

Stability of six-dimensional hyperstring braneworlds

Christophe Ringeval*

*Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2AZ, United Kingdom
Département de Physique Théorique, Université de Genève,
24 quai Ernest Ansermet, 1211 Genève 4, Switzerland*

Patrick Peter[†] and Jean-Philippe Uzan[‡]

*Institut d'Astrophysique de Paris, (GReCO), UMR 7095-CNRS, 98bis boulevard Arago, 75014 Paris, France
(Dated: February 1, 2008)*

We study a six-dimensional braneworld model with infinite warped extra dimensions in the case where the four-dimensional brane is described by a topological vortex of a $U(1)$ symmetry-breaking Abelian Higgs model in presence of a negative cosmological constant. A detailed analysis of the microscopic parameters leading to a finite volume space-time in the extra dimensions is numerically performed. As previously shown, we find that a fine-tuning is required to avoid any kind of singularity on the brane. We then discuss the stability of the vortex by investigating the scalar part of the gauge-invariant perturbations around this fine-tuned configuration. It is found that the hyperstring forming Higgs and gauge fields, as well as the background metric warp factors, cannot be perturbed at all, whereas transverse modes can be considered stable. The warped space-time structure that is imposed around the vortex thus appears severely constrained and cannot generically support nonempty universe models. The genericness of our conclusions is discussed; this will shed some light on the possibility of describing our space-time as a general six-dimensional warped braneworld.

PACS numbers: 04.50.+h, 11.10.Kk, 98.80.Cq

I. INTRODUCTION

Following the advent of string theory [1] and its implication that space may have more than the usual three dimensions (in the Kaluza-Klein way) came the suggestion that the extra dimensions could be much larger than previously expected, would it be because of a smaller value of the Planck energy in the bulk [2, 3, 4], or because of a large curvature in the (infinite) extra dimensions [5, 6]. A novel idea came into play with the assumption that we live on a hypersurface, a three-spatial dimensional “brane”, embedded in a larger dimensional warped space-time bulk [7, 8].

For any higher dimensional Universe model, it is essential to confine gravity since gravitation is experimentally tested to be three dimensional on many different scales, ranging from the millimeter [9] to a few Mega-parsecs [10, 11]. For a five-dimensional anti-de Sitter bulk, gravity was shown to be localized on the brane [5, 6] and to lead to a viable cosmological framework [12, 13, 14, 15, 16, 17, 18, 19, 20] provided the brane and bulk cosmological constants are adjusted by hand. The situation is not yet settled concerning the cosmological perturbations induced in the brane [21], although there are some indications that such models should satisfy more stringent constraints than previously expected [22]. In the case the brane is modeled as a domain wall-like topological defect, this fine-tuning trans-

forms into a tuning of the underlying parameters (masses, coupling constant and bulk cosmological constant) [23].

Many mechanisms have been proposed to confine the other known interactions and their associated particles: scalar [24], gauge bosons [25, 26, 27, 28, 29, 30, 31, 32, 33], and fermions [34]. In the latter case the mechanism relies on a generalization of the cosmic string case [35, 36, 37]. The fermions are trapped in the brane in the form of massless zero modes and some can even become massive, although their mass spectrum is not compatible [22] with the observed one [38]. It was also suggested [39] that the electroweak Higgs field, and thus the origin of electroweak symmetry breaking, could be understood from the existence of an extra dimension in the form of a transverse gauge field component. Most of these works were based on the simplifying assumption of a reflection symmetry with respect to the brane, although a more refined treatment, not assuming such a symmetry, appears possible [40, 41, 42].

Most of the relevant discussions on braneworld models have been restricted to the case of one spatial extra dimension, as advocated e.g. in the framework of the eleven dimensional realization of M -theory proposed by Hořava and Witten [43]. Moreover, the underlying brane model of the Universe is often assumed to be infinitely thin in the transverse direction, so that (i) the induced gravity stems essentially from Darmois – Israel junction conditions [12, 13, 14, 15, 16, 17, 18, 19, 20] and (ii) is mostly independent of the microstructure of the brane, if any. No such general condition is available in the less restrictive situation of more than one extra dimension that is the subject of the present work.

To study braneworlds with more than one extra dimen-

*Electronic address: c.ringeval@imperial.ac.uk

[†]Electronic address: peter@iap.fr

[‡]Electronic address: uzan@iap.fr

sion, it is necessary to specify the microstructure of the brane to fix a model, taking into account in particular the possibly finite thickness of the brane [44]. In particular, one needs to regularize long range self-interaction forces (including gravity). In the case the force derives from a potential (e.g. in linearized gravity), the self-interaction potential is well behaved outside the brane but will be singular on the brane in the thin brane limit. One needs to introduce a UV cutoff associated with the underlying microstructure [45]. The only case where such a procedure can be avoided is that of hypermembranes that can be satisfactorily treated without recourse to regularization.

Much work has been devoted to warped geometries in six dimensions [46, 47, 48, 49, 50, 51, 52]. When considering explicit realizations in terms of an underlying topological defect forming field model [53, 54, 55, 56], it seems that the six-dimensional case represents a limiting situation: for more than two extra dimensions, it was found that it is not possible to confine gravity on a strictly local brane although global topological defect configurations which allow gravity confinement might exist [57, 58]. At least two questions arise: are the properties of gravity dependent on the microscopic structure of the brane and is the chosen microscopic model consistent with M -theory? The second question has started to be addressed in Ref. [59], and it turns out that, for many purposes, it is possible to consider defectlike realizations of branes, as in the present article.

Before considering the possibility of trapping [60, 61] particle fields in a hyperstring embedded in an anti-de Sitter six-dimensional bulk space-time (adS_6), it is necessary to determine the background structure itself within a given field content, and decide whether it is possible to localize gravity in the Universe thus obtained, thereby generalizing the five-dimensional case. This article is devoted to this task and accordingly models the brane by a vortex of a $U(1)$ Abelian gauge-Higgs model. By means of a numerical exploration of the parameter space, we discuss different classes of solution exhibiting an anti-de Sitter space-time at infinity. It appears that they are generically associated with a conical or curvature singularity at the brane location, except for some fine-tuning between the model parameters. In the regular case, our approach agrees with the numerical results of Ref. [62].

Assuming this fine-tuning, we then go on to analyze the stability of the regular solution by studying the scalar modes of the gauge invariant perturbations, as originally suggested in Ref. [63]. Restricting our attention to the lowest angular momentum modes, we show that the only acceptable perturbations of the nonvanishing background quantities, *i.e.* the hyperstring forming fields and metric warp factors, are the vanishing ones. The conditions imposed on these perturbations to be acceptable being to be initially finite with respect to the background value away from the brane (a condition necessary in order to ensure that the bulk is close to anti-de Sitter space), and to be well-behaved on the brane. We also study the perturba-

tions of quantities which are not involved in the background configurations, as radial gauge field components and nondiagonal metric perturbations. It is found that this subset of perturbations is stable if the requirement of being bounded far from the hyperstring is relaxed (a condition which may not be required since there is not reference background fields for these perturbations).

The article is organized as follows: after setting the field theoretic framework both for the particles and gravity in Sec. II, we construct the Nielsen-Olesen ansatz for a three-dimensional vortex configuration, set and discuss the corresponding Euler-Lagrange field equations in Sec. III. We then show how to handle the boundary conditions in Sec. IV and solve numerically the field equations in Sec. V, insisting in particular on the numerous technical difficulties. This permits us to obtain the parameter range over which gravity is localized and exempt of singularity in the core. We then discuss some arguments leading to the suggestion that such a defect realization of a brane in six dimensions may be marginal in Sec. VI, and explicitly derive and discuss its allowed perturbations in Sec. VII before ending by some concluding remarks.

II. VORTEX CONFIGURATION IN ADS_6

We consider the action for a complex scalar field Φ coupled to gravity in a six-dimensional space-time

$$S = \int \left[\frac{1}{2\kappa_6^2} (R - 2\Lambda) + \mathcal{L}_{\text{mat}} \right] \sqrt{-g} d^6x, \quad (1)$$

where g_{AB} is the six-dimensional metric with signature $(+, -, -, -, -, -)$, R its Ricci scalar, Λ the six-dimensional cosmological constant and $\kappa_6^2 \equiv 32\pi^2 G_6/3$, G_6 being the six-dimensional gravity constant¹. The matter Lagrangian reads

$$\mathcal{L}_{\text{mat}} = \frac{1}{2} g^{AB} \mathcal{D}_A \Phi (\mathcal{D}_B \Phi)^* - V(\Phi) - \frac{1}{4} H_{AB} H^{AB}, \quad (2)$$

in which capital Latin indices $A, B \dots$ run from 0 to 5, H_{AB} is the electromagneticlike tensor defined by

$$H_{AB} = \partial_A C_B - \partial_B C_A, \quad (3)$$

where C_B is the connection 1-form. The $U(1)$ covariant derivative $\mathcal{D}_A$ is defined by

$$\mathcal{D}_A \equiv \partial_A - iqC_A, \quad (4)$$

where q is the charge. The potential of the scalar field Φ is chosen to break the underlying $U(1)$ symmetry and

¹ In D dimensions, we relate κ_D^2 to the D -dimensional gravitational constant G_D by $\kappa_D^2 = (D-2)\Omega^{[D-2]}G_D$, where $\Omega^{[D-1]} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface of the $(D-1)$ -sphere.

thereby allow for topological vortex (cosmic stringlike) configurations,

$$V(\Phi) = \frac{\lambda}{8} (|\Phi|^2 - \eta^2)^2, \quad (5)$$

where λ is a coupling constant and $\eta = \langle |\Phi| \rangle$ is the magnitude of the scalar field vacuum expectation values (VEV)².

Motivated by the brane picture, we choose the metric of the bulk space-time to be of the warped static form

$$ds^2 = g_{AB} dx^A dx^B = e^{\sigma(r)} \eta_{\mu\nu} dx^\mu dx^\nu - dr^2 - r^2 e^{\gamma(r)} d\theta^2, \quad (6)$$

where $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric of signature $(+, -, -, -)$, and (r, θ) the polar coordinates in the extra dimensions. Greek indices $\mu, \nu \dots$ run from 0 to 3 and describe the brane world sheet and we set $g_{\mu\nu} \equiv \exp[\sigma(r)] \eta_{\mu\nu}$. The action (1) with the ansatz (6) will admit static solutions depending only on r so that the general covariance along the four-dimensional (physical) space-time is left unbroken.

The Nielsen-Olesen like [64] ansatz for a generalized vortex configuration is taken to be of the form [65]

$$\Phi = \varphi(r) e^{in\theta} = \eta f(r) e^{in\theta}, \quad C_\theta = \frac{1}{q} [n - Q(r)] \quad (7)$$

where n is an integer, so that the only nonvanishing component of the electromagnetic tensor is $H_{\theta r} = Q'/q$. With such an ansatz, we shall now derive the relevant field equations and discuss their solutions.

III. EQUATIONS OF MOTION

With the metric given by Eq. (6), the nonvanishing Einstein tensor components reduce to

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{4} g_{\mu\nu} \left(6\sigma'' + \frac{6}{r} \sigma' + 6\sigma'^2 + 3\sigma' \gamma' \right. \\ &\quad \left. + 2\gamma'' + \gamma'^2 + \frac{4}{r} \gamma' \right), \\ G_{rr} &= -\frac{1}{2} \sigma' \left(3\sigma' + \frac{4}{r} + 2\gamma' \right), \\ G_{\theta\theta} &= -\frac{1}{2} r^2 e^\gamma (4\sigma'' + 5\sigma'^2), \end{aligned} \quad (8)$$

where a prime denotes differentiation with respect to r . Similarly, the matter stress-energy tensor,

$$T_{AB} \equiv 2 \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g^{AB}} - g_{AB} \mathcal{L}_{\text{mat}}, \quad (9)$$

has nonvanishing components provided by the Nielsen-Olesen ansatz (7) that are given by

$$\begin{aligned} T_{\mu\nu} &= g_{\mu\nu} \left[V + \frac{\eta^2}{2} \left(f'^2 + \frac{Q^2 f^2}{r^2} e^{-\gamma} \right) + \frac{1}{2} \frac{Q'^2}{q^2 r^2} e^{-\gamma} \right], \\ T_{rr} &= -V + \frac{\eta^2}{2} \left(f'^2 - \frac{Q^2 f^2}{r^2} e^{-\gamma} \right) + \frac{1}{2} \frac{Q'^2}{q^2 r^2} e^{-\gamma}, \\ T_{\theta\theta} &= r^2 e^\gamma \left[-V - \frac{\eta^2}{2} \left(f'^2 - \frac{Q^2 f^2}{r^2} e^{-\gamma} \right) + \frac{1}{2} \frac{Q'^2}{q^2 r^2} e^{-\gamma} \right]. \end{aligned} \quad (10)$$

It follows that the six-dimensional Einstein equations, with our conventions,

$$G_{AB} + \Lambda g_{AB} + \kappa_6^2 T_{AB} = 0, \quad (11)$$

can be cast in the form

$$\begin{aligned} \frac{3}{2} \ddot{\sigma} + \frac{3}{2} \dot{\sigma}^2 + \frac{3}{2\rho} \dot{\sigma} + \frac{3}{4} \ddot{\sigma} \dot{\gamma} + \frac{1}{2} \dot{\gamma} + \frac{1}{4} \dot{\gamma}^2 + \frac{1}{\rho} \dot{\gamma} &= -\frac{\Lambda}{|\Lambda|} \\ -\alpha \left[\beta (f^2 - 1)^2 + \dot{f}^2 + \frac{e^{-\gamma}}{\rho^2} \left(Q^2 f^2 + \frac{\dot{Q}^2}{\varepsilon} \right) \right] &, \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{3}{2} \ddot{\sigma} + \frac{2}{\rho} \dot{\sigma} + \ddot{\sigma} \dot{\gamma} &= -\frac{\Lambda}{|\Lambda|} - \alpha \left[\beta (f^2 - 1)^2 \right. \\ &\quad \left. - \dot{f}^2 + \frac{e^{-\gamma}}{\rho^2} \left(Q^2 f^2 - \frac{\dot{Q}^2}{\varepsilon} \right) \right], \end{aligned} \quad (13)$$

$$\begin{aligned} 2\ddot{\sigma} + \frac{5}{2} \dot{\sigma}^2 &= -\frac{\Lambda}{|\Lambda|} - \alpha \left[\beta (f^2 - 1)^2 \right. \\ &\quad \left. + \dot{f}^2 - \frac{e^{-\gamma}}{\rho^2} \left(Q^2 f^2 + \frac{\dot{Q}^2}{\varepsilon} \right) \right], \end{aligned} \quad (14)$$

where we have introduced the dimensionless radial coordinate

$$\rho \equiv \sqrt{|\Lambda|} r, \quad (15)$$

as well as the dimensionless parameters

$$\alpha \equiv \frac{1}{2} \kappa_6^2 \eta^2, \quad \beta \equiv \frac{1}{4} \frac{\lambda \eta^2}{|\Lambda|}, \quad \varepsilon \equiv \frac{q^2 \eta^2}{|\Lambda|}. \quad (16)$$

In Eqs. (12) to (14), we have introduced the convention that a dot refers to a differentiation with respect to the dimensionless radial coordinate ρ .

² Note, that, because of the unusual number of space-time dimensions, the fields have dimensions given by $[R] = M^2$, $[\Phi] = M^2$, $[\Lambda] = M^2$, $[\lambda] = M^{-2}$, $[\eta] = M^2$, $[\kappa_6^2] = M^{-4}$, $[C_A] = M^2$, $[H_{AB}] = M^3$ and $[q] = M^{-1}$ (M being a unit of mass). This can be further generalized in the D -dimensional case by: $[\Phi] = M^{(D-2)/2}$, $[\lambda] = M^{4-D}$, $[\eta] = M^{(D-2)/2}$, $[\kappa_D^2] = M^{2-D}$, $[C_A] = M^{D-2}$, $[H_{AB}] = M^{D-3}$ and $[q] = M^{2-D/2}$, $[R]$ and $[\Lambda]$ being unchanged.

The scalar field dynamics is given by the Klein-Gordon equation

$$\nabla_A \nabla^A \Phi = -\frac{\lambda}{2} (|\Phi|^2 - \eta^2) \Phi + q^2 C^2 \Phi + iq C^A \partial_A \Phi + iq \nabla_A (C^A \Phi), \quad (17)$$

which takes the reduced form

$$\ddot{f} + \left(2\dot{\sigma} + \frac{1}{2}\dot{\gamma} + \frac{1}{\rho}\right) \dot{f} = \frac{Q^2}{\rho^2} f e^{-\gamma} + 2\beta (f^2 - 1) f, \quad (18)$$

while the Maxwell equations

$$\nabla_A H^{AB} = -q^2 C^2 |\Phi|^2 + \frac{i}{2} q (\Phi \partial^B \Phi^* - \Phi^* \partial^B \Phi), \quad (19)$$

provide the single reduced equation for the only nonvanishing component of the gauge vector field

$$\ddot{Q} + \left(2\dot{\sigma} - \frac{1}{2}\dot{\gamma} - \frac{1}{\rho}\right) \dot{Q} = \varepsilon f^2 Q. \quad (20)$$

The set of equations (12, 13, 14, 18, 20) is a set of five differential equations for 4 unknown functions (σ , γ , f , Q). Indeed, we have a redundant equation due the Bianchi identities and one can check that the Higgs field equation (18) is recovered from the constraint equation (13) provided $\dot{f} \neq 0$. This system can be further simplified by remarking that Eq. (20) can also be written as

$$\frac{d}{d\rho} \left(\frac{e^{2\sigma}}{\sqrt{m}} \dot{Q} \right) = \frac{e^{2\sigma}}{\sqrt{m}} \varepsilon f^2 Q, \quad (21)$$

where

$$m \equiv \rho^2 v \equiv \rho^2 e^\gamma, \quad (22)$$

so that

$$\frac{d}{d\rho} \left(\frac{e^{2\sigma}}{\sqrt{m}} Q \dot{Q} \right) = \frac{e^{2\sigma}}{\sqrt{m}} (\varepsilon f^2 Q^2 + \dot{Q}^2). \quad (23)$$

On the other hand, combining Eqs. (12) and (14), one also finds that

$$\begin{aligned} \frac{d}{d\rho} [e^{2\sigma} \sqrt{m} (as + bl)] &= -e^{2\sigma} \sqrt{m} [(a+b) \mathcal{F} \\ &\quad + \frac{1}{2} (3b-a) \mathcal{V}_1 + 2b \mathcal{V}_2], \end{aligned} \quad (24)$$

for any set of arbitrary constants a and b , and where we have set the new functions

$$s \equiv \dot{\sigma}, \quad l \equiv \frac{\dot{m}}{m} = \frac{2}{\rho} + \dot{\gamma}, \quad (25)$$

as well as

$$\mathcal{F} \equiv \alpha \beta (f^2 - 1)^2 + \frac{\Lambda}{|\Lambda|}, \quad (26)$$

$$\mathcal{V}_1 \equiv \frac{2\alpha}{\varepsilon} \frac{\dot{Q}^2}{m}, \quad \mathcal{V}_2 \equiv 2\alpha \frac{f^2 Q^2}{m}. \quad (27)$$

Using the relation (24) with $a = -b$ leads to

$$\frac{d}{d\rho} [e^{2\sigma} \sqrt{m} (s-l)] = \frac{4\alpha e^{2\sigma}}{\sqrt{m}} \left(\frac{\dot{Q}^2}{\varepsilon} + f^2 Q^2 \right), \quad (28)$$

and combining this result with Eq. (23), one obtains

$$\frac{4\alpha}{\varepsilon} \frac{Q \dot{Q}}{m} = s - l + c, \quad (29)$$

where c is a remaining integration constant [66]. The equations of motion (12, 13, 14, 18, 20) end up being equivalent to the following five first-order differential equations

$$\dot{s} = -\frac{5}{2} s^2 + \frac{2\alpha}{\varepsilon} \frac{Qw}{m} s - \mathcal{F} + \frac{1}{2} \mathcal{V}_1 - \frac{c}{2} s, \quad (30)$$

$$\dot{m} = sm - \frac{4\alpha}{\varepsilon} Qw + cm, \quad (31)$$

$$\alpha \dot{f}^2 = \frac{5}{2} s^2 - \frac{4\alpha}{\varepsilon} \frac{Qw}{m} s + \mathcal{F} + \frac{\mathcal{V}_2}{2} - \frac{\mathcal{V}_1}{2} + cs, \quad (32)$$

$$\dot{Q} = w, \quad (33)$$

$$\dot{w} = \varepsilon Q f^2 - \frac{2\alpha}{\varepsilon} \frac{Qw^2}{m} - \frac{3}{2} sw + \frac{c}{2} w. \quad (34)$$

After discussing the behavior of these fields far from the vortex, *i.e.* far in the bulk, and on the brane itself in the following section, we shall solve numerically the field equations in order to determine the structure of the space-time and defect system.

IV. ASYMPTOTIC BEHAVIORS

By definition of the topological defect like configuration, we require that the Higgs field vanishes on the membrane itself, *i.e.* $\Phi = 0$ for $\rho = 0$, while it recovers its VEV, η , in the bulk. These requirements translate into the following boundary conditions for the function f :

$$f(0) = 0, \quad \lim_{\rho \rightarrow +\infty} f = 1. \quad (35)$$

The corresponding boundary conditions for the 1-form connection are given by

$$Q(0) = n, \quad \dot{Q}(0) = w(0) = 0, \quad \lim_{\rho \rightarrow +\infty} Q = 0. \quad (36)$$

In order to avoid any curvature singularity on the string, the Ricci scalar stemming from Eq. (1), namely

$$R = -|\Lambda| \left(\dot{\gamma} + 4\ddot{\sigma} + \frac{1}{2} \dot{\gamma}^2 + 5\dot{\sigma}^2 + 2\dot{\gamma}\dot{\sigma} + \frac{2\dot{\gamma}}{\rho} + \frac{4\dot{\sigma}}{\rho} \right), \quad (37)$$

has to be finite at $\rho = 0$ as the vortex is assumed to represent our physical four-dimensional space. As a result, the warp functions $\dot{\gamma}(0)$ and $\dot{\sigma}(0)$ have to vanish in the string core,

$$\dot{\gamma}(0) = \dot{\sigma}(0) = 0, \quad (38)$$

and the warp function l therefore scales near the string like

$$l(\rho) \sim \frac{2}{\rho} \Rightarrow m \propto \rho^2. \quad (39)$$

Note that we impose both functions to vanish on the string, and not merely the combination $\dot{\gamma} + 2\dot{\sigma}$ entering Eq. (37); this arises from the requirement that *all* geometrical quantities, e.g. $R^A_B R^B_A$ in which $\dot{\gamma}$ and $\dot{\sigma}$ enter with different coefficients, must be finite. A coordinate transformation along the brane allows to choose $\sigma(0) = 0$, while $\gamma(0)$ and v_0 , defined by

$$v_0 \equiv e^{\gamma(0)}, \quad (40)$$

are determined by the boundary conditions at infinity. Note that v_0 cannot be absorbed in a rescaling of the radial coordinate. To see this, it suffices to introduce a new coordinate, $\tilde{r}$ say, such that $r^2 e^{\gamma(0)} = \tilde{r}^2$, which is equivalent to defining $\gamma(\tilde{r}) = \gamma(r) - \gamma(0)$. This would induce a shift in the other warp function σ , shift that can however be taken care of by a rescaling of the vortex internal coordinates. This is not all though, because the last metric element $g_{rr} = -1$ gets modified into $g_{\tilde{r}\tilde{r}} = g_{rr} e^{\gamma(0)} = -e^{\gamma(0)}$. All the derivatives with respect to this new radial variable also acquire this numerical factor. In the Einstein tensor, given the symmetries of the vortex, this seems harmless as G_{AB} is also simply rescaled. However, the stress-energy tensor (10) is not so simply rescaled as it also involves non derivative terms (the Higgs field potential V and the gauge-Higgs coupling). On the other hand, v_0 can be absorbed by a rescaling of the angular coordinate $\tilde{\theta} = \sqrt{v_0} \theta$. In that case, the angular part of the metric (6) appears to be cylindrical in the hyperstring core, with however a missing angle

$$\Delta\tilde{\theta} = 2\pi(1 - \sqrt{v_0}). \quad (41)$$

The space-time geometry obtained for $v_0 \neq 1$ exhibits a conical singularity in the vortex core whose physical interpretation is the existence of an additional δ -like energy-momentum distribution (a Goto-Nambu hyperstring) lying at the center of the configuration. This interpretation remains valid provided $0 \leq v_0 \leq 1$, the other cases will be discussed in the next section. At this point, it is interesting to note that contrary to what is assumed in Ref. [62], the value of v_0 in our approach is completely determined as soon as the other boundary conditions are imposed and ends up being a function of the model parameters only. Setting $v_0 = 1$ afterward, to obtain a regular geometry in the hyperstring core, will allow us to recover the fine-tuning relation obtained in Ref. [62].

In the following, we derive analytical approximations of the fields at infinity and in the hyperstring core associated with an anti-de Sitter space-time at infinity. The influence of the model parameters on these solutions is discussed.

A. Far from the string

Asymptotically, the anti-de Sitter space-time is recovered provided

$$\lim_{\rho \rightarrow +\infty} \dot{s} = \lim_{\rho \rightarrow +\infty} \dot{l} = 0. \quad (42)$$

Denoting by an index ‘F’ (standing for “fixed”) the value of the fields at infinity, it follows from Eqs. (35), (36) and (42) that the adS_6 solution is a fixed point for the set of Eqs. (30 – 34) with $f_F = 1$, $w_F = Q_F = 0$ for the Higgs and gauge fields, and with the equations

$$-\frac{5}{2}s_F^2 + \frac{c}{2}s_F - \frac{\Lambda}{|\Lambda|} = 0, \quad (43)$$

$$m_F s_F + c m_F = 0, \quad (44)$$

$$\frac{5}{2}s_F^2 + 2c s_F + \frac{\Lambda}{|\Lambda|} = 0, \quad (45)$$

for the warp factors. The equations (43) and (45) can only be simultaneously satisfied for $c = 0$, since $s_F = 0$ would lead to $\Lambda = 0$. As a result, Eq. (44) requires that $m_F = 0$ and the asymptotic warp factors reduce to

$$l_F^2 = s_F^2 = -\frac{2}{5} \frac{\Lambda}{|\Lambda|}. \quad (46)$$

The anti-de Sitter solution is obtained for $\Lambda < 0$ so that $l_F = s_F = -\sqrt{2/5}$ and we can now verify that $\mathcal{V}_1$ and $\mathcal{V}_2$ effectively vanish. Indeed, from Eq. (31), the dominant behavior of m at infinity is given by

$$m \sim m_\infty e^{-\sqrt{2/5}\rho}, \quad (47)$$

while Eqs. (33) and (34) admit asymptotically the decaying solutions

$$Q \sim Q_\infty e^{-\ell_g \rho}, \quad w \sim -\ell_g Q, \quad (48)$$

with

$$\ell_g = \frac{3}{4} \sqrt{\frac{2}{5}} \left(\sqrt{\frac{40}{9} \varepsilon + 1} - 1 \right). \quad (49)$$

From Eq. (27), the gauge functions $\mathcal{V}_1$ and $\mathcal{V}_2$ vanish at infinity provided

$$2\ell_g > \sqrt{\frac{2}{5}} \Leftrightarrow \varepsilon > \frac{2}{5}. \quad (50)$$

This is the first restriction on the available parameter space.

From Eq. (18), the decaying branch of the Higgs field at infinity is given by

$$h \sim h_\infty e^{-\ell_h \rho} + \frac{Q_\infty^2}{4m_\infty} \frac{e^{-(2\ell_g - \sqrt{2/5})\rho}}{\beta - \left(\ell_g - \frac{1}{2} \sqrt{\frac{2}{5}} \right) \left(\ell_g + \frac{3}{4} \sqrt{\frac{2}{5}} \right)}, \quad (51)$$

where we have defined

$$h \equiv 1 - f, \quad (52)$$

and ℓ_h reads

$$\ell_h = \frac{1}{2} \sqrt{\frac{5}{2}} \left(\sqrt{\frac{32}{5} \beta + 1} - 1 \right). \quad (53)$$

From Eq. (51), it appears that the behavior of the Higgs field at infinity can be driven by the gauge field, provided $\ell_h > 2\ell_g - \sqrt{2/5}$. In this case, one can check that h remains positive definite ensuring that the Higgs field approaches its VEV from below and thus can support a topological defect configuration. Indeed, would h be negative asymptotically, the condition $f(0) = 0$ could only be realized if the sign of the derivative $\dot{f}$ changes at some intermediate point. From Eq. (18), it is clear that at the point $\dot{f} = 0$, the right-hand side is positive for $f > 1$. As a result $\dot{f}$ can only be positive and the profile of the Higgs field would remain always convex and greater than its VEV.

Similarly, the asymptotic expression for the warp factor can be obtained from Eq. (30). In terms of the new function

$$u \equiv s + \sqrt{\frac{2}{5}}, \quad (54)$$

one gets for the decaying branch, up to some fine-tuning (see Sect. VI)

$$u \sim \frac{4\alpha\beta}{2\ell_h + \sqrt{10}} h_\infty e^{-2\ell_h \rho} - \frac{\alpha \ell_g Q_\infty^2}{2\epsilon m_\infty} e^{-(2\ell_g - \sqrt{2/5}) \rho}. \quad (55)$$

If $2\ell_h < 2\ell_g - \sqrt{2/5}$, the convergence of the warp factors toward the anti-de Sitter solution is driven by the Higgs field and u remains positive definite since h_∞ is positive in that case [see Eq. (51)]. On the other hand, if $2\ell_h > 2\ell_g - \sqrt{2/5}$, the metric factor s behaves asymptotically like the gauge field and u remains definite negative. As the result, the surface $2\ell_h = 2\ell_g - \sqrt{2/5}$, i.e. from Eqs. (49) and (53),

$$\beta = -\frac{1}{10} + \frac{1}{4}\epsilon, \quad (56)$$

separates the parameter space $(\alpha, \epsilon, \beta)$ in two regions where the warp factor s approaches its anti-de Sitter value from above or below, respectively (see Fig. 1).

B. Near the string

The field behaviors near the string, to leading order in ρ , can be extracted from the equations of motion (30) to (34). We assume the truncated Taylor expansions

$$w \sim w_1 \rho^{\kappa_w}, \quad Q \sim n + \frac{w_1}{1 + \kappa_w} \rho^{1+\kappa_w}, \quad (57)$$

for the gauge fields, and

$$f \sim f_1 \rho^{\kappa_f}, \quad (58)$$

for the Higgs field with $\kappa_w \geq 1$ and $\kappa_f \geq 1$ in order that $\dot{w}$ and $\dot{f}$ remains finite in $\rho = 0$. Similarly, the warp factor s can be expanded around $\rho = 0$ as

$$s \sim s_1 \rho^{\kappa_s}, \quad (59)$$

with $\kappa_s \geq 1$ in order for the Ricci scalar to remains finite in the core [see Eq. (37)]. Setting $c = 0$ (see Sect. IV A) in Eq. (34), using Eqs. (39), (40), (57) and (58), yields $\kappa_w = 1$ and

$$v_0 = -\frac{2\alpha}{\epsilon} n w_1. \quad (60)$$

From Eq. (30) and making use of Eqs. (26) and (27), one gets

$$\kappa_s = 1, \quad s_1 = \frac{1}{2} \left(1 - \alpha\beta + \frac{\alpha}{\epsilon} \frac{w_1^2}{v_0} \right). \quad (61)$$

The Higgs field behavior is given by Eq. (32) and reads

$$\alpha \kappa_f^2 f_1^2 \rho^{2\kappa_f - 2} = 2s_1 + (\alpha\beta - 1) - \frac{\alpha}{\epsilon} \frac{w_1^2}{v_0} + \alpha \frac{f_1^2 n^2}{v_0} \rho^{2\kappa_f - 2}. \quad (62)$$

With s_1 given by the expression (61), this implies

$$\kappa_f = \frac{n}{\sqrt{v_0}}. \quad (63)$$

As a result, the behavior of the Higgs field around the string core is only determined by the asymptotic solutions at infinity through v_0 (and w_1), as announced above, and the requirement $\kappa_f \geq 1$ only provides the constraint

$$v_0 \leq n^2. \quad (64)$$

From a purely classical point of view, only $\dot{f}$ has to be well defined at $\rho = 0$. However, in the general situation, there always exists an integer $p \in \mathbb{N}$ such that $v_0 > n^2/p$, in which case all derivatives of the Higgs field, $f^{(k)}$, with $k \geq p$ are divergent in the core since the p^{th} derivative already is. This is so unless the bound is saturated, i.e. the equality $v_0 = n^2/p$ is strictly satisfied so that $\kappa_f = p$ and then all derivatives of order larger than p will strictly vanish. If the bound is not saturated however, the metric at the core exhibits the conical singularity on which the field reacts by introducing the aforementioned divergences. Note that, as discussed above, the conical singularity interpretation demands $v_0 \leq 1$, while for a vortex with $n > 1$, Eq. (64) allows v_0 to be larger than one. This unexpected regularity comes from the scaling properties of the field equations. Indeed, in Eqs. (30) through (34), the warp function m (and thus v) appears only through the ratio Q^2/m and w^2/m . As a result, a solution for a given winding number n and v_0 , is also solution of a winding number $\tilde{n} = pn$ and $\tilde{v}_0 = p^2 v_0$. This scaling permits

to solve the equations for a reduced set of parameters and yet obtain the complete spectrum of solutions. On the physical side, the kind of singularity appearing in the core for $1 < v_0 \leq n^2$ may be interpreted as an supercritical Goto-Nambu hyperstring lying in $\rho = 0$. On the other hand, the behavior of the fields leading to $v_0 > n^2$ requires divergences of $\dot{f}$ and the stress tensor no longer remains finite in the vortex core [see Eq. (10)]. In that case, the conditions $\kappa_w \geq 1$, $\kappa_f \geq 1$ and Eqs. (61) are no longer valid and the regularity requirements in Eq. (38) are no longer satisfied such that a curvature singularity appears in $\rho = 0$.

The previous analysis makes clear that the value of v_0 encodes the regularity of the matter fields and the geometry in the string core. By requiring the space-time to be of anti-de Sitter kind at infinity, we have shown that v_0 is a function of the model parameters only, as long as the boundary conditions (35), (36) and (46) can be satisfied. In the following, we numerically recover the field behavior expected from the asymptotic analysis, in particular the regular configurations in the core require a fine-tuning in the model parameters such as $v_0 = 1$.

V. NUMERICAL APPROACH AND PROBLEMS

Several technical difficulties appear in the numerical integration of the equations of motion (12) to (20). The first issue comes from the requirement of an anti-de Sitter space-time far from the core. Indeed, as mentioned in Sect. IV A, the metric factors as well as the Higgs and gauge fields admit growing modes at infinity which correspond to infinite-volume space-time in the extra dimensions (see Sect. VI and Appendix B). As the result, any direct numerical integration starting from the core toward the outer regions will necessary jump, due to the finite numerical accuracy, onto these growing solutions. This numerical instability can be overcome by performing a backward integration starting from a finite cutoff distance far the hyperstring toward the core. However, one has to pay attention to choose a convenient set of functions, as s and m rather than σ and γ to remove any explicit $1/\rho$ dependencies in the equations of motion. Indeed, the effect of the $1/\rho$ term in Eqs. (12) to (20) is to add flow turning points where the signs of the field derivatives change. As a result, growing behaviors would appear from these turning points whatever the initial conditions and the direction of integration. There is no such flow inversions in the closed system of Eqs. (30) to (34), and the exponential growth can be suppressed by integrating from the anti-de Sitter fixed point at infinity toward the hyperstring core. However, one has still to face the following difficulty. As pointed in Eq. (55), even the decaying branch of s at infinity admits two exponential decaying modes: one varying as $\exp(-2\ell_h\rho)$ and the other as $\exp[-(2\ell_g - \sqrt{2/5})\rho]$. As a result, a backward direct integration is numerically unstable with respect to one of these modes. In the same way that a

forward direct integration would jump onto the growing mode, such a backward numerical method would tend to select by numerical finite accuracy the strongest decaying exponential. For instance, if $2\ell_h > 2\ell_g - \sqrt{2/5}$, a backward method would be preferentially sensitive to the $\exp(-2\ell_h\rho)$ behavior (by moving toward the lower values of ρ , this mode blows up faster than the other), which is not the one in which we are interested. It is interesting to note that the numerical instability of the backward method disappears for $2\ell_h = 2\ell_g - \sqrt{2/5}$ where the two decaying modes are identified, which means that this method can only be efficient on a surface of the parameter space $(\alpha, \varepsilon, \beta)$ whose equation is given by Eq. (56). Note that a similar numerical instability occurs in Eq. (18) between the two corresponding decaying modes of Eq. (51).

In order to overcome these difficulties, we have chosen to use a finite difference numerical method instead of a direct method. In particular, the equations of motion (12) to (20) deriving from the action (1), we have implemented a successive over-relaxation method [67]. By discretizing the radial coordinate ρ , the action (1) can be expressed as a finite sum over ρ_i (the integer i indexing the discrete values taken by ρ over the radial grid) whose differentiation with respect to the fields evaluated at the discrete points leads to a system of finite difference equations corresponding to Eqs. (12), (14), (18) and (20). From a initial guess of the field and metric profiles along the radial grid, the values taken by the fields at ρ_i are corrected by a Newton's method to reduce the error with respect to the true solution. In this approach, the boundary conditions are part of the finite difference equations since they appear as the conditions satisfied by the fields at the first and last point of the ρ_i grid (see Appendix A in Ref. [68] for a relaxation method applied to a similar action). This procedure is stable provided the initial guessed profiles are not too far from the true solutions. The iterative corrections can be stopped when the discrete action remains stationary at the machine precision. We have also checked that the numerical solutions obtained in this way satisfy the constraint equation (13).

In order to probe the behavior of the solutions according to the model parameters, we have, in a first time, numerically allowed regular and conical solutions in the hyperstring core by requiring the boundary conditions (35), (36), (38), (39) and (46) to be satisfied. Along the lines drawn in the previous paragraph, we have used the relaxation method to compute the solutions of the field equations in the case of a unit winding vortex $n = 1$ and for 25^3 values of the parameters $(\alpha, \varepsilon, \beta)$. As expected from the asymptotic analysis (see Sect. IV) the hyperstring generically develops a conical singularity with $v_0 \neq 1$ (see Fig. 1). The regular solutions $v_0 = 1$ are obtained only for the parameters lying on the surface plotted in Fig. 3 which identifies to the fine-tuning surface previously found in Ref. [62].

However, under the previous boundary conditions, the relaxation procedure failed to converge in the regions of the parameter space which could have been associated

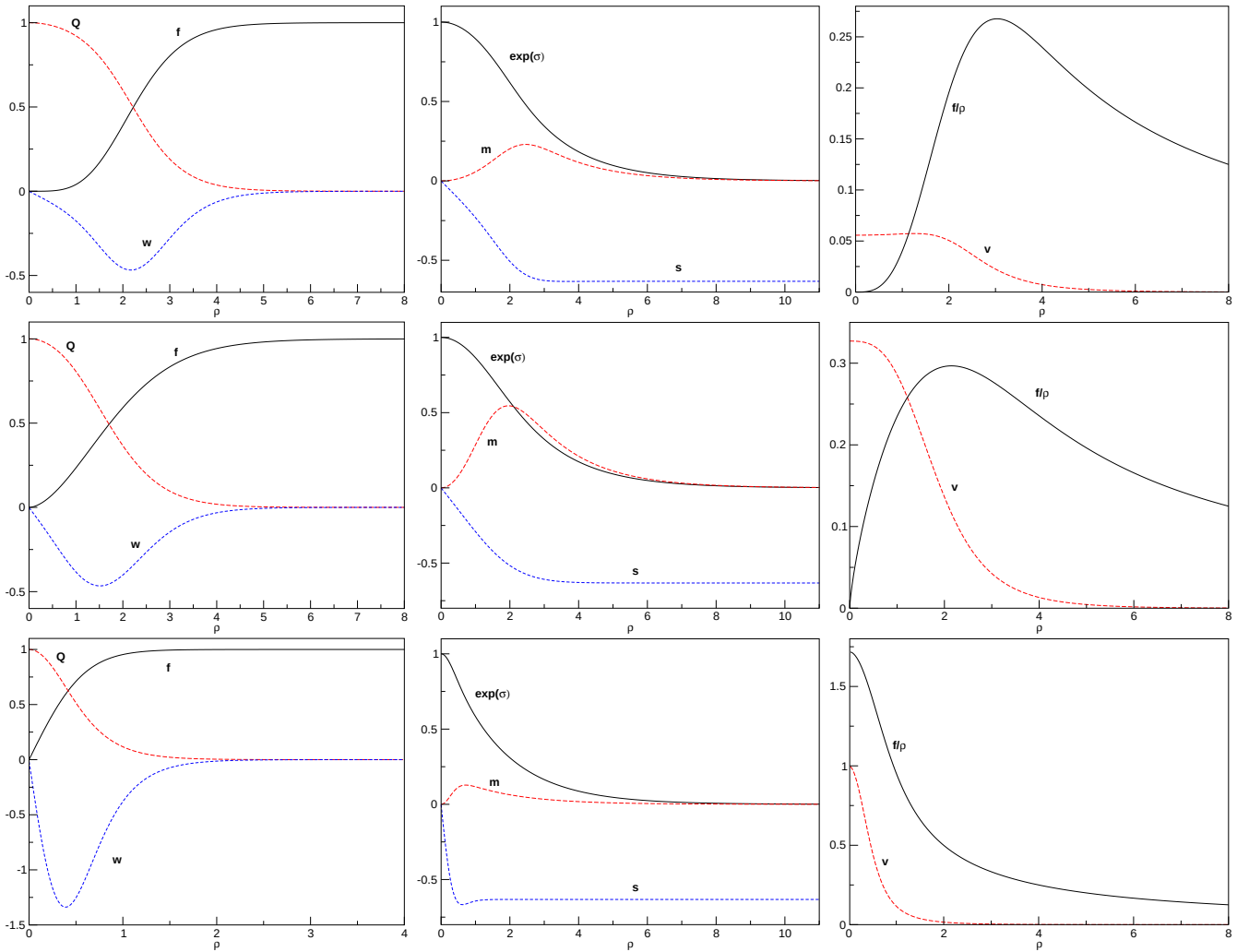


FIG. 1: Typical field solutions associated with an anti-de Sitter space-time at infinity in the regimes where $v_0 < 1/4$, $1/4 < v_0 < 1$ and $v_0 \sim 1$, obtained for $(\alpha, \varepsilon, \beta)$ equals to $(1.00, 5.00, 1.50)$, $(2.20, 5.00, 0.80)$ and $(1.29, 14.83, 6.39)$ respectively. In each case, the Higgs and gauge fields are plotted on the left picture, the warp factors are plotted in the middle one, while the right plot represents the behavior of f/ρ and $v \equiv m/\rho^2$. Note the behavior of the warp factor s (middle plots) for the parameters living above or under the plane (56) (see also Fig. 2 and Fig. 3).

with values of $v_0 > 1$ (see Fig. 2). For these regions, we have thus weakened the boundary conditions in order that Eqs. (35), (36) and (46) are still satisfied but with

$$\sigma(0) = 0, \quad m(0) = 0, \quad (65)$$

instead of Eqs. (38) and (39). In that case, the method converges and the numerical solutions exhibit a divergence in the Higgs field derivative while the derivative of the warp factors do no longer vanish in $\rho = 0$ (see Fig. 4). From Eq. (37), these solutions correspond to a curvature singularity in the hyperstring core and are not physical.

VI. FINE-TUNING AND STABILITY

The solutions we have obtained for the fields surrounding a branelike vortex requires a fine-tuning of the underlying microscopic parameters to be free of singularity in the string core and of finite volume in the extra dimensions. These conditions are indeed the minimal requirements for an acceptable warped braneworld model in six-dimensions. In the following we discuss qualitatively why such fine-tuning is expected as well as the differences appearing with respect to the domain wall model in a five-dimensional anti-de Sitter case [23]. Our analysis is analogous to the one developed in Ref. [53, 54] in the case of global vortex and leads to similar conclusions.

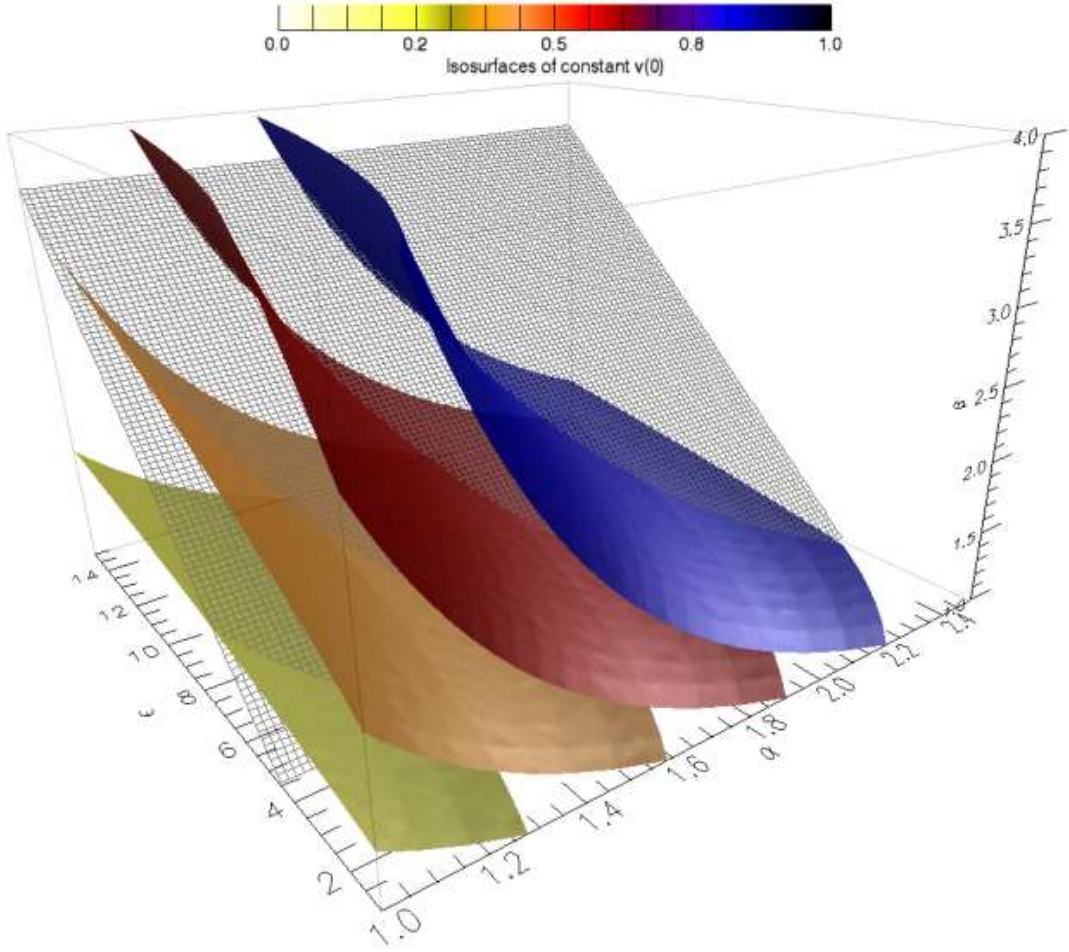


FIG. 2: Isosurfaces of constant v_0 in the parameter space $(\alpha, \varepsilon, \beta)$ associated with an anti-de Sitter space-time at infinity. From the left to the right, the four surfaces correspond to $v_0 = 0.25$, $v_0 = 0.5$, $v_0 = 0.75$ and $v_0 = 1$, respectively. The wired mesh represents the plane $\beta = -1/10 + \varepsilon/4$ (see Sect. IV A) which separates the parameter space in two regions. For $\beta > -1/10 + \varepsilon/4$ the gauge field drives the warp factors toward their anti-de Sitter value and s has a global minimum at a finite distance to the core whereas, in the other case, the Higgs field dominates at infinity and s always decreases toward its asymptotic value (see Fig. 1). On the left side of the $v_0 = 1$ isosurface the hyperstring exhibits a conical singularity in $\rho = 0$ whereas on the right side there is a curvature singularity (see Fig. 4). Only the surface $v_0 = 1$ ends up be associated with a regular configuration on the brane (see Fig. 3)

The asymptotic form of Eq. (30) reads

$$\dot{s} + \frac{5}{4}s^2 = \frac{1}{2}, \quad (66)$$

for which one obtains the general solution in the form

$$s = \sqrt{\frac{2}{5}} \times \frac{A_6 e^{c_6 \rho} - e^{-c_6 \rho}}{A_6 e^{c_6 \rho} + e^{-c_6 \rho}}, \quad (67)$$

where $c_6 = \sqrt{5/8}$, and A_6 is an arbitrary constant, to be matched with the vortex interior solution. In the asymptotic analysis of Sect. IV, this constant has been set to zero, but clearly, if $A_6 \neq 0$, one has

$$\lim_{\rho \rightarrow +\infty} s(\rho) = \sqrt{\frac{2}{5}}, \quad (68)$$

leading to an exponentially divergent warp factor for the metric (6).

Only the particular value $A_6 = 0$ can smoothly join the interior metric to a six-dimensional anti-de Sitter asymptotic space-time. In five dimensions, this choice happens to be imposed by the Einstein constraint [23], but in the case at hand, because of the extra degree of freedom provided by the other function γ , this is an explicit choice and not a mandatory consequence of the Einstein equations. In other words, the solution satisfying $\lim_{\rho \rightarrow \infty} s = -\sqrt{2/5}$ is also a point in the phase space from which all trajectories diverge. This is related to the fact that in the limit where the field contribution in the stress-energy tensor is negligible with respect to the bulk cosmological constant, Eqs. (12) and (14) are two dy-

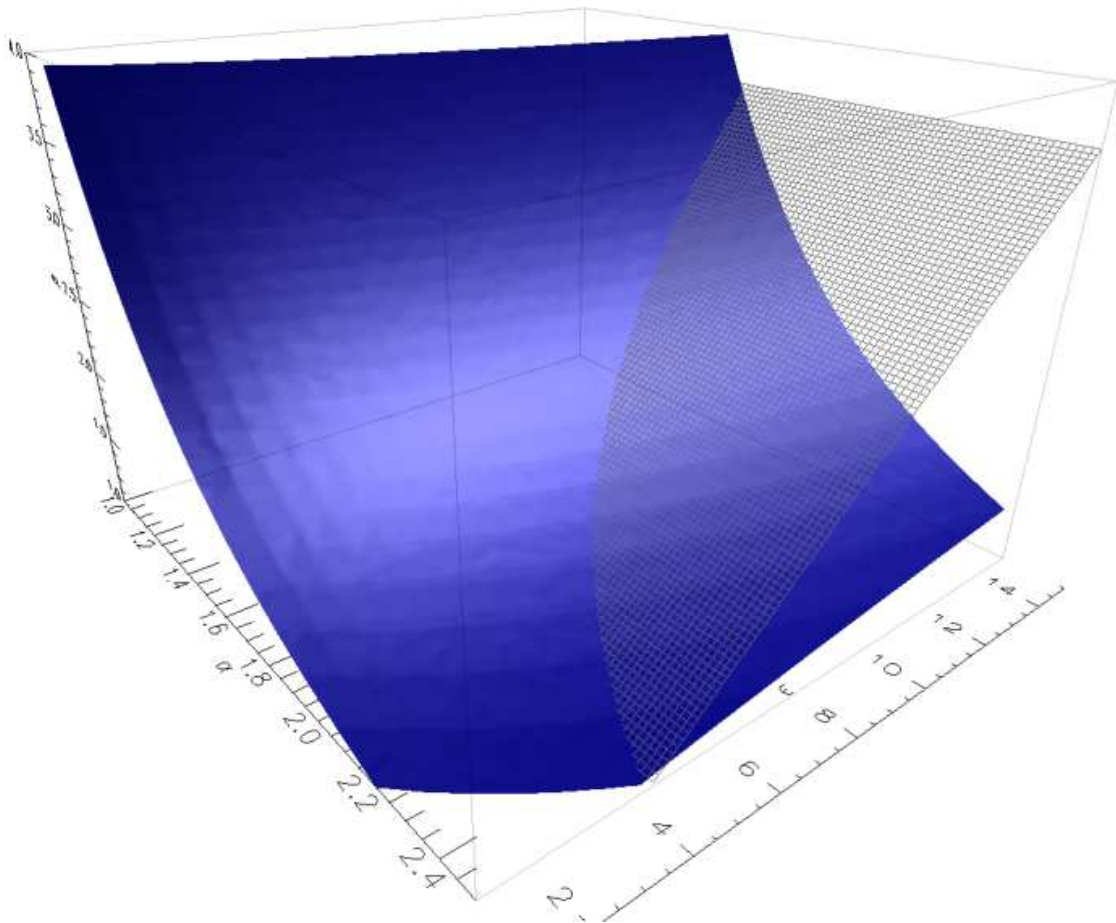


FIG. 3: The fine-tuning surface $v_0 = 1$ in the parameter space $(\alpha, \varepsilon, \beta)$ associated with an anti-de Sitter space-time at infinity and a regular geometry in the hyperstring core (for the $n = 1$ vortex). The wired mesh is the surface $\beta = -1/10 + \varepsilon/4$ (see Sect. IV A).

namical equations for the warp functions σ and γ , with Eq. (13) being a constraint. The two first order (in $\dot{\sigma}$ and $\dot{\gamma}$) dynamical equations require two constants of integration, of which the constraint fixes only one (together with the requirement of an anti-de Sitter asymptotic space-time). The solution with arbitrary (nonvanishing) A_6 is thus a valid solution, contrary to the five-dimensional case. However, any nonzero value of A_6 does not correspond to a solution with gravity localized on the vortex, being exponentially far from the adS_6 case (see also the Appendix B). One is thus led to conclude that the fine-tuning required in the 6D case is worse than in 5D since the physically relevant solution is a set of measure zero in the full set of solutions. This drives us to ask whether such a solution, although mathematically acceptable, can be reached by any dynamical evolution. In fact, as we show in the following section, no acceptable perturbation mode can be found for the regular vortex, except for some special modes, the transverse ones, which have no equivalent at zeroth order. In other words, the regular 6D vortex configurations cannot be subject to perturba-

tions in their background fields and in particular, cannot depart infinitesimally from adS_6 .

VII. GAUGE-INVARIANT PERTURBATIONS

As vector and tensor perturbations have been investigated elsewhere [60, 61], we shall concentrate on the scalar part [63] of the perturbations. Here and in what follows, the Scalar-Vector-Tensor decomposition is understood to be with respect to the four-dimensional vortex internal coordinates. Note that most of previous works were concentrating on zero modes, which were found to be “not normalizable”. In what follows, we consider massive modes and attention will be paid to the physical interpretation of the results.

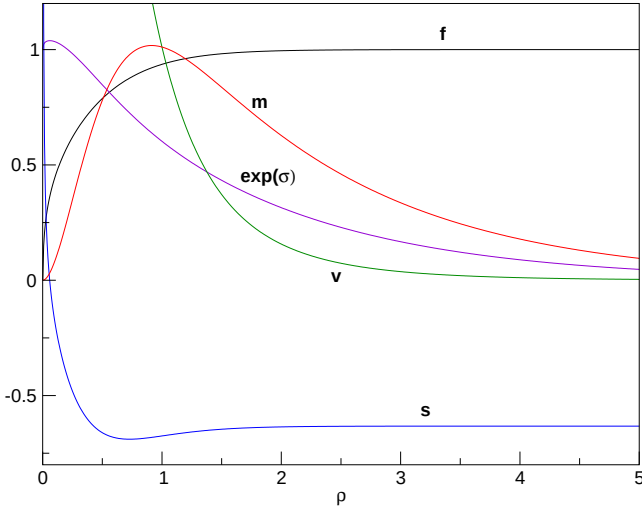


FIG. 4: Typical behavior of the Higgs field and metric factors for the model parameters living on the right-hand side of the $v_0 = 1$ isosurface, i.e. for large value of α (see Fig. 3). The derivatives of the Higgs field and the metric coefficient diverge in $\rho = 0$ leading to a curvature singularity [see Eq. (37)].

A. gauge-invariant variables

In order to conclude on the stability (or physical relevance as we shall see) of the configuration we obtained, it is necessary to perturb this background solution in a gauge-invariant way. The first order perturbation of the metric (6), when restricting attention to the scalar perturbations, reads

$$ds^2 = e^{\sigma(r)} [\eta_{\mu\nu} (1 + \psi) + \partial_\mu \partial_\nu E] dx^\mu dx^\nu - (1 + \xi) dr^2 - 2\zeta dr d\theta - r^2 e^{\gamma(r)} (1 + \omega) d\theta^2 - 2(\partial_\mu B dr + \partial_\nu C d\theta) dx^\mu, \quad (69)$$

where the scalar functions ψ , E , ξ , ζ , ω , B and C depend on all the coordinates (x^α, r, θ) and are assumed to be small.

A gauge transformation $x^A \rightarrow \tilde{x}^A = x^A + \epsilon^A$, with $\epsilon^\mu = \partial^\mu \epsilon$ (scalar transformations only) implies three gauge degrees of freedom, so we are left with four unknown functions to determine. The scalar functions transform under a gauge transformation as

$$\begin{aligned} \tilde{\psi} &= \psi - \sigma' \epsilon_r, & \tilde{E} &= E + 2e^{-\sigma} \epsilon, & \tilde{\xi} &= \xi - 2\epsilon_r', \\ \tilde{\zeta} &= \zeta - \frac{1}{2} \left[\epsilon_\theta' + \partial_\theta \epsilon_r - \left(\frac{2}{r} + \gamma' \right) \epsilon_\theta \right], \\ \tilde{\omega} &= \omega - \frac{2e^{-\gamma}}{r^2} \partial_\theta \epsilon_\theta - \left(\frac{2}{r} + \gamma' \right) \epsilon_r, \\ \tilde{B} &= B - \frac{1}{2} (\epsilon_r + \epsilon' - \sigma' \epsilon), & \tilde{C} &= C - \frac{1}{2} (\epsilon_\theta + \partial_\theta \epsilon). \end{aligned} \quad (70)$$

From these relations, we can derive the four gauge-

invariant variables

$$\begin{aligned} \Psi &\equiv \psi - \frac{1}{2} \sigma' (4B + e^\sigma E'), & \Xi &\equiv \xi - \partial_r (4B + e^\sigma E'), \\ \Upsilon &\equiv \zeta - \frac{1}{4} \partial_r (4C + e^\sigma \partial_\theta E) - \frac{1}{4} \partial_\theta (4B + e^\sigma E') \\ &\quad + \frac{1}{4} \left(\frac{2}{r} + \gamma' \right) (4C + e^\sigma \partial_\theta E), \\ \Omega &\equiv \omega - \frac{e^{-\gamma}}{r^2} \partial_\theta (4C + e^\sigma \partial_\theta E) \\ &\quad - \frac{1}{2} \left(\frac{2}{r} + \gamma' \right) (4B + e^\sigma \partial_r E). \end{aligned} \quad (71)$$

As in the usual cosmological case [69], these variables are identical to the original variables once the choice of longitudinal gauge ($E = B = C = 0$) is made. The transformation leading to this gauge, starting from an arbitrary gauge transformation, reads

$$\epsilon = -\frac{1}{2} e^\sigma E, \quad \epsilon_r = 2B + \frac{1}{2} e^\sigma E', \quad \epsilon_\theta = 2C + \frac{1}{2} e^\sigma \partial_\theta E, \quad (72)$$

and is unique. This gauge choice, which we shall for now on adopt, is thus complete for metric perturbations, but also for the matter ones. Indeed, in our model (2), the matter perturbations concern only the hyperstring forming scalar field Φ and its associated gauge field C_A . Note that in our framework, the location of the brane is given by the zeroes of the Higgs field and thus directly taken into account in its perturbations. Since we are only interested in scalar perturbations, the perturbed fields can be expanded as

$$\delta\Phi = \chi(r, \theta) e^{in\theta}, \quad \delta C_A = (\partial_\mu \vartheta, \vartheta_r, \vartheta_\theta), \quad (73)$$

where we have extracted the background winding phase $e^{in\theta}$ in the scalar field perturbations. Note that, for consistency, all the perturbations have to be invariant by a complete rotation around the hyperstring, and thus can be decomposed in discrete angular momentum modes around the string. Under the gauge transformation $x^A \rightarrow \tilde{x}^A = x^A + \epsilon^A$ these perturbations transform to

$$\begin{aligned} \tilde{\chi} &= \chi - \varepsilon_r \varphi' - in \frac{e^{-\gamma}}{r^2} \varepsilon_\theta \varphi, & \tilde{\vartheta} &= \vartheta - \frac{e^{-\gamma}}{r^2} C_\theta \varepsilon_\theta, \\ \tilde{\vartheta}_r &= \vartheta_r - \frac{e^{-\gamma}}{r^2} C_\theta \varepsilon_\theta' + 2 \frac{e^{-\gamma}}{r^2} C_\theta \left(\frac{1}{r} + \frac{\gamma'}{2} \right) \varepsilon_\theta, \\ \tilde{\vartheta}_\theta &= \vartheta_\theta - C_\theta' \varepsilon_r - \frac{e^{-\gamma}}{r^2} C_\theta \partial_\theta \varepsilon_\theta, \end{aligned} \quad (74)$$

and are therefore not invariant. Similarly to the metric tensor decomposition, we then define the gauge-invariant quantities through the relations

$$\mathcal{X} \equiv \chi - \frac{1}{2} \varphi' (4B + e^\sigma E') - \frac{ine^{-\gamma}}{2r^2} \varphi (4C + e^\sigma \partial_\theta E), \quad (75)$$

for the Higgs perturbations and

$$\Theta \equiv \vartheta - \frac{1}{2} \frac{e^{-\gamma}}{r^2} C_\theta (4C + e^\sigma \partial_\theta E), \quad (76)$$

$$\begin{aligned} \Theta_r \equiv \vartheta_r - \frac{1}{2} \frac{e^{-\gamma}}{r^2} C_\theta [4C' + (e^\sigma \partial_\theta E)'] \\ + \frac{e^{-\gamma}}{r^2} C_\theta \left(\frac{1}{r} + \frac{\gamma'}{2} \right) (4C + e^\sigma \partial_\theta E), \end{aligned} \quad (77)$$

$$\begin{aligned} \Theta_\theta \equiv \vartheta_\theta - \frac{1}{2} C'_\theta (4B + e^\sigma E') \\ - \frac{1}{2} \frac{e^{-\gamma}}{r^2} C_\theta (4\partial_\theta C + e^\sigma \partial_\theta^2 E), \end{aligned} \quad (78)$$

for the gauge field perturbations. They also match with the original variables in the longitudinal gauge ($E = B = C = 0$).

Note that since we are interested in perturbation theory, we have to keep in mind that all the perturbed physical quantities involved at some initial time have to be

close to the background solution. This implies in particular that we must impose on the physically meaningful perturbations to be initially bounded: of all the possible solutions of the perturbation equations which we discuss below, we shall retain only those for which neither long- nor short-distance, divergence appear. This, as it turns out, is extremely restrictive.

B. Perturbed Einstein equations

The Einstein equations, perturbed at first order, stem from Eq. (11). The perturbed metric tensor δg_{AB} is explicitly written in Eq. (69) and allows, by means of Eq. (73), the determination of the scalar part of the perturbed Einstein and stress-energy tensors. They are derived in Appendix A, and Eq. (11) leads, in terms of the gauge-invariant variables, to the following equations of motion

$$\begin{aligned} (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) \left(\frac{\Xi + \Omega}{2} + \Psi \right) + \frac{1}{2} e^\sigma \eta_{\mu\nu} \left\{ 3\Psi'' + 3 \frac{e^{-\gamma}}{r^2} \partial_\theta^2 \Psi + \Omega'' - 2 \frac{e^{-\gamma}}{r^2} \partial_\theta \Upsilon' + \frac{e^{-\gamma}}{r^2} \partial_\theta^2 \Xi + 3 \left(2\sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Psi' \right. \\ \left. - \left(\frac{3}{2} \sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Xi' + \left[\frac{3}{2} \sigma' + 2 \left(\frac{1}{r} + \frac{\gamma'}{2} \right) \right] \Omega' - 3 \frac{e^{-\gamma}}{r^2} \sigma' \partial_\theta \Upsilon \right. \\ \left. + \left[3\sigma'' + 3\sigma'^2 + 3\sigma' \left(\frac{1}{r} + \frac{\gamma'}{2} \right) + 2 \left(\frac{1}{r} + \frac{\gamma'}{2} \right)' + 2 \left(\frac{1}{r} + \frac{\gamma'}{2} \right)^2 \right] (\Psi - \Xi) \right\} \\ + \kappa_6^2 e^\sigma \eta_{\mu\nu} \left\{ - \frac{e^{-\gamma}}{r^2} \frac{Q'}{q} (\Theta'_\theta - \partial_\theta \Theta_r) + \varphi' \Sigma' + \frac{e^{-\gamma}}{r^2} \varphi Q^2 \Sigma + \frac{e^{-\gamma}}{r^2} \varphi Q \partial_\theta \Delta + \frac{dV}{d\varphi} \Sigma \right. \\ \left. - \frac{1}{2} \left(\frac{e^{-\gamma}}{r^2} \frac{Q'^2}{q^2} + \varphi'^2 \right) \Xi - \frac{1}{2} \frac{e^{-\gamma}}{r^2} \left(\frac{Q'^2}{q^2} + \varphi^2 Q^2 \right) \Omega \right. \\ \left. + \left[\frac{1}{2} \frac{e^{-\gamma}}{r^2} \left(\frac{Q'^2}{q^2} + \varphi^2 Q^2 \right) + \frac{1}{2} \varphi'^2 + V(\varphi) \right] \Psi - \frac{e^{-\gamma}}{r^2} q \varphi^2 Q \Theta_\theta \right\} = -\Lambda e^\sigma \Psi \eta_{\mu\nu}, \end{aligned} \quad (79)$$

for the (μ, ν) part of Eq. (11). The (μ, r) and (μ, θ) components lead to the following equations, respectively,

$$3\Psi' + \Omega' - \frac{e^{-\gamma}}{r^2} \partial_\theta \Upsilon - \left(\frac{3}{2} \sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Xi + \left(-\frac{1}{2} \sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Omega + 2\kappa_6^2 \left\{ -\frac{e^{-\gamma}}{r^2} \frac{Q'}{q} (\Theta_\theta - \partial_\theta \Theta) + \varphi' \Sigma' \right\} = 0, \quad (80)$$

and

$$\partial_\theta (3\Psi + \Xi) - \Upsilon' - \left(\sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Upsilon + 2\kappa_6^2 \left\{ \frac{Q'}{q} (\Theta_r - \Theta') + \varphi Q \Delta - q \varphi^2 Q \Theta \right\} = 0. \quad (81)$$

The purely bulk components of the first order perturbation of Eq. (11) read

$$\begin{aligned} \frac{1}{2} e^{-\sigma} \square (3\Psi + \Omega) - 2 \frac{e^{-\gamma}}{r^2} \partial_\theta^2 \Psi + 2 \frac{e^{-\gamma}}{r^2} \sigma' \partial_\theta \Upsilon - \left[3\sigma' + 2 \left(\frac{1}{r} + \frac{\gamma'}{2} \right) \right] \Psi' - \sigma' \Omega' \\ + \kappa_6^2 \left\{ - \frac{e^{-\gamma}}{r^2} \frac{Q'}{q} (\Theta'_\theta - \partial_\theta \Theta_r) + \varphi' \Sigma' - \frac{e^{-\gamma}}{r^2} \varphi Q^2 \Sigma - \frac{e^{-\gamma}}{r^2} \varphi Q \partial_\theta \Delta - \frac{dV}{d\varphi} \Sigma \right. \\ \left. - \left[\frac{1}{2} \frac{e^{-\gamma}}{r^2} \varphi^2 Q^2 + V(\varphi) \right] \Xi - \frac{1}{2} \frac{e^{-\gamma}}{r^2} \left(\frac{Q'^2}{q^2} - \varphi^2 Q^2 \right) \Omega + \frac{e^{-\gamma}}{r^2} q \varphi^2 Q \Theta_\theta \right\} - \Lambda \Xi = 0, \end{aligned} \quad (82)$$

for the (r, r) part,

$$\begin{aligned} \frac{1}{2}r^2e^\gamma e^{-\sigma}\square(3\Psi + \Xi) &- 2r^2e^\gamma\Psi'' + r^2e^\gamma\sigma'(\Xi' - 5\Psi') + \frac{1}{2}r^2e^\gamma(4\sigma'' + 5\sigma'^2)(\Xi - \Omega) \\ &+ \kappa_6^2\left\{-\frac{Q'}{q}(\Theta'_\theta - \partial_\theta\Theta_r) - r^2e^\gamma\varphi'\Sigma' + \varphi Q^2\Sigma + \varphi Q\partial_\theta\Delta - r^2e^\gamma\frac{dV}{d\varphi}\Sigma\right. \\ &\left.+ \frac{1}{2}\left(r^2e^\gamma\varphi'^2 - \frac{Q'^2}{q^2}\right)\Xi - r^2e^\gamma\left[\frac{1}{2}\varphi'^2 + V(\varphi)\right]\Omega - q\varphi^2Q\Theta_\theta\right\} - r^2e^\gamma\Lambda\Omega = 0, \end{aligned} \quad (83)$$

for the (θ, θ) component, while the mixed one (r, θ) leads to the equation

$$\begin{aligned} -\frac{1}{2}e^{-\sigma}\square\Upsilon + 2\partial_\theta\Psi' - \sigma'\partial_\theta\Xi + \left[\sigma' - 2\left(\frac{1}{r} + \frac{\gamma'}{2}\right)\right]\partial_\theta\Psi - \frac{1}{2}(4\sigma'' + 5\sigma'^2)\Upsilon \\ + \kappa_6^2\left\{\varphi Q\Delta' - \varphi'Q\Delta + \varphi'\partial_\theta\Sigma + \left[\frac{1}{2}\frac{e^{-\gamma}}{r^2}\left(\frac{Q'^2}{q^2} - \varphi^2Q^2\right) - \frac{1}{2}\varphi'^2 - V(\varphi)\right]\Upsilon\right. \\ \left.- q\varphi^2Q\Theta_r\right\} - \Lambda\Upsilon = 0, \end{aligned} \quad (84)$$

with “ $\square$ ” standing for the flat four-dimensional d'Alembertian, i.e.

$$\square = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2 - \nabla^2, \quad (85)$$

and where the perturbed Higgs field has been decomposed as

$$\mathcal{X} = \Sigma + i\Delta, \quad (86)$$

i.e. into its real and imaginary parts.

The perturbation (86), although true in general, does not make the most of the U(1) invariance of the theory (2). Indeed, with the definition (4) for the covariant derivative, the full theory is unchanged under the change $\Phi \rightarrow \Phi^{(N)} = e^{i\alpha(x_A)}\Phi$ provided the gauge vector field is simultaneously modified into $C_A \rightarrow C_A^{(N)} = C_A - (1/q)\partial_A\alpha$. For an infinitesimal U(1) gauge transformation, this leads to the transformation $\Sigma^{(N)} = \Sigma$ and $\Delta^{(N)} = \Delta + \alpha\varphi(r)$, so that choosing $\alpha = -\Delta/\varphi$ allows to restrict attention to purely real perturbations of

the scalar field perturbation $\mathcal{X}$; we shall accordingly call this choice the real gauge. Indeed, as we shall see explicitly, letting Δ arbitrary leads to equations involving only the U(1) gauge-invariant degrees of freedom, namely $\Theta - \Delta/(q\varphi)$, $\Theta_\theta - \partial_\theta\Delta/(q\varphi)$ and $\Theta_r - (1/q)(\Delta/\varphi)'$, which merely expresses the fact that by going to the real gauge, one can get rid of Δ .

C. Perturbed Maxwell equations

By means of Eq. (73), we can also derive the perturbed Maxwell equations stemming from Eq. (19), at first order in the fields. Since the Einstein equations impose to the stress-energy tensor to be conserved, the perturbed Maxwell equations are certainly already included in Eqs. (79) to (84). Nevertheless, they mainly involve the matter fields and may help to decouple the whole system. The (μ) component of the perturbed Faraday tensor gives

$$\Theta'_r - \Theta'' + \left(\sigma' + \frac{1}{r} + \frac{\gamma'}{2}\right)(\Theta_r - \Theta') + \frac{e^{-\gamma}}{r^2}\partial_\theta(\Theta_\theta - \partial_\theta\Theta) - q\varphi\Delta + q^2\varphi^2\Theta = 0, \quad (87)$$

while the (r) and (θ) bulk parts lead to

$$\begin{aligned} e^{-\sigma}\square(\Theta_r - \Theta') + \frac{e^{-\gamma}}{r^2}\partial_\theta(\Theta'_\theta - \partial_\theta\Theta_r) - \frac{e^{-\gamma}}{r^2}\frac{Q'}{q}\partial_\theta\left[2\Psi - \frac{1}{2}(\Xi + \Omega)\right] + \frac{e^{-\gamma}}{r^2}q\varphi^2Q\Upsilon - q\varphi\Delta' + q\varphi'\Delta \\ + q^2\varphi^2\Theta_r = 0, \end{aligned} \quad (88)$$

and

$$\begin{aligned} e^{-\sigma} \square (\Theta_\theta - \partial_\theta \Theta) &- (\Theta''_\theta - \partial_\theta \Theta'_r) - \left[2\sigma' - \left(\frac{1}{r} + \frac{\gamma'}{2} \right) \right] (\Theta'_\theta - \partial_\theta \Theta_r) + \frac{Q'}{q} \left[2\Psi' - \frac{1}{2} (\Xi' + \Omega') \right] \\ &- \left\{ \frac{Q''}{q} + \left[2\sigma' - \left(\frac{1}{r} + \frac{\gamma'}{2} \right) \right] \frac{Q'}{q} \right\} \Xi - 2q\varphi Q \Sigma - q\varphi \partial_\theta \Delta + q^2 \varphi^2 \Theta_\theta = 0, \end{aligned} \quad (89)$$

where use has been made of Eq. (20) to simplify the term otherwise proportional to Ω .

D. Perturbed Klein-Gordon equation

In the same way, the Klein-Gordon equation (17) can also be perturbed in terms of metric and matter fields. By means of Eqs. (73) and (86), its real and imaginary parts lead to two coupled equations,

$$\begin{aligned} e^{-\sigma} \square \Sigma &- \Sigma'' - \left(2\sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Sigma' - \frac{e^{-\gamma}}{r^2} \partial_\theta^2 \Sigma + 2 \frac{e^{-\gamma}}{r^2} Q \partial_\theta \Delta + \left[\frac{e^{-\gamma}}{r^2} Q^2 + \frac{\lambda}{2} (3\varphi^2 - \eta^2) \right] \Sigma \\ &- \varphi' \left(2\Psi' + \frac{\Omega' - \Xi'}{2} - \frac{e^{-\gamma}}{r^2} \partial_\theta \Upsilon \right) + \left[\varphi'' + \left(2\sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \varphi' \right] \Xi - \frac{e^{-\gamma}}{r^2} \varphi Q^2 \Omega - 2 \frac{e^{-\gamma}}{r^2} q \varphi Q \Theta_\theta = 0, \end{aligned} \quad (90)$$

and,

$$\begin{aligned} e^{-\sigma} \square \Delta &- \Delta'' - \left(2\sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Delta' - \frac{e^{-\gamma}}{r^2} \partial_\theta^2 \Delta - 2 \frac{e^{-\gamma}}{r^2} Q \partial_\theta \Sigma + \left[\frac{e^{-\gamma}}{r^2} Q^2 + \frac{\lambda}{2} (\varphi^2 - \eta^2) \right] \Delta \\ &+ \varphi Q \frac{e^{-\gamma}}{r^2} \left[\Upsilon' - \partial_\theta \left(2\Psi + \frac{\Xi - \Omega}{2} \right) \right] + q\varphi \left(-e^{-\sigma} \square \Theta + \Theta'_r + \frac{e^{-\gamma}}{r^2} \partial_\theta \Theta_\theta \right) \\ &+ \left[\varphi Q' + 2\varphi' Q + \varphi Q \left(2\sigma' - \frac{1}{r} - \frac{\gamma'}{2} \right) \right] \frac{e^{-\gamma}}{r^2} \Upsilon + \left[2\varphi' + \left(2\sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \varphi \right] q \Theta_r = 0, \end{aligned} \quad (91)$$

respectively.

As previously noted, the perturbed fields and geometry have to be invariant by a complete rotation around the string, *i.e.* they are 2π -periodic in the angular variable θ . Therefore, they can be decomposed in Fourier series with respect to θ , *e.g.* the perturbed Higgs fields is expanded as

$$\begin{aligned} \Delta(r, \theta) &= \sum_{p \in \mathbb{Z}} \Delta_p(r) e^{ip\theta}, \\ \Sigma(r, \theta) &= \sum_{p \in \mathbb{Z}} \Sigma_p(r) e^{ip\theta}, \end{aligned} \quad (92)$$

and similarly for all the other perturbations. Plugging Eq. (92), and analogous mode expansion for Ψ , Ξ , Ω , Θ , Θ_r and Θ_θ into Eqs. (79) to (91) clearly shows that each angular “ p -mode” decouples from the others. As a result, the time evolution of the perturbations can be focused on a particular angular mode p , the physical evolution ending up be given by their superposition. Although some redefinitions of the fields may separate the system of equations (79) to (91) into distinct subsets [70], such a situation already happens for the lowest angular mode $p = 0$. Indeed, the angular dependency of these zero modes disappear which is formally equivalent to nullify the “ ∂_θ ” operator in Eqs. (79) to (91). One gets two disjoint pieces of equations, namely Eqs. (79), (80), (82),

(83), (89) and (90) which only involve Ψ , Ω , Ξ , Θ_θ and Σ on one side, while Eqs. (81), (84), (87), (88) and (91) couple only Υ , Θ_r , Θ and Δ on the other side.

In fact the zero angular momentum modes, obtained for $p = 0$, represent cylindrical perturbations which strictly wind around the string as the background forming fields do. Moreover, since they correspond to the lowest angular momentum state, one may naturally expect them to be first excited in a generic modification of the vortex structure.

In order to study the stability of these zero-modes, the time evolution of the latter subset of perturbations will be thoroughly analyzed in the following section.

E. Stability of the transverse perturbations

The time evolution of the lowest angular momentum modes Υ_0 , Θ_0 , Θ_{r0} and Δ_0 is readily governed by the perturbed Einstein equations (81) and (84) together with the perturbed Maxwell equations (87) and (88), and the perturbed Higgs one (91). As noted before, due to implicit stress-energy tensor conservation in the Einstein equations, this system is not over-determined although it involves redundant equations. Moreover, since we are inter-

ested in perturbations which behave almost like the background vortex fields, we will only consider real perturbations of the hyperstring forming Higgs field, i.e. with $\Delta_0 = 0$ (in other words, we go to the real gauge). In terms of the dimensionless background fields and parameters [see Eqs. (7), (15) and (16)], together with the new dimensionless fields

$$\tilde{\Theta} = q\Theta, \quad \tilde{\Theta}_r = \frac{q}{\sqrt{|\Lambda|}}\Theta_r, \quad \tilde{Y} = \sqrt{|\Lambda|}\Upsilon, \quad (93)$$

the time evolution equations stemming from the Einstein and Maxwell equations read

$$\dot{\tilde{Y}} + \left(s + \frac{l}{2}\right) \tilde{Y} - \frac{4\alpha\dot{Q}}{\varepsilon} (\tilde{\Theta}_r - \dot{\tilde{\Theta}}) + 4\alpha f^2 Q \tilde{\Theta} = 0, \quad (94)$$

$$\left(e^{-\sigma} \tilde{M}^2 - 4\alpha \frac{f^2 Q^2}{m}\right) \tilde{Y} - 4\alpha f^2 Q \tilde{\Theta}_r = 0, \quad (95)$$

$$\dot{\tilde{\Theta}}_r - \ddot{\tilde{\Theta}} + \left(s + \frac{l}{2}\right) (\tilde{\Theta}_r - \dot{\tilde{\Theta}}) + \varepsilon f^2 \tilde{\Theta} = 0, \quad (96)$$

$$e^{-\sigma} \tilde{M}^2 (\tilde{\Theta}_r - \dot{\tilde{\Theta}}) - \varepsilon \frac{f^2 Q}{m} \tilde{Y} - \varepsilon f^2 \tilde{\Theta}_r = 0, \quad (97)$$

while the Higgs one becomes

$$\begin{aligned} \frac{Q}{m} \dot{\tilde{Y}} + e^{-\sigma} \tilde{M}^2 \tilde{\Theta} + \dot{\tilde{\Theta}}_r + \left[\dot{Q} + Q \left(2\frac{\dot{f}}{f} + 2s - \frac{l}{2}\right)\right] \frac{\tilde{Y}}{m} \\ + \left(2\frac{\dot{f}}{f} + 2s + \frac{l}{2}\right) \tilde{\Theta}_r = 0. \end{aligned} \quad (98)$$

A four-dimensional Fourier transform has been performed on the zero angular momentum perturbed fields with respect to the four-dimensional coordinates x^μ . The general perturbation solution is therefore a linear superposition of the d'Alembertian eigenmodes

$$\tilde{Y}_0(x^\mu, r) = \int \tilde{Y}(k^\mu, r) e^{-ik_\mu x^\mu} d^4 k, \quad (99)$$

with $\tilde{Y}(k^\mu, r)$ the solution of Eqs. (94) to (98), and similarly for the others perturbed quantities. The d'Alembertian eigenvalues ends up being

$$\square \longrightarrow -|\Lambda| \tilde{M}^2 = -\eta^{\mu\nu} k_\mu k_\nu, \quad (100)$$

and any perturbation with positive mass squared $\tilde{M}^2 \geq 0$ will be considered stable, whereas tachyonic modes, having $\tilde{M}^2 < 0$, will generate instabilities.

There are three variables $\tilde{\Theta}_r$, $\tilde{\Theta}$ and $\tilde{Y}$ for five equations, two of them being thus constraint equations. By means of Eq. (95), the metric perturbation $\tilde{Y}$ can be expressed in terms of $\tilde{\Theta}_r$ only

$$\tilde{Y} = \frac{4\alpha f^2 Q}{e^{-\sigma} \tilde{M}^2 - 4\alpha \frac{f^2 Q^2}{m}} \tilde{\Theta}_r, \quad (101)$$

while by means of Eq. (101), Eq. (97) gives the relation

$$\tilde{\Theta}_r = (\mathcal{P} + 1) \dot{\tilde{\Theta}}, \quad (102)$$

with

$$\mathcal{P} = \frac{\varepsilon f^2}{e^{-\sigma} \tilde{M}^2 - \varepsilon f^2 - 4\alpha \frac{f^2 Q^2}{m}}. \quad (103)$$

Finally, plugging the previous expressions for $\tilde{Y}$ and $\tilde{\Theta}_r$, given by Eq. (101) and Eq. (102), into Eq. (96), one gets a second order differential equation involving only the function $\tilde{\Theta}$, namely

$$\mathcal{P} \ddot{\tilde{\Theta}} + \left[\dot{\mathcal{P}} + \left(s + \frac{l}{2}\right) \mathcal{P}\right] \dot{\tilde{\Theta}} + \varepsilon f^2 \tilde{\Theta} = 0. \quad (104)$$

One can also verify that the two remaining equations (94) and (98) also lead to Eq. (104) when use is made of Eqs. (101) and (102), ensuring the consistency of the gauge choice $\Delta_0 = 0$. If this choice is relaxed, one can check that the new perturbed equations simply require $\tilde{\Theta}$ to be replaced by $\tilde{\Theta} - \Delta_0/\varphi$, and $\tilde{\Theta}_r$ by $\tilde{\Theta}_r - \partial_\rho(\Delta_0/\varphi)$ in Eqs. (101) to (103). The additional perturbed equation stemming from Eq. (91) ends up being equivalent to Eq. (104). As expected for a gauge degree of freedom, the field Δ_0 has therefore no dynamics and will not be considered in the following.

In order to conclude on the stability of the vortex solution with respect to these perturbations, let us consider the generic case of a real squared mass $\tilde{M}^2 \in \mathbb{R}$. In this case, and far from the string, Eq. (104) behaves as

$$\ddot{\tilde{\Theta}} - \sqrt{\frac{5}{2}} \dot{\tilde{\Theta}} + \tilde{M}^2 \exp\left(\sqrt{\frac{2}{5}} \rho\right) \tilde{\Theta} \simeq 0, \quad (105)$$

and through the change of variable $z = \exp(\rho/\sqrt{10})$ and function $\tilde{\Theta} = \exp(\sqrt{5/8}\rho) \tilde{T}$, Eq. (105) reads

$$\frac{1}{z} \frac{d}{dz} \left(z \frac{d\tilde{T}}{dz} \right) + \left(10\tilde{M}^2 - \frac{25}{4z^2} \right) \tilde{T} = 0, \quad (106)$$

whose solutions are known, see, e.g. Eq. (8.491) in Ref. [71]. This gives

$$\tilde{\Theta} \propto \exp\left(\sqrt{\frac{5}{8}} \rho\right) \times \mathcal{Z}_{5/2} \left[\sqrt{10} |\tilde{M}| \exp\left(\frac{\rho}{\sqrt{10}}\right) \right], \quad (107)$$

in which $\mathcal{Z}_{5/2}$ is a Bessel function of order 5/2 and of its argument in brackets for $\tilde{M}^2 > 0$, and a modified Bessel function for $\tilde{M}^2 < 0$. As a result, for any positive mass squared, $\tilde{M}^2 > 0$, the solution given in Eq. (107) behaves, asymptotically far from the vortex, as an oscillatory exponentially divergent quantity whose amplitude scales as

$$|\tilde{\Theta}|_\infty \propto \exp\left(\sqrt{\frac{2}{5}} \rho\right). \quad (108)$$

Such solutions appear to be unbounded far from the string and are, strictly speaking, not well-defined: the

first order perturbation equations can be derived from the action expanded to second order in these perturbations, which one would thus expect to be finite, since the volume of the extra dimensions is finite (this is the very reason for choosing anti-de Sitter in the first place). But this action contains a term $\propto \int \sqrt{-g} e^{-\sigma} \tilde{\Theta}^2$ (the exponential stemming from the unperturbed contravariant metric coefficient) which diverges exponentially. In spite of this issue, the solutions (107) can be given a physical meaning in the framework of perturbation theory since the energy they contain to first order is actually finite, and the corresponding gravitational potential $\tilde{Y}$ vanishes asymptotically [see Eq. (101)]. In this respect, these positive squared mass perturbations are physically admissible solutions.

In the case $\tilde{M}^2 < 0$, the Bessel function in Eq. (107) is of the modified kind and admits an asymptotically exponential of exponential decaying behavior: one of the two degrees of freedom of Eq. (107) has to be fixed to ensure the decrease of the $\tilde{M}^2 < 0$ solution asymptotically. However, this does not mean that tachyonic modes exist inside the system since it is also necessary that these perturbations are well-defined in the string core. We shall accordingly turn attention to the interior solution.

In the hyperstring core, the function $\mathcal{P}$ can be expanded as

$$\mathcal{P} = \frac{\varepsilon f_1^2}{\tilde{M}^2 - 4\alpha f_1^2} \rho^2 + \mathcal{O}(\rho^4), \quad (109)$$

where use has been made of the background field behaviors in the string core. In the limit $\rho \rightarrow 0$, Eq. (104) becomes

$$\rho^2 \ddot{\tilde{\Theta}} + 3\rho \dot{\tilde{\Theta}} + \rho^2 (\tilde{M}^2 - 4\alpha f_1^2) \tilde{\Theta} \simeq 0, \quad (110)$$

whose solutions are

$$\tilde{\Theta} \propto \frac{1}{\rho} \mathcal{Z}_1 \left(\sqrt{|\tilde{M}^2 - 4\alpha f_1^2|} \rho \right). \quad (111)$$

Again, $\mathcal{Z}_1$ refers to the two independent Bessel functions J_1 and Y_1 provided $\tilde{M}^2 - 4\alpha f_1^2 > 0$, while it designs the two modified one, I_1 and K_1 , in the other case (which is the case for $\tilde{M}^2 < 0$ in which we are interested). Therefore, there always is a divergent solution in the string core, behaving as $1/\rho^2$, together with a well-defined one going to a constant value. As a result, the $\tilde{M}^2 < 0$ decreasing solution far from the string can not generically match with the well-defined one in the string core, the required degree of freedom being already fixed to ensure the asymptotic normalizability. Although there is thus no tachyonic continuum spectrum, the matching between the two well-defined solutions at infinity and in the string core could happen for some peculiar values of the negative mass squared, making a discrete spectrum of unstable modes. In the following, we show that this is not the case.

The equation (104) can be rewritten in the form of a zero mode Schrödinger equation for the quantity $u \equiv [\rho|\mathcal{P}| \exp(\sigma + \gamma/2)]^{1/2} \tilde{\Theta}$, namely

$$-\ddot{u} + V_M(\rho)u = 0, \quad (112)$$

with the potential

$$V_M(\rho) = W^2 + \dot{W} + \varepsilon f^2 + 4\alpha \frac{f^2 Q^2}{m} - \tilde{M}^2 e^{-\sigma}, \quad (113)$$

with

$$W = \frac{1}{2} \left[\frac{\dot{\mathcal{P}}}{\mathcal{P}} + \left(s + \frac{l}{2} \right) \right], \quad (114)$$

the superpotential-like of the Schrödinger equation (112) [72].

It is immediately clear from Eq. (113) that the potential is asymptotically dominated by the last, exponentially increasing, term. As a result, a confinement for the corresponding scalar mode is achieved provided this last term is positive, hence requiring a negative squared mass. This would seem to imply the existence of tachyonic modes on the hyperstring. However, a closer examination of the potential (displayed on Fig. 5) actually shows that even in the negative squared mass case, V_M is, numerically, positive definite: the associated Schrödinger equation has only strictly positive eigenvalues, and in particular no zero mode. This can be seen analytically in the following way: we first note that Eq. (112) can be written as $[\mathcal{A}^\dagger \mathcal{A} + Z^2(\rho)] u = 0$, where Z can be obtained from the potential (113) and the definition of the operator

$$\mathcal{A} \equiv \frac{d}{d\rho} + W(\rho), \quad \rightarrow \quad \mathcal{A}^\dagger \equiv -\frac{d}{d\rho} + W(\rho), \quad (115)$$

i.e.

$$Z^2 = \varepsilon f^2 + \frac{4\alpha f^2 Q^2}{m} - \tilde{M}^2 e^{-\sigma},$$

which is indeed a positive definite function of the core distance. Then, if one demands the perturbation u to be bounded, as implied by the fact that the energy contained in the mode be finite, then one is seeking zero mode bound (normalizable) states of the Schrödinger equation (112). It is however immediately clear that since the operator $\mathcal{A}^\dagger \mathcal{A}$ has only nonnegative eigenvalues, then the spectrum of $\mathcal{A}^\dagger \mathcal{A} + Z^2(\rho)$ must be positive definite; thus, as announced, there are no zero modes solution of Eq. (112), and hence no instability. It is interesting to notice that the nonnormalizability property at the hyperstring core also stems from the remark that the potential (113) diverges as $V_M \sim \rho^{-2}$ near the core.

Thus, the tachyonic solutions cannot be considered as physical perturbations of the system. We are led to the conclusion that, for the subset of fields considered in this section, the hyperstring is stable with respect to transverse perturbations. To decide on the overall stability of the vortex, one needs to clarify the role of the complementary subset of modes; This is done in the following section.

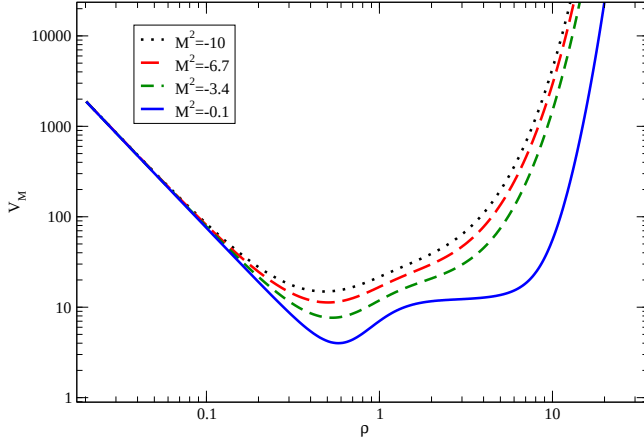


FIG. 5: Characteristic shape of the potential $V_M(\rho)$ given by Eq. (113) for a typical set of values $(\alpha, \beta, \varepsilon) = (1.77, 2.66, 11.56)$ on the fine-tuning surface for which $v_0 = 1$ and different values of the negative squared mass $\tilde{M}^2$. It is clear from this figure that the potential is positive definite, and its minimum increases with $|\tilde{M}^2|$. Hence, there is no zero mode for the perturbation equation, and therefore no instability zone.

F. Sausage perturbation modes

We now turn to the second decoupled subset of cylindrical perturbation modes which respect the axial symmetry, hence their name. Provided we are interested in the zero angular momentum modes, the time evolution of the perturbations Ψ , Ξ , Ω , Σ and Θ_θ is determined by the Eqs. (79), (80), (82), (83), (89) and (90). Some rapid simplifications can be performed. First, from the $\mu \neq \nu$ part of the Einstein equation (79), one gets

$$\Omega = -2\Psi - \Xi. \quad (116)$$

This expression can thus be used to simplify the above mentioned equations for $\mu = \nu$. In terms of dimensionless quantities the perturbed Einstein equations simplify into two dynamical equations

$$\ddot{\Psi} - \ddot{\Xi} + \left(3s - \frac{l}{2}\right) \dot{\Psi} - \left(3s + \frac{3}{2}l\right) \dot{\Xi} + (2\mathcal{V}_1 + 2\mathcal{V}_2) \Psi + (2\mathcal{F} + \mathcal{V}_1 + 2\mathcal{V}_2) \Xi = \mathcal{S}_{\mu\mu}, \quad (117)$$

$$4\ddot{\Psi} + 10s\dot{\Psi} - 2s\dot{\Xi} + \left(3e^{-\sigma}\tilde{M}^2 - 2\mathcal{V}_1 - 2\mathcal{V}_2\right) \Psi + \left(e^{-\sigma}\tilde{M}^2 + 2\mathcal{F} - \mathcal{V}_1 - 2\mathcal{V}_2\right) \Xi = \mathcal{S}_{\theta\theta}, \quad (118)$$

and two constraint equations

$$\dot{\Psi} - \dot{\Xi} + (s - l)\Psi - (s + l)\Xi = \mathcal{S}_{\mu r}, \quad (119)$$

$$2(s + l)\dot{\Psi} - 2s\dot{\Xi} + \left(e^{-\sigma}\tilde{M}^2 - 2\mathcal{V}_1 + 2\mathcal{V}_2\right) \Psi + \left(-e^{-\sigma}\tilde{M}^2 + 2\mathcal{F} - \mathcal{V}_1 + 2\mathcal{V}_2\right) \Xi = \mathcal{S}_{rr}, \quad (120)$$

where the matter source field functions $\mathcal{S}$ stand for

$$\begin{aligned} \mathcal{S}_{\mu\mu} = & 4\alpha \frac{Q\dot{Q}}{\varepsilon m} \dot{\tilde{\Theta}}_\theta - 4\alpha f \dot{f} \dot{\tilde{\Sigma}} + 2(\mathcal{V}_1 + \mathcal{V}_2) \tilde{\Theta}_\theta \\ & - \left[4\alpha \dot{f}^2 + 2\mathcal{V}_2 + 2f \frac{d\mathcal{F}}{df}\right] \tilde{\Sigma}, \end{aligned} \quad (121)$$

$$\begin{aligned} \mathcal{S}_{\theta\theta} = & -4\alpha \frac{Q\dot{Q}}{\varepsilon m} \dot{\tilde{\Theta}}_\theta - 4\alpha f \dot{f} \dot{\tilde{\Sigma}} - 2(\mathcal{V}_1 + \mathcal{V}_2) \tilde{\Theta}_\theta \\ & - \left[4\alpha \dot{f}^2 - 2\mathcal{V}_2 + 2f \frac{d\mathcal{F}}{df}\right] \tilde{\Sigma}, \end{aligned} \quad (122)$$

$$\mathcal{S}_{\mu r} = \frac{4\alpha Q\dot{Q}}{\varepsilon m} \tilde{\Theta}_\theta - 4\alpha f \dot{f} \tilde{\Sigma}, \quad (123)$$

$$\begin{aligned} \mathcal{S}_{rr} = & -4\alpha \frac{Q\dot{Q}}{\varepsilon m} \dot{\tilde{\Theta}}_\theta + 4\alpha f \dot{f} \dot{\tilde{\Sigma}} - 2(\mathcal{V}_1 - \mathcal{V}_2) \tilde{\Theta}_\theta \\ & - \left[-4\alpha \dot{f}^2 + 2\mathcal{V}_2 + 2f \frac{d\mathcal{F}}{df}\right] \tilde{\Sigma}, \end{aligned} \quad (124)$$

in which $\tilde{\Theta}_\theta \equiv q\Theta_\theta/Q$ and $\tilde{\Sigma} \equiv \Sigma/\varphi$.

The perturbed Klein-Gordon and Maxwell equations, also expressed in terms of dimensionless quantities read

$$\begin{aligned} \ddot{\tilde{\Sigma}} + \left(2\frac{\dot{f}}{f} + 2s + \frac{l}{2}\right) \dot{\tilde{\Sigma}} + \left(e^{-\sigma}\tilde{M}^2 - 4\beta f^2\right) \tilde{\Sigma} \\ = \frac{\dot{f}}{f} \left(\dot{\Xi} - \dot{\Psi}\right) + 2\left[\frac{Q^2}{m} + \beta f(f^2 - 1)\right] \Xi \\ + 2\frac{Q^2}{m} \left(\Psi - \tilde{\Theta}_\theta\right), \end{aligned} \quad (125)$$

and

$$\begin{aligned} \ddot{\tilde{\Theta}}_\theta + \left(2\frac{\dot{Q}}{Q} + 2s - \frac{l}{2}\right) \dot{\tilde{\Theta}}_\theta + e^{-\sigma}\tilde{M}^2 \tilde{\Theta}_\theta \\ = 3\frac{\dot{Q}}{Q} \dot{\Psi} - \varepsilon f^2 \left(\Xi + 2\tilde{\Sigma}\right). \end{aligned} \quad (126)$$

It is interesting to note at this point that the equations of motion for the five-dimensional domain wall having this kind of perturbation modes are directly obtained from Eqs. (117) to (125) by setting $\Omega = 0$ and removing any dependencies in l and Q .

As previously noted, due to the Bianchi identities some of these equations are redundant. Indeed, Eq. (117) is readily obtained by differentiation of Eq. (119), up to the background Einstein, Klein-Gordon and Maxwell equations (30) to (34). Similarly, differentiating Eq. (120), and using Eqs. (119), (125) and (126) to express $\dot{\mathcal{S}}_{rr}$ in terms of $\mathcal{S}_{\theta\theta}$ and the metric perturbations leads to Eq. (118), also up to the background Einstein, Klein-Gordon and Maxwell equations, and provided $l \neq 0$. As a result, only Eqs. (116), (119), (120), (125) and (126)

are relevant for this subset of matter and metric perturbations, with the constraint that the solutions are regular at the point where l vanishes. However this system remains fully coupled and no simple second order differential equation on one perturbation variable can be

obtained, as it was the case for the transverse modes [see Eq. (104)].

To study the stability properties of the hyperstring with respect to the sausage modes we derive in the following their behavior in the string core and asymptotically.

The previous system can be recast into a set of first order differential equations,

$$2l\dot{\Psi} = \left[-e^{-\sigma}\tilde{M}^2 + 2\mathcal{V}_1 - 2\mathcal{V}_2 + 2s(s-l) \right] \Psi + \left[e^{-\sigma}\tilde{M}^2 + \mathcal{V}_1 - 2\mathcal{V}_2 - 2\mathcal{F} - 2s(s+l) \right] \Xi \\ + \left[4\alpha f \dot{f} \left(\frac{\dot{f}}{f} + 2s \right) - 2\mathcal{V}_2 - 2f \frac{d\mathcal{F}}{df} \right] \tilde{\Sigma} + 4\alpha f \dot{f} \tilde{X} + \left[-2\mathcal{V}_1 + 2\mathcal{V}_2 - 2s(s-l) \right] \tilde{\Theta} - (s-l)\tilde{W}, \quad (127)$$

$$2l\dot{\Xi} = \left[-e^{-\sigma}\tilde{M}^2 + 2\mathcal{V}_1 - 2\mathcal{V}_2 + 2(s+l)(s-l) \right] \Psi + \left[e^{-\sigma}\tilde{M}^2 + \mathcal{V}_1 - 2\mathcal{V}_2 - 2\mathcal{F} - 2(s+l)^2 \right] \Xi \\ + \left[4\alpha f \dot{f} \left(\frac{\dot{f}}{f} + 2s + 2l \right) - 2\mathcal{V}_2 - 2f \frac{d\mathcal{F}}{df} \right] \tilde{\Sigma} + 4\alpha f \dot{f} \tilde{X} + \left[-2\mathcal{V}_1 + 2\mathcal{V}_2 - 2(s+l)(s-l) \right] \tilde{\Theta} - (s-l)\tilde{W}, \quad (128)$$

for the metric perturbations

$$\dot{\tilde{\Sigma}} = \tilde{X}, \quad (129)$$

$$\dot{\tilde{X}} = \left[2\frac{Q^2}{m} + \frac{\dot{f}}{f}(s-l) \right] \Psi + \left[2\frac{Q^2}{m} - \frac{\dot{f}}{f}(s+l) + \frac{1}{2\alpha} \frac{d\mathcal{F}}{df} \right] \Xi + \left[-e^{-\sigma}\tilde{M}^2 + 4\alpha\dot{f}^2 + 4\beta f^2 \right] \tilde{\Sigma} \\ - \left[2\frac{\dot{f}}{f} + 2s + \frac{1}{2}l \right] \tilde{X} - \left[2\frac{Q^2}{m} + \frac{\dot{f}}{f}(s-l) \right] \tilde{\Theta}, \quad (130)$$

for the Higgs field perturbations and

$$\dot{\tilde{\Theta}} = \tilde{W}, \quad (131)$$

$$2l\dot{\tilde{W}} = 3\frac{\dot{Q}}{Q} \left[-e^{-\sigma}\tilde{M}^2 + 2\mathcal{V}_1 - 2\mathcal{V}_2 + 2s(s-l) \right] \Psi + \left\{ -2\varepsilon l f^2 + 3\frac{\dot{Q}}{Q} \left[e^{-\sigma}\tilde{M}^2 + \mathcal{V}_1 - 2\mathcal{V}_2 - 2\mathcal{F} - 2s(s+l) \right] \right\} \Xi \\ + \left\{ -4\varepsilon l f^2 + 3\frac{\dot{Q}}{Q} \left[4\alpha f \dot{f} \left(\frac{\dot{f}}{f} + 2s \right) - 2\mathcal{V}_2 - 2f \frac{d\mathcal{F}}{df} \right] \right\} \tilde{\Sigma} + 12\alpha \frac{\dot{Q}}{Q} f \dot{f} \tilde{X} \\ + \left\{ -2le^{-\sigma}\tilde{M}^2 + 3\frac{\dot{Q}}{Q} [-2\mathcal{V}_1 + 2\mathcal{V}_2 - 2s(s-l)] \right\} \tilde{\Theta} - \left[\frac{\dot{Q}}{Q}(3s+l) + 2l(2s - \frac{1}{2}l) \right] \tilde{W}, \quad (132)$$

for the gauge field.

From Eqs. (127), (128) and making use of the asymptotic behaviors of the background fields (see Sect. IV), the metric sausage perturbation modes at infinity verify

$$\dot{\Psi} \simeq \sqrt{\frac{5}{2}} e^{\sqrt{2/5}\rho} \tilde{M}^2 \frac{\Psi}{2} - \left(\sqrt{\frac{5}{2}} e^{\sqrt{2/5}\rho} \tilde{M}^2 + \sqrt{\frac{2}{5}} \right) \frac{\Xi}{2}, \quad (133)$$

and

$$\dot{\Xi} \simeq \sqrt{\frac{5}{2}} e^{\sqrt{2/5}\rho} \tilde{M}^2 \frac{\Psi}{2} - \left(\sqrt{\frac{5}{2}} e^{\sqrt{2/5}\rho} \tilde{M}^2 - 3\sqrt{\frac{2}{5}} \right) \frac{\Xi}{2}. \quad (134)$$

From Eq. (133), one gets

$$\Xi \simeq \frac{e^{\sqrt{2/5}\rho} \tilde{M}^2 \Psi - 2\sqrt{2/5} \dot{\Psi}}{e^{\sqrt{2/5}\rho} \tilde{M}^2 + 2/5}, \quad (135)$$

while Eq. (134) yields

$$\dot{\Psi} - \sqrt{\frac{5}{2}} \dot{\Psi} + e^{\sqrt{2/5}\rho} \tilde{M}^2 \Psi \simeq 0. \quad (136)$$

From Eq. (105) and (107), the metric perturbations Ψ , and therefore Ξ , diverge at infinity as $\exp(\sqrt{2/5}\rho)$ as long as $\tilde{M}^2 > 0$. This behavior is not admissible in

the framework of perturbation theory. As can be seen in Eq. (69), the metric perturbations Ψ and Ξ have to be small compared with their corresponding background values, themselves are of order unity, at least initially. Otherwise, mathematically speaking, it is not consistent to expand the equations of motion to first order. Physically, this means we would start from a space which is infinitely far from the background one.

Similar conclusions also hold for the matter fields: from Eqs. (129) to (132), one gets asymptotically for the Higgs field perturbations

$$\ddot{\tilde{\Sigma}} - \sqrt{\frac{5}{2}} \dot{\tilde{\Sigma}} + e\sqrt{2/5}\rho \tilde{M}^2 \tilde{\Sigma} \simeq 0, \quad (137)$$

and, using Eq. (48),

$$\ddot{\tilde{\Theta}} - \left(2\ell_g + \frac{3}{2}\sqrt{\frac{2}{5}}\right) \tilde{\Theta} + e\sqrt{2/5}\rho \tilde{M}^2 \tilde{\Theta} \simeq -3\ell_g \dot{\tilde{\Sigma}}, \quad (138)$$

for the gauge field. For $\tilde{M}^2 > 0$, both of these equations have only divergent behavior at infinity, as $\exp(\sqrt{2/5}\rho)$ for the Higgs perturbations and as $\exp[(2\ell_g + \sqrt{2/5})\rho/2]$ for the gauge field perturbations [see Eq. (50)]. Also for the matter fields, the $\tilde{M}^2 > 0$ solutions are not admissible since it would physically means that the Higgs field is infinitely far from its vacuum expectation value $f = 1$.

As a result, the asymptotic study of the sausage modes shows that the hyperstring cannot be stable with respect to these perturbations. As this stage, either there are instabilities if there exist some $\tilde{M}^2 < 0$ modes which are well-defined in the hyperstring core and match the decreasing solution at infinity (see Sect. VII E), or the configuration is not perturbable at all, i.e. the only acceptable solution is vanishing perturbations. To explore this point, we discuss the behavior of the solutions in the hyperstring core.

The physical solutions we are interested in have to be well-defined at $\rho = 0$. In particular, the geometry can only be regular provided $\dot{\Psi}(0) = \dot{\Xi}(0) = 0$ [see Eq. (37) and discussion below for the background case]. Moreover, the sausage perturbations are required to be small with respect to their corresponding background values, i.e. Ψ , Ξ , $\tilde{\Sigma}$ and $\tilde{\Theta}$ have to be finite in the core. Assuming these fields can be expanded in Laurent series around $\rho = 0$, the previous constraints yield

$$\begin{aligned} \Psi &\sim \psi_0 + \sum_{n=2}^{\infty} \psi_n \rho^n, & \Xi &\sim \xi_0 + \sum_{n=2}^{\infty} \xi_n \rho^n, \\ \tilde{\Sigma} &\sim \sum_{n=0}^{\infty} \sigma_n \rho^n, & \tilde{\Theta} &\sim \sum_{n=0}^{\infty} \theta_n \rho^n, \end{aligned} \quad (139)$$

where ψ_n , ξ_n , σ_n and θ_n are real numbers. Plugging these expansions into Eqs. (127), (128), (129), (130), (131) and (132), where the derivatives with respect to ρ are readily obtained from Eq. (139), leads to a set of coupled algebraic relations for the coefficients ψ_n , ξ_n , σ_n and θ_n . By using the behaviors of the background fields in the hyperstring core obtained in Sect. IV B, we find that this hierarchy is fully determined by the knowledge of the three parameters ψ_0 , σ_0 and θ_2 . As a result, the regular solutions in the hyperstring core generate a three-dimensional subspace of the six-dimensional full space of solutions. It is therefore necessary to fix three degrees of freedom to get regular solutions in $\rho = 0$.

From Eqs. (135), (136), (137) and (138), we see that three degrees of freedom must also be fixed to ensure that the sausage perturbations are asymptotically well-defined (one for the metric perturbations Ψ and Ξ , one for the Higgs field perturbation $\tilde{\Sigma}$ and one for the gauge field perturbation $\tilde{\Theta}$). Moreover, from Eqs. (127), (128) and (132), there will be no jump in the derivative of the perturbations at $l = 0$ provided

$$\begin{aligned} &\left(-e^\sigma \tilde{M}^2 + 2\mathcal{V}_1 - 2\mathcal{V}_2 + 2s^2\right) \Psi(\rho_c) + \left(-e^\sigma \tilde{M}^2 + \mathcal{V}_1 - 2\mathcal{V}_2 - 2\mathcal{F} - 2s^2\right) \Xi(\rho_c) \\ &+ \left[4\alpha f \dot{f} \left(\frac{\dot{f}}{f} + 2s\right) - 2\mathcal{V}_2 - 2f \frac{d\mathcal{F}}{df}\right] \tilde{\Sigma}(\rho_c) + 4\alpha f \dot{f} \tilde{X}(\rho_c) + [-2\mathcal{V}_1 + 2\mathcal{V}_2 - 2s^2] \tilde{\Theta}(\rho_c) - s\tilde{W}(\rho_c) = 0, \end{aligned} \quad (140)$$

where the background fields are evaluated at $\rho = \rho_c$, the vanishing point of l .

Three degrees of freedom are thus fixed to ensure a convergent behavior of the perturbations at infinity, plus another one for the regularity at $l = 0$. Two degrees of freedom are left, which is not sufficient, according to the previous discussion, to ensure regularity of the solutions in the hyperstring core. Even if one tunes the mass $\tilde{M}^2$ to keep only the solutions regular at $\rho = 0$, the con-

vergent solutions at infinity (and regular at $l = 0$) will not generically match with the regular ones in the vortex core. We have also numerically verified that no exceptional hidden symmetry realizes this matching for a wide range of masses. Nevertheless, note that another parameters of the model could be used to realize the matching between the regular solutions in the core and at infinity. Indeed, the background fine-tuning between α , β and ε (see Fig. 3) is a surface in the three-dimensional param-

ters space and one cannot exclude that instabilities could marginally occur on a curve along this surface.

In conclusion, the only acceptable sausage perturbations modes are the vanishing ones, *i.e.* the hyperstring cannot be perturbed at all in the subset of matter and metric perturbations which correspond to nonvanishing background fields. In the following, we discuss the physical meaning of this result.

From a geometric point of view, it is well known that the generic space-time generated by a cosmic string in presence of a cosmological constant is of infinite-volume [73] (see Appendix B for a six-dimensional analogous derivation). This is precisely why we have to fine tune the model parameters to obtain a finite-volume space-time with decreasing warp factors and no singularity in the core. As can be seen from the metric (6), the obtained space-time geometry leads to vanishing proper length circles around the hyperstring at infinity. This kind of geometry implies the existence of a point where $l(\rho_c) = 0$ which is precisely the stationary point of proper length circles. From $\rho < \rho_c$ the proper perimeter of a circle around the hyperstring increases with respect to the radius, whereas for $\rho > \rho_c$ it decreases toward 0 (see Fig. 1). To link this structure to the usual conical geometry generated by cosmic strings, one may imagine a missing angle starting from zero in the hyperstring core toward 2π at infinity. This is not really surprising since we have required the hyperstring to generate an anti-de Sitter space-time at infinity, or naively, the fine-tuning allows to pass from a cylindrical symmetry in the core to a spherical one asymptotically. Now, it is clear that disturbing the fields around the values which lead to such fine-tuned gravitational configuration is not necessary allowed. And this is precisely our result. The only allowed perturbations concern the transverse modes which have not equivalent at zero order. On the contrary, all perturbations of the background fields, those generating this fine-tuned space-time, are forbidden. Interestingly, the natural behavior of the perturbations far the string for $\tilde{M}^2 > 0$, as $\exp(\sqrt{2/5}\rho)$, looks like the generic metric coefficients which appears when there is no fine-tuning (see Appendix B). Although one might expect the hyperstring to relax toward this generic configuration, no conclusion can be drawn from the perturbations theory since the generic space-time configuration, with infinite-volume, is not at all “close to” the studied fine-tuned one. To end this section, it is worth pointing out that the previous conclusion is valid in the physical motivated framework where the vortex remains regular in $\rho = 0$. This hypothesis has allowed us to set the regular expansions in Eq. (139). On the other hand, as shown for the background fields (see Sect IV and Sect. V), there is a dense set of solutions associated with an anti-de Sitter space-time at infinity which exhibit conical, or curvature singularities, in the vortex core (see Fig. 4). This suggest that an allowed evolution of the fine-tuned regular vortex might be the birth of a singularity in $\rho = 0$ (see Appendix B).

VIII. CONCLUSIONS

In the braneworld framework, one issue is to determine how to model the brane, and in particular to investigate whether its internal structure influences the properties of gravity and of the other fields living on the brane. Among other solutions, more interests has been focused on the possibility for the brane to be realized by a topological defect [7, 8, 59].

In five dimensions, it has been found that there always exists a domain wall solution that confines gravity, which moreover is symmetric with respect to both sides of the brane provided the usual relationship between the bulk and the brane cosmological constants is satisfied. This relationship translates into a fine-tuning of the underlying microphysics parameters [23]. As far as the gravitational sector is concerned, the properties of the braneworld are mostly independent of the internal structure of the brane. It is then possible to find various confinement mechanisms that lead to the existence of bosonic [24, 25, 26, 27, 28, 29, 30, 31, 32, 33] as well as fermionic [34] zero modes that can be made massive [23, 39] (although with a spectrum not yet compatible with accelerator data). In short, a five dimensional topological model of our Universe is, for the time being, an open possibility both from the cosmological and particles physics points of view.

In six dimensions, the general machinery used to study five-dimensional reflection symmetric braneworld does not apply, mainly because of the necessity to regularize the long range gravitational self-interaction [45]. A way around is to specify a complete model determining the internal structure of the brane in order to grasp some features of six-dimensional braneworld models. Indeed, one will then need to discuss the genericity of the conclusions drawn on a particular microphysics. For instance, there exists a vortexlike brane configuration on which gravity was shown to be localizable [62, 63]. We have shown in this article that such vortexlike branes are generically associated with a singularity in the core, except when a fine-tuning between the model parameters is assumed [62].

In order to address the stability issue of the fine-tuned solution, we have performed a full gauge-invariant perturbation theory around the regular six-dimensional vortex background solution. Focusing on the scalar perturbations, we showed that the hyperstring forming fields and nonvanishing metric parts cannot be perturbed at all. This result comes from the requirement that the hyperstring generates a finite volume space-time with infinite extra dimension and remains regular in the core. Any perturbation of the background fields would destroy this configuration and are not allowed. As a result, a nonempty universe, where additional observable fields would generate such perturbations, is severely constrained, if not altogether ruled out. Indeed all induced modifications of the string and gauge forming fields, as time and radial metric factors are forbidden, even at the

perturbation level. In other words, such nonempty universe should not couple to Higgs and gauge fields, and should not modify at all the gravity in the radial extra dimension. Therefore, the physical status of such a configuration seems rather unclear, rather artificial to say the least, and the possibility of having such a 6D vortex-brane realization in nature very dubious.

A priori, these conclusions are specific to the case at hand. In particular they might be argued to depend on the field content of the underlying theory. However, since the discussion involved the gravitational sector, it could be conjectured that six-dimensional braneworld with anti-de Sitter bulk and infinite extra dimension, cannot similarly be perturbed without exhibiting singularities. This would imply that nonempty multi-dimensional braneworld models could only have one (large) extra dimension of the warped form (unless some extra structure, such as a 4-brane at a fixed and finite location away from the 3-brane is added [74]).

Acknowledgments

We wish to thank Brandon Carter, Christos Charmousis, Ruth Durrer, Massimo Giovannini, Seif Randjbar-Daemi, Mikhail Shaposnikov and Jérémie Vinet for enlightening discussions. We are specially indebted to Harvey Meyer who explained in details the numerical method used in Ref. [62].

APPENDIX A: PERTURBED QUANTITIES

In this appendix, we derive all the gauge-invariant parts of the various tensors necessary for the stability analysis.

1. Metric tensor

According to Eq. (69), the scalar perturbed metric tensor reads, in terms of gauge-invariant variables,

$$\begin{aligned} \delta g_{\mu\nu} &= e^\sigma \eta_{\mu\nu} \Psi, & \delta g_{\mu r} &= 0, & \delta g_{\mu\theta} &= 0, \\ \delta g_{r\theta} &= -\Upsilon, & \delta g_{rr} &= -\Xi, & \delta g_{\theta\theta} &= -r^2 e^\gamma \Omega. \end{aligned} \quad (\text{A1})$$

By means of

$$\delta g^{AB} = -g^{AC} g^{BD} \delta g_{BD}, \quad (\text{A2})$$

one can get the inverse perturbed metric tensor

$$\begin{aligned} \delta g^{\mu\nu} &= -e^{-\sigma} \eta^{\mu\nu} \Psi, & \delta g^{\mu r} &= 0, & \delta g^{\mu\theta} &= 0, \\ \delta g^{r\theta} &= \frac{e^{-\gamma}}{r^2} \Upsilon, & \delta g^{rr} &= \Xi, & \delta g^{\theta\theta} &= \frac{e^{-\gamma}}{r^2} \Omega. \end{aligned} \quad (\text{A3})$$

The perturbed Riemann tensor can also be expressed as a function of the perturbed metric tensor through the perturbed Christoffel symbols

$$\delta R^A_{BCD} = -\delta \Gamma^A_{BC;D} + \delta \Gamma^A_{BD;C}, \quad (\text{A4})$$

where the covariant derivatives with respect to the unperturbed metric have been noted with a semicolon, and the perturbed connections are given by

$$\delta \Gamma^A_{BC} = \frac{1}{2} g^{AD} (\delta g_{DB;C} + \delta g_{DC;B} - \delta g_{BC;D}). \quad (\text{A5})$$

2. Einstein tensor

From the perturbed Riemann tensor, the perturbed Einstein tensor can be expressed in terms of gauge-invariant variables by means of

$$\delta G_{AB} = \delta R_{AB} - \frac{1}{2} R \delta g_{AB} - \frac{1}{2} g_{AB} \delta R, \quad (\text{A6})$$

where the perturbed Ricci scalar is

$$\delta R = g^{AB} \delta R_{AB} + \delta g^{AB} R_{BD}. \quad (\text{A7})$$

After some (tedious) calculations one gets

$$\begin{aligned} \delta G_{\mu\nu} &= (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) \left(\frac{\Xi + \Omega}{2} + \Psi \right) + \frac{1}{2} e^\sigma \eta_{\mu\nu} \left\{ 3\Psi'' + 3 \frac{e^{-\gamma}}{r^2} \partial_\theta^2 \Psi + \Omega'' - 2 \frac{e^{-\gamma}}{r^2} \partial_\theta \Upsilon' + \frac{e^{-\gamma}}{r^2} \partial_\theta^2 \Xi \right. \\ &\quad \left. + 3 \left(2\sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Psi' - \left(\frac{3}{2} \sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Xi' + \left[\frac{3}{2} \sigma' + 2 \left(\frac{1}{r} + \frac{\gamma'}{2} \right) \right] \Omega' - 3 \frac{e^{-\gamma}}{r^2} \sigma' \partial_\theta \Upsilon \right\} + G_{\mu\nu} (\Psi - \Xi), \end{aligned} \quad (\text{A8})$$

for the purely brane part, while the mixed ones read

$$\begin{aligned}\delta G_{\mu r} &= \frac{1}{2} \partial_\mu \left\{ 3\Psi' + \Omega' - \frac{e^{-\gamma}}{r^2} \partial_\theta \Upsilon - \left(\frac{3}{2} \sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Xi + \left(-\frac{1}{2} \sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Omega \right\}, \\ \delta G_{\mu \theta} &= \frac{1}{2} \partial_\mu \left\{ \partial_\theta (3\Psi + \Xi) - \Upsilon' - \left(\sigma' + \frac{1}{r} + \frac{\gamma'}{2} \right) \Upsilon \right\},\end{aligned}\tag{A9}$$

and the purely bulk components are

$$\begin{aligned}\delta G_{rr} &= \frac{1}{2} e^{-\sigma} \square (3\Psi + \Omega) - 2 \frac{e^{-\gamma}}{r^2} \partial_\theta^2 \Psi + 2 \frac{e^{-\gamma}}{r^2} \sigma' \partial_\theta \Upsilon - \left[3\sigma' + 2 \left(\frac{1}{r} + \frac{\gamma'}{2} \right) \right] \Psi' - \sigma' \Omega', \\ \delta G_{\theta\theta} &= \frac{1}{2} r^2 e^\gamma e^{-\sigma} \square (3\Psi + \Xi) - 2r^2 e^\gamma \Psi'' + r^2 e^\gamma \sigma' (\Xi' - 5\Psi') + \frac{1}{2} r^2 e^\gamma (4\sigma'' + 5\sigma'^2) (\Xi - \Omega), \\ \delta G_{r\theta} &= -\frac{1}{2} e^{-\sigma} \square \Upsilon + 2\partial_\theta \Psi' - \sigma' \partial_\theta \Xi + \left[\sigma' - 2 \left(\frac{1}{r} + \frac{\gamma'}{2} \right) \right] \partial_\theta \Psi - \frac{1}{2} (4\sigma'' + 5\sigma'^2) \Upsilon.\end{aligned}\tag{A10}$$

where $\square$ is the brane d'Alembertian defined above [Eq. (85)].

and to first order in metric and field perturbations, using Eq. (73), this perturbed matter Lagrangian reads

3. Stress-energy tensor

In terms of the underlying fields, the stress-energy tensor stemming from Eq. (9) reads

$$\begin{aligned}T_{AB} &= \frac{1}{4} (\mathcal{D}_A \Phi)^\dagger (\mathcal{D}_B \Phi) + \frac{1}{4} (\mathcal{D}_B \Phi)^\dagger (\mathcal{D}_A \Phi) \\ &\quad - g^{CD} H_{AC} H_{BD} - g_{AB} \mathcal{L}_{\text{mat}},\end{aligned}\tag{A11}$$

with $\mathcal{L}_{\text{mat}}$ given by Eq. (2); to zeroth order, this is given by

$$\mathcal{L}_{\text{mat}} = -\frac{1}{2} \frac{e^{-\gamma}}{r^2} \frac{Q'^2}{q^2} - \frac{1}{2} \varphi'^2 - \frac{1}{2} \frac{e^{-\gamma}}{r^2} \varphi^2 Q^2 - V(\varphi),\tag{A12}$$

$$\begin{aligned}\delta \mathcal{L}_{\text{mat}} &= \frac{e^{-\gamma}}{r^2} \frac{Q'}{q} (\Theta'_\theta - \partial_\theta \Theta_r) + \frac{1}{2} \frac{e^{-\gamma}}{r^2} \frac{Q'^2}{q^2} (\Xi + \Omega) \\ &\quad + \frac{1}{2} \varphi'^2 \Xi + \frac{1}{2} \frac{e^{-\gamma}}{r^2} \varphi^2 Q^2 \Omega + \frac{e^{-\gamma}}{r^2} q \varphi^2 Q \Theta_\theta \\ &\quad - \left[\varphi' \frac{\mathcal{X}' + \mathcal{X}'^\dagger}{2} + \frac{e^{-\gamma}}{r^2} \varphi Q^2 \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} \right. \\ &\quad \left. + \frac{e^{-\gamma}}{r^2} \varphi Q \partial_\theta \frac{\mathcal{X} - \mathcal{X}^\dagger}{2i} + \frac{dV}{d\varphi} \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} \right].\end{aligned}\tag{A13}$$

The purely brane part of the perturbed stress-energy tensor is therefore given by

$$\begin{aligned}\delta T_{\mu\nu} &= e^\sigma \eta_{\mu\nu} \left\{ -\frac{e^{-\gamma}}{r^2} \frac{Q'}{q} (\Theta'_\theta - \partial_\theta \Theta_r) + \left[\varphi' \frac{\mathcal{X}' + \mathcal{X}'^\dagger}{2} + \frac{e^{-\gamma}}{r^2} \varphi Q^2 \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} + \frac{e^{-\gamma}}{r^2} \varphi Q \partial_\theta \frac{\mathcal{X} - \mathcal{X}^\dagger}{2i} + \frac{dV}{d\varphi} \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} \right] \right. \\ &\quad - \left(\frac{1}{2} \frac{e^{-\gamma}}{r^2} \frac{Q'^2}{q^2} + \frac{1}{2} \varphi'^2 \right) \Xi - \left(\frac{1}{2} \frac{e^{-\gamma}}{r^2} \frac{Q'^2}{q^2} + \frac{1}{2} \frac{e^{-\gamma}}{r^2} \varphi^2 Q^2 \right) \Omega + \left[\frac{1}{2} \frac{e^{-\gamma}}{r^2} \frac{Q'^2}{q^2} + \frac{1}{2} \varphi'^2 + \frac{1}{2} \frac{e^{-\gamma}}{r^2} \varphi^2 Q^2 + V(\varphi) \right] \Psi \\ &\quad \left. - \frac{e^{-\gamma}}{r^2} q \varphi^2 Q \Theta_\theta \right\},\end{aligned}\tag{A14}$$

while the mixed components are

$$\begin{aligned}\delta T_{\mu r} &= \partial_\mu \left\{ -\frac{e^{-\gamma}}{r^2} \frac{Q'}{q} (\Theta_\theta - \partial_\theta \Theta) + \varphi' \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} \right\}, \\ \delta T_{\mu \theta} &= \partial_\mu \left\{ \frac{Q'}{q} (\Theta_r - \Theta') + \varphi Q \frac{\mathcal{X} - \mathcal{X}^\dagger}{2i} - q \varphi^2 Q \Theta \right\},\end{aligned}\tag{A15}$$

and the bulk ones

$$\begin{aligned}
\delta T_{rr} &= -\frac{e^{-\gamma} Q'}{r^2 q} (\Theta'_\theta - \partial_\theta \Theta_r) + \left[\varphi' \frac{\mathcal{X}' + \mathcal{X}'^\dagger}{2} - \frac{e^{-\gamma}}{r^2} \varphi Q^2 \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} - \frac{e^{-\gamma}}{r^2} \varphi Q \partial_\theta \frac{\mathcal{X} - \mathcal{X}^\dagger}{2i} - \frac{dV}{d\varphi} \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} \right] \\
&\quad - \left[\frac{1}{2} \frac{e^{-\gamma}}{r^2} \varphi^2 Q^2 + V(\varphi) \right] \Xi - \left(\frac{1}{2} \frac{e^{-\gamma}}{r^2} \frac{Q'^2}{q^2} - \frac{1}{2} \frac{e^{-\gamma}}{r^2} \varphi^2 Q^2 \right) \Omega + \frac{e^{-\gamma}}{r^2} q \varphi^2 Q \Theta_\theta, \\
\delta T_{\theta\theta} &= -\frac{Q'}{q} (\Theta'_\theta - \partial_\theta \Theta_r) + \left[-r^2 e^\gamma \varphi' \frac{\mathcal{X}' + \mathcal{X}'^\dagger}{2} + \varphi Q^2 \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} + \varphi Q \partial_\theta \frac{\mathcal{X} - \mathcal{X}^\dagger}{2i} - r^2 e^\gamma \frac{dV}{d\varphi} \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} \right] \\
&\quad + \left(\frac{1}{2} r^2 e^\gamma \varphi'^2 - \frac{1}{2} \frac{Q'^2}{q^2} \right) \Xi - r^2 e^\gamma \left[\frac{1}{2} \varphi'^2 + V(\varphi) \right] \Omega - q \varphi^2 Q \Theta_\theta, \\
\delta T_{r\theta} &= \left[\varphi Q \frac{\mathcal{X}' - \mathcal{X}'^\dagger}{2i} - \varphi' Q \frac{\mathcal{X} - \mathcal{X}^\dagger}{2i} + \varphi' \partial_\theta \frac{\mathcal{X} + \mathcal{X}^\dagger}{2} \right] + \left[\frac{1}{2} \frac{e^{-\gamma}}{r^2} \frac{Q'^2}{q^2} - \frac{1}{2} \varphi'^2 - \frac{1}{2} \frac{e^{-\gamma}}{r^2} \varphi^2 Q^2 - V(\varphi) \right] \Upsilon - q \varphi^2 Q \Theta_r.
\end{aligned} \tag{A16}$$

4. Faraday tensor

In order to directly derive the perturbed Maxwell equations from Eq. (19), we have used the following perturbed Faraday tensor whose purely brane components vanish

$$\delta H_{\mu\nu} = 0 = \delta H^{\mu\nu}, \tag{A17}$$

and with the mixed parts

$$\begin{aligned}
\delta H_{\mu r} &= \partial_\mu (\Theta_r - \Theta'), \\
\delta H^{\mu r} &= -e^{-\sigma} \eta^{\mu\nu} \partial_\nu (\Theta_r - \Theta'), \\
\delta H_{\mu\theta} &= \partial_\mu (\Theta_\theta - \partial_\theta \Theta), \\
\delta H^{\mu\theta} &= -\frac{e^{-(\sigma+\gamma)}}{r^2} \eta^{\mu\nu} \partial_\nu (\Theta_\theta - \partial_\theta \Theta).
\end{aligned} \tag{A18}$$

The only nonvanishing purely bulk components end up being

$$\begin{aligned}
\delta H_{r\theta} &= \Theta'_\theta - \partial_\theta \Theta_r, \\
\delta H^{r\theta} &= \frac{e^{-\gamma}}{r^2} \left[\frac{Q'}{q} (\Xi + \Omega) + \Theta'_\theta - \partial_\theta \Theta_r \right].
\end{aligned} \tag{A19}$$

Owing to these formulas, one can calculate the perturbed Faraday tensor divergence involved in Eq. (19) by means of

$$\delta H^{AB}{}_{;A} = \partial_A \delta H^{AB} + \Gamma^A{}_{DA} \delta H^{DB}, \tag{A20}$$

where we have used the antisymmetry property of δH_{AB} . The perturbed left-hand side of Eq. (19) can be, in turn, expressed in terms of the perturbed matter fields by means of Eqs. (4) and (73) to give the perturbed Maxwell equations (87) to (89).

APPENDIX B: THE 6D CYLINDRICAL SOLUTION

In this appendix, we closely follow the analysis of Ref. [73], generalized to the six-dimensional case, and show that the only gravity confining solution to Einstein

equations in the presence of a negative cosmological constant is the one used throughout this article.

We start with the most general static, cylindrically symmetric line element for a hyperstring inside which one assumes also rotation invariance, namely

$$ds^2 = g_3(r) dt^2 - dr^2 - g_1(r) (dx^2 + dy^2 + dz^2) - g_2(r) d\theta^2. \tag{B1}$$

Note that if one also demands Lorentz invariance along the vortex world sheet, then one would restrict attention to the subset of solutions for which $g_1(r) = g_3(r)$, if it exists.

Setting $u^2 \equiv g_1^3 g_2 g_3 = -\det(g_{AB})$, Einstein equations take the form

$$\left(\frac{u}{g_i} g'_i \right)' + \Lambda u = 0, \tag{B2}$$

where a prime indicates a derivative with respect to r , and

$$3 \frac{g'_1 g'_2}{g_1 g_2} + 3 \frac{g'_1 g'_3}{g_1 g_3} + \frac{g'_2 g'_3}{g_2 g_3} + 3 \left(\frac{g'_1}{g_1} \right)^2 + 4\Lambda = 0, \tag{B3}$$

which can be combined to yield the simple equation for the metric determinant, namely

$$u'' + \frac{5}{2} \Lambda u = 0. \tag{B4}$$

With $\Lambda < 0$, the general solution of Eq. (B4) is

$$u = u_+ e^{r/\tau_\Lambda} + u_- e^{-r/\tau_\Lambda} \tag{B5}$$

with $r_\Lambda^{-2} = -5\Lambda/2$. We shall come back shortly to this solution.

Eq. (B4) can be integrated, with the result that

$$u'^2 = \frac{u^2}{r_\Lambda^2} + K^2, \tag{B6}$$

where K is an arbitrary real constant, *i.e.* $K^2 > 0$. This latter requirement stems from the fact that, in order to

have actual cylindrical coordinates with $0 \leq \theta \leq 2\pi$ and to avoid a singularity at the symmetry point $r = 0$, one must in general impose that $\lim_{r \rightarrow 0} u(r) = 0$.

Expanding and integrating Eqs. (B2) lead to the solution

$$\frac{g'_i}{g_i} = \frac{KK_i}{u} + \frac{2u'}{5u}, \quad (\text{B7})$$

where the otherwise arbitrary constants K_i are related through

$$3K_1 + K_2 + K_3 = 0,$$

coming from the definition of u in terms of g_i , and

$$2K_1(K_2 + K_3) + K_2K_3 = -\frac{8}{5},$$

which is nothing but a rewriting of Eq. (B3).

Now, with $u(0) = 0$, the solution (B5) or a direct integration of (B6) gives

$$u = Kr_\Lambda \sinh\left(\frac{r}{r_\Lambda}\right), \quad (\text{B8})$$

and therefore the metric functions read

$$g_i(r) = g_i^{(0)} \left[\sinh\left(\frac{r}{r_\Lambda}\right) \right]^{\frac{2}{5}} \left[\tanh\left(\frac{r}{2r_\Lambda}\right) \right]^{K_i}, \quad (\text{B9})$$

where the $g_i^{(0)}$ are three constants of integrations satisfying

$$\left[g_1^{(0)} \right]^3 g_2^{(0)} g_3^{(0)} = \frac{2K^2}{5\Lambda}$$

and $[u'(0)]^2 = K^2$. Eq. (B9) is but the 6 dimensional generalization for nonvanishing cosmological constant of the Kasner metric [75, 76] already obtained in Ref. [73].

For the purpose of confining gravity, one needs a space with finite transverse volume $\mathcal{V}_{(2)}^\perp \equiv \int d\theta dr \sqrt{-g} = 2\pi \int u(r) dr$, i.e. one demands that $u(r)$ goes asymptotically to zero faster than $1/r$. The solution (B8) leads however to $\mathcal{V}_{(2)}^\perp = 2\pi K \cosh(r/r_\Lambda)|_0^\infty$, which is unbounded unless $K \rightarrow 0$, in which case it is undefined. The general solution is thus useless as one needs to impose $K = 0$. However, with $u_+ = 0$ in Eq. (B5), the transverse volume remains finite, so gravity ends up being actually confined on the hyperstring core. In this case, since $u(0) \neq 0$, the solution is singular at $r = 0$, and is thus incomplete, being unable to describe the vortex location itself. The warped metric

$$ds^2 = e^{-r/r_\Lambda} (\eta_{\mu\nu} dx^\mu dx^\nu - d\theta^2) - dr^2 \quad (\text{B10})$$

is thus only asymptotically valid. Note that this is in fact not problematic in the framework of the Abelian Higgs vortex since the point at $r = 0$ cannot be described by the vacuum Einstein equations.

An interesting point concerning the finite-volume metric (B1) is that, seen from far away from the vortex, the warp factor can be interpreted as a missing angle, just like in the simpler case of a Nambu-Goto string in Minkowski space. However, in the case at hand, the requirements of both cylindrical and maximal symmetries now demands a missing angle of 2π , as can be seen by evaluating the diameter of a circle $r = \text{const.}$, at a distance $r \rightarrow \infty$ and noting that this diameter vanishes with the warp factor.

One can also note that changing the radial coordinates into

$$r \rightarrow \bar{r} = \bar{r}_\Lambda \exp\left(-\frac{r}{2r_\Lambda}\right), \quad \theta \rightarrow \bar{\theta} = \frac{\theta}{\bar{r}_\Lambda},$$

with $\bar{r}_\Lambda = 2r_\Lambda = \sqrt{10/|\Lambda|}$, transforms the metric into

$$ds^2 = -\left(\frac{\bar{r}}{\bar{r}_\Lambda}\right)^2 (\eta_{\mu\nu} dx^\mu dx^\nu) - \bar{r}^2 d\bar{\theta}^2 - \left(\frac{\bar{r}_\Lambda}{\bar{r}}\right)^2 d\bar{r}^2. \quad (\text{B11})$$

With these new coordinates, the hyperstring is located at $\bar{r} = \bar{r}_\Lambda$, and $\bar{\theta}$ is a compact coordinate, varying between 0 and $2\pi/\bar{r}_\Lambda$.

Finally, it is worth pointing out that when a time dependence is allowed for in either (B10) or (B11), and metric variables are assumed separable, with $g_i(r) \rightarrow g_i(r)\Pi_i(t)$, then one still has a valid solution provided $\Pi_1 = \Pi_3 = C(t - t_0)^2 = \Pi_2^{-1}$, with C and t_0 arbitrary constants of integrations. This solution, which is also of constant scalar curvature $R = 3\Lambda$, corresponds, to a shrinking extra dimension (a circle located at a given coordinate distance from the vortex gets smaller with time) and an expanding brane interior. As discussed above, in the framework of the present article, this solution is only asymptotically valid. Nevertheless, according to the background field solutions (see Sect. IV), this shrinking might be associated with a growing conical singularity in the vortex core, i.e. with a time-dependent v_0 . It is a peculiarity of this number of dimensions that the scale factor in the brane in this model evolves as $\propto t^2$, which, recalling the time coordinate to be conformal time, is the expected evolution of a matter-dominated universe.

- [2] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. **B429**, 263 (1998), hep-ph/9803315.
- [3] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Rev. **D59**, 086004 (1999), hep-ph/9807344.
- [4] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. **B436**, 257 (1998), hep-ph/9804398.
- [5] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), hep-ph/9905221.
- [6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999), hep-th/9906064.
- [7] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **B125**, 136 (1983).
- [8] M. Visser, Phys. Lett. **B159**, 22 (1985), hep-th/9910093.
- [9] J. C. Long et al., Nature **421**, 922 (2003), hep-ph/0210004.
- [10] J.-P. Uzan and F. Bernardeau, Phys. Rev. **D64**, 083004 (2001), hep-ph/0012011.
- [11] S. W. Allen, S. Eddi, and A. C. Fabian, Mon. Not. Roy. Astron. Soc. **324**, 877 (2001), astro-ph/0008517.
- [12] P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000), hep-th/9905012.
- [13] C. Csaki, M. Graesser, C. F. Kolda, and J. Terning, Phys. Lett. **B462**, 34 (1999), hep-ph/9906513.
- [14] J. M. Cline, C. Grojean, and G. Servant, Phys. Rev. Lett. **83**, 4245 (1999), hep-ph/9906523.
- [15] P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. **B477**, 285 (2000), hep-th/9910219.
- [16] P. Kraus, JHEP **12**, 011 (1999), hep-th/9910149.
- [17] T. Shiromizu, K.-i. Maeda, and M. Sasaki, Phys. Rev. **D62**, 024012 (2000), gr-qc/9910076.
- [18] E. E. Flanagan, S. H. H. Tye, and I. Wasserman, Phys. Rev. **D62**, 044039 (2000), hep-ph/9910498.
- [19] R. Maartens, D. Wands, B. A. Bassett, and I. Heard, Phys. Rev. **D62**, 041301 (2000), hep-ph/9912464.
- [20] V. A. Rubakov, Phys. Usp. **44**, 871 (2001), hep-ph/0104152.
- [21] A. Riazuelo, F. Vernizzi, D. Steer, and R. Durrer (2002), hep-th/0205220.
- [22] C. Ringeval, T. Boehm, and R. Durrer (2003), hep-th/0307100.
- [23] C. Ringeval, P. Peter, and J.-P. Uzan, Phys. Rev. **D65**, 044016 (2002), hep-th/0109194.
- [24] B. Bajc and G. Gabadadze, Phys. Lett. **B474**, 282 (2000), hep-th/9912232.
- [25] G. R. Dvali and M. A. Shifman, Nucl. Phys. **B504**, 127 (1997), hep-th/9611213.
- [26] G. R. Dvali and M. A. Shifman, Phys. Lett. **B396**, 64 (1997), hep-th/9612128.
- [27] S. L. Dubovsky, V. A. Rubakov, and P. G. Tinyakov, Phys. Rev. **D62**, 105011 (2000), hep-th/0006046.
- [28] G. R. Dvali, G. Gabadadze, and M. A. Shifman, Phys. Lett. **B497**, 271 (2001), hep-th/0010071.
- [29] P. Dimopoulos, K. Farakos, A. Kehagias, and G. Koutsoumbas, Nucl. Phys. **B617**, 237 (2001), hep-th/0007079.
- [30] M. J. Duff, J. T. Liu, and W. A. Sabra, Nucl. Phys. **B605**, 234 (2001), hep-th/0009212.
- [31] I. Oda (2001), hep-th/0103052.
- [32] K. Ghoroku and A. Nakamura, Phys. Rev. **D65**, 084017 (2002), hep-th/0106145.
- [33] E. K. Akhmedov, Phys. Lett. **B521**, 79 (2001), hep-th/0107223.
- [34] A. Neronov, Phys. Rev. **D65**, 044004 (2002), gr-qc/0106092.
- [35] R. Jackiw and C. Rebbi, Phys. Rev. **D13**, 3398 (1976).
- [36] C. Ringeval, Phys. Rev. **D64**, 123505 (2001), hep-ph/0106179.
- [37] C. Ringeval, *Fermionic currents flowing along extended objects* (Phd Thesis, University Paris 6, 2002), hep-ph/0211126.
- [38] S. Eidelman, K. Hayes, K. Olive, M. Aguilar-Benitez, C. Amsler, D. Asner, K. Babu, R. Barnett, J. Beringer, P. Burchat, et al., Physics Letters B **592**, 1+ (2004), URL <http://pdg.lbl.gov>.
- [39] G. R. Dvali, S. Randjbar-Daemi, and R. Tabbash, Phys. Rev. **D65**, 064021 (2002), hep-ph/0102307.
- [40] B. Carter and J.-P. Uzan, Nucl. Phys. **B606**, 45 (2001), gr-qc/0101010.
- [41] R. A. Battye, B. Carter, A. Mennim, and J.-P. Uzan, Phys. Rev. **D64**, 124007 (2001), hep-th/0105091.
- [42] B. Carter, J.-P. Uzan, R. A. Battye, and A. Mennim, Class. Quant. Grav. **18**, 4871 (2001), gr-qc/0106038.
- [43] P. Horava and E. Witten, Nucl. Phys. **B460**, 506 (1996), hep-th/9510209.
- [44] P. Tinyakov and K. Zuleta, Phys. Rev. **D64**, 025022 (2001), hep-th/0103062.
- [45] B. Carter, R. A. Battye, and J.-P. Uzan, Commun. Math. Phys. **235**, 289 (2003), hep-th/0204042.
- [46] C. P. Burgess, J. M. Cline, N. R. Constable, and H. Firouzjahi, JHEP **01**, 014 (2002), hep-th/0112047.
- [47] I. I. Kogan, S. Mouslopoulos, A. Papazoglou, and G. G. Ross, Phys. Rev. **D64**, 124014 (2001), hep-th/0107086.
- [48] M. Gogberashvili and P. Midodashvili, Phys. Lett. **B515**, 447 (2001), hep-ph/0005298.
- [49] M. Gogberashvili, Phys. Lett. **B484**, 124 (2000), hep-ph/0001109.
- [50] M. Gogberashvili and P. Midodashvili, Europhys. Lett. **61**, 308 (2003), hep-th/0111132.
- [51] A. G. Cohen and D. B. Kaplan, Phys. Lett. **B470**, 52 (1999), hep-th/9910132.
- [52] P. Kanti, R. Madden, and K. A. Olive, Phys. Rev. **D64**, 044021 (2001), hep-th/0104177.
- [53] R. Gregory, Phys. Rev. Lett. **84**, 2564 (2000), hep-th/9911015.
- [54] R. Gregory and C. Santos, Phys. Rev. **D56**, 1194 (1997), gr-qc/9701014.
- [55] T. Nihei, Phys. Rev. **D62**, 124017 (2000), hep-th/0005014.
- [56] T. Gherghetta and M. E. Shaposhnikov, Phys. Rev. Lett. **85**, 240 (2000), hep-th/0004014.
- [57] T. Gherghetta, E. Roessl, and M. E. Shaposhnikov, Phys. Lett. **B491**, 353 (2000), hep-th/0006251.
- [58] E. Roessl and M. Shaposhnikov, Phys. Rev. **D66**, 084008 (2002), hep-th/0205320.
- [59] N. D. Antunes, E. J. Copeland, M. Hindmarsh, and A. Lukas, Phys. Rev. **D68**, 066005 (2003), hep-th/0208219.
- [60] M. Giovannini, Phys. Rev. **D66**, 044016 (2002), hep-th/0205139.
- [61] M. Giovannini, Phys. Rev. **D65**, 124019 (2002), hep-th/0204235.
- [62] M. Giovannini, H. Meyer, and M. E. Shaposhnikov, Nucl. Phys. **B619**, 615 (2001), hep-th/0104118.
- [63] M. Giovannini, J.-V. Le Be, and S. Riederer, Class. Quant. Grav. **19**, 3357 (2002), hep-th/0205222.
- [64] N. K. Nielsen and P. Olesen, Nucl. Phys. **B291**, 829 (1987).
- [65] P. Peter, Phys. Rev. **D45**, 1091 (1992).

- [66] Y. Verbin, Phys. Rev. **D59**, 105015 (1999), hep-th/9809002.
- [67] S. L. Adler and T. Piran, Rev. Mod. Phys. **56**, 1 (1984).
- [68] C. Ringeval and J.-W. Rombouts, Phys. Rev. **D71**, 044001 (2005), hep-th/0411282.
- [69] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992).
- [70] S. Randjbar-Daemi and M. Shaposhnikov, Nucl. Phys. **B645**, 188 (2002), hep-th/0206016.
- [71] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York and London, 1965).
- [72] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rept. **251**, 267 (1995), hep-th/9405029.
- [73] B. Linet, J. Math. Phys. **27**, 1817 (1986).
- [74] J. M. Cline, J. Descheneau, M. Giovannini, and J. Vinet, JHEP **06**, 048 (2003), hep-th/0304147.
- [75] E. Kasner, Tans. Am. Math. Soc. **27**, 101 (1925).
- [76] P. Peter, Class. Quant. Grav. **11**, 131 (1994).