A conjecture about age inequalities

Eric Deleersnijder, 19 February 2020

Abstract. *The age of a passive tracer is evaluated by means of CART's equations (www.climate.be/cart) as the time elapsed since touching for the last time the departure (open) boundary of the domain of interest. The tracer particles are discarded on the arrival (open) boundary. On the latter, Dirichlet boundary conditions or Neumann (zero diffusive flux) conditions can be prescribed. It is hypothesized that the age ensuing from the previous types of arrival boundary conditions is, at any time and position, smaller than or equal to that obtained by imposing Neumann boundary conditions on the arrival boundary. This conjecture, which is seen to hold true in a simplistic flow problem, has yet to be demonstrated. Assuming that this conjecture holds valid, a tentative physical explanation of it is suggested.*

Domain geometry and flow properties

Let Ω denote the domain of interest, whose boundary is surface Γ, with outward unit normal $\mathbf{n}$ (Figure 1). This surface consists of an impermeable part (Γ^{imp}), a departure boundary (Γ^{dep}) and an arrival one (Γ^{arv}), with $\Gamma = \Gamma^{imp} \cup \Gamma^{dep} \cup \Gamma^{arv}$. The fluid velocity is $\mathbf{v}(t,\mathbf{x})$, where t and $\mathbf{x}$ are the time and position-vector, respectively.

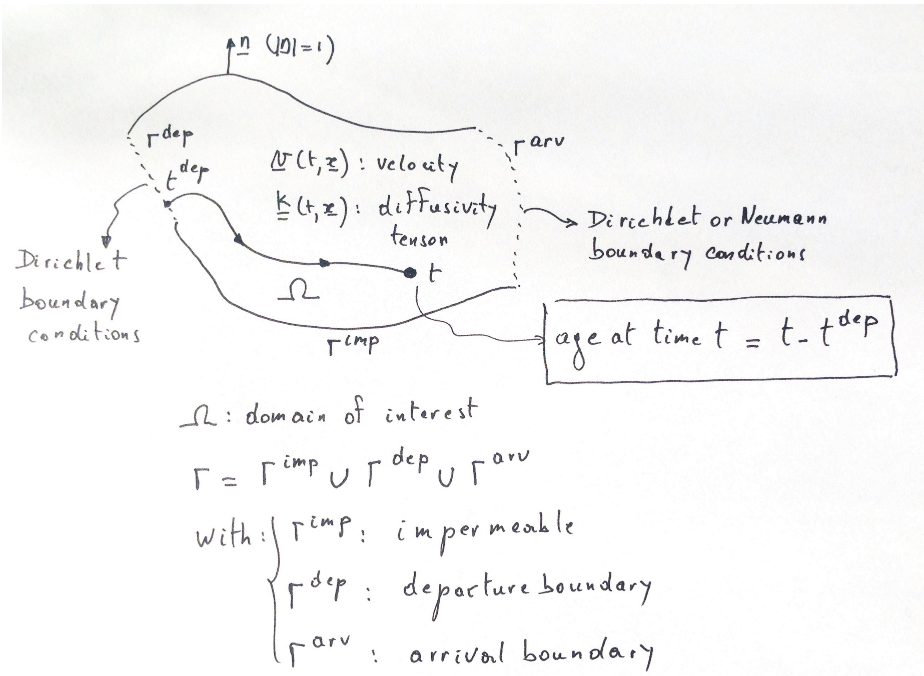

Figure 1. Illustration of the geometry of the domain of interest and some of the flow properties. The age of a passive tracer is the time elapsed since touching for the last time departure boundary Γ^{dep}. The arrival boundary (Γ^{arv}) is the surface where the tracer particles are discarded, which can be achieved by having recourse to various types of boundary conditions.

The Boussinesq approximation holds valid, implying that the velocity is divergence-free ($\nabla \bullet \mathbf{v} = 0$). The velocity satisfies boundary condition

$$\left[\mathbf{v} \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^{imp}} = 0 \quad . \tag{1}$$

Although it may not be not strictly required, the velocity is likely to satisfy $\mathbf{v} \bullet \mathbf{n} \leq 0$ (resp. $\mathbf{v} \bullet \mathbf{n} \geq 0$) on the departure (resp. arrival) boundary. On the other hand, unresolved fluxes are parameterised with the help of diffusivity tensor $\mathbf{K}(t,\mathbf{x})$, which must be symmetric and positive definite (e.g. Deleersnijder 2012).

Age-based diagnostic strategy

To diagnose the flow in the domain of interest and, to a certain degree, the exchanges with its environment, a passive tracer is taken into consideration. Its concentration, $C(t,\mathbf{x})$, is the solution of partial differential problem

$$\begin{cases} \dfrac{\partial C}{\partial t} = -\nabla \bullet (C\mathbf{v} - \mathbf{K} \bullet \nabla C) \\ C(0,\mathbf{x}) = 0 \,, \; \left[(-\mathbf{K} \bullet \nabla C) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^{imp}} = 0 \,, \; \left[C(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^{dep}} = 1 \,, \; \left[C(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^{arv}} = 0 \end{cases} \tag{2}$$

We will evaluate the age of this tracer as the time elapsed since touching for the last time the departure boundary. To do so, the age concentration, $\alpha(t,\mathbf{x})$, must be calculated, which is governed by[1]

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} = -\nabla \bullet (\alpha\mathbf{v} - \mathbf{K} \bullet \nabla \alpha) + C \\ \alpha(0,\mathbf{x}) = 0 \,, \; \left[(-\mathbf{K} \bullet \nabla \alpha) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^{imp}} = 0 \,, \; \left[\alpha(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^{dep} \cup \Gamma^{arv}} = 0 \end{cases} \tag{3}$$

Then, the age of the tracer is

$$a(t,\mathbf{x}) = \frac{\alpha(t,\mathbf{x})}{C(t,\mathbf{x})} \quad . \tag{4}$$

Owing to the Dirichlet boundary conditions prescribed on the arrival boundary, both the concentration and age concentration are zero on this surface, implying that the age, the ratio of the latter to the former, is an indeterminate form, i.e. 0/0. However, as demonstrated in Deleersnijder et al. (2020), the age satisfies

$$\left[(-\mathbf{K} \bullet \nabla a) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^{arv}} = 0 \quad , \tag{5}$$

implying that the age is most likely to have a finite value on the arrival boundary — as well as at any point of the domain of interest.

Alternative arrival boundary conditions

Because of the Dirichlet boundary conditions imposed on the arrival boundary, the tracer flux leaving the domain through this boundary is entirely of a diffusive nature. An alternative to this approach consists in prescribing that the diffusive flux be zero (i.e. imposing a Neumann

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[1] The age related equations used herein are those of CART (Constituent-oriented Age and Residence time Theory). Relevant references may be found on the web at address http://www.climate.be/cart

boundary condition instead of a Dirichlet one) so that the flux through this boundary will be entirely advective. The related concentration and age concentration (identified below by primes) are to be obtained by tackling the following partial differential problems:

$$\begin{cases} \dfrac{\partial C'}{\partial t} = -\nabla \bullet (C'\mathbf{v} - \mathbf{K} \bullet \nabla C') \\ C'(0,\mathbf{x}) = 0 \,, \; \left[(-\mathbf{K} \bullet \nabla C') \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^{imp} \cup \Gamma^{arv}} = 0 \,, \; \left[C'(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^{dep}} = 1 \end{cases} \tag{6}$$

and

$$\begin{cases} \dfrac{\partial \alpha'}{\partial t} = -\nabla \bullet (\alpha'\mathbf{v} - \mathbf{K} \bullet \nabla \alpha') + C' \\ \alpha'(0,\mathbf{x}) = 0 \,, \; \left[(-\mathbf{K} \bullet \nabla \alpha') \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^{imp} \cup \Gamma^{arv}} = 0 \,, \; \left[\alpha'(t,\mathbf{x})\right]_{\mathbf{x} \in \Gamma^{dep}} = 0 \end{cases} \tag{7}$$

Then, the alternative age is

$$a'(t,\mathbf{x}) = \frac{\alpha'(t,\mathbf{x})}{C'(t,\mathbf{x})} \; . \tag{8}$$

One-dimensional, steady-state illustration

For illustration purposes, we consider a one-dimensional flow in a domain defined by inequalities $0 \le x \le L$, where x is the along-flow space coordinate and L is the length of the domain (Figure 2). There is a departure (resp. arrival) open boundary at $x = 0$ (resp. $x = L$) (Deleersnijder et al 2020).

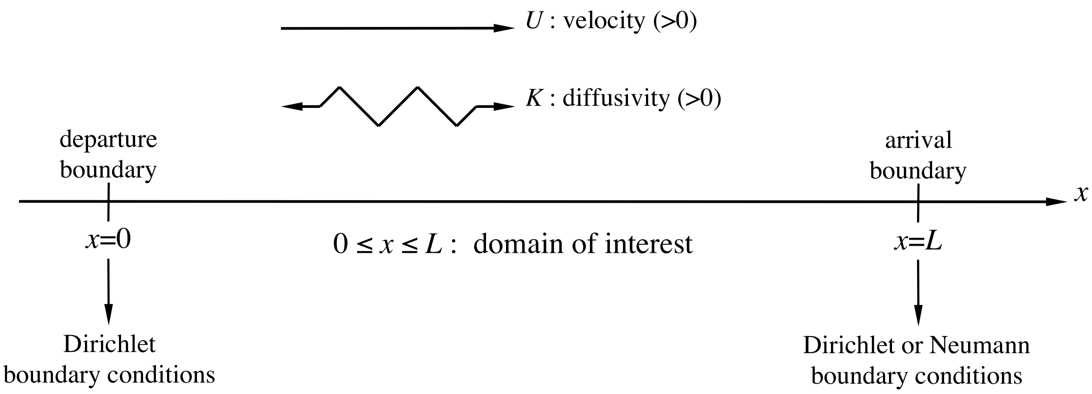

Figure 2. Schematic representation of a finite-sized domain ($x \in [0, L]$) with a departure boundary at x=0 and an arrival one at $x = L$. Dirichlet boundary conditions are prescribed on the boundaries. An alternative treatment of the arrival boundary leads to the implementation of Neumann (zero-diffusive flux) boundary conditions at $x = L$.

For the sake of simplicity, we focus on steady-state solutions. Accordingly, concentration $C(x)$ and age concentration $\alpha(x)$ obey

$$\begin{cases} 0 \;=\; -U\dfrac{dC}{dx} \;+\; K\dfrac{d^2C}{dx^2} \\ C(0)=1\,, \;\; C(L)=0 \end{cases} \tag{9}$$

and

$$\begin{cases} 0 \;=\; -U\dfrac{d\alpha}{dx} \;+\; K\dfrac{d^2\alpha}{dx^2} \;+\; C \\ \alpha(0)=0\,, \;\; \alpha(L)=0 \end{cases} \tag{10}$$

where positive constant U and K denote the water velocity and the diffusivity, respectively.

The concentration and age concentration are (Figure 3)

$$C(x) \;=\; \frac{e^{Pe}-e^{Ux/K}}{e^{Pe}-1} \tag{11}$$

and

$$\alpha(x) \;=\; \frac{e^{Pe}+e^{Ux/K}}{e^{Pe}-1}\frac{x}{U} \;-\; \frac{2e^{Pe}(e^{Ux/K}-1)}{(e^{Pe}-1)^2}\frac{L}{U} \quad , \tag{12}$$

where dimensionless parameter $Pe = UL/K$ is the Peclet number, i.e. the ratio of the timescale characterising diffusion (L^2/K) and that associated with advection (L/U). In the vicinity of the departure boundary ($x=0$), the age, $a(x)=\alpha(x)/C(x)$, admits asymptotic expansion

$$a(x) \sim \frac{e^{2Pe}-2Pe\,e^{Pe}-1}{(e^{Pe}-1)^2}\frac{x}{U} \;, \quad x \to 0 \quad , \tag{13}$$

which, unsurprisingly, simplifies to $a(x) \sim x/U$ in the limit $Pe \to \infty$. As for the arrival boundary ($x=L$), the age tends to a finite value with a zero gradient (Deleersnijder et al. 2020),

$$a(x) \sim \underbrace{\frac{Pe(e^{Pe}+1)-2(e^{Pe}-1)}{e^{Pe}-1}\frac{K}{U^2}}_{=a(L)} \;-\; \frac{(L-x)^2}{6K} \;, \quad x \to L \tag{14}$$

with $a(L) \to L/U$ as $Pe \to \infty$. The larger the Peclet number, the closer the solutions are to their zero diffusion counterparts, i.e. a unit value of concentration, with the age concentration and age equal to x/U. Such solutions cannot satisfy the boundary conditions prescribed at $x=L$. This is why the correct concentration and age concentration exhibit a boundary layer adjacent to the outgoing boundary when $Pe >> 1$ (Figure 3).

The alternative diagnostic strategy referred to above consists in prescribing Neumann boundary conditions at $x=L$. Accordingly, the differential problems to be solved read

$$\begin{cases} 0 \;=\; -U\dfrac{dC'}{dx} \;+\; K\dfrac{d^2C'}{dx^2} \\ C'(0)=1\,, \;\; \left[-K\dfrac{dC'}{dx}\right]_{x=L} = 0 \end{cases} \tag{15}$$

and

$$\begin{cases} 0 = -U\dfrac{d\alpha'}{dx} + K\dfrac{d^2\alpha'}{dx^2} + C' \\ \alpha'(0)=0\,, \quad \left[-K\dfrac{d\alpha'}{dx}\right]_{x=L} = 0 \end{cases} \tag{16}$$

Then, the alternative concentration, age concentration and age are

$$C'(x)=1 \;, \quad \alpha'(x) = \frac{x}{U} - \frac{e^{Ux/K}-1}{e^{Pe}}\frac{K}{U^2} = a'(x) \quad . \tag{17}$$

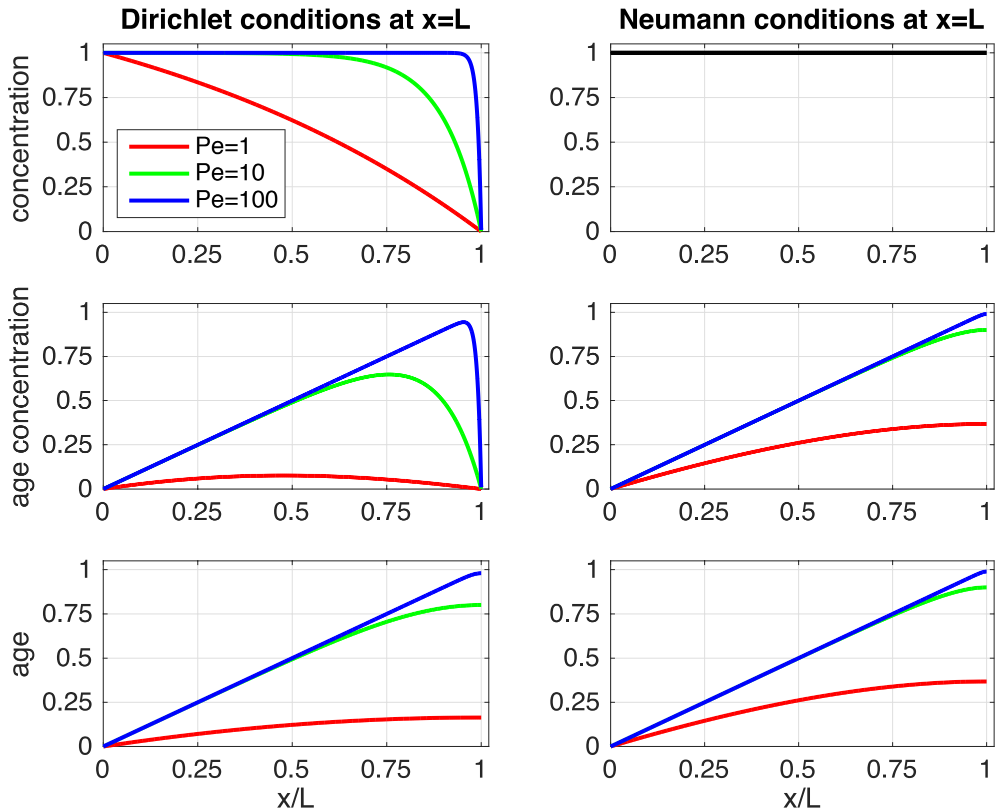

Figure 3. Illustration for various values of the Peclet number ($Pe = UL/K$) of the one-dimensional, steady-state concentrations, age concentrations and ages obtained by imposing Dirichlet (left column) and Neumann (right column) boundary conditions on the arrival boundary ($x = L$). The same boundary conditions are prescribed on the departure boundary ($x = 0$). Dimensionless age concentration and age are displayed, which are obtained by dividing the dimensional age and age concentration by advective timescale L/U .

For markedly different reasons, the derivative of both ages is zero on the arrival boundary ($x = L$). On the other hand, it can be proven rigorously that the abovementioned solutions satisfy the following inequalities (Figure 3)

$$C(x) \le C'(x)\,, \quad \alpha(x) \le \alpha'(x)\,, \quad a(x) \le a'(x) \quad . \tag{18}$$

Theorem to be demonstrated and tentative physical interpretation

Unless I am gravely mistaken, at any time and position, the concentrations and age concentrations are such that

$$C(t,\mathbf{x}) \leq C'(t,\mathbf{x}) \tag{19}$$

and

$$\alpha(t,\mathbf{x}) \leq \alpha'(t,\mathbf{x}) \tag{20}$$

Presumably, proving that (19) and (20) hold valid could be achieved by having recourse to "energy methods" such as those used in Deleersnijder (2019).

In view of the steady-state, one-dimensional solution derived above, I feel inclined to believe that the ages satisfy

$$\boxed{a(t,\mathbf{x}) \leq a'(t,\mathbf{x})} \tag{21}$$

Would it be possible to prove[2] that (21) holds true or else find a counterexample, in which case (21) would not always be valid? Would this property still hold valid for a tracer undergoing a first-order decay process (e.g. a radioactive tracer)?

If conjecture (21) is true, then a tentative physical explanation thereof might be as follows: by virtue of inequality (20), the age content[3] of the domain is greater for the alternative diagnostic strategy, which might be the reason why the related age is greater than that ensuing from Dirichlet boundary being imposed on the arrival boundary. This reasoning is, however, not flawless. Indeed, the mass of tracer present in the domain is also larger when the alternative approach is adopted. In other words, things are not as simple as they seem: when attempting to interpret age results, one should beware of a reasoning having recourse to additive quantities and, above all, keep in mind that age is an intensive variable as opposed to the concentration and the age concentration as well as the integrals over the domain of them.

A more convincing explanation might be based on a slightly different (and somewhat more sophisticated) line of argument. It is because of the boundary conditions imposed on the departure boundary that the alternative concentration is greater that than obtained under Dirichlet boundary conditions. In turn, the alternative age concentration is larger for two reasons, i.e. the implementation of Neumann boundary conditions on the departure boundary and the larger ageing term (the last term in the right-hand side of the age concentration equation, which is equal to the concentration). As a consequence, when adopting the alternative approach the increase of the age concentration may be relatively larger than that undergone by the concentration. This may be why the alternative age eventually is greater than the other one. This reasoning has yet to be thoroughly assessed by having recourse to appropriate mathematical developments.

References

Deleersnijder E., 2012, *Homogenisation of a passive tracer concentration in an isolated domain*, Working note, Université catholique de Louvain, Louvain-la-Neuve, Belgium, 7

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[2] I surmise that the proof should resort to the problem's Green's function.
[3] The age content of a particle is the product of its mass and its age.

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Reward

As usual in such circumstances[4], I will offer a bottle of Champagne (or any other delicious beverage) to the first person who will demonstrate the above theorem or else build a relevant counterexample, thereby showing that inequality (21) is not always valid.

[4] For instance, see https://www.giss.nasa.gov/research/briefs/hall_02/