

Investment decisions and normalization with incomplete markets: a remark*

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Abstract

Profit maximization is not a well defined objective when markets are incomplete. Several criteria of investment choice have therefore been put forward in the literature, some of which crucially hinge upon aggregation of shareholders' preferences, as is the case with the criteria proposed by Drèze (1974) and Grossman and Hart (1979). This note shows that these criteria are normalization dependent, i.e. their outcome depends on the good chosen to express individuals' marginal rates of substitution.

Key words: investment decisions, normalization, incomplete markets.

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1 Introduction

The usual assumption about firms in a competitive, frictionless economy is that they act in the interest of shareholders.

The meaning of this assumption is clear when all commodities (at all possible dates and/or states of nature) are priced and exchanged against each other, as in the model of general equilibrium of Arrow and Debreu (1954) and McKenzie (1954). In the Arrow-Debreu-McKenzie model, in which all agents maximize their preferences subject to a unique, intertemporal, budget constraint, firms simply have to maximize profits in order to maximize shareholders' wealth. As all goods (in an extended sense) are priced, the concept of profit is well defined, and no problem of shareholders' disagreement might never arise.

The same is true when agents cannot trade all (contingent) goods against one another, but can freely redistribute wealth across periods and/or states of the world by using a set of financial instruments, as in Arrow (1963). Also in this case people will face a vector of implicit or explicit prices that they can use to evaluate unambiguously cash flows accruing to them in the different contingencies.

When markets are incomplete, on the other hand, the contingent and/or asset prices observable on the market are not sufficient to price consumption in all possible contingencies, and agents can only use their personal, subjective prices (i.e. their marginal rates of substitution at different date-event pairs) to make this evaluation. Unfortunately, these subjective prices will not in general coincide across agents, which in turn implies that the evaluations of investment plans will be different across shareholders.

This creates an important problem for defining a sensible objective function for firms, if we assume that firms should operate on behalf of their shareholders, and this is a problem empirically signalled by the presence of "proxy fights" taking place in many corporations.

As explained by Drèze (1982), three different approaches have been tried to solve the indeterminacy problem which has just been hinted at.

One approach tried to exploit the information contained in prices of marketed shares. This was the avenue followed by Ekern and Wilson (1974), Leland (1974), Radner (1974) and others. This approach is also known as the "spanning" approach, as it is fundamental that all possible production plans by firms be "spanned by" (obtainable as a linear combination of) other firms' production plans. That the "spanning" condition is non generic has been demonstrated by DeMarzo (1993).

A second approach tries to derive firms' behaviour from shareholders' preferences, as they get reflected in individual state prices, in the following way: firms use a convex combination of shareholders' marginal utilities of income (of initial shareholders, as in Grossman and Hart (1979), or final shareholders, as in Drèze (1974)) to evaluate profits.

A third approach, followed by Radner (1972,1980), Sondermann (1974), Drèze (1980), Leland (1972) and others, impute to firms a sort of utility functions to maximize.

These three approaches are nested into one another. The first, for instance,

is compatible with the last one (which might be termed, after DeMarzo (1993), the “value maximizing approach”) as firms’ utility functions might be taken to coincide with those of one of their shareholders (the dominant shareholder, for example).

In this note the second approach will be taken into account, to show how the criteria that have been put forward to solve the indeterminacy problem are themselves indeterminate, in the sense that they depend on the particular normalization that is adopted. The outcome, in terms of investment choices, will be different according to the good in terms of which agents’ implicit prices are expressed. In other words, the production plan which is chosen if agents discount future consumption streams to date 0 will differ from the one that would be chosen if they discounted consumption to a different date/event pair. This reminds us very much of a similar problem encountered in the literature on imperfect competition (see Gabszewicz and Vial (1972)), but with an important difference: the problem arises, in the case of incomplete markets, only when one tries to aggregate shareholders’ preferences, in one or another way. In the case of imperfect competition, on the contrary, the problem arises even in the case of a single ownership.

The rest of the work is organized as follows: section 2 contains some quick, but hopefully useful, remarks on the problem of asset valuation with and without complete markets. Section 3 plugs this problem in the context of a multi-agent problem, and illustrates the so called “shareholder constrained efficient” rule for selecting investment decisions on the part of firms. Section 4 will show that such a rule, as any other depending on the aggregation of agents’ preferences, crucially depends on the normalization of preferences. Section 5 concludes.

2 Asset valuation with incomplete markets

A convenient way to look at the problem of asset valuation (and therefore also investment valuation) is that of considering agents’ utility maximization problem in an intertemporal and/or uncertain set-up. Let us imagine the following simple exchange economy, extending over $T + 1$ periods and featuring only one good per period. There is also one asset, which pays fixed quantities (a vector y) of the good at the various dates. The asset can be traded at one date only (i.e. it is not retraded).

Suppose an agent h in this economy solves the problem:

$$\begin{aligned} \max_x \quad & u^h(x) \\ \text{s.t.} \quad & p_0 x_0^h + (q - y_0)\theta^h = p_0 w^h + q\bar{\theta}^h \\ & p_t x_t^h = p_t w_t + p_t y_t \bar{\theta}^h \quad \forall t = 1, \dots, T \end{aligned}$$

where $x = (x_0, x_1, \dots, x_t, \dots, x_T)$, θ^h and $\bar{\theta}^h$ denote respectively the final and initial holding of the asset, q is the asset price p_0 and p_t the spot prices for the physical good available at date 0 and date t .

Spot prices at all dates other than the first play no role, as there is only one good per period and trade does not occur; therefore they will all be normalized to one. Notice, moreover, that the purchase of the asset takes place at time 0. This is tantamount to assuming that the opportunity cost of buying the asset is expressed in terms of forgone consumption at date 0. Let us compute the first order condition of this problem with respect to the asset. We obtain:

$$\frac{\partial u^h}{\partial x_0^h} q = \frac{\partial u^h}{\partial x_1^h} y_1 + \frac{\partial u^h}{\partial x_2^h} y_2 + \dots + \frac{\partial u^h}{\partial x_t^h} y_t + \dots + \frac{\partial u^h}{\partial x_T^h} y_T$$

or

$$q = \left(\frac{\partial u^h}{\partial x_1^h} / \frac{\partial u^h}{\partial x_0^h}\right) y_1 + \left(\frac{\partial u^h}{\partial x_2^h} / \frac{\partial u^h}{\partial x_0^h}\right) y_2 + \dots + \left(\frac{\partial u^h}{\partial x_t^h} / \frac{\partial u^h}{\partial x_0^h}\right) y_t + \dots + \left(\frac{\partial u^h}{\partial x_T^h} / \frac{\partial u^h}{\partial x_0^h}\right) y_T.$$

At an interior optimum the price of the asset has to be equal to the weighted average of its returns at the various dates, the weights being the components of the gradient of the utility function, divided by the marginal utility of consumption at time 0 (this is what we'll refer in the sequel as "normalized" gradient). The right hand side of the previous expression can also be taken to represent the marginal willingness to pay for the asset, which is expressed in terms of units of consumption at date 0. There is no compelling reason, however, to express agent h 's marginal willingness to pay in terms of consumption at date 0. We might wish to express it in terms of consumption at any time t . In this case the first order condition for a maximum would read:

$$q = \left(\frac{\partial u^h}{\partial x_1^h} / \frac{\partial u^h}{\partial x_t^h}\right) y_1 + \left(\frac{\partial u^h}{\partial x_2^h} / \frac{\partial u^h}{\partial x_t^h}\right) y_2 + \dots + y_t + \dots + \left(\frac{\partial u^h}{\partial x_T^h} / \frac{\partial u^h}{\partial x_t^h}\right) y_T,$$

where the weight attached to date t is obviously one.

The previous expression corresponds to the first order condition of the problem:

$$\begin{aligned} \max_x \quad & u^h(x) \\ \text{s.t. } \quad & x_s^h = w_s^h + \theta^h y_s \quad \forall s \neq t \\ & x_t^h + (q - y_t) \theta^h = w_t + \bar{\theta}^h q \end{aligned}$$

This corresponds to the problem in which agent h pays for the asset at time t , instead of at time 0.

The marginal willingness to pay for an asset also represents the agent's valuation of a small additional quantity of this asset in his portfolio. This is why it has been often used to evaluate marginal changes of asset returns with respect to a given vector of returns.

3 The case of many agents

Let us now introduce an economy with many agents; each of them will assess the value of an asset following the procedure illustrated in the previous section, i.e. taking the weighted average of the asset's returns, the weights being his/her marginal rates of substitution.

It is certainly helpful to compare the case in which markets are complete with that in which markets are incomplete. In the first case all agents have the possibility of trading as many independent assets as there are contingencies (in a temporal or uncertainty sense) to redistribute wealth across date-event pairs. In the second case the set of assets will be more limited and in any case will not allow agents to effect any redistribution of wealth, but only some.

In the first case the first order conditions of agent h 's maximization problem will be, in vector form:

$$\bar{q} = Du^h Y \quad (3.1)$$

where $\bar{q}$ is the vector of asset prices, Du^h is the gradient of h 's utility function, normalized with respect to, say, consumption at date 0 and Y is the matrix of asset returns.

Comparing the case of complete markets to that of incomplete markets is essentially the same as comparing condition (3.1) in the case in which Y is a full rank matrix with condition (3.1) when Y is not a full rank matrix. In the former case we obtain:

$$Du^h = \bar{q} Y^{-1} \quad (3.2)$$

which means that all agents' normalized gradients will be aligned across agents, as they only depend upon "objective elements", such as asset prices and asset returns.

When matrix Y is not invertible, the former expression can no longer be obtained; Equation (3.2) possesses a possibly multidimensional infinity of solutions. Normalized gradients will not necessarily coincide across agents, and indeed they will not generically, as was shown in the contributions of Duffie and Shafer (1985) and DeMarzo (1993).

What can managers in a corporation do, in such a case? Whose marginal rates of substitution should they use to evaluate any investment plan? Many possible answers have been put forward, in terms of criteria that a corporation might possibly adopt. One such criterion is that of "shareholder constrained efficiency", introduced by Drèze (1974), modified by Grossman and Hart (1979) and later on proposed by Geanakoplos et al. (1990) for an economy with production and incomplete markets.

Definition 1 (see Geanakoplos et al. (1990)): a production plan y^* is said to be "shareholder constrained efficient" if there does not exist another plan $\hat{y} = y^* + dy$, technologically feasible, and a vector of income transfers $\tau \equiv (\tau_1, \dots, \tau_S)$, with $\sum_{i \in S} \tau^i = 0$, S being the set of final shareholders, such that:

$$V^i(\hat{y}, \tau^i) \geq V^i(y^*, 0), \forall i \in S \quad (3.3)$$

with at least one strict inequality ($V^i(\cdot)$ being the indirect utility associated to the plan y and the transfer τ^i).

The right hand side of (3.3) represents the level of satisfaction reached by shareholder i at the “status quo”. The left hand side represents the level of satisfaction he would get if plan $\hat{y}$ were implemented, and if he were in addition given the transfer τ^i . It should be clear from the beginning that these transfers of numeraire are meant to take place *ex post*, therefore not influencing equilibrium prices (we could alternatively say that these transfers are in “utils”, which cannot be traded).

To put the notion in a slightly different way, which will be made clearer in the sequel, we could say that a plan y^* is shareholders constrained efficient if the following is true: no group (possibly singletons) of shareholders would propose a marginal change of the plan y^* if they had to compensate all the shareholders suffering a loss from the change. It seems evident that a notion of veto power is embodied in the definition (but one can think of a “ k -percent” shareholders’ constrained efficiency definition).

In the following section it will be shown that the outcome of such a rule is normalization dependent, in the sense which was made clear in the previous section.

4 Normalization matters

Let us consider the following, simple example consisting of one firm owned by two shareholders, living for three periods. The intertemporal technology of the firm is a linear one, of the type:

$$Y = \left\{ y \in R^3 \mid y_0 = -k, y_1 = -\frac{1}{2}k, y_2 = 2\sqrt{k}, k \geq 0 \right\}.$$

Therefore, this firm produces a positive output in period 2 using inputs in the other two periods.

There is only one good per period, and we will refer equivalently to good 0, 1 and 2 to indicate, respectively, consumption in period 0, period 1 and period 2.

Markets are incomplete, in that there are no assets which allow agents to redistribute freely their wealth across periods of time.

Shareholders 1 and 2 own a quota of respectively θ and $(1 - \theta)$ of this firm. They have preferences represented by the utility functions:

$$\begin{aligned} u^1 &= x_0^1 + 2x_1^1 + x_2^1 \\ u^2 &= x_0^2 + x_1^2 + 3x_2^2 \end{aligned}$$

and they have no initial endowments.

The linearity of the example has the only purpose of simplifying calculations, and entails no loss of generality. In fact, we are not allowing agents in this economy to trade shares, as trade would generically lead to a corner solution, with one agent disinvesting his ownership quota and the other buying the entirety of the firm. We want to exclude this occurrence, in which the problem of shareholders' disagreement vanishes. What we intend to show, however, is that “utility maximizing” rules, like those proposed by Drèze or by Grossman-Hart, generate outcomes which depend on the particular normalization which has been adopted, and this can be demonstrated in a non-trading set-up, as the present one.

Simple calculations show that the “shareholder efficient rule” would be, in the case of normalization with respect to good 0 (i.e. compensation taking place in terms of the good available in period 0):

$$k = \left(\frac{3-2\theta}{1+\frac{1}{2}(1+\theta)} \right)^2. \quad (4.1)$$

If, alternatively, the utility gradients were normalized with respect to good 1, the optimal decision would be:

$$k = \left(\frac{3-\frac{5}{2}\theta}{\frac{3}{2}-\frac{1}{2}\theta} \right)^2. \quad (4.2)$$

Finally, if gradients are normalized with respect to consumption at time 2, we obtain:

$$k = \left(\frac{1}{\frac{1}{2}+\frac{3}{2}\theta} \right)^2. \quad (4.3)$$

It is evident that the three normalization rules lead to three different outcomes. In fact, the following graph shows that the three rules can be uniformly ranked according to the optimal k , as a function of the ownership quota, θ (the boldface number above each curve indicates the normalization adopted in the computation).

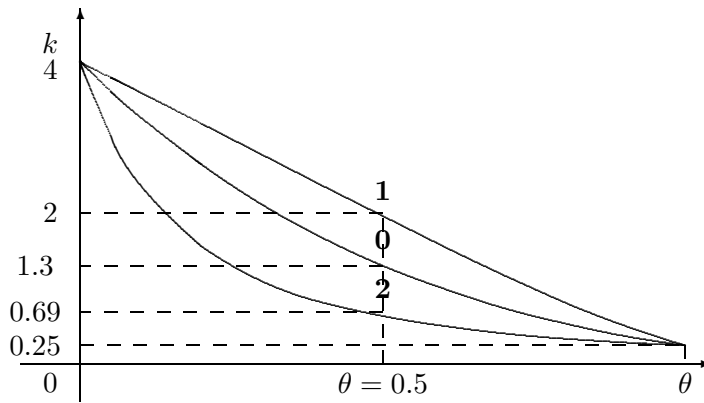


Figure 1

What drives the result is that agents differ in marginal rates of substitution between consumption at different dates. Let us consider, for example, the shareholders' efficient plan corresponding to $\theta = 0.5$, when gradients are normalized with respect to the good available in period 0. From (4.1) we obtain $k = 64/49$. If the gradients of the utility functions were normalized with respect to consumption in period 1, the sum of the marginal willingness to pay would be positive, and equal to 0,281. The reason of this is very clear. For the second agent nothing changes with respect to the first case: his marginal utility for period 1 consumption is, in fact, equal to that for consumption at time 0. This implies, in turn, that a marginal increase in k is equivalent, from agent 2's standpoint, to the same additional quantity of good 0 and good 1. For agent 1, on the contrary, the marginal utility of consumption at date 1 is twice as much as that of consumption at date 0, which means that he will demand, in terms of good 1, half of the compensation he demanded in terms of good 0. That's why agent 2 is left with a positive surplus, which makes that $k = 64/49$ is not a shareholders' constrained efficient plan. In fact, and this should be clear from figure 1, the shareholder constrained efficient plan corresponding to $\theta = 0.5$ is associated to a higher k .

Quite the contrary occurs when the gradients of utility functions are normalized with respect to consumption in period 2; in this case it becomes much more expensive for the second agent to compensate the first for his marginal loss; therefore the sum of agents' valuations of a marginal change in production becomes negative, and equal to $-0,375$. As a consequence of this, we can observe from figure 1 that the shareholder efficient plan is associated to a lower value of k . Needless to say, if marginal rates of substitution were equal across agents, as in the case of complete markets, the three possible normalization would lead to exactly the same results.

In the example proposed in this section we have imposed a difference in shareholders' marginal rates of substitution by fixing the coefficients of the linear utility functions, but the same would have happened with more general utility functions, given market incompleteness.

Figure 2 and figure 3 plot the utility levels of the two agents at the "shareholder constrained efficient" k , always as a function of the ownership quota. For each agent the three normalization rules can be uniformly ranked in terms of utility levels.

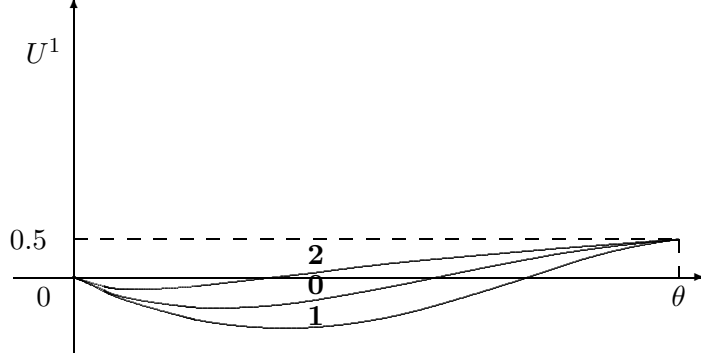


Figure 2

From the graphs it is also clear that no rule Pareto dominates the other two. The best normalization rule is the third for agent 1, the second for agent 2.

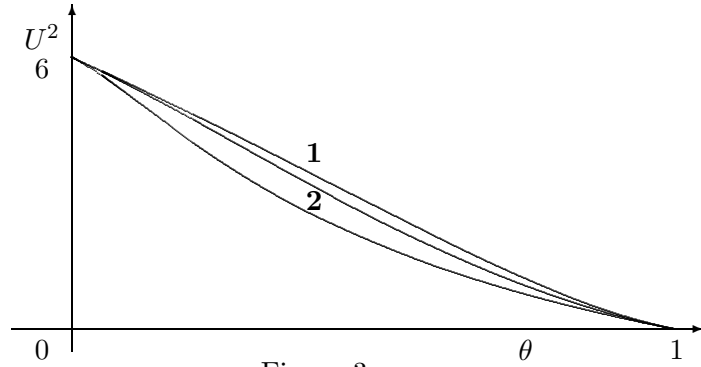


Figure 3

5 Final remarks

One of the most interesting and controversial issues in the domain of incomplete markets is that concerning the behaviour (in terms of investments) of competitive corporations. If markets are incomplete, the usual goal of profit maximization becomes ambiguous, in that marginal rates of substitution across date-event pairs are not equalized across agents. Many criteria have been proposed that are based on the aggregation of individuals' marginal rates of substitutions, such as the so-called Drèze criterion or the Grossman-Hart criterion, or the generalizations proposed by DeMarzo (1993). In this note it is explained, and shown by means of a simple example, that all those criteria are normalization dependent, in that the investment plan selected by firms depends on the particular normalization chosen for individuals' marginal rates of substitution.

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