OPTIMAL USE OF A REPLENISHABLE RESOURCE IN A MODEL OF CAPITAL ACCUMULATION

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1. Introduction

We consider an optimal growth model in which the flow from a stock of replenishable resource is an essential factor in a production function which combines the flow with a capital stock to produce an all purpose good which may be consumed or saved to augment the capital stock. We assume that the stock of replenishable resource follows a natural growth law which is a concave function of the stock. We seek the optimal time path of the flow of the replenishable resource and consumption out of production so as to maximise over an infinite horizon the integral of time discounted utility of consumption. We examine the equilibrium towards which the solution of the differential system determining the optimal path tends and discuss the stability of this equilibrium. We also examine the effect of including the stock of replenishable resource in the utility function.

The type of replenishable resource we have in mind might include forests, biological resources or more generally hybrid factors of production as discussed by Kemp [1974]. The problem of determining the optimal use of a replenishable resource in models which abstract from the question of capital accumulation has already been discussed by Kemp [1974], Neher [1974] and Smith [1977]. The model of Sampson [1976], after appropriate reinterpretation of variables, has some common features with the one we present here however our natural growth law is far more general. The problem of including a non-replenishable resource in an optimal growth model has been treated by Stiglitz [1974].

2. The Model and the Necessary Conditions

Denoting by $K(t)$, the capital stock, $R(t)$, the stock of replenishable resource, $E(t)$, the flow of replenishable resource into production and $C(t)$, consumption, all at time t and assuming a constant population, the resource owner's decision problem may be posed as the following optimal control problem:-

(1) $$\max_{C,E} \int_0^\infty e^{-\rho t}\, U(C)\, dt ,$$

subject to the dynamic constraints,

(2) $$\dot{K} = Y(K,E) - C,$$

(3) $$\dot{R} = \phi(R) - E ,$$

where $Y(K,E) = K^\alpha E^\beta$ $(\alpha,\beta>0,\ \alpha+\beta = 1)$ is a constant returns to scale production function, $\phi(R)$ is the natural growth function of the replenishable resource and is assumed to have the form displayed in figure 1 i.e. $\phi(R)$ is concave with $\phi(0)=\phi(\bar{A}) = 0$, $\phi'(A^*) = 0$. We impose the following conditions on the utility function

(4) $$U_c>0, \quad U_{cc}<0 ,$$

where the subscripts denote differentiation with respect to C. Of the dynamic constraints, (2) shows output Y being divided between consumption (C) and investment ($\dot{K}$), whilst (3) shows the natural growth rate of the stock of replenishable resource reduced by the amount of the flow of resource into production.

From the present value Hamiltonian

$$H = e^{-\rho t}[U(C)+\lambda_1(Y-C)+\lambda_2(\phi(R)- E)], \tag{5}$$

where $e^{-\rho t}\lambda_1(t)$ and $e^{-\rho t}\lambda_2(t)$ are, respectively, the shadow prices of the capital stock and the resource stock, we find the necessary conditions for the maximisation of (1) subject to (2) and (3) to be:-

$$\lambda_1 = U_c \quad , \tag{6}$$

$$\lambda_2 = \lambda_1 Y_E \quad , \tag{7}$$

$$\hat{\lambda}_1 = \rho - Y_K \quad , \tag{8}$$

$$\hat{\lambda}_2 = \rho - \phi_R . \tag{9}$$

We have introduced the notation ^ whereby $\dot{f} = df/dt$ and $\hat{f} = \dot{f}/f$ for any time function $f(t)$, also Y_E, Y_K denote the partial derivatives $\frac{\partial Y}{\partial E}$, $\frac{\partial Y}{\partial K}$ respectively and ϕ_R denotes the derivative $\phi'(R)$.

Substitution of (6) into (8) yields

$$\widehat{(U_c \, e^{-\rho t})} = - Y_K \quad , \tag{10}$$

which is just the classical Ramsey savings rule, Ramsey [1928], that along the optimal path the discounted instantaneous marginal utility declines at a rate equal to the marginal product of capital.

Substitution of (7) and (8) into (9) yields

$$\hat{Y}_E = Y_K - \phi_R \tag{11}$$

which states that along the optimal path resource should be employed up to the point where the rate of return on resource use ($\hat{Y}_E$) equals the rate of return on capital reduced by the change in the natural growth rate. Note that for the corresponding model with a <u>non</u>-replenishable resource, so that $\phi_R = 0$, (11) reduces to

$$\hat{Y}_E = Y_K \ , \tag{12}$$

as obtained by Stiglitz [1974].

3. <u>The Equilibrium and its Stability</u>

The differential equations determining the optimal path are (2) and (3) for K and R and (8) and (9) for λ_1 and λ_2 with C being given as a function of λ_1 through equation (6) whilst from (7) E is given by

$$E = \left(\frac{\lambda_1 \beta}{\lambda_2}\right)^{1/\alpha} K \ . \tag{13}$$

Hence we can write the system of differential equations as

$$\dot{K} = \left(\frac{\lambda_1 \beta}{\lambda_2}\right)^{\beta/\alpha} K - C(\lambda_1) \ , \tag{14.a}$$

$$\dot{R} = \phi(R) - \left(\frac{\lambda_1 \beta}{\lambda_2}\right)^{1/\alpha} K \ , \tag{14.b}$$

$$\dot{\lambda}_1 = \lambda_1 \left(\rho - \alpha \ \left(\frac{\lambda_1 \beta}{\lambda_2}\right)^{\beta/\alpha}\right) \tag{14.c}$$

$$\dot{\lambda}_2 = \lambda_2 (\rho - \phi_R) \ . \tag{14.d}$$

Considering the differential equations (2), (3), (8) and (9) we see that the equilibrium of the differential equations defining the optimal path is given by

(15.a) $\bar{Y} = \bar{C}$,

(15.b) $\phi(\bar{R}) = \bar{E}$,

(15.c) $\bar{Y}_K = \rho$,

(15.d) $\bar{\phi}_R = \rho$,

where we use bars to denote equilibrium values. We see immediately that for the assumed form of $\phi(R)$, equation (15.d) determines a unique value of $\bar{R}$ and furthermore with reference to figure 1,

(16) $0 < \bar{R} < A^*$,

provided that the discount rate is positive and satisfies $\rho < \phi'(0)$. It then follows from (15.b and c) that $\bar{E}$ and $\bar{K}$ will be unique and hence the equilibrium defined by equation (15) is unique. At this equilibrium the level of the stock of replenishable resource is determined by the discount factor, being higher the lower the discount factor, but always below the level which would yield a maximum natural rate of growth of the stock. We also see that at the equilibrium all output is consumed and the rate of return on capital equals the discount factor.

We now turn to a discussion of the stability of the equilibrium defined by equation (15) however before doing so we point out that $\bar{\lambda}_1, \bar{\lambda}_2$ the equilibrium values of λ_1 and λ_2 are calculated from $\overline{C}$, $\overline{E}$ and $\overline{K}$ via equations (6) and (7) and furthermore that

$$(17) \qquad \frac{\overline{\lambda}_1 \beta}{\overline{\lambda}_2} = \left(\frac{\rho}{\alpha}\right)^{\alpha/\beta} ,$$

a relationship which follows from (7), (13) and (15.c). We note that the initial values K(0) and R(0) are assumed given whilst the initial values C(0) and E(0) (and hence $\lambda_1(0)$ and $\lambda_2(0)$) are unknown at the outset, part of our task will be to specify how these two unknown initial values should be chosen.

Let V denote the column vector $(K-\overline{K}, R-\overline{R}, \lambda_1-\overline{\lambda}_1, \lambda_2-\overline{\lambda}_2)^T$ so that linearising the differential system (14) about its equilibrium (15) we obtain the linear system

$$(18) \qquad \dot{V} = JV ,$$

where J is the Jacobian matrix of the differential system (14) evaluated at the equilibrium (15). Putting $\varepsilon = \rho/\alpha$ we find that

$$(19) \quad J = \begin{bmatrix} \varepsilon & 0 & \frac{\beta}{\alpha}\frac{\overline{K}}{\overline{\lambda}_1}\varepsilon - \frac{1}{\overline{U}_{CC}} & -\frac{\beta}{\alpha}\frac{\overline{K}}{\overline{\lambda}_2}\varepsilon \\ -\varepsilon^{1/\beta} & \rho & \frac{-\overline{K}}{\alpha\overline{\lambda}_1}\varepsilon^{1/\beta} & \frac{\overline{K}}{\alpha\overline{\lambda}_2}\varepsilon^{1/\beta} \\ 0 & 0 & -\beta\varepsilon & \varepsilon^{1/\beta} \\ 0 & -\overline{\lambda}_2\,\overline{\Phi}_{RR} & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of the matrix J turns out to be

(20) $H(\gamma) \equiv |J-\gamma I| = \gamma^4 - 2\rho\gamma^3 + A\gamma^2 - B\gamma + \frac{\lambda_2 \phi_{RR}}{U_{cc}} \varepsilon^{2/\beta} = 0$

where

(21.a) $A = \rho^2 \left(1 - \frac{\beta}{\alpha^2}\right) + \phi_{RR} \frac{K}{\alpha} \varepsilon^{1/\beta}$,

(21.b) $B = -\left(\beta\ \alpha\ \varepsilon^3 - \phi_{RR}\ K\ \varepsilon^{(1+\beta)/\beta}\right)$,

so that for the assumed functional form of $\phi(R)$ we must have

(21.c) $B < 0.$

The roots of (20), γ_i $(i=1,\ldots,4)$, will be the eigenvalues of the matrix J and it follows from well known properties of the characteristic polynomial that

(22.a) $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 2\rho > 0$,

indicating that at least one eigenvalue has positive real part,

(22.b) $\gamma_1\gamma_2\gamma_3\gamma_4 = \frac{\lambda_2 \bar{\phi}_{RR}}{\bar{U}_{cc}} \varepsilon^{2/\beta} > 0$,

which in conjunction with (22.a) indicates that either (i) all eigenvalues have positive real parts, or (ii) there are two with positive and two with negative real parts, and finally

(22.c) $\left.\begin{matrix}\text{The sum of the product of the} \\ \text{roots taken 3 at a time}\end{matrix}\right\} = B < 0$,

indicating that at least one eigenvalue has negative real part. We can finally conclude from (22.a-c) that the matrix J has two eigenvalues with positive real parts and two eigenvalues with negative real parts.

The solution of the linear system (18) may be written

$$V(t) = \sum_{i=1}^{4} a_i \psi_i e^{\gamma_i t} \quad , \tag{23}$$

where the $\psi_i (i=1,\ldots,4)$ are the eigenvectors of the matrix J and the a_i $(i=1,\ldots,4)$ are arbitrary constants. Now we recall our remark following equation (17) that two of the initial values are known and two are unknown and since the matrix J has two eigenvalues with positive real parts and two eigenvalues with negative real parts we have just the right proportion of known and unknown initial values to equate to zero the two arbitrary constants attached to the two positive exponentials in (23) and determine uniquely the two arbitrary constants attached to the two negative exponentials in (23). For the given initial conditions we have thus determined a unique path converging to the equilibrium (15).

4. Utility Dependent of the Stock of Resource

In this section we investigate the effect on our previous analysis of making the utility function dependent not only on consumption but also on the stock of replenishable resource. In including this latter variable we seek to capture the effect of the conservation motive.

Thus we consider the following control problem:-

$$\max_{C,E} \int_0^\infty e^{-\rho t} U(C,R)\, dt \tag{24}$$

subject to the dynamic constraints (2) and (3). The only assumptions that we impose on the utility function at this stage are

$$U_C > 0,\ U_R > 0,\ U_{CC} < 0,\ U_{RR} < 0\ . \tag{25}$$

The necessary conditions are derived in exactly the same manner as in section 2 and we shall note only the differences. Thus equation (6) is replaced by

$$\lambda_1 = U_C(C,R)\ , \tag{26}$$

where we emphasise the appearance of the additional argument R on the right-hand side, equation (9) is replaced by

$$\hat{\lambda}_2 = \rho - \phi_R - \frac{U_R}{U_C\, Y_E}\ , \tag{27}$$

whilst equations (7) and (8) remain unchanged. Substitution of (26) into (8) will still yield the Ramsey savings rule, with the modification that the marginal utility of consumption now depends on both C and R. Substitution of (7) and (8) into (27) gives a modified version of equation (11) in the form

$$\hat{Y}_E = Y_K - \phi_R - \frac{U_R}{U_C\, Y_E}\ , \tag{28}$$

stating that along the optimal path the resource should be employed up to the point where the rate of return on resource use ($\hat{Y}_E$) equals the rate of return on capital reduced not only by the change in the natural growth rate but also reduced by a term which increases as the ratio of the marginal utility of the resource stock to the marginal utility of consumption increases and decreases as the marginal product of the resource increases.

Following manipulations analogous to those leading to the differential system (14) we find that (14.a and d) become, respectively,

(29.a) $$\dot{K} = \left(\frac{\lambda_1 \beta}{\lambda_2}\right)^{\beta/\alpha} K - C(\lambda_1, R) \quad ,$$

(29.b) $$\dot{\lambda}_2 = \lambda_2(\rho - \phi_R) - U_R \quad ,$$

where we emphasise that on the right hand side of (29.a) C is now a function of both λ_1 and R. The differential equations (14.b and c) remain unchanged and together with (29.a and b) define the optimal path.

The equilibrium towards which the differential equations determining the optimal path tend is given by equations (15.a-c) and

(30) $$\bar{\phi}_R = \rho - \frac{\bar{U}_R}{\bar{U}_c \; \bar{Y}_E}$$

in place of (15.d). First we point out that $\bar{R}$, the equilibrium value of the resource stock will be higher than its corresponding value when $U_R = 0$ and it is now quite possible that

(31) $$\bar{R} > A^* \ .$$

The system of equations (15.a-c) and (30) will not in general determine a unique equilibrium since equation (30) cannot in general be expected to yield a unique value of $\bar{R}$ as was the case in section 3. The possibility of multiple equilibria makes a general discussion of the stability of the equilibria along the lines of section 3 impossible without assuming some specific forms for U(C,R). We are still able to calculate the Jacobian matrix J of the linearised system (18) and since only 5 elements differ from the expression given in (19) we merely note the differences, which are

(32.a) $$J_{12} = \overline{U}_{cR}/\overline{U}_{RR} \ ,$$

(32.b) $$J_{22} = \overline{\phi}_R$$

(32.c) $$J_{42} = -\overline{\lambda}_2 \ \overline{\phi}_{RR} - (\overline{U}_{RR}\overline{U}_{cc} - \overline{U}_{cR}^2)/\overline{U}_{cc} \equiv \Delta \ ,$$

(32.d) $$J_{43} = -\overline{U}_{cR}/\overline{U}_{cc} \ ,$$

(32.e) $$J_{44} = \rho - \overline{\phi}_R \ ,$$

where we use J_{ij} to denote the ijth element of the matrix J. We do not bother to write down the characteristic polynomial since the expressions for the coefficients are unwieldy and do not have unambiguous signs as was the case for all but one of the coefficients in equation (20). However we do observe that

(33) $$\text{trace } (J) = 2\rho > 0$$

still holds, indicating that J has at least one eigenvalue with positive real part. The expression for the determinant J turns out to be

$$(34)\qquad |J| = -\varepsilon^{(1+\beta)/\beta}\left\{\left(\frac{\bar{U}_R}{\bar{U}_c} - \frac{\bar{U}_{cR}}{\bar{U}_{cc}}\right)\left(\bar{\phi}_R + \varepsilon^{\alpha/\beta}\frac{\bar{U}_{cR}}{\bar{U}_{RR}}\right) + \varepsilon^{\alpha/\beta}\frac{\Delta}{\bar{U}_{cc}}\right\}$$

and clearly the sign of $|J|$ is ambiguous for a general form of $U(C,R)$ and will probably vary from one equilibrium to another when multiple equilibria exist. We recall from our discussion on stability at the end of section 3 that in order that any particular equilibrium have the desired stability property it is necessary that

$$(35)\qquad |J| > 0$$

since $|J| = \gamma_1\gamma_2\gamma_3\gamma_4$.

As an example we use the particular functional form

$$(36)\qquad U(C,R) = \log\,(C + \gamma R)$$

with $\gamma > 0$. Equations (15.a-c) and (30) yield a unique equilibrium with $\bar{R}$ given by

$$(37)\qquad \bar{\phi}_R = \rho - \frac{\gamma}{\beta}\left(\frac{\rho}{\alpha}\right)^{\alpha/\beta},$$

and the expression for $|J|$ reduces to

$$(38)\qquad |J| = \frac{\bar{\lambda}_2\bar{\phi}_{RR}}{\bar{U}_{cc}}\left(\frac{\rho}{\alpha}\right)^{2/\beta} > 0$$

Equations (33) and (38) together imply that the matrix J has either four eigenvalues with positive real parts or two eigenvalues with positive real parts and two with negative real parts. In section 3 we eliminated the possibility of four eigenvalues with positive parts by consideration of the sign of B in equation (22.c), however for the utility function given by equation (36) the calculation of B turns out to be extremely tedious and we prefer an alternative approach. In figure 2 we illustrate the λ_1, λ_2 phase plane with R held at some constant value close to $\bar{R}$, since in this phase plane we observe a saddle-point equilibrium it is impossible that all the eigenvalues of J have positive real parts. Thus we are able to apply the same arguments as at the end of section 3 to demonstrate that there is a unique path converging to the equilibrium. Again we emphasise that our analysis is only a local analysis.

5. Conclusion

For the model postulated in section 2 we have found that optimal consumption is determined by the Ramsey savings rule, equation (10), as in the standard optimal growth model, while the optimal flow of replenishable resource is determined by the efficiency condition (11). In the equilibrium towards which the differential system determining the optimal path tends, the stock of replenishable resource will be such as to equate the slope of the natural growth rate function to the discount factor, the rate of return on capital equals the discount factor and all output will be consumed, furthermore the equilibrium will be unique under the assumptions of our model. Finally we have shown that, locally at least, it will always be possible to choose the unknown initial values so as to obtain a unique optimal path converging to the equilibrium.

When the stock of replenishable resource is included in the utility function the optimal consumption is still determined by the Ramsey savings rule but the efficiency condition (11) is modified to (28) by an extra term depending on the ratio of the marginal utility of resource stock to marginal utility of consumption and also on the marginal product of the resource. The differential system determining the optimal path can now have multiple equilibria depending on the particular utility function chosen. In this case the equilibrium towards which the system moves depends on the initial values.

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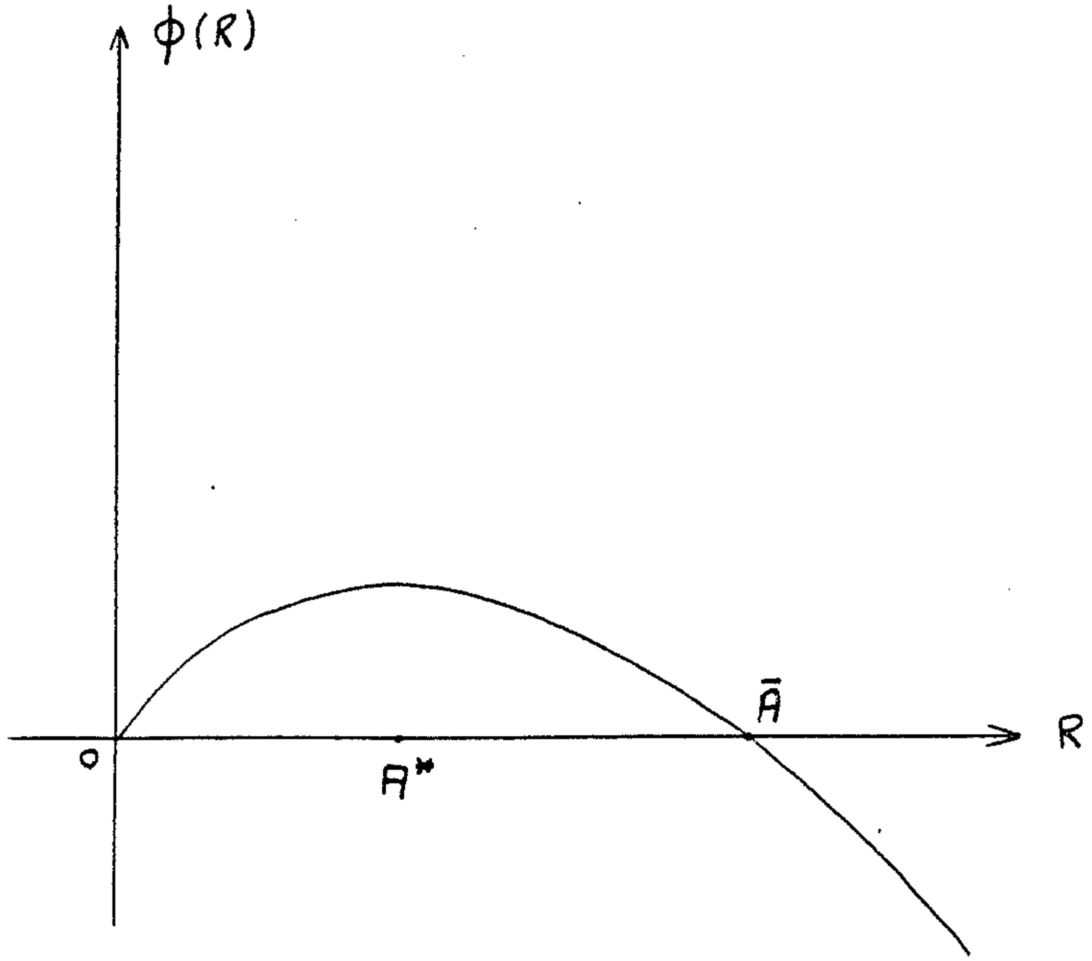
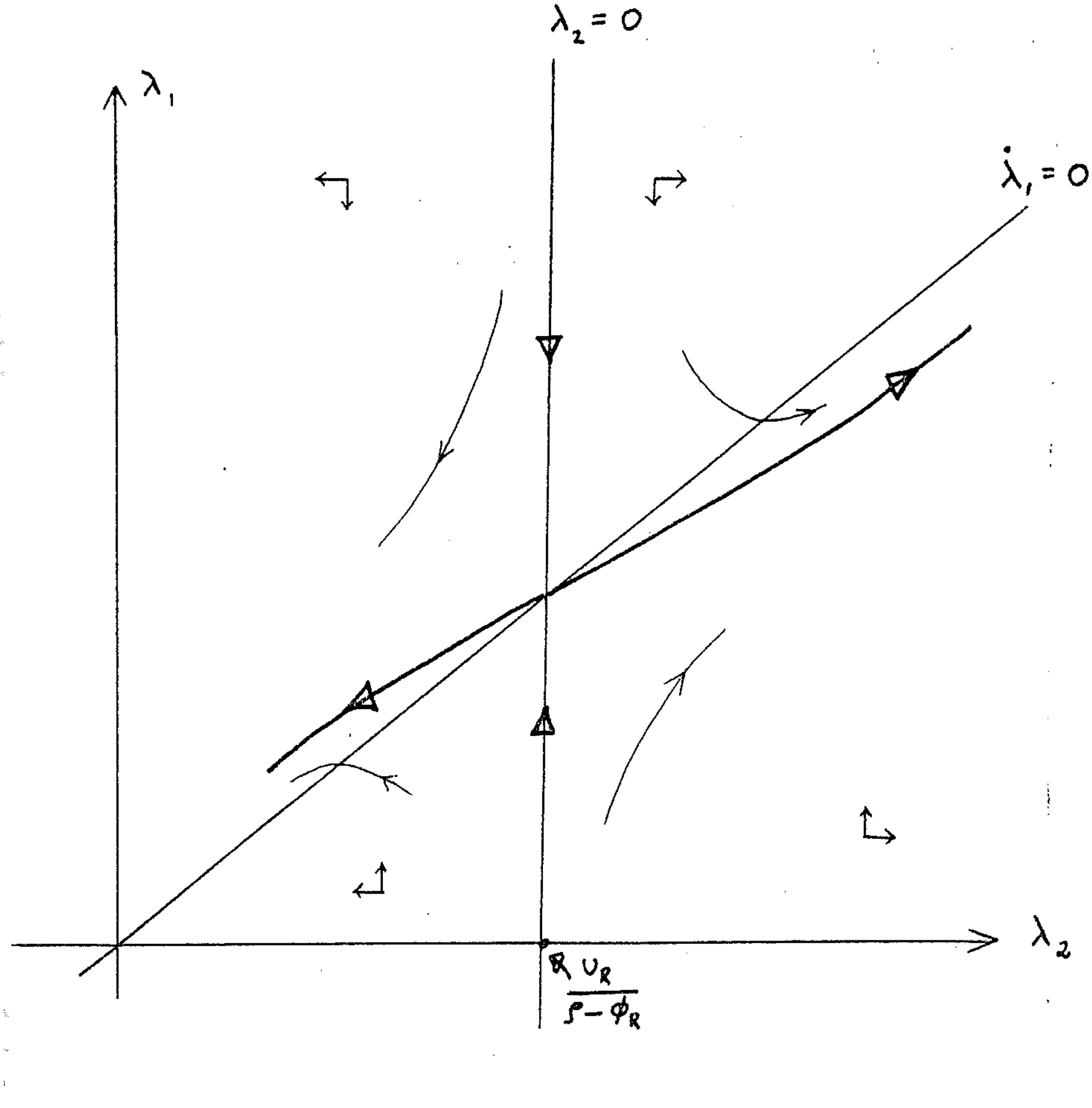


Figure I



$\dot{\lambda}_2 = 0$
λ_1
$\dot{\lambda}_1 = 0$
λ_2
$\frac{R\, v_R}{\rho - \phi_R}$

Figure 2