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GENERALIZED INCREASING CONVEX AND DIRECTIONALLY CONVEX ORDERS

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Abstract

In this paper, the componentwise increasing convex order, the upper orthant order, the upper orthant convex order and the increasing directionally convex order for random vectors are generalized to hierarchical classes of integral stochastic order relations. The elements of the generating classes of functions possess non-negative partial derivatives up to some given degrees. Some properties of these new stochastic order relations are studied. A particular attention is paid to the comparison of weighted sums of the respective components of ordered random vectors. By providing a unified derivation of standard multivariate stochastic orderings, the present paper shows how some well-known results derive from a common principle.

Key words and phrases: integral stochastic orders; supermodular functions; directionally convex functions.

Classification: 60E15 (Inequalities; stochastic orderings)

1 Introduction and motivation

Stochastic orderings among random vectors have a wide field of applications in probability and statistics; see, e.g., Shaked & Shanthikumar (1994, 2007) and Denuit et al. (2005). Typically, these orders help to identify simpler, computationally more tractable, stochastic models providing bounds on various quantities of interest.

In this paper, we consider non-negative random variables. Since all the order relations considered in this paper are invariant under shifts, all the results remain valid as long as the support of the random variables is bounded from below.

A number of stochastic orderings can be defined by reference to some class of measurable functions. Specifically, let X and Y be two random variables. Then, X is said to be smaller than Y in the $\preceq_*$ ordering associated with the class $\mathcal{U}_*^{[1]}$ of real-valued functions defined on (a subset of) the real line $\mathbb{R}$ if

$$\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)] \text{ for all } g \in \mathcal{U}_*^{[1]}, \tag{1.1}$$

provided that the expectations exist. Whitt (1986) gave the name of integral stochastic orderings to the stochastic order relations defined by means of (1.1). The extension to n-dimensional random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ is then clear: $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the $\preceq_*$ ordering associated with the class $\mathcal{U}_*^{[n]}$ of real-valued functions defined on (a subset of) the n-dimensional real space $\mathbb{R}^n$ if

$$\mathbb{E}[g(\boldsymbol{X})] \leq \mathbb{E}[g(\boldsymbol{Y})] \text{ for all } g \in \mathcal{U}_*^{[n]}, \tag{1.2}$$

provided that the expectations exist.

Among standard integral stochastic orderings used for comparing pairs of random variables, the stochastic dominance (denoted by "$\preceq_{\text{st}}$") is obtained from (1.1) when $\mathcal{U}_*^{[1]}$ is the class of all the non-decreasing functions and the increasing convex order (denoted by "$\preceq_{\text{icx}}$") when $\mathcal{U}_*^{[1]}$ is the class of all the non-decreasing and convex functions. Corresponding orders for random vectors are obtained from (1.2).

In addition to these standard order relations, we also consider the univariate s-increasing convex orders, $s = 1, 2, \ldots$, introduced by Denuit et al. (1998) and defined as follows: given two random variables X and Y, X is said to be smaller than Y in the s-increasing convex sense, denoted as $X \preceq_{s-\text{icx}} Y$, if (1.1) holds with for $\mathcal{U}_*^{[1]}$ the class $\mathcal{U}_{s-\text{icx}}^{[1]}$ of the regular s-increasing convex functions g, i.e. those functions g such that $\frac{d^k}{dx^k} g \geq 0$ for $k = 1, 2, \ldots, s$. Clearly, $\preceq_{1-\text{icx}} \Leftrightarrow \preceq_{\text{st}}$ and $\preceq_{2-\text{icx}} \Leftrightarrow \preceq_{\text{icx}}$.

Let us now present some order relations for random vectors. To this end, let us denote as $F_{\boldsymbol{X}}$ the joint distribution function of a random vector $\boldsymbol{X}$ of dimension n with components $X_1, X_2, \ldots, X_n$ and as $\overline{F}_{\boldsymbol{X}}$ its multivariate survival function, that is,

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \Pr[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n]$$

and

$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) = \Pr[X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n].$$

Let $\boldsymbol{Y}$ be another n-dimensional random vector with distribution function $F_{\boldsymbol{Y}}$ and survival function $\overline{F}_{\boldsymbol{Y}}$. If the inequality

$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) \leq \overline{F}_{\boldsymbol{Y}}(\boldsymbol{x}) \text{ is valid for all } \boldsymbol{x} \tag{1.3}$$

then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the upper orthant order (denoted by $\boldsymbol{X} \preceq_{\text{uo}} \boldsymbol{Y}$). The reason for this terminology is that sets of the form $\{\boldsymbol{x} \in \mathbb{R}^n | x_1 > a_1, x_2 > a_2, \ldots, x_n > a_n\}$, for some fixed $\boldsymbol{a}$, are called upper orthants. Moreover, the relation $\preceq_{\text{uo}}$ can be defined by means of (1.2) with for $\mathcal{U}_*^{[n]}$ the class $\underline{\mathcal{U}}_{\text{uo}}$ of all the indicator functions for upper orthants. A general account of the properties of $\preceq_{\text{uo}}$ can be found in Shaked & Shanthikumar (2007, Section 6.G.1). Denoting as $\mathcal{U}_{\text{uo}}^{\text{prod}}$ the class of all the products $\prod_{i=1}^n g_i$ where each $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, it is well-known that $\boldsymbol{X} \preceq_{\text{uo}} \boldsymbol{Y}$ if, and only if, (1.2) holds with $\mathcal{U}_*^{[n]} = \mathcal{U}_{\text{uo}}^{\text{prod}}$.

If $\boldsymbol{X} \preceq_{\text{uo}} \boldsymbol{Y}$ holds then (1.2) in fact applies to a much larger class of functions than $\underline{\mathcal{U}}_{\text{uo}}$ or $\mathcal{U}_{\text{uo}}^{\text{prod}}$. Specifically, define the forward difference operator Δ_{i,h_i} as

$$\Delta_{i,h_i} g(\boldsymbol{x}) = g(x_1, \ldots, x_{i-1}, x_i + h_i, x_{i+1}, \ldots, x_n) - g(x_1, \ldots, x_n)$$

with $\Delta_{i,h_i}^0 g(\boldsymbol{x}) = g(\boldsymbol{x})$. Now, define $\overline{\mathcal{U}}_{\text{uo}}$ as the class of all the functions g such that

$$\Delta_{1,h_1}^{k_1} \circ \cdots \circ \Delta_{n,h_n}^{k_n} g(\boldsymbol{x}) \geq 0$$

for all $\boldsymbol{x}$ and for every $k_1, \ldots, k_n \in \{0, 1\}$ such that $k_1 + \cdots + k_n \geq 1$, and for every $h_1, \ldots, h_n \geq 0$. Note that all the indicator functions of upper orthants belong to $\overline{\mathcal{U}}_{\text{uo}}$, that is, $\underline{\mathcal{U}}_{\text{uo}} \subset \overline{\mathcal{U}}_{\text{uo}}$. Indeed, we get with $g(\boldsymbol{x}) = \mathbb{I}[x_1 > a_1, \ldots, x_n > a_n]$ for some fixed $\boldsymbol{a}$ that

$$\Delta_{1,h_1}^{k_1} \circ \cdots \circ \Delta_{n,h_n}^{k_n} g(\boldsymbol{x}) = \prod_{i|k_i=0} \mathbb{I}[x_i > a_i] \times \prod_{i|k_i=1} \Big(\mathbb{I}[x_i > a_i - h_i] - \mathbb{I}[x_i > a_i] \Big) \geq 0.$$

Now, it can be shown that every $g \in \overline{\mathcal{U}}_{\text{uo}}$ is the uniform limit of a sequence $g_1, g_2, \ldots$ of functions expressible as non-negative linear combinations of indicator functions of upper orthants. It follows that $\boldsymbol{X} \preceq_{\text{uo}} \boldsymbol{Y}$ if, and only if, (1.2) holds with $\mathcal{U}_*^{[n]} = \overline{\mathcal{U}}_{\text{uo}}$.

Let us denote as $\mathcal{U}_{\text{uo}}$ the class of all the differentiable functions g such that $\frac{\partial^{k_1+\ldots+k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} g \geq 0$ for every integer $k_1, \ldots, k_n \in \{0, 1\}$ such that $k_1 + \ldots + k_n \geq 1$. It can be shown that $\mathcal{U}_{\text{uo}} \subset \overline{\mathcal{U}}_{\text{uo}}$. Since $\mathcal{U}_{\text{uo}}$ is dense in $\overline{\mathcal{U}}_{\text{uo}}$, then $\boldsymbol{X} \preceq_{\text{uo}} \boldsymbol{Y}$ if, and only if, (1.2) holds with $\mathcal{U}_*^{[n]} = \mathcal{U}_{\text{uo}}$.

To sum up, we thus have the following equivalent characterizations of $\preceq_{\text{uo}}$:

$$\begin{aligned} \boldsymbol{X} \preceq_{\text{uo}} \boldsymbol{Y} \quad &\Leftrightarrow \quad \text{(1.2) holds with } \mathcal{U}_*^{[n]} = \underline{\mathcal{U}}_{\text{uo}} \\ &\Leftrightarrow \quad \text{(1.2) holds with } \mathcal{U}_*^{[n]} = \mathcal{U}_{\text{uo}}^{\text{prod}} \\ &\Leftrightarrow \quad \text{(1.2) holds with } \mathcal{U}_*^{[n]} = \mathcal{U}_{\text{uo}} \\ &\Leftrightarrow \quad \text{(1.2) holds with } \mathcal{U}_*^{[n]} = \overline{\mathcal{U}}_{\text{uo}}. \end{aligned}$$

This chain of equivalences allows us to identify rather "small" generating classes of functions for $\preceq_{\text{uo}}$, the largest one as well as some intermediate cases. Small ones, such as $\underline{\mathcal{U}}_{\text{uo}}$ offer convenient ways to establish that a $\preceq_{\text{uo}}$ ranking indeed holds in some specific cases. Intermediate ones allow us to take advantage of particular features of the test functions, such as differentiability for $\mathcal{U}_{\text{uo}}$ and product form for $\mathcal{U}_{\text{uo}}^{\text{prod}}$. Finally, the maximal generator $\overline{\mathcal{U}}_{\text{uo}}$ represents the largest class of functions such that the implication holds in (1.2). It is particularly useful in applications, once a ranking in the $\preceq_{\text{uo}}$-sense has been shown to hold.

When two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are ordered with respect to some multivariate stochastic order relation, it is often interesting to know what kind of univariate order holds between the sums of the respective components of $\boldsymbol{X}$ and $\boldsymbol{Y}$, or between more general functions of these components. In that respect, Boutsikas & Vaggelatou (2002, Section 3.2) studied which stochastic order relation holds between the sums of the components of two random vectors ordered in the $\preceq_{\text{uo}}$-sense. They established that the following implication holds true:

$$\boldsymbol{X} \preceq_{\text{uo}} \boldsymbol{Y} \Rightarrow \sum_{i=1}^{n} X_i \preceq_{n-\text{icx}} \sum_{i=1}^{n} Y_i. \tag{1.4}$$

Formula (1.4) indicates that an $\preceq_{\text{uo}}$ ordering between random vectors translates into an $\preceq_{n-\text{icx}}$ ordering between the sums of their respective components. The validity of (1.4) can easily be established as follows. For any function g in $\mathcal{U}_{n-\text{icx}}^{[1]}$, the function $\boldsymbol{x} \mapsto g(x_1 + x_2 + \ldots + x_n)$ clearly belongs to $\mathcal{U}_{\text{uo}}$ whence the announced implication (1.4) follows.

Now, let us consider another multivariate stochastic order relation, close to $\preceq_{\text{uo}}$ and defined as follows. If the inequality

$$\int_{x_1}^{\infty} \int_{x_2}^{\infty} \ldots \int_{x_n}^{\infty} \overline{F}_{\boldsymbol{X}}(u_1, u_2, \ldots, u_n)\, du_1 du_2 \cdots du_n$$
$$\leq \int_{x_1}^{\infty} \int_{x_2}^{\infty} \ldots \int_{x_n}^{\infty} \overline{F}_{\boldsymbol{Y}}(u_1, u_2, \ldots, u_n)\, du_1 du_2 \cdots du_n \quad \text{holds true for all } \boldsymbol{x}$$

then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the upper orthant-convex order (denoted by $\boldsymbol{X} \preceq_{\text{uocx}} \boldsymbol{Y}$). See Shaked & Shanthikumar (2007, Section 7.A.9). Denote as $\mathcal{U}_{\text{uocx}}^{\text{prod}}$ the class of all the products $\prod_{i=1}^{n} g_i$ where each $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing and convex. It is well-known that $\boldsymbol{X} \preceq_{\text{uocx}} \boldsymbol{Y}$ if, and only if, (1.2) holds with $\mathcal{U}_* = \mathcal{U}_{\text{uocx}}^{\text{prod}}$. Clearly, $\boldsymbol{X} \preceq_{\text{uo}} \boldsymbol{Y} \Rightarrow \boldsymbol{X} \preceq_{\text{uocx}} \boldsymbol{Y}$.

Moreover, the two results recalled above for $\preceq_{\text{uo}}$ extend to $\preceq_{\text{uocx}}$. Specifically, denoting as $\mathcal{U}_{\text{uocx}}$ the class of all the differentiable functions g such that $\frac{\partial^{k_1+\ldots+k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} g \geq 0$ for every $k_1, \ldots, k_n \in \{0, 1, 2\}$ such that $k_1 + \ldots + k_n \geq 1$. Then, $\boldsymbol{X} \preceq_{\text{uocx}} \boldsymbol{Y}$ if, and only if, (1.2) holds with $\mathcal{U}_* = \mathcal{U}_{\text{uocx}}$. Coming back to the implication (1.4), it can be shown that the analogue for $\preceq_{\text{uocx}}$ becomes

$$\boldsymbol{X} \preceq_{\text{uocx}} \boldsymbol{Y} \Rightarrow \sum_{i=1}^{n} X_i \preceq_{2n-\text{icx}} \sum_{i=1}^{n} Y_i. \tag{1.5}$$

Compared to (1.4), an $\preceq_{\text{uocx}}$ ordering between random vectors translates into an $\preceq_{2n-\text{icx}}$ ordering between the sums of their respective components.

These results suggest that $\preceq_{\text{uo}}$ and $\preceq_{\text{uocx}}$ are two members of the same hierarchical family of stochastic order relations. We will show in Section 3 that this is indeed the case, and we will study some of the properties of this family. As $\preceq_{\text{uo}}$ can be characterized through (1.2) with for $\mathcal{U}_*^{[n]}$ the class $\mathcal{U}_{\text{uo}}^{\text{prod}}$ of products of elements in $\mathcal{U}_{1-\text{icx}}^{[1]}$ and as $\preceq_{\text{uocx}}$ can be characterized through (1.2) with for $\mathcal{U}_*^{[n]}$ the class $\mathcal{U}_{\text{uocx}}^{\text{pod}}$ of products of elements in $\mathcal{U}_{2-\text{icx}}^{[1]}$, it is natural to investigate the order relation resulting from products of elements in $\mathcal{U}_{s_i-\text{icx}}^{[1]}$ for some positive integers s_i. This is done in Section 3 and it turns out that this hierarchical class of stochastic order relations inherits many of the properties of $\preceq_{\text{uo}}$ and $\preceq_{\text{uocx}}$.

Of course, there are many other multivariate stochastic order relations for comparing pairs of random vectors. Suppose for instance that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are such that (1.2) holds with for $\mathcal{U}_*^{[n]}$ the class $\mathcal{U}_{\text{iccx}}$ of all the non-decreasing functions g that are convex in each argument when the other arguments are held fixed, provided the expectations exist. Then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the increasing componentwise convex order (denoted by $\boldsymbol{X} \preceq_{\text{iccx}} \boldsymbol{Y}$). See Section 7.A.7 in Shaked & Shanthikumar (2007). The order $\preceq_{\text{iccx}}$ can typically be used to compare random vectors with independent components. Specifically, let $X_1, X_2, \ldots, X_n$ be a set of independent random variables and let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables. If $X_i \preceq_{\text{icx}} Y_i$ for $i = 1, 2, \ldots, n$, then $\boldsymbol{X} \preceq_{\text{iccx}} \boldsymbol{Y}$. The order resulting from the componentwise s-increasing convex functions is investigated in Section 2.

Another closely related stochastic order relation is the increasing directionally convex one. Recall that a function g is said to be supermodular if the inequality $g(\boldsymbol{x}) + g(\boldsymbol{y}) \leq g(\boldsymbol{x} \wedge \boldsymbol{y}) + g(\boldsymbol{x} \vee \boldsymbol{y})$ is valid for every $\boldsymbol{x}$ and $\boldsymbol{y}$, where the operators $\wedge$ and $\vee$ denote coordinatewise minimum and maximum, respectively. If g has second partial derivatives then it is supermodular if, and only if, $\frac{\partial^2}{\partial x_i \partial x_j} g \geq 0$ for all $i \neq j$. Many examples of supermodular functions can be found in Chapter 6 of Marshall & Olkin (1979). Now, the function g is said to be directionally convex if it is supermodular and coordinatewise convex. If g is twice differentiable then it is directionally convex if, and only if, all its second derivatives are non-negative.

If (1.2) holds with for $\mathcal{U}_*^{[n]}$ the class $\mathcal{U}_{\text{idircx}}$ of all the non-decreasing directionally convex functions g then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the increasing directionally convex order (denoted by $\boldsymbol{X} \preceq_{\text{idircx}} \boldsymbol{Y}$). See Section 7.A.8 in Shaked & Shanthikumar (2007). Clearly, the chain of inclusions $\mathcal{U}_{\text{uocx}} \subset \mathcal{U}_{\text{idircx}} \subset \mathcal{U}_{\text{iccx}}$ guarantees the chain of implications

$$\boldsymbol{X} \preceq_{\text{iccx}} \boldsymbol{Y} \Rightarrow \boldsymbol{X} \preceq_{\text{idircx}} \boldsymbol{Y} \Rightarrow \boldsymbol{X} \preceq_{\text{uocx}} \boldsymbol{Y}.$$

As we did above for $\preceq_{\text{uo}}$ and $\preceq_{\text{uocx}}$, we could investigate the order relation holding between the sum of the components of random vectors ordered in the $\preceq_{\text{idircx}}$-sense. In this case, we get

$$\boldsymbol{X} \preceq_{\text{idircx}} \boldsymbol{Y} \Rightarrow \sum_{i=1}^{n} X_i \preceq_{2-\text{icx}} \sum_{i=1}^{n} Y_i. \tag{1.6}$$

The validity of (1.6) can easily be established as follows. For any function g in $\mathcal{U}_{2-\text{icx}}^{[1]}$, it is easily seen that the function $\boldsymbol{x} \mapsto g(x_1 + x_2 + \ldots + x_n)$ belongs to $\mathcal{U}_{\text{idircx}}$ whence the announced implication (1.6) follows. Compared to (1.4)-(1.5), we see that the degree of the order relation between $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} Y_i$ does not depend on the dimension n in (1.6).

Our aim in this paper is to embed $\preceq_{\text{uo}}$, $\preceq_{\text{uocx}}$, $\preceq_{\text{iccx}}$ and $\preceq_{\text{idircx}}$ in three hierarchical classes of stochastic order relations:

- the multivariate s-increasing convex orders comprising $\preceq_{\text{uo}}$ and $\preceq_{\text{uocx}}$ as special cases for $s = 1$ and 2, respectively;
- the componentwise s-increasing convex orders extending $\preceq_{\text{iccx}}$ corresponding to $s = 2$; and

- the s-increasing directionally convex orders generalizing $\preceq_{\text{idircx}}$ corresponding to $s = 2$.

This will help to understanding the structure underlying these standard stochastic order relations, and also to realize the important role played by the number n of dimensions relative to the degree s of the orders. We propose a unified approach to defining $\preceq_{\text{iccx}}$, $\preceq_{\text{uo}}$, $\preceq_{\text{uocx}}$, and $\preceq_{\text{idircx}}$ as integral stochastic orderings generated by classes of functions possessing non-negative partial derivatives of specific degrees. This clearly shows how well-known results derive from a common principle and stresses the differences inherent to these standard multivariate stochastic orders.

After having discussed some of the properties of these classes of stochastic order relations, we compare positive linear combinations of the respective components of ordered random vectors. This allows us to extend (1.4)-(1.5) to the whole class of the multivariate s-increasing convex order, and (1.6) to the whole class of the s-increasing directionally convex orders. In the former case, the dimension n drives the degree of the resulting stochastic inequality between $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$ whereas in the latter case, this degree is not affected by n.

2 Componentwise s-increasing convex order

Let us define $\mathcal{U}_{s-\text{iccx}}$ as the class of all the functions g that are s-increasing convex in each component, once the others are kept fixed. Then, we say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ in the s-increasing componentwise convex order (denoted by $\boldsymbol{X} \preceq_{s-\text{iccx}} \boldsymbol{Y}$) if (1.2) holds with for $\mathcal{U}_*^{[n]}$ the class $\mathcal{U}_{s-\text{iccx}}$. Clearly, $\preceq_{2-\text{iccx}} \Leftrightarrow \preceq_{\text{iccx}}$ and $\preceq_{1-\text{iccx}}$ is the multivariate stochastic dominance $\preceq_{\text{st}}$. The definition we just gave can easily be extended to different degree of higher convexity for each dimension if needed, by considering test functions g that are s_i-increasing convex in the ith component, once the others are kept fixed, $i = 1, 2, \ldots, n$.

As $\preceq_{\text{iccx}}$, the order $\preceq_{s-\text{iccx}}$ can typically be used to compare random vectors with independent components, as shown in the next result. The proof is rather standard and can be found in Shaked & Shanthikumar (2007) for the $\preceq_{\text{iccx}}$ order, so that we do not repeat it here. It clearly shows why the $\preceq_{s-\text{iccx}}$ order is tailored to compare vectors with independent components.

Property 2.1. *Let $X_1, X_2, \ldots, X_n$ be a set of independent random variables and let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables. If $X_i \preceq_{s-icx} Y_i$ for $i = 1, 2, \ldots, n$, then $\boldsymbol{X} \preceq_{s-iccx} \boldsymbol{Y}$.*

3 Multivariate s-increasing convex order

Let $\boldsymbol{s} = (s_1, \ldots, s_n)$ be a vector of positive integers. In view of the theory developed in the univariate case by Denuit, Lefèvre & Shaked (1998) and in the bivariate case by Denuit, Lefèvre & Mesfioui (1999), we say that the n-dimensional random vector $\boldsymbol{X}$ is smaller than the n-dimensional random vector $\boldsymbol{Y}$ in the $\boldsymbol{s}$-increasing convex order (denoted by $\boldsymbol{X} \preceq_{\boldsymbol{s}-\text{icx}} \boldsymbol{Y}$) if (1.2) holds with for $\mathcal{U}_*^{[n]}$ the class $\mathcal{U}_{\boldsymbol{s}-\text{icx}}$ of all the functions g such that

$$\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}} g \geq 0 \text{ for } k_i = 0, 1, \ldots, s_i, \;\; i = 1, 2, \ldots, n, \;\; k_1 + k_2 + \cdots + k_n \geq 1.$$

A particular case of interest is when $s_1 = s_2 = \ldots = s_n = s$, where we denote as $\preceq_{s-\text{icx}}$ the resulting multivariate stochastic order relation.

This order can be characterized as follows.

Proposition 3.1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors and assume that $s_i \geq 1$ for $i = 1, 2, \ldots, n$. Then, $\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y}$ if, and only if, (1.2) holds with for $\mathcal{U}_*^{[n]}$ the class $\underline{\mathcal{U}}_{\boldsymbol{s}-icx}$ of all the functions of the form*

$$\prod_{i\in\mathcal{S}} x_i^{k_i} \prod_{j\in\overline{\mathcal{S}}} (x_j - t_j)_+^{s_j-1}$$

where $\mathcal{S}$ and $\overline{\mathcal{S}}$ form a partition of $\{1, 2, \ldots, n\}$, $k_i = 0, 1, \ldots, s_i - 1$, $t_j \geq 0$.

Proof. Consider $g \in \mathcal{U}_{\boldsymbol{s}-\text{icx}}$. By Taylor's expansion of g viewed as a function of x_1 around 0 for fixed $x_2, \ldots, x_n$, we obtain

$$g(\boldsymbol{x}) = \sum_{i_1=0}^{s_1-1} \frac{\partial^{i_1} g(0, x_2, \ldots, x_n)}{\partial x_1^{i_1}} \frac{x_1^{i_1}}{i_1!} + \int_0^{+\infty} \frac{(x_1 - t_1)_+^{s_1-1}}{(s_1 - 1)!} \frac{\partial^{s_1} g(t_1, x_2, \ldots, x_n)}{\partial x_1^{s_1}} dt_1$$

and by applying Taylor's expansion to $\partial^{i_1} g(0, x_2, \ldots, x_n)/\partial x_1^{i_1}$ and $\partial^{s_1} g(t_1, x_2, \ldots, x_n)/\partial x_1^{s_1}$ viewed as functions of x_2 around 0, we get

$$\begin{aligned} g(\boldsymbol{x}) &= \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} \frac{\partial^{i_1+i_2} g(0, 0, x_3 \ldots, x_n)}{\partial x_1^{i_1} \partial x_2^{i_2}} \frac{x_1^{i_1}}{i_1!} \frac{x_2^{i_2}}{i_2!} \\ &+ \sum_{i_2=0}^{s_2-1} \int_0^{+\infty} \frac{(x_1 - t_1)_+^{s_1-1} x_2^{i_2}}{(s_1 - 1)! i_2!} \frac{\partial^{s_1+i_2} g(t_1, 0, x_3 \ldots, x_n)}{\partial x_1^{s_1} \partial x_2^{i_2}} dt_1 \\ &+ \sum_{i_1=0}^{s_1-1} \int_0^{+\infty} \frac{(x_2 - t_2)_+^{s_2-1} x_1^{i_1}}{(s_2 - 1)! i_1!} \frac{\partial^{i_1+s_2} g(0, t_2, x_3 \ldots, x_n)}{\partial x_1^{i_1} \partial x_2^{s_2}} dt_2 \\ &+ \int_0^{+\infty} \int_0^{+\infty} \frac{(x_1 - t_1)_+^{s_1-1} (x_2 - t_2)_+^{s_2-1}}{(s_1 - 1)!(s_2 - 1)!} \frac{\partial^{s_1+s_2} g(t_1, t_2, x_3 \ldots, x_n)}{\partial x_1^{s_1} \partial x_2^{s_2}} dt_1 dt_2. \end{aligned}$$

By repeating this argument component by component, we get the next general expansion formula:

$$g(\boldsymbol{x}) = \sum_{\mathcal{S}} \sum_{i\in\mathcal{S}} \sum_{k_i=0}^{s_i-1} \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{i\in\mathcal{S}} \frac{x_i^{k_i}}{k_i!} \prod_{j\in\overline{\mathcal{S}}} \frac{(x_j - t_j)_+^{s_j-1}}{(s_j - 1)!} \frac{\partial^{\sum_{i\in\mathcal{S}} k_i + \sum_{j\in\overline{\mathcal{S}}} s_j}}{\prod_{i\in\mathcal{S}} \partial x_i^{k_i} \prod_{j\in\overline{\mathcal{S}}} \partial x_j^{s_j}} g(\boldsymbol{t}_{\overline{\mathcal{S}}}) \prod_{j\in\overline{\mathcal{S}}} dt_j$$

where $\boldsymbol{t}_{\overline{\mathcal{S}}} = \sum_{i\in\overline{\mathcal{S}}} t_i \boldsymbol{e}_i$ and $\boldsymbol{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. It follows by Fubini's theorem that

$$\begin{aligned} \mathbb{E}[g(\boldsymbol{X})] &= \sum_{\mathcal{S}} \sum_{i\in\mathcal{S}} \sum_{k_i=0}^{s_i-1} \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{\mathbb{E}\left[\prod_{i\in\mathcal{S}} X_i^{k_i} \prod_{j\in\overline{\mathcal{S}}} (X_j - t_j)_+^{s_j-1}\right]}{\prod_{i\in\mathcal{S}} k_i! \prod_{j\in\overline{\mathcal{S}}} (s_j - 1)!} \\ &\qquad \frac{\partial^{\sum_{i\in\mathcal{S}} k_i + \sum_{j\in\overline{\mathcal{S}}} s_j}}{\prod_{i\in\mathcal{S}} \partial x_i^{k_i} \prod_{j\in\overline{\mathcal{S}}} \partial x_j^{s_j}} g(\boldsymbol{t}_{\overline{\mathcal{S}}}) \prod_{j\in\overline{\mathcal{S}}} dt_j \end{aligned}$$

which ends the proof. $\square$

Remark 3.2. Starting from $\overline{F}_{\boldsymbol{X}}^{[1,\ldots,1]} = \overline{F}_{\boldsymbol{X}}$, let us define the integrated right tails of $\boldsymbol{X}$ as

$$\overline{F}_{\boldsymbol{X}}^{[k_1,\ldots,k_i+1,\ldots,k_n]}(x_1,\ldots,x_i,\ldots,x_n) = \int_{x_i}^{+\infty} \overline{F}_{\boldsymbol{X}}^{[k_1,\ldots,k_i,\ldots,k_n]}(x_1,\ldots,\xi_i,\ldots,x_n)d\xi_i. \tag{3.1}$$

It can be shown by induction that

$$\overline{F}_{\boldsymbol{X}}^{[\boldsymbol{k}]}(\boldsymbol{t}) = \frac{\mathbb{E}\left[\prod_{i=1}^n (X_i - t_i)_+^{k_i-1}\right]}{\prod_{i=1}^n (k_i-1)!}. \tag{3.2}$$

Therefore, Proposition 3.1 can be equivalently expressed in terms of integrated right tails.

Let us now derive the largest class of functions for which the implication in (1.2) holds for $\preceq_{\boldsymbol{s}-\mathrm{icx}}$.

Proposition 3.3. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors and assume that $s_i \geq 1$ for $i = 1, 2, \ldots, n$. Then, $\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y}$ if, and only if, (1.2) holds with for $\mathcal{U}_*^{[n]}$ the class $\overline{\mathcal{U}}_{\boldsymbol{s}-icx}$ of all the functions g such that*

$$\Delta_{1,h_1}^{k_1} \circ \cdots \circ \Delta_{n,h_n}^{k_n} g(\boldsymbol{x}) \geq 0 \textit{ for all } \boldsymbol{x}$$

for every $k_i = 0, 1, \ldots, s_i$, $h_1, \ldots, h_n \geq 0$.

The next result is a direct consequence of Proposition 3.1. It ensures that we only have to consider product of test functions to establish that $\preceq_{\boldsymbol{s}-\mathrm{icx}}$ holds.

Proposition 3.4. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors. Then, $\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y}$ if, and only if, (1.2) holds with for $\mathcal{U}_*^{[n]}$ the class $\mathcal{U}_{\boldsymbol{s}-icx}^{prod}$ of all the products $\prod_{i=1}^n g_i$ where each $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to $\mathcal{U}_{s_i-icx}$.*

Recall that the n-dimensional random vector $\boldsymbol{X}$ is said to be Positively Upper Orthant Dependent (PUOD, in short) when the inequality

$$\Pr[\boldsymbol{X} > \boldsymbol{x}] \geq \prod_{i=1}^n \Pr[X_i > x_i], \text{ holds for all } \boldsymbol{x}. \tag{3.3}$$

If the reverse inequality holds in (3.3), then $\boldsymbol{X}$ is said to be Negatively Upper Orthant Dependent (NUOD, in short). Defining a random vector $\boldsymbol{X}^\perp$ with the same univariate marginals as $\boldsymbol{X}$ but with independent components, $\boldsymbol{X}$ is PUOD when $\boldsymbol{X}^\perp \preceq_{\mathrm{uo}} \boldsymbol{X}$. We easily deduce from Proposition 3.1 that

$$\boldsymbol{X}^\perp \preceq_{\boldsymbol{s}-\mathrm{icx}} \boldsymbol{Y}^\perp \Leftrightarrow X_i \preceq_{s_i-\mathrm{icx}} Y_i \text{ for } i = 1, 2, \ldots, n. \tag{3.4}$$

This result can be strengthened as follows.

Property 3.5. *(i) Let $\boldsymbol{X}$ be NUOD, $\boldsymbol{Y}$ with independent components. Then*

$$X_i \preceq_{s_i-icx} Y_i, \ \ i = 1, 2, \ldots, n \Leftrightarrow \boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y}.$$

(ii) Let $\boldsymbol{X}$ be PUOD, $\boldsymbol{Y}$ with independent components. Then

$$Y_i \preceq_{s_i-icx} X_i, \ \ i = 1, 2, \ldots, n \Leftrightarrow \boldsymbol{Y} \preceq_{\boldsymbol{s}-icx} \boldsymbol{X}.$$

4 The s-increasing directionally convex order

In view of the definition of $\preceq_{\text{idircx}}$, we say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ in the s-increasing directionally convex order (denoted by $\boldsymbol{X} \preceq_{s-\text{idircx}} \boldsymbol{Y}$) if (1.2) holds with for $\mathcal{U}_*^{[n]}$ the class $\mathcal{U}_{s-\text{idircx}}$ of all the functions g such that

$$\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_n^{k_n}} g \geq 0 \text{ for all the non-negative integers } k_1, k_2, \ldots, k_n$$
$$\text{such that } 1 \leq k_1 + k_2 + \cdots + k_n \leq s.$$

For $s \geq n$, this new stochastic order relation is closely related to the higher-degree increasing convex order discussed in the preceding section. To see this, let us define $\mathcal{R}_s$ as the set of all the non-negative integers $r_1, \ldots, r_n$ such that $1 \leq r_1 + \cdots + r_n = s$. The following technical lemma relates $\mathcal{U}_{s-\text{idircx}}$ to the classes of functions generating the $\boldsymbol{s}$-increasing convex orders studied in the preceding section.

Lemma 4.1. *If $s \geq n$ then the generator of $\preceq_{s-idircx}$ can be written in terms of the generators of the orderings $\preceq_{\boldsymbol{r}-icx}$, $\boldsymbol{r} \in \mathcal{R}_s$, as follows:*

$$\mathcal{U}_{s-idircx} = \bigcap_{\boldsymbol{r}\in\mathcal{R}_s} \mathcal{U}_{\boldsymbol{r}-icx}.$$

Proof. We first prove that $\mathcal{U}_{s-\text{idircx}} \subset \mathcal{U}_{\boldsymbol{r}-\text{icx}}$ for all $\boldsymbol{r} \in \mathcal{R}_s$. To this end, let us consider $g \in \mathcal{U}_{s-\text{idircx}}$ and $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathcal{R}_s$. Now, for any $k_1 \leq r_1, \ldots, k_n \leq r_n$ such that $1 \leq k_1 + \cdots + k_n$, we have $1 \leq k_1 + \cdots + k_n \leq r_1 + \cdots + r_n = s$. Since $g \in \mathcal{U}_{s-\text{idircx}}$, it then follows that

$$\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} g \geq 0 \text{ for all such } k_1, \ldots, k_n,$$

which implies that $g \in \mathcal{U}_{\boldsymbol{r}-\text{icx}}$ for all $\boldsymbol{r} \in \mathcal{R}_s$, and thus $g \in \bigcap_{\boldsymbol{r}\in\mathcal{R}_s} \mathcal{U}_{\boldsymbol{r}-\text{icx}}$. Consequently, $\mathcal{U}_{s-\text{idircx}} \subset \bigcap_{\boldsymbol{r}\in\mathcal{R}_s} \mathcal{U}_{\boldsymbol{r}-\text{icx}}$.

Now, consider $g \in \bigcap_{\boldsymbol{r}\in\mathcal{R}_s} \mathcal{U}_{\boldsymbol{r}-\text{icx}}$ and let $k_1, \ldots, k_n$ be such that $1 \leq k_1 + \cdots + k_n \leq s$. We chose $r_1^* = k_1, \ldots, r_{n-1}^* = k_{n-1}$ and $r_n^* = s - (k_1 + \cdots + k_{n-1})$. By construction, $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*) \in \mathcal{R}_s$ with $k_1 = r_1^*, \ldots, r_{n-1}^* = k_{n-1}$ and $k_n \leq r_n^*$. Since $g \in \bigcap_{\boldsymbol{r}\in\mathcal{R}_s} \mathcal{U}_{\boldsymbol{r}-\text{icx}}$, we have in particular that $g \in \mathcal{U}_{\boldsymbol{r}^*-\text{icx}}$. Hence,

$$\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} g \geq 0 \text{ for all such } k_1, \ldots, k_n,$$

which allows us to conclude $g \in \mathcal{U}_{s-\text{idircx}}$ and ends the proof. □

Now, let us define the following classes of functions:

$$\begin{aligned}
\underline{\mathcal{U}}_{s-\text{idircx}} &= \bigcap_{\boldsymbol{r}\in\mathcal{R}_s} \underline{\mathcal{U}}_{\boldsymbol{r}-\text{icx}} \\
\overline{\mathcal{U}}_{s-\text{idircx}} &= \bigcap_{\boldsymbol{r}\in\mathcal{R}_s} \overline{\mathcal{U}}_{\boldsymbol{r}-\text{icx}} \\
\mathcal{U}^{\text{prod}}_{s-\text{idircx}} &= \bigcap_{\boldsymbol{r}\in\mathcal{R}_s} \mathcal{U}^{\text{prod}}_{\boldsymbol{r}-\text{icx}}.
\end{aligned}$$

We are in a position to state the following characterizations of $\preceq_{s-\text{idircx}}$.

Proposition 4.2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors and assume that $s \geq n$. Then,*

$$\begin{aligned} \boldsymbol{X} \preceq_{s-idircx} \boldsymbol{Y} &\Leftrightarrow (1.2) \text{ holds with } \mathcal{U}_*^{[n]} = \underline{\mathcal{U}}_{s-idircx} \\ &\Leftrightarrow (1.2) \text{ holds with } \mathcal{U}_*^{[n]} = \overline{\mathcal{U}}_{s-idircx} \\ &\Leftrightarrow (1.2) \text{ holds with } \mathcal{U}_*^{[n]} = \mathcal{U}_{s-idircx}^{prod}. \end{aligned}$$

Remark 4.3. Note that the condition $s \geq n$ is necessary to get characterizations of the stochastic ordering $\preceq_{s-\mathrm{idircx}}$. When $s < n$, it is not possible to obtain a dense subclass for the generating class $\mathcal{U}_{s-\mathrm{idircx}}$ of this ordering. This explains why in dimension 3, the supermodular order does not possess any characterization involving subclasses of the (regular) supermodular functions. If $s < n$, we face the same problem as with the supermodular order. In dimension 2, this order coincides with the $\preceq_{\mathrm{uo}}$ order (restricted to random couples with identical univariate marginals) and can be characterized as explained in the introduction. For dimensions 3 and over, there is no dense subclass of the supermodular functions that can be used to characterize this integral stochastic order.

5 Properties

We now establish various properties of the stochastic order relations introduced above. Firstly, as $\mathcal{U}_{\boldsymbol{t}-\mathrm{icx}} \subset \mathcal{U}_{\boldsymbol{s}-\mathrm{icx}}$ provided $\boldsymbol{t} \geq \boldsymbol{s}$, that is, $t_i \geq s_i$ for all $i = 1, 2, \ldots, n$, and $\mathcal{U}_{t-\mathrm{idircx}} \subset \mathcal{U}_{s-\mathrm{idircx}}$ for all integer $t \geq s$, we deduce that the s-increasing convex and s-increasing directionally convex orders form hierarchical classes of stochastic order relations, i.e. stochastic inequalities weaken as the degree of the order increases. This is formally stated in the next result.

Property 5.1. *For any $\boldsymbol{t} \geq \boldsymbol{s}$, we have $\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y} \Rightarrow \boldsymbol{X} \preceq_{\boldsymbol{t}-icx} \boldsymbol{Y}$. Similarly, for any $t \geq s$, we have $\boldsymbol{X} \preceq_{s-idircx} \boldsymbol{Y} \Rightarrow \boldsymbol{X} \preceq_{t-idircx} \boldsymbol{Y}$.*

These stochastic orders are clearly closed under mixtures. This is stated in the next property, together with some direct consequences involving sums and other functions of random vectors.

Property 5.2. **(i)** *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors and let $\boldsymbol{\Theta}$ be a m-dimensional random vector. Then,*

$$[\boldsymbol{X}|\boldsymbol{\Theta} = \boldsymbol{\theta}] \preceq_{\boldsymbol{s}-icx} [\boldsymbol{Y}|\boldsymbol{\Theta} = \boldsymbol{\theta}] \text{ for all } \boldsymbol{\theta} \Rightarrow \boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y}$$

and

$$[\boldsymbol{X}|\boldsymbol{\Theta} = \boldsymbol{\theta}] \preceq_{s-idircx} [\boldsymbol{Y}|\boldsymbol{\Theta} = \boldsymbol{\theta}] \text{ for all } \boldsymbol{\theta} \Rightarrow \boldsymbol{X} \preceq_{s-idircx} \boldsymbol{Y}.$$

(ii) *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors such that $\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y}$ and let $\boldsymbol{Z}$ be a m-dimensional random vector, which is independent from $\boldsymbol{X}$ and $\boldsymbol{Y}$. Then,*

$$\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y} \Rightarrow (\psi_1(X_1, \boldsymbol{Z}), \ldots, \psi_n(X_n, \boldsymbol{Z})) \preceq_{\boldsymbol{s}-icx} (\psi_1(Y_1, \boldsymbol{Z}), \ldots, \psi_n(Y_n, \boldsymbol{Z}))$$

whenever $\psi_i(x, \boldsymbol{z})$, $i = 1, 2, \ldots, n$, *viewed as a functions of* x, *with* $\boldsymbol{z}$ *fixed, is* s_i-*increasing convex for any* $\boldsymbol{z}$. *In particular, consider the* n-*dimensional random vectors* $\boldsymbol{X}$, $\boldsymbol{Y}$ *and* $\boldsymbol{Z}$ *such that* $\boldsymbol{Z}$ *is independent from both* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$. *Then,*

$$\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y} \Rightarrow \boldsymbol{X} + \boldsymbol{Z} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y} + \boldsymbol{Z}$$

and

$$\boldsymbol{X} \preceq_{s-idircx} \boldsymbol{Y} \Rightarrow \boldsymbol{X} + \boldsymbol{Z} \preceq_{s-idircx} \boldsymbol{Y} + \boldsymbol{Z}.$$

(iii) *Consider two sets of independent* n-*dimensional random vectors,* $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_m$ *and* $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_m$, *say. Then,*

$$\boldsymbol{X}_i \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y}_i \text{ for } i = 1, 2, \ldots, m \Rightarrow \sum_{i=1}^{m} \boldsymbol{X}_i \preceq_{\boldsymbol{s}-icx} \sum_{i=1}^{m} \boldsymbol{Y}_i.$$

and

$$\boldsymbol{X}_i \preceq_{s-idircx} \boldsymbol{Y}_i \text{ for } i = 1, 2, \ldots, m \Rightarrow \sum_{i=1}^{m} \boldsymbol{X}_i \preceq_{s-idircx} \sum_{i=1}^{m} \boldsymbol{Y}_i.$$

(iv) *Consider two sequences of independent* n-*dimensional random vectors,* $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ *and* $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots$, *say, and the positive integer-valued random variable* N, *independent from the* $\boldsymbol{X}_i$*'s and the* $\boldsymbol{Y}_j$*'s. Then,*

$$\boldsymbol{X}_i \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y}_i \text{ for all } i \Rightarrow \sum_{i=1}^{N} \boldsymbol{X}_i \preceq_{\boldsymbol{s}-icx} \sum_{i=1}^{N} \boldsymbol{Y}_i$$

and

$$\boldsymbol{X}_i \preceq_{s-idircx} \boldsymbol{Y}_i \text{ for all } i \Rightarrow \sum_{i=1}^{N} \boldsymbol{X}_i \preceq_{s-idircx} \sum_{i=1}^{N} \boldsymbol{Y}_i.$$

Aggregating independent random variables ordered in the univariate s-increasing convex sense produces random vectors ordered in both the $\preceq_{s-\text{icx}}$ sense and $\preceq_{s-\text{idircx}}$ sense. This comes from the fact that $\preceq_{s-\text{iccx}}$ implies both $\preceq_{s-\text{icx}}$ and $\preceq_{s-\text{idircx}}$. This result stays valid when blocks of random variables are concatenated, as shown next. Note that we take $s_1 = \ldots = s_n = s$ in this case.

Property 5.3. *Let* $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\ldots$, $\boldsymbol{X}_m$ *be independent* k_i-*dimensional random vectors where all the* k_i*'s are greater or equal to 2. Let* $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, $\ldots$, $\boldsymbol{Y}_m$ *be another set of independent* k_i-*dimensional random vectors. If* $\boldsymbol{X}_i \preceq_{s-icx} \boldsymbol{Y}_i$ *for* $i = 1, 2, \ldots, m$, *then*

$$(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_m) \preceq_{s-icx} (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_m).$$

Similarly, if $\boldsymbol{X}_i \preceq_{s-idircx} \boldsymbol{Y}_i$ *for* $i = 1, 2, \ldots, m$, *then*

$$(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_m) \preceq_{s-idircx} (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_m).$$

Proof. Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be two independent k_1- and k_2-dimensional random vectors, and let $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ be two other independent k_1- and k_2-dimensional random vectors. Suppose that $\boldsymbol{X}_1 \preceq_{s-\mathrm{icx}} \boldsymbol{Y}_1$ and $\boldsymbol{X}_2 \preceq_{s-\mathrm{icx}} \boldsymbol{Y}_2$. Then, for any $g \in \mathcal{U}_{s-\mathrm{icx}}$, we have that

$$\begin{aligned}
\mathbb{E}[g(\boldsymbol{X}_1, \boldsymbol{X}_2)] &= \mathbb{E}\Big[\mathbb{E}[g(\boldsymbol{X}_1, \boldsymbol{X}_2)|\boldsymbol{X}_2]\Big] \\
&\leq \mathbb{E}\Big[\mathbb{E}[g(\boldsymbol{Y}_1, \boldsymbol{X}_2)|\boldsymbol{X}_2]\Big] \\
&= \mathbb{E}[g(\boldsymbol{Y}_1, \boldsymbol{X}_2)] \\
&\leq \mathbb{E}[g(\boldsymbol{Y}_1, \boldsymbol{Y}_2)],
\end{aligned}$$

where the first inequality follows from the fact that $g(\boldsymbol{x}_1, \boldsymbol{x}_2)$ viewed as a function of $\boldsymbol{x}_1$, with $\boldsymbol{x}_2$ fixed, is s-increasing convex, and the second inequality follows in a similar manner. The announced result then follows from the above by mathematical induction. □

If we fix the values of some arguments of an $\boldsymbol{s}$-increasing convex or of an s-increasing directionally convex function, we still have an $\boldsymbol{s}$-increasing convex or an s-increasing directionally convex function. This ensures that the integral stochastic order relations generated by these classes of functions are closed under marginalization, as formally stated in the next result.

Property 5.4. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors. If $\boldsymbol{X} \preceq_{s-icx} \boldsymbol{Y}$, then for any $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$,*

$$(X_i)_{i \in \mathcal{I}} \preceq_{s-icx} (Y_i)_{i \in \mathcal{I}},$$

where $(X_i)_{i \in \mathcal{I}}$ denotes the $\#\mathcal{I}$-dimensional random vector formed with the components of $\boldsymbol{X}$ with indices in $\mathcal{I}$. Similarly, if $\boldsymbol{X} \preceq_{s-idircx} \boldsymbol{Y}$, then for any $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$ with $\#\mathcal{I} = p \geq 2$,

$$(X_i)_{i \in \mathcal{I}} \preceq_{s-idircx} (Y_i)_{i \in \mathcal{I}}.$$

Moreover, $X_i \preceq_{s-icx} Y_i$ when $\mathcal{I} = \{i\}$.

6 Comparison of univariate transforms, with special emphasis on non-negative linear combinations

Let us now consider the relationship between multivariate stochastic order relations and univariate stochastic dominance between positive linear combinations of their respective components. Having two ordered random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, we would like to investigate the degree of the stochastic dominance relation holding between $\sum_{i=1}^n \alpha_i X_i$ and $\sum_{i=1}^n \alpha_i Y_i$ with non-negative $\alpha_1, \ldots, \alpha_n$.

Property 6.1. *(i) Consider a non-negative function Ψ in $\mathcal{U}_{\boldsymbol{s}-icx}$. Then,*

$$\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y} \Rightarrow \Psi(\boldsymbol{X}) \preceq_{(\sum_{i=1}^n s_i)-icx} \Psi(\boldsymbol{Y}).$$

In particular,

$$\boldsymbol{X} \preceq_{\boldsymbol{s}-icx} \boldsymbol{Y} \Rightarrow \sum_{i=1}^n \alpha_i X_i \preceq_{(\sum_{i=1}^n s_i)-icx} \sum_{i=1}^n \alpha_i Y_i \text{ for any } \alpha_1, \ldots, \alpha_n \geq 0.$$

(ii) Consider a non-negative function Ψ in $\mathcal{U}_{s-idiricx}$. Then,

$$\boldsymbol{X} \preceq_{s-idircx} \boldsymbol{Y} \Rightarrow \Psi(\boldsymbol{X}) \preceq_{s-icx} \Psi(\boldsymbol{Y}).$$

In particular,

$$\boldsymbol{X} \preceq_{s-idircx} \boldsymbol{Y} \Rightarrow \sum_{i=1}^{n} \alpha_i X_i \preceq_{s-icx} \sum_{i=1}^{n} \alpha_i Y_i \text{ for any } \alpha_1, \ldots, \alpha_n \geq 0.$$

Proof. To establish (i), let us consider g in $\mathcal{U}_{(\sum_{i=1}^n s_i)-\text{icx}}$. Then, it is easily seen that $g \circ \Psi$ belongs to $\mathcal{U}_{\boldsymbol{s}-\text{icx}}$ so that $\boldsymbol{X} \preceq_{\boldsymbol{s}-\text{icx}} \boldsymbol{Y}$ ensures that the stochastic inequality $\Psi(\boldsymbol{X}) \preceq_{(\sum_{i=1}^n s_i)-\text{icx}} \Psi(\boldsymbol{Y})$ holds between the random variables $\Psi(\boldsymbol{X})$ and $\Psi(\boldsymbol{Y})$. Considering $\Psi(\boldsymbol{x}) = \sum_{i=1}^n \alpha_i x_i$ with $\alpha_1, \ldots, \alpha_n \geq 0$, we clearly see that $\Psi \in \mathcal{U}_{\boldsymbol{s}-\text{icx}}$. The proof of (ii) is similar. □

It is even possible to get order relations for sums of functions of the components of the respective random vectors. Indeed, since

$$\boldsymbol{X} \preceq_{\boldsymbol{s}-\text{icx}} \boldsymbol{Y} \Rightarrow (\psi_1(X_1), \psi_2(X_2), \ldots, \psi_n(X_n)) \preceq_{\boldsymbol{s}-\text{icx}} (\psi_1(Y_1), \psi_2(Y_2), \ldots, \psi_n(Y_n))$$

for all non-negative $\psi_i \in \mathcal{U}_{s_i-\text{icx}}$ we also have

$$\boldsymbol{X} \preceq_{s-\text{icx}} \boldsymbol{Y} \Rightarrow \sum_{i=1}^{n} \psi_i(X_i) \preceq_{(\sum_{i=1}^n s_i)-\text{icx}} \sum_{i=1}^{n} \psi_i(Y_i).$$

Similarly, as

$$\boldsymbol{X} \preceq_{s-\text{idircx}} \boldsymbol{Y} \Rightarrow (\psi_1(X_1), \psi_2(X_2), \ldots, \psi_n(X_n)) \preceq_{s-\text{idircx}} (\psi_1(Y_1), \psi_2(Y_2), \ldots, \psi_n(Y_n))$$

for all non-negative $\psi_i \in \mathcal{U}_{s-\text{icx}}$ we also have

$$\boldsymbol{X} \preceq_{s-\text{idircx}} \boldsymbol{Y} \Rightarrow \sum_{i=1}^{n} \psi_i(X_i) \preceq_{s-\text{icx}} \sum_{i=1}^{n} \psi_i(Y_i).$$

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