(t)$ the state trajectory of (24) that corresponds to the initial condition ξ_0, the following identity*

$$\lim_{\|\xi_0\|_{C(0,1)}\to0}\frac{\|R(\xi,x^e)\|_{L^\infty([0,t_0];L^2(0,1))}}{\|\xi_0\|_{L^2(0,1)}} = 0$$

holds, where the remainder $R(\xi,x^e)$ is given by $e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)$.

Proof. The notations $\|\cdot\|_{L^2(0,1)} =: |\cdot|_0, \|\cdot\|_{C(0,1)} =: |\cdot|_1$, $\|\cdot\|_{L^\infty([0,t_0];L^2(0,1))} =: |\cdot|_0^\infty$ and $\|\cdot\|_{L^\infty([0,t_0];C(0,1))} =: |\cdot|_1^\infty$ are used in the following for the sake of readability. Using the definitions of $\tilde{g}$ and of its Gâteaux derivative gives the following:

$$\begin{aligned}
&\left|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)\right|_0^\infty \\
&:= \Bigg|e^{-\frac{Pe}{2}z}(\alpha(\delta-\eta(\xi))e^{\frac{-\mu}{1+\eta(\xi)}} \\
&-\alpha(\delta-x^e)e^{\frac{-\mu}{1+x^e}}) - \left(\alpha\frac{\mu(\delta-x^e)}{(1+x^e)^2}e^{\frac{-\mu}{1+x^e}}\xi - \alpha e^{\frac{-\mu}{1+x^e}}\xi\right)\Bigg|_0^\infty \\
&= \alpha\Bigg|e^{-\frac{Pe}{2}z}(\delta-x^e)\left[e^{\frac{-\mu}{1+\eta(\xi)}} - e^{\frac{-\mu}{1+x^e}}\right] \\
&-\frac{\mu(\delta-x^e)}{(1+x^e)^2}e^{\frac{-\mu}{1+x^e}}\xi - \left[e^{\frac{-\mu}{1+\eta(\xi)}} - e^{\frac{-\mu}{1+x^e}}\right]\xi\Bigg|_0^\infty.
\end{aligned}$$

Observe that

$$\begin{aligned}
&e^{\frac{-\mu}{1+\eta(\xi)}} - e^{\frac{-\mu}{1+x^e}} = e^{\frac{-\mu}{1+x^e}}\left(e^{\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}} - 1\right) \\
&= e^{\frac{-\mu}{1+x^e}}\left(\sum_{n=0}^{+\infty}\left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}\right]^n\frac{1}{n!} - 1\right) \\
&= e^{\frac{-\mu}{1+x^e}}\left(\sum_{n=2}^{+\infty}\left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}\right]^n\frac{1}{n!} + \frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}\right).
\end{aligned}$$

In that way, by using the triangular inequality and the fact that $|e^{\frac{-\mu}{1+x^e}}|_\infty \leq 1$, one obtains[8]

$$\begin{aligned}&\left|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)\right|_0^\infty \\ &\leq \alpha\left|\mu(\delta - x^e)\frac{1}{(1+\eta(\xi))(1+x^e)}\xi\right. \\ &\left.\sum_{n=2}^{+\infty}\left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}\right]^{n-1}\frac{1}{n!}\right|_0^\infty \\ &+\alpha\mu|\delta - x^e|_\infty \cdot \left|\frac{e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)^2}\xi\right|_0^\infty \\ &+\alpha\left|\sum_{n=1}^{+\infty}\left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}\right]^n\frac{1}{n!}\xi\right|_0^\infty.\end{aligned}$$

By using $|\delta - x^e|_\infty \leq \delta$ and Lemma 4.1, it holds

$$\begin{aligned}&\left|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)\right|_0^\infty \\ &\leq \alpha\mu\delta\kappa|\xi|_0^\infty\left(\frac{e^{\mu\kappa e^{\frac{Pe}{2}}|\xi|_1^\infty} - \mu\kappa e^{\frac{Pe}{2}}|\xi|_1^\infty - 1}{\mu\kappa e^{\frac{Pe}{2}}|\xi|_1^\infty}\right) \\ &+\alpha\mu\delta e^{\frac{Pe}{2}}|\xi|_1^\infty|\xi|_0^\infty + \alpha|\xi|_0^\infty\left(e^{\mu e^{\frac{Pe}{2}}\kappa|\xi|_1^\infty} - 1\right), \qquad (41)\end{aligned}$$

see (♦) in the Appendix for further details. Moreover, by using Lemma 4.3 and equation (40), the inequalities $|\xi|_0^\infty \leq e^{kt_0}|\xi_0|_0 := \lambda_{t_0}|\xi_0|_0$ and $|\xi|_1^\infty \leq \gamma_{t_0}|\xi_0|_1$ are satisfied. Since the functions $\frac{e^x - x - 1}{x}$ and $e^x - 1$ are increasing functions of x, one obtains

$$\begin{aligned}&\left|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)\right|_0^\infty \\ &\leq \alpha\mu\delta\kappa\lambda_{t_0}|\xi_0|_0\frac{e^{\mu\kappa e^{\frac{Pe}{2}}\gamma_{t_0}|\xi_0|_1} - \mu\kappa e^{\frac{Pe}{2}}\gamma_{t_0}|\xi_0|_1 - 1}{\mu\kappa e^{\frac{Pe}{2}}\gamma_{t_0}|\xi_0|_1} \\ &+\alpha\mu\delta e^{\frac{Pe}{2}}\gamma_{t_0}\lambda_{t_0}|\xi_0|_1|\xi_0|_0 + \alpha\lambda_{t_0}|\xi_0|_0(e^{\mu e^{\frac{Pe}{2}}\kappa\gamma_{t_0}|\xi_0|_1} - 1).\end{aligned}$$

Consequently,

$$\begin{aligned}&\lim_{|\xi_0|_1\to 0}\frac{\left|e^{-\frac{Pe}{2}z}(\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e)e^{\frac{Pe}{2}z}\xi)\right|_0^\infty}{|\xi_0|_0} \\ &\leq \lim_{|\xi_0|_1\to 0}\left[\alpha\mu\delta\kappa\lambda_{t_0}\frac{e^{\mu\kappa e^{\frac{Pe}{2}}\gamma_{t_0}|\xi_0|_1} - \mu\kappa e^{\frac{Pe}{2}}\gamma_{t_0}|\xi_0|_1 - 1}{\mu\kappa e^{\frac{Pe}{2}}\gamma_{t_0}|\xi_0|_1}\right. \\ &\left.+\alpha\mu\delta e^{\frac{Pe}{2}}\gamma_{t_0}\lambda_{t_0}|\xi_0|_1 + \alpha\lambda_{t_0}(e^{\mu e^{\frac{Pe}{2}}\kappa\gamma_{t_0}|\xi_0|_1} - 1)\right] = 0. \blacksquare\end{aligned}$$

Similar arguments lead to

$$\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|f(\hat{\xi} + \xi^e) - f(\xi^e) - df(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];Y)}}{\|\hat{\xi}_0\|_Y} = 0,$$

providing that $(S(t))_{t\geq 0}$ is $Y-$Fréchet differentiable. This together with equalities (36) and (39) makes sure that Assumption 2.3 is satisfied. Assumption 2.4 is deduced from Remark 4.2 and (37). Note also that Assumption 3.1 is satisfied according to Remark 4.2.

By applying Lemma 3.1, one has that the nonlinear C_0-semigroup $(S(t))_{t\geq 0}$ is $C(0,1)$ to $L^2(0,1)$ Fréchet differentiable at 0. Looking at Theorem 3.1 leads to the following theorem. The proof is an immediate consequence of Theorem 3.1, Lemmas 4.1, 4.2, 4.3 and 4.4.

Theorem 4.1. *Consider the nonlinear PDE (24) that describes the time evolution of the temperature in a non-isothermal axial dispersion tubular reactor. In the case where the reactor exhibits one equilibrium profile, the latter is $(C(0,1), L^2(0,1))-$locally exponentially stable for the nonlinear system (24). In the case of three equilibria the pattern $(C(0,1), L^2(0,1))-$"locally exponentially stable - locally unstable - locally exponentially stable" is highlighted, which is called bistability.*

5. Numerical simulations

This section is devoted to the illustration of Theorem 4.1. A set of parameters is chosen in order to highlight three equilibria. The initial condition is chosen sufficiently small in the $C(0,1)-$norm and the diffusion is assumed to be large enough.

We choose $\xi_0(z) = \omega(\sin(\pi z) + \frac{2\pi}{Pe})$ as initial condition, where ω is a weighting factor that can make the $C(0,1)-$norm of ξ_0 as small as desired. Remark that this choice of initial condition satisfies $\frac{d\xi_0}{dz}(0) = \frac{Pe}{2}\xi_0(0)$ and $\frac{d\xi_0}{dz}(1) = -\frac{Pe}{2}\xi_0(1)$. The boundary conditions are then satisfied, i.e. $\xi_0 \in D(A)$.

An explicit computation of the $C(0,1)-$norm of ξ_0 gives

$$\|\xi_0\|_{C(0,1)} := \sup_{z\in[0,1]}|\xi_0(z)| = \frac{\omega}{Pe}(Pe + 2\pi). \qquad (42)$$

By choosing for instance $\omega = \epsilon Pe$, $\epsilon > 0$, (42) becomes $\|\xi_0\|_{C(0,1)} = \epsilon Pe + 2\pi\epsilon$, which can be made as small as desired by considering ϵ small.

As mentioned in Theorem 4.1, in the case of three equilibria, these are alternatively locally exponentially stable with the pattern "stable - unstable - stable". In order to illustrate the latter, the parameters have been chosen as follows : $\mu = 10, \delta = 1, v = 1.1 * 10^{-3}$ and $D = 10^{-3}$. Hence, $Pe = 1.1$. The parameter ω is fixed to $Pe * 10^{-2}$ such that $\|\xi_0\|_{C(0,1)} = 0.07383$. The state trajectories $\xi(t,z)$ and their $L^2(0,1)-$norm are represented in Figures 2, 4, 6, 3, 5 and 7 for the different equilibrium profiles, respectively. By looking at Figure 4, one can observe that the state trajectory does not diverge. The state trajectory does not go to $+\infty$ for t large but to another profile than the null function, which means that the system possesses at least one space dependent profile to which ξ is attracted.

[8]A preliminary step is found in the Appendix (♣).

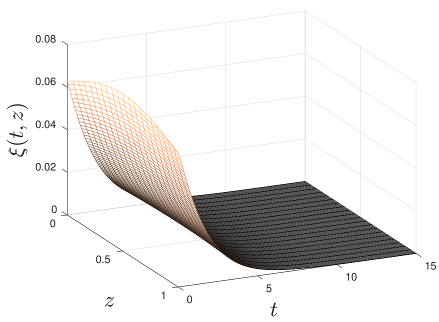

Figure 2: State trajectory ξ for $\mu = 10, \delta = 1$, first equilibrium.

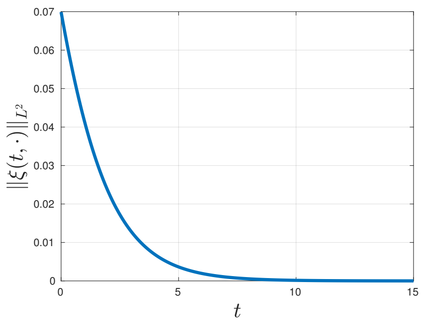

Figure 3: L^2-norm of the state trajectory ξ for $\mu = 10, \delta = 1$, first equilibrium.

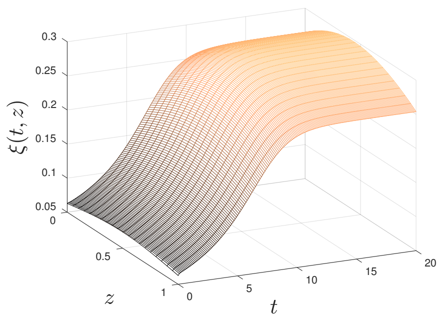

Figure 4: State trajectory ξ for $\mu = 10, \delta = 1$, second equilibrium.

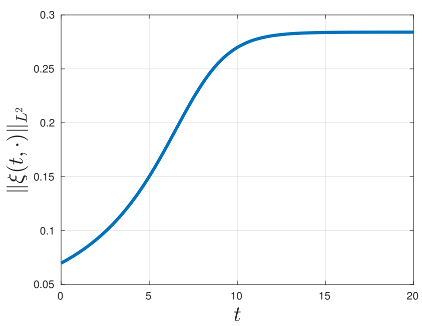

Figure 5: L^2-norm of the state trajectory ξ for $\mu = 10, \delta = 1$, second equilibrium.

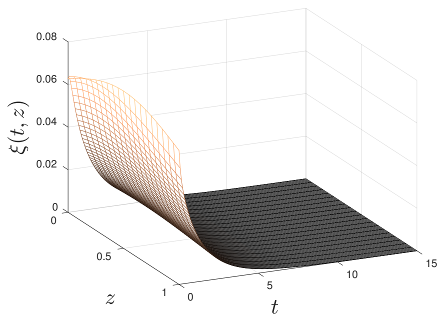

Figure 6: State trajectory ξ for $\mu = 10, \delta = 1$, third equilibrium.

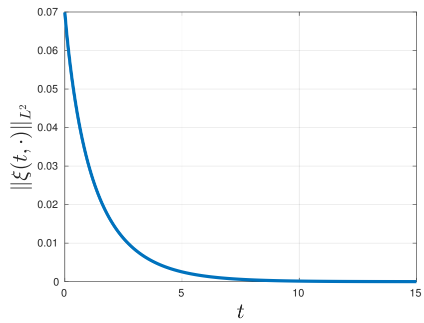

Figure 7: L^2-norm of the state trajectory ξ for $\mu = 10, \delta = 1$, third equilibrium.

6. Conclusion and future works

In this paper, it was shown how to link exponential stability or instability of an equilibrium profile of a nonlinear system with the exponential stability or instability of the corresponding linearized model around that equilibrium. The theoretical analysis goes along the lines of previous works in (Al Jamal et al., 2014; Al Jamal and Morris, 2018), which cannot be applied as such in the case of the specific systems that are considered here. In particular, our main result (Theorem 3.1) states that the exponential stability of an equilibrium profile of the nonlinear system is guaranteed whenever exponential stability holds for its linearization. This result is based on an extended concept of Fréchet differentiability of the nonlinear semigroup generated by the semilinear operator in the dynamics. It is also explained how our approach extends the works of (Al Jamal et al., 2014; Al Jamal and Morris, 2018).
Furthermore, a nonlinear PDE model describing a chemical process is considered: the main result is applied in order to show that in the case when the model exhibits only one equilibrium profile the latter is always locally exponentially stable for the nonlinear model. When three equilibria are highlighted, bistability is reached. That is, the equilibria are alternatively locally exponentially stable with the pattern "stable - unstable - stable", see Theorem 4.1. Numerical simulations supported the results.
A first perspective would be the investigation of non adiabatic conditions inside of the reactor. This leads us to

consider (15) with the additional term $-\gamma x_1 + \gamma x_w, \gamma > 0$ in the PDE associated with temperature. Remember that x_1 denotes the temperature and x_w the coolant temperature which is seen as a control variable here. Since the homogeneous system is considered in the analysis, this variable is identically 0. Hence, it suffices to study the system in which $-\gamma x_1$ is incorporated in the PDE associated with x_1. This can be seen as the linear perturbation operator $-\gamma \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = -\gamma P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where P is the projection operator defined as $P : L^2(0,1) \times L^2(0,1) \to L^2(0,1) \times L^2(0,1), P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and which norm is obviously less than 1. Note also that P is a positive operator, that is $\langle Px, x \rangle \geq 0$ for all $x \in L^2(0,1) \times L^2(0,1)$. Hence, in the case where the adiabatic reactor exhibits a stable equilibrium, adding the term $-\gamma P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ improves its stability margin by a negative constant Λ which satisfies $-\gamma \leq \Lambda \leq 0$. This equilibrium remains exponentially stable. Further analysis could be done in the case of an unstable equilibrium by using these "perturbation" arguments.

Moreover, the question of different Peclet numbers could be a possible way of extension of the present work. This would lead first to the study of exponential stability of the Gâteaux linearized version of (15) which remains still an open question.

Future works aim also at stating a general theoretical framework in order to design control laws to stabilize nonlinear distributed parameter systems.

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Appendix

(♣) The relations

$$\left| e^{-\frac{Pe}{2}z} (\tilde{g}(\eta(\xi)) - \tilde{g}(x^e) - d\tilde{g}(x^e) e^{\frac{Pe}{2}z}\xi) \right|_0^\infty =$$

$$\alpha \left| e^{-\frac{Pe}{2}z} (\delta - x^e) e^{\frac{-\mu}{1+x^e}} \sum_{n=2}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!} \right.$$

$$+ e^{-\frac{Pe}{2}z} (\delta - x^e) \, e^{\frac{-\mu}{1+x^e}} \frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)}$$

$$\left. - \frac{\mu(\delta - x^e)}{(1+x^e)^2} e^{\frac{-\mu}{1+x^e}} \xi - e^{\frac{-\mu}{1+x^e}} \sum_{n=1}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!} \xi \right|_0^\infty$$

$$\leq \alpha \left| e^{\frac{-\mu}{1+x^e}} \right|_\infty \cdot \left| e^{-\frac{Pe}{2}z} (\delta - x^e) \frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} * \right.$$

$$* \sum_{n=2}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^{n-1} \frac{1}{n!} + \mu(\delta - x^e)\xi \left[\frac{1}{(1+\eta(\xi))(1+x^e)} \right.$$

$$\left. \left. - \frac{1}{(1+x^e)^2} \right] - \sum_{n=1}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^n \frac{1}{n!} \xi \right|_0^\infty$$

hold.

(♦) Integration in Bochner spaces and Corollary 4.1 imply the following:

$$\left\| \mu(\delta - x^e) \frac{1}{(1+\eta(\xi))(1+x^e)} \xi * \right.$$

$$\left. * \sum_{n=2}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^{n-1} \frac{1}{n!} \right\|_{L^\infty([0,t_0];L^2(0,1))}$$

$$\leq \mu\delta\kappa \sum_{n=2}^{+\infty} \frac{1}{n!} \left\| \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^{n-1} \xi \right\|_{L^\infty([0,t_0];L^2(0,1))}$$

$$\leq \mu\delta\kappa \sum_{n=2}^{+\infty} \frac{\mu^{n-1}(e^{\frac{Pe}{2}})^{n-1}\kappa^{n-1}}{n!} \left\| \xi^{n-1}\xi \right\|_{L^\infty([0,t_0];L^2(0,1))}$$

$$\leq \mu\delta\kappa \sum_{n=2}^{+\infty} \frac{(\mu e^{\frac{Pe}{2}}\kappa)^{n-1}}{n!} \|\xi\|^{n-1}_{L^\infty([0,t_0];C(0,1))} \|\xi\|_{L^\infty([0,t_0];L^2(0,1))}$$

Noting that

$$\sum_{n=2}^{+\infty} \frac{k^{n-1}}{n!} = \frac{1}{k} \sum_{n=2}^{+\infty} \frac{k^n}{n!} = \frac{1}{k} \left(\sum_{n=0}^{+\infty} \frac{k^n}{n!} - k - 1 \right) = \frac{e^k - k - 1}{k}$$

leads to

$$\left\| \mu(\delta - x^e) \frac{1}{(1+\eta(\xi))(1+x^e)} \xi * \right.$$

$$\left. * \sum_{n=2}^{+\infty} \left[\frac{\mu e^{\frac{Pe}{2}z}\xi}{(1+\eta(\xi))(1+x^e)} \right]^{n-1} \frac{1}{n!} \right\|_{L^\infty([0,t_0];L^2(0,1))}$$

$$\leq \mu\delta\kappa \frac{e^{\mu\kappa e^{\frac{Pe}{2}} \|\xi\|_{L^\infty([0,t_0];C(0,1))}} - \mu\kappa e^{\frac{Pe}{2}} \|\xi\|_{L^\infty([0,t_0];C(0,1))} - 1}{\mu\kappa e^{\frac{Pe}{2}} \|\xi\|_{L^\infty([0,t_0];C(0,1))}} *$$

$$* \|\xi\|_{L^\infty([0,t_0];L^2(0,1))} .$$

Inequality (41) is obtained by using similar arguments.

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