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## Split Extensions, Actions and Crossed Modules in the categories of Hopf Algebras

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# Résumé

Le but de cette dissertation est de développer et renforcer les interactions entre deux champs de recherche actifs en mathématiques : la théorie des catégories et la théorie des algèbres de Hopf.

La théorie des catégories permet une approche unificatrice des structures mathématiques à priori, non reliées. En particulier, la théorie des catégories semi-abéliennes, dont l'exemple représentatif est donné par la catégorie des groupes, offre un cadre propice pour, entre autres, développer la théorie des commutateurs et l'homologie non-abélienne. D'autre part, les algèbres de Hopf décrivent la symétrie des espaces non-commutatifs, similairement aux groupes décrivant les symétries des espaces classiques. Les algèbres de Hopf peuvent être vues comme une linéarisation de la notion de groupe.

Le premier objectif de cette thèse est d'étudier la catégorie des algèbres de Hopf cocommutatives sur un corps quelconque. Nous montrons que cette catégorie est semi-abélienne. Cela nous permet de comparer des notions et théorèmes existant dans le monde des algèbres de Hopf avec des concepts et résultats correspondants dans le contexte semi-abélien. Dans cette thèse, nous étudions certains exemples de ce type d'interaction pour les algèbres de Hopf cocommutatives : la notion d'action, de produit semi-direct, de commutateur, de module croisé, de carré croisé et le lemme de Zassenhaus.

Le second objectif de cette thèse est d'approfondir notre connaissance de la catégorie des algèbres de Hopf quelconques. En particulier, nous développons une notion adéquate d'extension scindée d'algèbres de Hopf (et de bigèbres). Cette notion est adéquate pour plusieurs raisons. En particulier, les extensions scindées d'algèbres de Hopf (et de bigèbres)

sont équivalentes aux actions d'algèbres de Hopf (et de bigèbres). De plus, le lemme des cinq courts scindés est satisfait quand on le restreint à ces extensions particulières, ce qui ouvre la voie à d'autres développements en algèbre homologique.

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# Introduction

## State of the art

### Semi-abelian categories

In 1945, S. Eilenberg and S. Mac Lane [36] introduced the concepts of category, functor and natural transformation, which turned out to be important in the development of a new approach to several branches of mathematics. This new field of mathematics, called category theory, focuses on the interactions between mathematical objects rather than the objects themselves. Here, “objects” can refer to a lot of different things: sets, groups, matrices, topological spaces, etc. Categories formalize the notion of a “transformation” between objects independently of the type of objects considered. As an example, if we take as objects the members of a family, category theorists would be interested in the relationships between the members of this family. This innovative approach offers new insights and, often, surprising new results. The abstraction of the category theory allows us to forget about the mathematical structures behind a construction and focus on the properties of this construction.

Let us make this statement more explicit by giving an example. In mathematics, we can define the cartesian product of two sets, the direct product of two groups, the intersection of two relations on the same object and the tensor product of two cocommutative coalgebras. At a first sight, these constructions seem totally distinct from one to another. Nevertheless, when we look at them from the perspective of category theory, a beautiful common scheme appears: each of them has the universal property of a categorical product. Given two objects  $A$  and  $B$

in  $\mathcal{C}$ , the categorical product of  $A$  and  $B$  is a unique (up to isomorphism) diagram

$$\begin{array}{ccc} & A \times B & \\ \pi_A \swarrow & & \searrow \pi_B \\ A & & B \end{array}$$

such that for any  $f: C \rightarrow A$  and any  $g: C \rightarrow B$ , there exists a unique morphism  $\psi: C \rightarrow A \times B$  such that  $\pi_A \cdot \psi = f$  and  $\pi_B \cdot \psi = g$ . We can represent this property in a diagrammatic way by saying that the following triangles commute

$$\begin{array}{ccccc} & & A \times B & & \\ & \pi_A \swarrow & \uparrow \psi & \searrow \pi_B & \\ A & & & & B \\ & \nwarrow f & & \nearrow g & \\ & & C & & \end{array}$$

This observation allows us to deal with the cartesian product of two sets, the direct products of two groups, the intersection of two relations on the same object, and the tensor product of two cocommutative coalgebras as a unique construction. It is a simple example of the power of unification of category theory.

In the fifties, the important notion of abelian category was introduced [67, 23, 53]. Among the examples of abelian categories, there are abelian groups (which is the “model example”), vector spaces, modules on a ring. This notion offers an ideal framework to develop a good notion of homology and to capture some typical properties of abelian groups. A short definition of an abelian category is given by asking that the category has finite products, a zero object, and that any morphism factors into a normal epimorphism followed by a normal monomorphism. This last condition is verified for the abelian groups since the image  $f(A)$  of a morphism of abelian groups  $f: A \rightarrow B$  is a normal subgroup in  $B$ . For a morphism in the category of general groups, the image is not a normal subgroup of the codomain. This implies that the category of groups is not abelian. For a long time, categorists have been trying to complete

the following schematic picture and looked for a good notion of category that could capture the properties of the category of groups, similarly to the way the notion of abelian category captures the properties of abelian groups.

|                    |        |
|--------------------|--------|
| Abelian groups     | Groups |
| Abelian categories | ?      |

After several attempts, the theory of semi-abelian categories was introduced by [58]. Other representative examples of semi-abelian categories are given by the category of associative rings and the category of Lie algebras. Semi-abelian categories provide a good context to obtain similar properties and results as in groups: Noether's isomorphism theorems, the Snake Lemma, the Split Short Five Lemma, the Jordan-Hölder theorem, the Zassenhaus Lemma, etc. More precisely, they offer us a general framework to develop and study the (co)homology of algebraic structures [41], commutator theory [16], radical and torsion theories [17], actions [10, 19] and crossed modules [57]. The notion of semi-abelian category can be seen as a non-additive version of the classical one of abelian category. More precisely, the additivity in the definition of abelian categories is replaced by the protomodularity (in the sense of [11]) and the zero object in the definition of semi-abelian categories. It is also interesting to observe that in a semi-abelian category, there is no notion of biproduct. There is no isomorphism between the binary coproduct of two objects and their product. Instead, the canonical morphism  $A + B \rightarrow A \times B$  is a normal epimorphism [58].

## Hopf algebras

In 1941, H. Hopf introduced the notion of Hopf algebras when he was studying the homology of Lie groups.

A Hopf algebra can be interpreted as a “linear” version of a group. It relates to a vector space, as a group does to a set. Diagrammatically, the notion of Hopf algebra completes the following figure.

|               |        |
|---------------|--------|
| Sets          | Groups |
| Vector spaces | ?      |

A Hopf algebra is a vector space, endowed with operations satisfying suitable axioms which are similar to the ones of groups. More precisely, a group is a set,  $G$ , endowed with an associative multiplication,  $m: G \times G \rightarrow G$ , a unit element,  $u_G \in G$ , and an inverse,  $(\cdot)^{-1}: G \rightarrow G$ . Diagrammatically, the inverse condition of a group can be expressed as follows

$$\begin{array}{ccccc}
 & & G \times G & \xrightleftharpoons[(\cdot)^{-1} \times 1_G]{1_G \times (\cdot)^{-1}} & G \times G \\
 & \nearrow \Delta & & & \searrow m \\
 G & \xrightarrow{e} & \{*\} & \xrightarrow{u} & G.
 \end{array} \quad (0.1)$$

where  $\Delta(g) = (g, g)$  is the diagonal map,  $e$  sends any element to the singleton and  $u: \{*\} \rightarrow G$ , sends the singleton to the neutral element  $u_G \in G$ . By “linearization” of this structure, we obtain a vector space over a field  $K$  endowed with a multiplication  $m: A \otimes A \rightarrow A$ , a unit  $u_A: K \rightarrow A$ , algebra morphisms  $\Delta: A \rightarrow A \otimes A$  and  $\epsilon_A: A \rightarrow K$ , called comultiplication and counit, and a morphism  $S: A \rightarrow A$ , called antipode, playing the role of taking inverses in a group. These morphisms have to satisfy some natural conditions (see Section 1.3), and in particular they make the following diagrams commute

$$\begin{array}{ccccc}
 & & A \otimes A & \xrightleftharpoons[S \otimes 1_A]{1_A \otimes S} & A \otimes A \\
 & \nearrow \Delta & & & \searrow m \\
 A & \xrightarrow{\epsilon_A} & K & \xrightarrow{u_A} & A.
 \end{array} \quad (0.2)$$

Such a structure is called a Hopf algebra over a field  $K$ .

The concept of a group is closely related to the idea of symmetry. More precisely, the notion of a group acting on a set conceptualizes the permutations (hence symmetries) of this set. Let us illustrate this notion with a few examples. The symmetric group  $S_n$  acts on the set  $\{1, 2, \dots, n\}$  by permuting the elements of the set. The dihedral group  $D_{2n}$  (generated by a rotation of  $2\pi/n$  and a reflection), acts on a regular  $n$ -sided polygon via the isomorphic transformations of a regular  $n$ -sided polygon. The groups  $Gl(n, \mathbb{R})$ , consisting of all  $n \times n$  real invertible matrices, and  $\mathbb{A}(n, \mathbb{R}) = \{x \mapsto Ax + b \mid x, b \in \mathbb{R}^n, A \in Gl(n, \mathbb{R})\}$  act on  $\mathbb{R}^n$ . The

group  $\mathbb{R}^m$  acts on itself by translation. Similarly, Hopf algebras can be understood as the algebraic objects describing the symmetries of non-commutative spaces, just as groups describe those of usual geometries. Indeed, there are several canonical ways to construct Hopf algebras out of groups. Firstly, for a given group  $G$  and a field  $K$ , one can endow the group algebra  $K[G]$  with a Hopf algebra structure. Moreover, if  $G$  acts on a set  $X$  then the vector space generated by the element of  $X$ ,  $K[X]$ , becomes a module (coalgebra) over the Hopf algebra  $K[G]$ . Secondly, for an algebraic group  $G$ , its coordinate algebra  $\mathcal{O}(G)$  is a Hopf algebra and if  $G$  acts on an algebraic variety  $X$ , then  $\mathcal{O}(X)$  becomes in a natural way a  $\mathcal{O}(G)$ -comodule algebra. Henceforth, actions of groups are directly related to (co)actions of Hopf algebras, which are one of the main subjects of this thesis.

Let us give a basic example by looking at the quantum plane,

$$K_q[x, y] = \frac{K\langle x, y \rangle}{I_q},$$

where  $K$  is a field,  $K\langle x, y \rangle$  is the free algebra on the indeterminates  $x$  and  $y$ ,  $I_q$  is the two-sided ideal generated by  $xy - qyx$ . Note that when  $q = 1$ , the algebra  $K_q[x, y]$  is the coordinate algebra of the affine plane of dimension two.

Just as  $Gl(2, \mathbb{R})$  acts on the plane  $\mathbb{R}^2$  and the Hopf algebra  $\mathcal{O}(Gl(2, \mathbb{R}))$  coacts on  $\mathcal{O}(\mathbb{R}^2) = \mathbb{R}[x, y]$ , one can consider a Hopf algebra  $Gl_q(2)$ , which coacts on  $K_q[x, y]$ . This Hopf algebra  $Gl_q(2)$  is called a “quantum group” and is obtained by deforming the multiplication of  $\mathcal{O}(Gl(2, \mathbb{R}))$  using the parameter  $q$  (see [61] for an introduction to quantum groups).

Another motivation to study Hopf algebras comes from representation theory. In classical representation theory of groups, one of the important properties is that the tensor product of two representations and the vector space of linear maps between two representations are again representations of the same group. In fact, one can observe that for any Hopf algebra one has the same phenomenon. This can be expressed by saying that the category of modules over a Hopf algebra is a closed monoidal category admitting a closed strict monoidal forgetful functor to the category of vector spaces. This closed monoidal structure on the category of modules over a Hopf algebra can be described explicitly by

means of the comultiplication, counit and antipode and quite remarkably by Tannaka duality, this procedure can be reversed, which allows us to (re)construct the comultiplication, counit and antipode out of the monoidal structure on its category of modules.

Apart from the link with group theory, as already commented upon above, the theory of Hopf algebras has been developed intensively and examples showed up in several fields of mathematics including Lie theory (the universal enveloping algebra of a Lie algebra can be endowed with the structure of a Hopf algebra), Galois theory [77], combinatorics [2], quantum mechanics [69], and knot invariants [63].

### **An intersection between category theory and Hopf algebras**

This thesis will develop and strengthen the interactions between these two active research fields of mathematics: categorical algebra and the theory of Hopf algebras.

Although both fields of Hopf algebras and semi-abelian categories are strongly motivated by group theory, until recently, there was little interaction between these two scientific communities, and they flourished independently of each other. However, the category  $\mathbf{Hopf}_{K, \text{coc}}$  of cocommutative Hopf algebras is known to share many exactness properties with the categories of groups and Lie algebras. It was already noted by H. Yanagihara in [96, 97] that some classical isomorphism theorems from group theory have their natural counterpart for cocommutative Hopf algebras (this work was partly based on a theorem by Newman [79]). More recently, it was shown in [48] that the category  $\mathbf{Hopf}_{K, \text{coc}}$  of cocommutative Hopf algebras over a field of characteristic zero is semi-abelian. This fact implies that many of the exactness properties of the category of cocommutative Hopf algebras follow directly from the axioms of semi-abelian categories, and this has opened the way to explore some new connections between categorical algebra and Hopf algebra theory [49, 93, 45]. This result initiated a new line of research based on interactions between Hopf algebras and semi-abelian categories.

An aspect of this research comes from the possibility of applying categorical concepts and results to obtain a deeper understanding of Hopf

algebra theory. Existing concepts and theorems within the realm of Hopf algebras, which have been introduced in an “ad hoc” way, can be compared to corresponding categorical concepts and theorems, which have a wider range of applications. For instance, the notions of “split extensions”, “actions” and “semi-direct products” have been studied both in Hopf algebras [4, 5, 76] and in category theory [10, 18]. Similarly, the notion of a “crossed module” has been defined and investigated independently within semi-abelian category theory [57] and Hopf algebra theory [70, 43, 7, 44].

One of the goals of this thesis is to study these notions at the frontier between Hopf algebras and category theory.

As explained, Hopf algebra theory and semi-abelian category theory are both mathematical fields closely related to group theory. It is therefore natural to observe the notions of actions, split extensions and crossed modules of groups.

## Group actions

We first introduce the notions of actions and split extensions of groups and describe the equivalence between them.

As mentioned before, an action of a group is an interesting notion that describes the symmetries of a space. One can consider actions of groups on different mathematical structures such as sets, vector spaces, or groups. For the purpose of this thesis, we are interested in the actions of a group on a group.

An action of a group  $(G, u_G)$  on a group  $(X, u_X)$  is a map  $\triangleright: G \times X \rightarrow X: (g, x) \mapsto g \triangleright x$  such that for any  $g, g' \in G$  and  $x, x' \in X$  the following identities hold

$$\begin{aligned} (gg' \triangleright x) &= (g \triangleright (g' \triangleright x)), \\ (u_G \triangleright x) &= x, \\ g \triangleright (xx') &= (g \triangleright x)(g \triangleright x'). \end{aligned}$$

A split extension of groups is a diagram of the form

$$0 \longrightarrow \text{Ker}(f) \xrightarrow{k} G \xleftarrow[s]{s} B \longrightarrow 0 \quad (0.3)$$

where  $G$  and  $B$  are groups,  $\text{Ker}(f) = \{x \in G \mid f(x) = u_B\}$  is the kernel of  $f$ ,  $k$  is the inclusion,  $s$  and  $f$  are group morphisms, and  $f \cdot s = 1_B$ .

Split extensions of groups correspond to group actions. From the split extension (0.3), we obtain the following action

$$\triangleright: B \times \text{Ker}(f) \rightarrow \text{Ker}(f): (b, x) \mapsto s(b)k(x)s(b)^{-1}. \quad (0.4)$$

In the other direction, from  $\triangleright: B \times X \rightarrow X$ , we can construct (via the semi-direct product) the following split extension

$$0 \longrightarrow \text{Ker}(f) \xrightarrow{i_1} \text{Ker}(f) \rtimes B \xleftarrow[p_2]{i_2} B \longrightarrow 0$$

where  $p_2(x, b) = b$ ,  $i_1(x) = (x, 1)$  and  $i_2(b) = (1, b)$ . The semi-direct product  $\text{Ker}(f) \rtimes B$  is the set  $\text{Ker}(f) \times B$  endowed with the following group structure for  $x, y \in \text{Ker}(f)$  and  $b, c \in B$

$$(x, b)(y, c) = (x(b \triangleright y), bc).$$

These constructions lead to the following result:

**Proposition 0.0.1.** *There is an equivalence of categories between the category of actions of groups on groups and the category of split extensions of groups.*

This result allows us to decompose certain groups into two simpler groups. We give some examples.

Let  $D_{2n}$  be the dihedral group, the group of symmetries of a regular  $n$ -sided polygon. This group has order  $2n$ , possesses  $n$  rotations and  $n$  reflections, and is generated by a rotation of  $2\pi/n$  (denoted by  $\theta$ ) and a reflection (denoted by  $r$ ). The following sequence is a split extension of groups:

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{k} D_{2n} \xleftarrow[f]{s} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 ,$$

where  $\mathbb{Z}/n\mathbb{Z}$  is the group of integers modulo  $n$ , the morphism  $k: \mathbb{Z}/n\mathbb{Z} \rightarrow D_{2n}$  is defined via  $k(\bar{1}) = \theta$ , the morphism  $f: D_{2n} \rightarrow \mathbb{Z}/2\mathbb{Z}$  is given by  $f(\theta) = \bar{0}$ ,  $f(r) = \bar{1}$ , and the morphism  $s: \mathbb{Z}/2\mathbb{Z} \rightarrow D_{2n}$  is defined as  $s(\bar{1}) = r$  and  $s(\bar{0}) = Id$ . Hence, we have the following group isomorphism:  $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ .

The semi-direct product allows us to give another interpretation of the group of all the affine automorphisms of  $\mathbb{R}^n$ :  $\mathbb{A}(n, \mathbb{R}) = \{x \mapsto Ax + b \mid x, b \in \mathbb{R}^n, A \in Gl(n, \mathbb{R})\}$ . We have a split extension of groups

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{k} \mathbb{A}(n, \mathbb{R}) \xrightleftharpoons[f]{s} Gl(n, \mathbb{R}) \longrightarrow 0 ,$$

where  $k: \mathbb{R}^n \rightarrow \mathbb{A}(n, \mathbb{R}): b \mapsto (x \mapsto x + b)$ ,  $f: \mathbb{A}(n, \mathbb{R}) \rightarrow Gl(n, \mathbb{R}): (x \mapsto Ax + b) \mapsto A$ , and  $s: Gl(n, \mathbb{R}) \rightarrow \mathbb{A}(n, \mathbb{R}): A \mapsto (x \mapsto Ax)$ . Hence, we have  $\mathbb{A}(n, \mathbb{R}) \cong \mathbb{R}^n \rtimes Gl(n, \mathbb{R})$ .

More generally, in any semi-abelian category, there is an equivalence between (internal) actions and split extensions ([10] and [19]). This result allows us to link two different notions.

The equivalence of Proposition 0.0.1 is also the building block to obtain an equivalence between crossed modules and internal groupoids in the category of groups.

But first, let us answer the following question: “What is a crossed module of groups?”

## Crossed modules of groups

Motivated by the study of the properties of the boundary map

$$\partial: \pi_2(X, A, x) \rightarrow \pi_1(A, x),$$

relating the second relative homotopy group and the first homotopy group of a pair of pointed topological spaces, J.H.C Whitehead introduced the notion of crossed module in 1949 [94]. A crossed module of groups is given by a morphism of groups

$$d: X \rightarrow B,$$

endowed with an action of  $B$  on  $X$  ( $\triangleright: B \times X \rightarrow X$ ) in a way that is “compatible” with the action given by conjugation

$$\begin{aligned} d(b \triangleright x) &= bd(x)b^{-1}, \\ d(x') \triangleright x &= x'xx'^{-1}, \end{aligned} \tag{0.5}$$

(see Definition 3.2.8). Later on, crossed modules attracted a lot of attention as an algebraic structure in its own right. R. Brown and C. Spencer [85] proved the equivalence between crossed modules of groups and internal groupoids in the category of groups (a result that they credit to Verdier). On the one hand, a crossed module of groups can be seen as the “normalization” of an internal groupoid. Indeed, if

$$\begin{array}{ccc} & \xrightarrow{\delta} & \\ A_1 & \xleftarrow{\iota} & A_0 \\ & \xrightarrow{\gamma} & \end{array} \tag{0.6}$$

is the reflexive graph underlying an internal groupoid structure in the category of groups ( $\delta$  is the “domain morphism”,  $\gamma$  the “codomain morphism”,  $\iota$  the “identity morphism”, see Definition 1.6.7), its normalization

$$\gamma|_{\text{Ker}(\delta)}: \text{Ker}(\delta) \rightarrow A_0,$$

endowed with the action given in (0.4) is a crossed module of groups. On the other hand, we can construct an internal groupoid from a crossed module of groups thanks to the semi-direct product. When  $d: X \rightarrow B$  is a crossed module of groups, the following diagram

$$\begin{array}{ccc} & \xrightarrow{p_2} & \\ X \rtimes B & \xleftarrow{i_2} & B \\ & \xrightarrow{p_1} & \end{array},$$

where  $p_2$  is the projection on  $B$ ,  $i_2$  is the inclusion of  $B$  and  $p_1$  is defined as  $p_1(x, b) = d(x)b$  is underlying a unique internal groupoid structure in the category of groups. These constructions yield the following result.

**Proposition 0.0.2.** [85] *There is an equivalence between the category of crossed modules and the category of internal groupoids in groups.*

This equivalence allows one to deal with the concept of internal groupoid in an alternative way, that is useful for computations [39].

Note that, similarly to the case of internal groupoids, reflexive graphs also have a “crossed” counterpart. There exists an equivalence between precrossed modules (which are defined as crossed modules without the Peiffer condition (6.9)) and the reflexive graphs in groups.

**Proposition 0.0.3.** *There is an equivalence of categories between the category of precrossed modules and the category of reflexive graphs in groups.*

In [62] R. Lavendhomme and J.-R. Roisin were interested in the cohomology of algebraic structures, and they showed similar equivalences in the context of Lie algebras and rings. G. Guin-Waléry and J.L. Loday in [55, 64] studied the relation between crossed modules and the so-called *1-cat-groups*, which are reflexive graphs as in (0.6) such that the kernels of  $\delta$  and  $\gamma$  have a trivial commutator:  $[Ker(\delta), Ker(\gamma)] = \{1\}$ .

More recently, G. Janelidze discovered that the theory of semi-abelian categories offers an adequate framework to “internalize” the notions of precrossed module and of crossed module. In 2003, besides defining the categorical concepts of internal precrossed module and crossed module, he proved two equivalences in any semi-abelian category [57]: one between internal crossed structures and internal groupoids, and another one between internal precrossed structures and internal reflexive graphs. In the category of groups and the category of Lie algebras, he recovers the main results of [85] and [62] on this topic.

## Crossed squares of groups

In the category of groups (or in any semi-abelian category), the construction of crossed modules can be repeated, since the category of internal groupoids in a semi-abelian category is semi-abelian [46, 15]. One obtains a “2-dimensional” version of the equivalence between internal crossed modules and internal groupoids, providing an equivalence between internal double crossed modules and internal double groupoids. The authors of [55, 64] highlighted a relation between what they called *crossed squares of groups* and *2-cat-groups*. A few years later G. Ellis introduced the notion of *cat<sup>2</sup>-Lie algebra* (and, more generally, of *cat<sup>n</sup>-Lie algebra*) and obtained analogous results for Lie algebras [37]. This higher dimensional

equivalence (in groups) allows one to consider a double groupoid in the category of groups

$$\begin{array}{ccc}
 G_{\{1,2\}} & \begin{array}{c} \xrightarrow{\gamma_2} \\ \xleftarrow{\iota_2} \\ \xrightarrow{\delta_2} \end{array} & G_2 \\
 \begin{array}{c} \downarrow \delta_1 \\ \uparrow \iota_1 \\ \downarrow \gamma_1 \end{array} & & \begin{array}{c} \downarrow \delta'_1 \\ \uparrow \iota'_1 \\ \downarrow \gamma'_1 \end{array} \\
 G_1 & \begin{array}{c} \xrightarrow{\gamma'_2} \\ \xleftarrow{\iota'_2} \\ \xrightarrow{\delta'_2} \end{array} & G_\emptyset
 \end{array}$$

as a commutative square with *compatible* actions – called *crossed square of groups* –

$$\begin{array}{ccc}
 L & \xrightarrow{\lambda} & M \\
 \downarrow \lambda' & & \downarrow \mu \\
 M' & \xrightarrow{\mu'} & P.
 \end{array}$$

The quite complicated structure of double groupoids is equivalent to a commutative square of groups with some suitably defined and compatible actions, which turns out to be easier to handle in some relevant cases. In particular, in [39], G. Ellis used the notion of crossed square of groups to cover the homotopy classification of maps from a 3-dimensional connected CW-space  $X$  to some other CW-space. In [22], R. Brown and J.L. Loday proved a Van Kampen theorem for crossed squares of groups.

### Actions and split extensions of monoids and magmas

We recalled in this introduction an equivalence between actions and split extensions in the category of groups (Proposition 0.0.1). This equivalence holds in any semi-abelian category.

Unfortunately, the category of general Hopf algebras is not a semi-abelian category [45]. Nevertheless, the category of monoids will offer us a strategy to obtain a variation of this equivalence.

In the category of monoids, even if there is no equivalence between split extensions and monoid actions, we can observe that some split sequences have “good properties” while others do not. We illustrate this observation with the following examples [20]

$$0 \longrightarrow \mathbb{N} \xrightarrow{i_2} \mathcal{O}_{\mathbb{N}} \xrightleftharpoons[\pi_1]{\delta} \mathbb{N} \longrightarrow 0, \quad (0.7)$$

$$0 \longrightarrow \{0\} \xrightarrow{i} \mathcal{O}_{\mathbb{N}} \xrightleftharpoons[\pi_2]{\delta} \mathbb{N}, \quad (0.8)$$

where  $\mathcal{O}_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\}$ ,  $\pi_1(n, m) = n$ ,  $\pi_2(n, m) = m$ ,  $i(0) = (0, 0)$ ,  $i_2(n) = (0, n)$  and  $\delta(p) = (p, p)$ .

We observe that both sequences are split epimorphisms of monoids, given with their kernels. For the sequence (0.7), similarly to the case of groups, we can construct an isomorphism between  $\mathcal{O}_{\mathbb{N}}$  and  $\text{Ker}(\pi_1) \rtimes \mathbb{N} = \mathbb{N} \times \mathbb{N}$  (the equality holds since the action of  $\mathbb{N}$  on  $\mathbb{N}$  is the trivial one). Unfortunately, the sequence (0.8) does not satisfy a similar property since  $\mathcal{O}_{\mathbb{N}}$  is not isomorphic to  $\mathbb{N}$ . We can also notice that in the sequence (0.7),  $i_2$  and  $\delta$  are jointly epimorphic. On the contrary, the morphisms  $i$  and  $\delta$  of the sequence (0.8) are not.

Not all split epimorphisms of monoids are “good split extensions”, in the sense that they do not have properties similar to the ones in groups. This observation was one of the motivations for the authors of [20, 72] to introduce the notion of Schreier split epimorphism. They proved in [20], that there exists a restricted equivalence between the actions of monoids and the “Schreier split epimorphisms”. The terminology “Schreier split epimorphism” comes from a paper by A. Patchkoria [82], who worked on a notion of a Schreier internal category in the category of monoids and proved that the category of Schreier internal categories in the category of monoids is equivalent to the category of crossed semimodules.

A further step was done in a recent paper on split extensions of unitary magmas, where a suitable notion of split extension of magmas was introduced and shown to correspond to actions [47]. Another example is the study of split extensions in the full subcategory of preordered monoids whose preorder is determined by the corresponding positive cone [73].

## Our contribution

The purpose of this thesis is to study categorical concepts and properties of Hopf algebras. We explore, among others, the concepts of split extensions, actions, commutators, crossed modules and crossed squares.

To achieve this goal we first examine the category of cocommutative Hopf algebras over a field  $K$  and subsequently, the category of (non-associative) Hopf algebras in a symmetric monoidal category. These categories provide us with two different contexts to investigate properties and categorical notions of Hopf algebras.

In **Chapter 1**, we recall several definitions and results and fix the notation.

The remainder of the thesis is divided into two parts. In the first part, we focus on cocommutative Hopf algebras over a field.

Recently, it was observed in [48] that, when the base field  $K$  has characteristic zero, the category  $\mathbf{Hopf}_{K,\text{coc}}$  is semi-abelian (in the sense of [58]). The proof given in [48] depends on the Milnor-Moore theorem providing a canonical decomposition of any cocommutative Hopf algebra as a semi-direct product (also called smash product in the literature) of a group Hopf algebra acting on a primitively generated Hopf algebra [75]. This theorem holds when the characteristic of the field  $K$  is zero so that the fact that  $\mathbf{Hopf}_{K,\text{coc}}$  is semi-abelian could only be proved under this additional assumption.

In **Chapter 2**, we prove that the category of cocommutative Hopf algebras over *any* field is semi-abelian. From this result follows a well-known theorem of M. Takeuchi [89] saying that the category of commutative and cocommutative Hopf algebras is abelian. These results appeared in a joint paper with M. Gran and J. Vercruysse in *Journal of Pure and Applied Algebra*, [50]. The fact that  $\mathbf{Hopf}_{K,\text{coc}}$  is semi-abelian allows us to obtain new results in the category of cocommutative Hopf algebras. More precisely, we give an explicit description of the Huq commutator of two normal subobjects, [50]. We also provide an explicit version of the Zassenhaus Lemma for the cocommutative Hopf algebras. This latter result appeared in a joint paper with Olivette Ngaha Ngaha in *Theory and Applications of Categories*, [81].

In **Chapter 3**, we investigate the notions of crossed modules and of crossed squares of cocommutative Hopf algebras. First, we recall a well-known equivalence between actions of Hopf  $K$ -algebras and split extensions of Hopf  $K$ -algebras, similar to Proposition 0.0.1 (see [76], [50], [7], [93]). We use this equivalence to prove that the notion of crossed modules introduced in [70] or in [43] (their definitions coincide in the case of cocommutative Hopf algebras) is equivalent to the notion of internal crossed modules in the sense of G. Janelidze [57]. Since  $\mathbf{Hopf}_{K, \text{coc}}$  is semi-abelian (as proved in Chapter 2), we know that such a construction exists. By showing that the definition of crossed modules of cocommutative Hopf algebras of [70] (or [43]) is equivalent to the one of internal groupoid of cocommutative Hopf algebras, we have an alternative and explicit description of the internal crossed modules in the sense of G. Janelidze in the category of cocommutative Hopf  $K$ -algebras, [50]. It follows from the equivalence between crossed modules of cocommutative Hopf algebras and internal groupoids that the category of crossed modules of cocommutative Hopf  $K$ -algebras is semi-abelian [15, 46]. Hence, the construction of crossed modules can be repeated. Inspired by the results in groups, we define crossed squares of cocommutative Hopf  $K$ -algebras and show the equivalence between this notion and the internal double groupoids of cocommutative Hopf  $K$ -algebras. This new definition of Hopf crossed squares generalizes the notions of crossed squares of groups and of Lie algebras. These results about Hopf crossed squares appeared in an article in the *Journal of Algebra* [86].

After a better understanding of the category of cocommutative Hopf  $K$ -algebras, we move to more general Hopf algebras. In the second part of this thesis, to deepen the understanding of the notions of split extensions, actions and precrossed modules in Hopf algebras, we investigate non-associative non-cocommutative Hopf algebras in any symmetric monoidal category.

Inspired by [47], we study the non-associative bialgebras and Hopf algebras. In particular, cocommutative (non-associative) bialgebras over a symmetric monoidal category  $\mathcal{C}$  can be seen as internal monoids (or magmas) in the category of cocommutative coalgebras over  $\mathcal{C}$ . Hence, the observations about the internalized magmas in a category with finite

products given in [47] can be applied and give an appropriate notion of split extension for cocommutative non-associative bialgebras.

Now a few natural questions come to mind: “What happens in the non-cocommutative case?”, “How to define split extensions of general Hopf algebras that are equivalent to actions of Hopf algebras?” and “How to define precrossed modules of general Hopf algebras?”

In **Chapter 4**, by using the semi-direct product of non-associative bialgebras, we define split extensions of (non-associative) bialgebras. We prove several properties satisfied by these extensions, in particular, that they are exact sequences and that they are equivalent to the actions of (non-associative) bialgebras. The validity of the Split Short Five Lemma is an important property of any semi-abelian category. Even if the category of bialgebras is not a protomodular category [45], we prove that for the particular split extensions of non-associative bialgebras the Split Short Five Lemma holds. All the definitions and results for associative bialgebras follow from the non-associative case.

In **Chapter 5**, we use a similar method as in Chapter 4 and we define split extensions of non-associative Hopf algebras that are equivalent to the actions of non-associative Hopf algebras, in any symmetric monoidal category. As a corollary, we obtain a similar result for associative Hopf algebras. We give several enlightening examples of split extensions of Hopf algebras, and we compare our definition to some notions of split extensions of Hopf algebras appearing in the literature [5, 7].

The paper [87] containing all the definitions and results obtained in Chapter 4 and Chapter 5 will appear in *Applied Categorical Structures*.

We observed that in groups (or in any semi-abelian category) the actions form an underlying structure for the definition of precrossed modules. Similarly, split extensions are a part of the definition of reflexive graphs. It seems then natural to use the results and definitions in Chapter 5, to study the precrossed modules of (non-associative) Hopf algebras, and how they can be related to internal structures.

In any semi-abelian category, there is an equivalence between a “pre-crossed” construction and an “internal” construction. Even if the category of (non-associative) Hopf algebras is not semi-abelian [45], we can still expect that a variation of this kind of equivalence exists. The main

example we have in mind is the category of monoids in which there is an equivalence between crossed semimodules and Schreier internal categories [82].

In **Chapter 6**, based on the definitions of actions of (non-associative) Hopf algebras and split extensions of (non-associative) Hopf algebras, we define the precrossed modules of (non-associative) Hopf algebras and the *S-reflexive graphs* and show they are equivalent. A similar equivalence for the associative Hopf algebras follows from this result and, in particular, we recover the equivalence for groups, Proposition 0.0.2.

This chapter ends our journey through Hopf algebras and category theory. In the last chapter, we include some concluding remarks and a discussion of future research.



# Chapter 1

## Preliminaries

In this chapter, we give some definitions and fix notation about the required knowledge for the rest of the document.

### 1.1 Basic constructions in a category

We recall some basics constructions defined in any category. We refer to [66] for a more complete explanation of the basis of category theory.

**Definition 1.1.1.** *In any category  $\mathcal{C}$ , an initial object is an object  $I$  in  $\mathcal{C}$  such that for any object  $X$  in  $\mathcal{C}$  there exists a unique morphism from  $I$  to  $X$ .*

*Dually, an object  $T$  of  $\mathcal{C}$  is called terminal, if, for any object  $X \in \mathcal{C}$ , there exists a unique morphism from  $X$  to  $T$ .*

*An object which is both initial and terminal is called a zero object. If such an object exists, we say that the category is pointed.*

*The zero morphism between two objects is the unique morphism that factors through the zero object.*

The category of vector spaces over a field  $K$   $\text{Vect}_K$ , the category of groups  $\text{Grp}$ , and the category of (non necessarily unitary) algebras  $\text{Alg}_K$  are pointed categories, whose the zero object is the trivial vector space,

the trivial group, and the trivial algebra, respectively. The category of sets **Set** is not pointed, the initial object is the empty set and the terminal object is a singleton.

An important construction is the binary product.

**Definition 1.1.2.** *Let  $A$  and  $B$  be two objects in a category  $\mathcal{C}$ , the binary product of  $A$  and  $B$  is an object of  $\mathcal{C}$ , denoted by  $A \times B$ , endowed with two morphisms,  $\pi_A: A \times B \rightarrow A$  and  $\pi_B: A \times B \rightarrow B$  such that for any  $f: C \rightarrow A$  and any  $g: C \rightarrow B$ , there exists a unique morphism  $\psi: C \rightarrow A \times B$  such that the two following triangles commute*

$$\begin{array}{ccccc} A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \\ & \nwarrow f & \uparrow \psi & \nearrow g & \\ & & C & & \end{array}$$

Let us give some examples of the categorical product. In  $\mathbf{Vect}_K$ , the product is given by the direct sum endowed with the projections. In **Set**, the product is given by the cartesian product endowed with the projections.

We give the definition of the equalizer of two arrows in a category.

**Definition 1.1.3.** *The morphism  $e: E \rightarrow A$ , in  $\mathcal{C}$ , is the equalizer of  $f, g: A \rightarrow B$  if  $f \cdot e = g \cdot e$  and if for any morphism  $h: X \rightarrow A$  such that  $f \cdot h = g \cdot h$  there exists a unique morphism  $\tilde{h}: X \rightarrow E$  such that  $e \cdot \tilde{h} = h$*

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \xrightarrow[f]{g} & B \\ & \nearrow \tilde{h} & \nearrow h & & \\ & X & & & \end{array}$$

This notion of equalizers allows us to define the useful concept of kernels in a pointed category.

**Definition 1.1.4.** *In a pointed category  $\mathcal{C}$ , the kernel of an arrow  $f: A \rightarrow B$  is the equalizer of  $f$  and the zero morphism  $0: A \rightarrow B$ .*

In  $\mathbf{Vect}_K$  and  $\mathbf{Grp}$ , the categorical kernels coincide with the classical kernels.

Another important notion is the pullback. The pullbacks can be constructed via the product and the equalizer, and it is defined as follows.

**Definition 1.1.5.** *Let  $f: A \rightarrow C$  and  $g: B \rightarrow C$  be two morphisms of  $\mathcal{C}$ , the pullback of these arrows is given by the object  $A \times_C B$  endowed with two projections  $p_A, p_B$  making the following diagram commutes*

$$\begin{array}{ccc} A \times_C B & \xrightarrow{p_B} & B \\ p_A \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C, \end{array}$$

and satisfying the universal property that for any  $a: X \rightarrow A$  and any  $b: X \rightarrow B$  such that  $g \cdot b = f \cdot a$ , there exists an arrow  $(a, b): X \rightarrow A \times_C B$  making the two following triangles commute

$$\begin{array}{ccccc} X & & & & \\ & \searrow^{(a,b)} & & \searrow^b & \\ & A \times_C B & \xrightarrow{p_B} & B & \\ & p_A \downarrow & & \downarrow g & \\ & A & \xrightarrow{f} & C. & \\ & \swarrow_a & & & \end{array}$$

In  $\mathbf{Set}$ , the object part of a pullback of two functions  $f: A \rightarrow C$  and  $g: B \rightarrow C$  is a subset of the cartesian product  $A \times B$  given by  $\{(a, b) \in A \times B \mid f(a) = g(b)\}$ .

In particular, we call the pullback of the morphism  $f: A \rightarrow B$  along itself a *kernel pair*. More explicitly, let  $f: A \rightarrow B$  be a morphism of  $\mathcal{C}$ , the kernel pair of  $f$  is the following pullback

$$\begin{array}{ccc} Eq(f) & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B. \end{array} \tag{1.1}$$

We described several limits in a category: binary products, equalizers, pullbacks. In particular, a category is called (finitely) complete when it has all the (finite) limits. It is equivalent for a category to be (finitely) complete or to have equalizers and all (binary) products or to have pullbacks and a terminal object.

Dually a category is (finitely) cocomplete whenever it has all (finite) colimits. We give an explicit description of the dual version of the notion of equalizers: coequalizers.

**Definition 1.1.6.** *The morphism  $q: B \rightarrow Q$ , in  $\mathcal{C}$ , is the coequalizer of  $f, g: A \rightarrow B$  if  $q \cdot f = q \cdot g$  and if for any morphism  $h: B \rightarrow X$  such that  $h \cdot f = h \cdot g$  there exists a unique morphism  $\hat{h}: Q \rightarrow X$  such that  $\hat{h} \cdot q = h$*

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{q} & Q \\
 & \xrightarrow{g} & & & \downarrow \hat{h} \\
 & & & \searrow h & X.
 \end{array}$$

Similarly to the definition of a kernel, the cokernel of an arrow is defined as the coequalizer of this arrow and the zero morphism.

**Definition 1.1.7.** *In a pointed category  $\mathcal{C}$ , the cokernel of an arrow  $f: A \rightarrow B$  is the coequalizer of  $f$  and the zero morphism  $0: A \rightarrow B$ .*

In the class of arrows of a category, some of them are particularly important in categorical algebra.

We say that an arrow  $f: A \rightarrowtail B$  is an *epimorphism* if for any pair of morphisms  $u, v: B \rightarrow C$  such that  $u \cdot f = v \cdot f$ , it implies that  $u = v$ . We say that an arrow  $f: A \rightarrowtail B$  is a *monomorphism* if for any pair of morphisms  $u, v: Y \rightarrow A$  such that  $f \cdot u = f \cdot v$  it implies that  $u = v$ . The morphism  $f: A \rightarrow B$  is an *isomorphism* if there exists a morphism  $f^{-1}: B \rightarrow A$  such that  $f \cdot f^{-1} = 1_B$  and  $f^{-1} \cdot f = 1_A$ .

**Definition 1.1.8.** *A morphism  $f: A \rightarrow B$  is a strong epimorphism if for any commutative diagram*

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & \swarrow t & \downarrow \\
C & \xrightarrow{m} & D,
\end{array}$$

where  $m: C \rightarrowtail D$  is a monomorphism, there exists a unique morphism  $t: B \rightarrow C$  such that the two triangles commute.

A morphism that is the coequalizer of two arrows of  $\mathcal{C}$  is called *regular epimorphism*.

A morphism that is the cokernel of an arrow is called *normal epimorphism*.

A morphism  $f: A \rightarrow B$  is called *split epimorphism* if there exists  $s: B \rightarrow A$  in  $\mathcal{C}$  such that  $f \cdot s = 1_B$ .

A pair of morphisms  $(f_1, f_2): X \rightarrow Y$  are called *jointly epimorphic* if given two morphisms  $u, v: Y \rightarrow Z$  such that  $u \cdot f_i = v \cdot f_i$  for  $i \in \{1, 2\}$ , it follows that  $u = v$ .

Note that in a category, the following implications hold

$$\begin{aligned}
&\text{split epimorphism} \Rightarrow \text{regular epimorphism}, \\
&\text{regular epimorphism} \Rightarrow \text{strong epimorphism}.
\end{aligned}$$

In a category with binary products, we have that

$$\text{strong epimorphism} \Rightarrow \text{epimorphism}.$$

In a pointed category, it is true that

$$\text{normal epimorphism} \Rightarrow \text{regular epimorphism}.$$

Dually, we have the notion of strong monomorphism, regular monomorphism, normal monomorphism, split monomorphism, and jointly monomorphic.

**Definition 1.1.9.** A morphism  $m: C \rightarrow D$  is a *strong monomorphism* if for any commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & \swarrow t & \downarrow \\
C & \xrightarrow{m} & D.
\end{array}$$

where  $f: C \twoheadrightarrow D$  is an epimorphism, there exists a unique morphism  $t: B \rightarrow C$  such that the two triangles commute.

A morphism that is the equalizer of two arrows of  $\mathcal{C}$  is called regular monomorphism.

A morphism that is the kernel of an arrow is called normal monomorphism.

A morphism  $f: A \rightarrow B$  is called split monomorphism if there exists  $s: B \rightarrow A$  in  $\mathcal{C}$  such that  $s \cdot f = 1_A$ .

A pair of morphisms  $(f_1, f_2): X \rightarrow Y$  are called jointly monomorphic (or jointly monic morphisms) if given two morphisms  $u, v: Z \rightarrow X$  such that  $f_i \cdot u = f_i \cdot v$  for  $i \in \{1, 2\}$ , it follows that  $u = v$ .

Note that an isomorphism in  $\mathcal{C}$  is a monomorphism and an epimorphism. But the converse is not true. Indeed, in the category of rings, the inclusion  $\mathbb{Z} \rightarrow \mathbb{Q}$  is a monomorphism and an epimorphism but it is clear that it is not an isomorphism.

**Lemma 1.1.10.** A morphism  $f: A \rightarrow B$  is an isomorphism if and only if it  $f$  is both a monomorphism and a strong epimorphism.

**Definition 1.1.11.** In a pointed category  $\mathcal{C}$ , a short exact sequence, also called extension, is given by two morphisms in  $\mathcal{C}$ :  $\kappa: A \rightarrow B$  and  $\alpha: B \rightarrow C$  such that  $\kappa$  is the kernel of  $\alpha$  and  $\alpha$  is the cokernel of  $\kappa$ . It is written as

$$0 \longrightarrow A \xrightarrow{\kappa} B \xrightarrow{\alpha} C \longrightarrow 0.$$

**Definition 1.1.12.** A short exact sequence  $0 \rightarrow X \xrightarrow{\kappa} A \xrightleftharpoons[\alpha]{e} B \rightarrow 0$  is called split short exact sequence if  $\alpha$  is a split epimorphism with section  $e$ , i.e.  $\alpha \cdot e = 1_B$ .

## 1.2 Monoidal categories

This section is devoted to *monoidal categories* [66]. Roughly speaking a monoidal category is a category endowed with a tensor product that is associative and unital (up to coherent isomorphisms). The main example of monoidal categories is the category of vector spaces equipped with the tensor product.

**Definition 1.2.1.** [66] A monoidal category is given by the 6-tuple  $(\mathcal{C}, \otimes, I, a, l, r)$ , where

- $\mathcal{C}$  is a category,
- $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  a bifunctor,
- $I \in \mathcal{C}$  is the identity object,
- $a: \otimes \cdot (1_{\mathcal{C}} \times \otimes) \rightarrow \otimes \cdot (\otimes \times 1_{\mathcal{C}})$  is a natural isomorphism, called the *associator*,
- $l: \otimes \cdot (I \times 1_{\mathcal{C}}) \rightarrow \mathcal{C}$  and  $r: \otimes \cdot (1_{\mathcal{C}} \times I) \rightarrow \mathcal{C}$  are natural isomorphisms called the *left and right unit* respectively.

These morphisms make the following diagrams commute

$$\begin{array}{ccc}
 A \otimes (B \otimes (C \otimes D)) & \xrightarrow{a_{A,B,C \otimes D}} & (A \otimes B) \otimes (C \otimes D) \xrightarrow{a_{A \otimes B,C,D}} ((A \otimes B) \otimes C) \otimes D \\
 \downarrow 1_A \otimes a_{B,C,D} & & \uparrow a_{A,B,C} \otimes 1_D \\
 A \otimes ((B \otimes C) \otimes D) & \xrightarrow{a_{A,B \otimes C,D}} & (A \otimes (B \otimes C)) \otimes D \\
 \downarrow 1_A \otimes l_B & & \downarrow r_A \otimes 1_B \\
 A \otimes (I \otimes B) & \xrightarrow{a_{A,I,B}} & (A \otimes I) \otimes B \\
 \searrow 1_A \otimes l_B & & \swarrow r_A \otimes 1_B \\
 & A \otimes B. &
 \end{array}$$

By Mac Lane's coherence theorem [65], there is a monoidal equivalence between any monoidal category and a monoidal category where  $a, l$

and  $r$  are identities. Therefore, from now on, we will omit the natural isomorphisms  $a$ ,  $l$  and  $r$  and a monoidal category will be denoted by the triple  $(\mathcal{C}, \otimes, I)$ .

**Definition 1.2.2.** A braided monoidal category is a 4-tuple  $(\mathcal{C}, \otimes, I, \sigma)$  where  $(\mathcal{C}, \otimes, I)$  is a monoidal category and  $\sigma$  is a braiding. A braiding consists of a family of natural isomorphisms  $\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X$  satisfying

$$\begin{aligned}\sigma_{X \otimes Y, Z} &= (\sigma_{X,Z} \otimes 1_Y) \cdot (1_X \otimes \sigma_{Y,Z}), \\ \sigma_{X, Y \otimes Z} &= (1_Y \otimes \sigma_{X,Z}) \cdot (\sigma_{X,Y} \otimes 1_Z).\end{aligned}$$

A braided monoidal category is called *symmetric* when

$$\sigma_{Y,X}^{-1} = \sigma_{X,Y}. \quad (1.2)$$

In this thesis, we omit the indexes of the braiding when this does not cause any confusion.

**Examples 1.2.3.** (1) The four-tuple  $(\mathbf{Set}, \times, \{*\}, \sigma)$ , where  $\sigma$  is the twist morphism,  $\sigma(x, y) = (y, x)$  for any element  $x$  of a set  $X$  and any element  $y$  of a set  $Y$ , forms a symmetric monoidal category.

(2) The four-tuple  $(\mathbf{Vect}_K, \otimes, K, \sigma)$  where  $\sigma$  is the twist morphism defined by  $\sigma(x \otimes y) = y \otimes x$  for any  $x \otimes y \in X \otimes Y$ , is a symmetric monoidal category.

(3) Any category with finite products and the terminal object, forms a symmetric monoidal category. These categories are called *cartesian (monoidal) categories*.

### 1.3 Hopf algebras in a symmetric monoidal category

In this section, we recall how to construct algebras, coalgebras, bialgebras, and Hopf algebras in a (symmetric) monoidal category.

An *algebra* in a monoidal category  $(\mathcal{C}, \otimes, I)$  is given by an object  $A \in \mathcal{C}$  endowed with a morphism  $m: A \otimes A \rightarrow A$ , called the multiplication, and

a morphism  $u_A: I \rightarrow A$  called the unit. The algebras that we consider in this thesis are unital i.e. the following conditions hold

$$m \cdot (u_A \otimes 1_A) = 1_A = m \cdot (1_A \otimes u_A), \quad (1.3)$$

$$\begin{array}{ccccc} A & \xrightarrow{u_A \otimes 1_A} & A \otimes A & \xleftarrow{1_A \otimes u_A} & A \\ & \searrow & \downarrow m & \swarrow & \\ & & A & & \end{array}$$

**Convention:** The unit morphism of an algebra  $A$  is denoted by  $u_A: I \rightarrow A$ . In Chapter 2 and Chapter 3, we will work with unital algebras in the monoidal category of vector spaces, in this case, the unit element will (by slight abuse of notation) also be denoted by  $u_A \in A$ . The identity morphism over an object  $B$ , is noted  $1_B: B \rightarrow B$ .

The algebra  $A$  is called associative whenever it satisfies

$$m \cdot (m \otimes 1_A) = m \cdot (1_A \otimes m) \quad (1.4)$$

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes 1_A} & A \otimes A \\ \downarrow 1_A \otimes m & & \downarrow m \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

A *morphism of algebras*  $f: A \rightarrow B$  is a morphism in  $\mathcal{C}$  such that the following two diagrams commute

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ m \downarrow & & \downarrow m \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{ccc} I & \xrightarrow{u_A} & A \\ & \searrow u_B & \downarrow f \\ & & B \end{array}$$

A *Coalgebra* is the dual notion of the notion of algebra. In other words, a coalgebra over  $(\mathcal{C}, \otimes, I)$  is an object  $C \in \mathcal{C}$  with a morphism  $\Delta: C \rightarrow C \otimes C$ , called comultiplication, and a morphism  $\epsilon_C: C \rightarrow I$ , called counit. From now on, the coalgebras will always be counital and coassociative

$$(\epsilon_C \otimes 1_C) \cdot \Delta = 1_C = (1_C \otimes \epsilon_C) \cdot \Delta \quad (1.5)$$

$$(\Delta \otimes 1_C) \cdot \Delta = (1_C \otimes \Delta) \cdot \Delta \quad (1.6)$$

as expressed by the commutativity of the following diagrams

$$\begin{array}{ccc} C & \xleftarrow{\epsilon_C \otimes 1_C} & C \otimes C & \xrightarrow{1_C \otimes \epsilon_C} & C \\ & \searrow & \uparrow \Delta & \swarrow & \\ & & C & & \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow 1_C \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes 1_C} & C \otimes C \otimes C \end{array}$$

Similarly, a *morphism of coalgebras*  $g: C \rightarrow D$  is a morphism in  $\mathcal{C}$  such that the following two diagrams commute

$$\begin{array}{ccc} D \otimes D & \xleftarrow{g \otimes g} & C \otimes C \\ \Delta \uparrow & & \uparrow \Delta \\ D & \xleftarrow{g} & C \end{array} \quad \begin{array}{ccc} I & \xleftarrow{\epsilon_D} & D \\ \epsilon_C \swarrow & & \uparrow g \\ & & C \end{array}$$

We also recall that a *bialgebra* in a symmetric monoidal category (or even just in a braided monoidal category)  $(\mathcal{C}, \otimes, I, \sigma)$  is a 5-tuple  $(B, m, u_B, \Delta, \epsilon_B)$  where  $(B, m, u_B)$  is an algebra,  $(B, \Delta, \epsilon_B)$  is a coalgebra and  $\Delta, \epsilon_B$  are algebra morphisms (which is equivalent to asking that  $m$  and  $u_B$  are coalgebra morphisms) i.e. the following conditions hold

$$\Delta \cdot m = (m \otimes m) \cdot (1_B \otimes \sigma \otimes 1_B) \cdot (\Delta \otimes \Delta), \quad (1.7)$$

$$\begin{array}{ccc} B \otimes B & \xrightarrow{\quad m \quad} & B \\ \downarrow \Delta \otimes \Delta & & \downarrow \Delta \\ B \otimes B \otimes B \otimes B & \xrightarrow{1_B \otimes \sigma \otimes 1_B} B \otimes B \otimes B \otimes B \xrightarrow{m \otimes m} & B \otimes B \end{array}$$

$$\Delta \cdot u_B = u_B \otimes u_B, \quad (1.8)$$

$$\begin{array}{ccc} I & \xrightarrow{u_B} & B \\ \searrow u_B \otimes u_B & & \downarrow \Delta \\ & & B \otimes B \end{array}$$

$$\epsilon_B \cdot m = \epsilon_B \otimes \epsilon_B, \quad (1.9)$$

$$\begin{array}{ccc}
B \otimes B & \xrightarrow{m} & B \\
& \searrow \epsilon_B \otimes \epsilon_B & \downarrow \epsilon_B \\
& & I
\end{array}$$

$$\epsilon_B \cdot u_B = 1_I. \quad (1.10)$$

$$\begin{array}{ccc}
I & \xrightarrow{u_B} & B \\
& \searrow & \downarrow \epsilon_B \\
& & I
\end{array}$$

Moreover, a morphism in  $\mathcal{C}$  is a *morphism of bialgebras* if it is a morphism of algebras and coalgebras.

A *Hopf algebra* is a 7-tuple  $(A, m, u_A, \Delta, \epsilon_A, S)$ , where  $(A, m, u_A, \Delta, \epsilon_A)$  is a bialgebra and  $S$  is an anti-homomorphism of coalgebras and algebras ( $S \cdot m = m \cdot \sigma \cdot (S \otimes S)$  and  $(S \otimes S) \cdot \sigma \cdot \Delta = \Delta \cdot S$ ), called the antipode, making the following diagrams commute

$$\begin{array}{ccccc}
& & A \otimes A & \xrightleftharpoons[S \otimes 1_A]{1_A \otimes S} & A \otimes A \\
& \nearrow \Delta & & & \searrow m \\
A & \xrightarrow{\epsilon_A} & I & \xrightarrow{u_A} & A.
\end{array} \quad (1.11)$$

A *morphism of Hopf algebras* is a morphism  $f: A \rightarrow B$  of bialgebras such that the antipode is preserved:  $S \cdot f = f \cdot S$ .

**Remark 1.3.1.** Note that, when the Hopf algebra is associative,  $S$  is automatically an antihomomorphism of coalgebras and algebras. Moreover, a bialgebra morphism between associative Hopf algebras necessarily preserves the antipode.

**Definition 1.3.2.** A Hopf algebra  $(A, m, u_A, \Delta, \epsilon_A, S)$  is cocommutative if its underlying coalgebra is cocommutative, meaning that the comultiplication map  $\Delta$  satisfies  $\sigma \cdot \Delta = \Delta$ .

**Examples 1.3.3.** (1) In the symmetric monoidal category  $(\text{Set}, \times, \{*\}, \sigma)$ , every object has a unique coalgebra structure with  $\Delta$  being the diagonal

and  $\epsilon$  the morphism sending every element to the singleton. Hence, a non-associative bialgebra (or algebra) is a magma, an associative bialgebra (or algebra) is a monoid, and an associative Hopf algebra is a group.

(2) In the symmetric monoidal category  $(\mathbf{Vect}_K, \otimes, K, \sigma)$  of vector spaces over a field  $K$ , we recover the notion of  $K$ -algebras,  $K$ -coalgebras,  $K$ -bialgebras, and Hopf  $K$ -algebras.

(3) In [25], a symmetric monoidal category was introduced such that (regular) Hom-algebras, Hom-coalgebras, and Hom-Hopf algebras (see [71]) coincide with the associative algebras, coalgebras, and Hopf algebras in this symmetric monoidal category.

(4) In [24], the authors showed that Turaev's Hopf group-coalgebras (see [92]) are associative Hopf algebras in a symmetric monoidal category of families of vector spaces, which they called the Turaev category.

(5) (Non-coassociative, associative) bialgebras and Hopf algebras in any symmetric monoidal category  $\mathcal{C}$  can be seen as (non-associative, coassociative) bialgebras and Hopf algebras in  $\mathcal{C}^{\text{op}}$ , the opposite category, which is still a symmetric monoidal category.

(6) The coquasi-bialgebras and quasi-bialgebras are respectively examples of non-associative bialgebras in  $\mathbf{Vect}_K$  and  $\mathbf{Vect}_K^{\text{op}}$ , see [68] for an introduction about these structures. The coquasi-Hopf algebras have different antipode conditions, but under some specific assumptions, it is possible to see them as non-associative Hopf algebras. An example which is both a non-associative Hopf algebra, as we defined and a coquasi-Hopf algebra is the structure of octonions see [3].

To end this section, we would like to describe an important monoidal category, the category of  $A$ -modules, where  $A$  is a bialgebra.

**Definition 1.3.4.** Let  $A$  be an algebra in the symmetric monoidal category  $(\mathcal{C}, \otimes, I, \sigma)$ , we say that  $B \in \mathcal{C}$  is an  $A$ -module if there exists a morphism in  $\mathcal{C}$ ,  $\triangleright : A \otimes B \rightarrow B$  such that these two conditions are satisfied

$$\triangleright \cdot (m \otimes 1_B) = \triangleright \cdot (1_A \otimes \triangleright), \quad (1.12)$$

$$\triangleright \cdot (u_A \otimes 1_B) = 1_B. \quad (1.13)$$

**Definition 1.3.5.** A morphism between two  $A$ -modules  $\triangleright: A \otimes B \rightarrow B$  and  $\triangleright': A \otimes B' \rightarrow B'$  is a morphism  $f: B \rightarrow B'$  in  $\mathcal{C}$  such that

$$f \cdot \triangleright = \triangleright' \cdot (1_A \otimes f).$$

When  $A$  is a bialgebra in  $(\mathcal{C}, \otimes, I, \sigma)$ , the category of  $A$ -modules and  $A$ -module morphisms, denoted  ${}_A\mathbf{Mod}$ , inherits a monoidal structure from  $(\mathcal{C}, \otimes, I, \sigma)$ . Indeed, let  $M, N$  be  $A$ -modules then  $M \otimes N$  is an  $A$ -module via the following application:

$$(\triangleright \otimes \triangleright) \cdot (1_A \otimes \sigma \otimes 1_N) \cdot (\Delta \otimes 1_M \otimes 1_N): A \otimes M \otimes N \rightarrow M \otimes N. \quad (1.14)$$

Moreover, the unit object of  $\mathcal{C}$  is also the unit object in  ${}_A\mathbf{Mod}$  via the following morphism in  $\mathcal{C}$

$$\epsilon_A \otimes 1_I: A \otimes I \rightarrow I.$$

If  $\mathcal{C}$  is moreover a closed monoidal category (which means that for any  $X \in \mathcal{C}$ , the endofunctor  $- \otimes X$  has a right adjoint  $[X, -]$ ), and  $A$  is Hopf algebra, then  ${}_A\mathbf{Mod}$  moreover inherits a closed structure from  $\mathcal{C}$ .

Furthermore, by restricting to  $A$ -modules satisfying an additional condition, one obtains a symmetric monoidal category, as explained in the following proposition. This additional condition is inspired by the definition of the crossed modules of Hopf algebras given in [70]. This condition is a special instance of the condition in the definition of the Yetter-Drinfeld modules (see Remark 1.3.7 2)). The bialgebras and Hopf algebras in this category will be useful later to obtain an appropriate notion of action of bialgebras and Hopf algebras.

**Proposition 1.3.6.** Let  $A$  be a bialgebra (or a Hopf algebra) in a symmetric monoidal category  $(\mathcal{C}, \otimes, I, \sigma)$ , then the full subcategory of  ${}_A\mathbf{Mod}$ , whose objects are the  $A$ -modules,  $\triangleright: A \otimes B \rightarrow B$  satisfying the condition

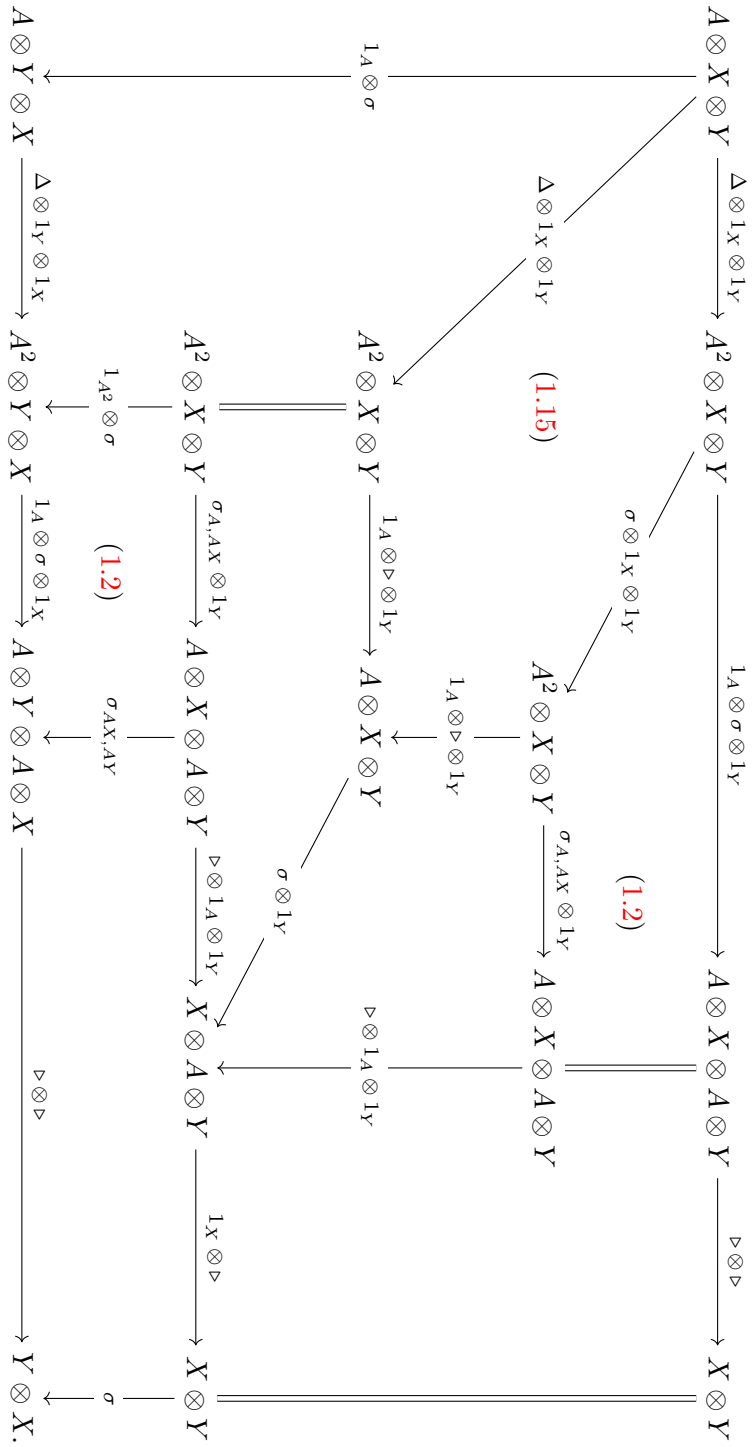
$$(1_A \otimes \triangleright) \cdot (\Delta \otimes 1_B) = (1_A \otimes \triangleright) \cdot (\sigma \otimes 1_B) \cdot (\Delta \otimes 1_B), \quad (1.15)$$

is a symmetric monoidal category. We denote this category by  ${}_A\overline{\mathbf{Mod}}$ .

*Proof.* The isomorphism  $\sigma_{X,Y}$  is a morphism of  $A$ -modules since it satisfies the equation

$$\sigma_{X,Y} \cdot \triangleright = \triangleright \cdot (1_A \otimes \sigma_{X,Y}),$$

Figure 1.1:  $\sigma_{X,Y}$  is a morphism of  $A$ -modules



which is shown in the diagram Figure 1.1. Moreover, it is easy to check that if  $M, N$  are  $A$ -modules satisfying the condition (1.15), then  $M \otimes N$  and  $I$  are  $A$ -modules satisfying (1.15) via the linear map defined by (1.14).  $\square$

We will define bialgebras and Hopf algebras in this category in the next chapters.

**Remark 1.3.7.** 1) If we consider non-associative  $A$ -modules, i.e. we remove the condition (1.12) in the definition of  $A$ -module, the category defined in Proposition 1.3.6 is still a symmetric monoidal category in which we can also define bialgebras and Hopf algebras. We denote this category  ${}_A\overline{\text{Mod}}$ .

2) We note that the category defined in Proposition 1.3.6,  ${}_A\overline{\text{Mod}}$ , where  $A$  is Hopf algebra in  $(\mathcal{C}, \otimes, I, \sigma)$ , is a full subcategory of the Yetter-Drinfeld modules category over  $A$  [84]. An object in  ${}_A\overline{\text{Mod}}$  is a Yetter-Drinfeld module with trivial coaction  $(1_B \otimes u_A: B \rightarrow B \otimes A)$ .

3) We also note that if  $A$  is a cocommutative Hopf algebra, then the condition (1.15) becomes trivial and  ${}_A\text{Mod} = {}_A\overline{\text{Mod}}$ .

## 1.4 Algebraic categories

In this section, we recall the definitions of some categories with good algebraic properties: regular, Barr-exact, protomodular, Mal'tsev, semi-abelian, and action representable categories.

We start with the notion of a regular category. A key element in the definition of a regular category is the regular epimorphism - monomorphism factorization.

**Definition 1.4.1.** A finitely complete category  $\mathcal{C}$  is regular if any arrow  $f: A \rightarrow B$  factors as a regular epimorphism  $p: A \rightarrow D$  followed by a monomorphism  $i: D \rightarrow B$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow p & \nearrow i \\ & D & \end{array} \quad (1.16)$$

and if, moreover, the regular epimorphisms are pullback-stable. It means that given a pullback

$$\begin{array}{ccc} A \times_C B & \xrightarrow{p_2} & B \\ p_1 \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

the morphism  $p_1$  is a regular epimorphism whenever  $g$  is.

The categories **Set**, **Grp**, and **Vect** are examples of regular categories. In the category of groups **Grp**, the decomposition “regular epimorphism - monomorphism” is given via the following factorization

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \pi \searrow & & \nearrow i \\ & A/Ker(f) \cong f(A) & \end{array}$$

where  $\pi: A \rightarrow A/Ker(f)$  is the canonical surjection and  $i$  is the inclusion. It is also easy to show that surjective morphisms (which are the regular epimorphisms in **Grp**) are stable under pullbacks. The category of topological spaces, **Top**, is not a regular category [8].

The following reformulation of the notion of a regular category will be useful:

**Lemma 1.4.2.** *Let  $\mathcal{C}$  be a finitely complete category. Then  $\mathcal{C}$  is a regular category if and only if*

1. *any arrow in  $\mathcal{C}$  factors as a regular epimorphism followed by a monomorphism;*
2. *given any regular epimorphism  $f: A \rightarrow B$  and any object  $E$ , the induced arrow  $1_E \times f: E \times A \rightarrow E \times B$  is a regular epimorphism;*
3. *regular epimorphisms are stable under pullbacks along split monomorphisms.*

*Proof.* It is well-known that the properties (1), (2) and (3) hold in any regular category.

Conversely, if these properties hold, we need to prove that regular epimorphisms are pullback stable. Given any pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ \downarrow p_1 & & \downarrow f \\ E & \xrightarrow{p} & B \end{array} \quad (1.17)$$

where  $f$  is a regular epimorphism, consider the diagram

$$\begin{array}{ccc} E \times_B A & \xrightarrow{e} & E \times A \\ \downarrow p_1 & & \downarrow 1_E \times f \\ E & \xrightarrow{(1_E, p)} & E \times B \end{array}$$

where  $e: E \times_B A \rightarrow E \times A$  is the equalizer of  $p \cdot \pi_1$  and  $f \cdot \pi_2$ , and the arrows  $\pi_1: E \times A \rightarrow E$  and  $\pi_2: E \times A \rightarrow A$  are the product projections. This diagram is a pullback, and it then follows that  $p_1$  is a regular epimorphism since it is the pullback of the regular epimorphism  $1_E \times f$  along the split monomorphism  $(1_E, p)$ .  $\square$

Our next aim is to introduce a particular class of regular categories, called Barr-exact categories [6], for example the category of groups. The notion of Barr-exactness for a category is defined via the concept of equivalence relation that we recall below.

A *relation* on an object  $A$  in  $\mathcal{C}$  is a triple  $(R, r_0, r_1)$ , where  $R$  is an object of  $\mathcal{C}$  and  $r_0, r_1: R \rightarrow A$  is a pair of jointly monic morphisms in  $\mathcal{C}$  (see Definition 1.1.9). A relation  $(R, r_0, r_1)$  on  $A$  is called:

- *reflexive* if there is a (unique) morphism  $\delta: A \rightarrow R$  such that  $r_0 \cdot \delta = 1_A = r_1 \cdot \delta$ ;
- *symmetric* if there is a (unique) morphism  $\varsigma: R \rightarrow R$  such that  $r_0 \cdot \varsigma = r_1$  and  $r_1 \cdot \varsigma = r_0$ ;
- *transitive* if there is a (unique) morphism  $\tau: R \times_A R \rightarrow R$  such that  $r_0 \cdot \tau = r_0 \cdot \pi_1$  and  $r_1 \cdot \tau = r_1 \cdot \pi_2$ , where  $(R \times_A R, \pi_1, \pi_2)$  is the pullback of  $r_0$  and  $r_1$ .

- an *equivalence relation* if it is reflexive, symmetric, and transitive.

A kernel pair (as defined in Equation (1.1)) is an example of an equivalence relation. Indeed, the universal property of the pullback

$$\begin{array}{ccc} Eq(f) & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B. \end{array}$$

will imply that  $(Eq(f), p_1, p_2)$  is an equivalence relation.

**Definition 1.4.3.** A regular category  $\mathcal{C}$  is *exact* (in the sense of Barr [6]) if any equivalence relation  $R$  in  $\mathcal{C}$  is effective, i.e. it is the kernel pair of some morphism in  $\mathcal{C}$ . This property means that for any equivalence relation  $(R, r_0, r_1)$  there is a morphism  $f: A \rightarrow B$  in  $\mathcal{C}$  such that the following square is a pullback:

$$\begin{array}{ccc} R & \xrightarrow{r_1} & A \\ r_0 \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B. \end{array}$$

The categories **Set**, **Grp**, and **Vect** are examples of exact categories. In **Grp**, let  $(R, r_0, r_1)$  be an equivalence relation on a group  $G$ , then the quotient morphism  $\pi: G \rightarrow G/R$  is a group morphism and  $(Eq(\pi), p_1, p_2) = (R, r_0, r_1)$ . This observation implies that **Grp** is an exact category.

We recall now the definitions of a protomodular category (in the pointed case) and a Mal'tsev category. It is well-known that any finitely complete protomodular category is a Mal'tsev category (see [9], for instance).

**Definition 1.4.4.** A pointed category is *protomodular*, in the sense of [11], if the Split Short Five Lemma holds in  $\mathcal{C}$ . This means that for any diagram of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{\kappa_1} & A_1 & \begin{smallmatrix} \xleftarrow{e_1} \\ \xrightarrow{\alpha_1} \end{smallmatrix} & B_1 \longrightarrow 0 \\ & & \downarrow v & & \downarrow p & & \downarrow g \\ 0 & \longrightarrow & X_2 & \xrightarrow{\kappa_2} & A_2 & \begin{smallmatrix} \xleftarrow{e_2} \\ \xrightarrow{\alpha_2} \end{smallmatrix} & B_2 \longrightarrow 0 \end{array}$$

where  $\kappa_i$  is the kernel of  $\alpha_i$  and  $\alpha_i \cdot e_i = 1_{B_i}$  for any  $i \in \{1, 2\}$ , then  $p$  is an isomorphism whenever  $v$  and  $g$  are.

**Definition 1.4.5.** [26, 27] A finitely complete category  $\mathcal{C}$  is a Mal'tsev category when any (internal) reflexive relation in  $\mathcal{C}$  is an equivalence relation.

The following property of pointed, regular and protomodular categories, which will be useful later, was proven in [13, Proposition 7].

**Proposition 1.4.6.** Let  $\mathcal{C}$  be a pointed, regular, and protomodular category. Let consider the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \xrightarrow{k} & A & \xrightarrow{f} & B \longrightarrow 0 \\
 & & \downarrow v & & \downarrow p & & \downarrow g \\
 0 & \longrightarrow & K' & \xrightarrow{k'} & A' & \xrightarrow{f'} & B'
 \end{array}
 \quad (2)$$

where  $k'$  is the kernel of  $f'$  and the upper row is exact. If  $v$  is an isomorphism then (2) is a pullback.

We are now ready to give the definition of a semi-abelian category.

**Definition 1.4.7.** A semi-abelian category is an exact category  $\mathcal{C}$ , that is also pointed, finitely cocomplete and protomodular.

One of the main examples of semi-abelian categories is the category of groups. Thanks to the observations made about the regularity and the exactness property of  $\mathbf{Grp}$ , it is easy to show that the category of groups is semi-abelian.

Among the examples of semi-abelian categories there are also the categories of Lie algebras, (associative) rings, loops, crossed modules, compact groups, Heyting semilattices,  $\mathbb{C}^*$ -algebras, the dual of the category of pointed sets. Moreover, any abelian category is semi-abelian. More precisely, a semi-abelian  $\mathcal{C}$  is abelian if and only if  $\mathcal{C}$  is additive. In particular, we have the following proposition

**Proposition 1.4.8.** [58] A category  $\mathcal{C}$  is abelian if and only if  $\mathcal{C}$  and  $\mathcal{C}^{op}$  are semi-abelian.

In this thesis, we will add several new cases to this list of examples of semi-abelian categories (see Theorem 2.2.8, Corollary 2.2.10 and Corollary 3.2.7)

As explained in [58], semi-abelian categories are suitable to define and study (co)homology of non-abelian structures [41] and to define and develop a categorical approach to commutator [16] and radical and torsion theories [17]. In a semi-abelian category, there is also a natural notion of semi-direct product, internal action [10], and crossed module [57]. We refer to [58, 9] for more details about the basic notions and properties of semi-abelian categories.

We give another characterization of a semi-abelian category using one of the so-called “old axioms” (see 3.7 in [58], for instance).

**Proposition 1.4.9.** *A pointed, regular, protomodular category that has coproducts, is a semi-abelian category provided that the image of a normal monomorphism under a normal epimorphism is again a normal monomorphism. This condition means that for  $p, q$  two normal epimorphisms and  $u, v$  two monomorphisms such that the following square commutes*

$$\begin{array}{ccc} D & \xrightarrow{q} & D \\ w \downarrow & & \downarrow v \\ A & \xrightarrow{p} & B \end{array}$$

*then  $v$  is normal whenever  $w$  is.*

In this thesis, we will also add a new example in the list of examples of action representable categories (see Proposition 2.2.11).

**Definition 1.4.10.** *A pointed protomodular category  $\mathcal{C}$  is called action representable (equivalently, that it has representable object actions in the sense of [10]) if the following property holds: given any object  $X$  in  $\mathcal{C}$ , there is a split exact sequence,*

$$0 \longrightarrow X \xrightarrow{\tilde{k}} \overline{X} \begin{array}{c} \xleftarrow{\tilde{s}} \\ \xrightarrow{\tilde{f}} \end{array} [X] \longrightarrow 0 \quad (1.18)$$

where  $\tilde{k}$  is the kernel of  $\tilde{f}$ , that is universal in the sense that, given any other split short exact sequence in  $\mathcal{C}$  where  $k$  is the kernel of  $f$

$$0 \longrightarrow X \xrightarrow{k} A \begin{smallmatrix} \xleftarrow{s} \\ \xrightarrow{f} \end{smallmatrix} B \longrightarrow 0$$

there is a unique  $\chi: B \rightarrow [X]$  (and  $\bar{\chi}: A \rightarrow \bar{X}$ ) such that the following diagram of split exact sequences commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{k} & A & \begin{smallmatrix} \xleftarrow{s} \\ \xrightarrow{f} \end{smallmatrix} & B \longrightarrow 0 \\ & & \parallel & & \downarrow \bar{\chi} & & \downarrow \chi \\ 0 & \longrightarrow & X & \xrightarrow{i_1} & \bar{X} & \begin{smallmatrix} \xleftarrow{i_2} \\ \xrightarrow{p_2} \end{smallmatrix} & [X] \longrightarrow 0. \end{array}$$

The object  $[X]$  in (1.18) is called a split extension classifier for  $X$ .

In the action representable category  $\mathbf{Grp}$  of groups, the object  $[X]$  is simply given by the group  $\mathbf{Aut}(X)$  of automorphisms of the group  $X$ . In the category  $\mathbf{Lie}_K$  of Lie algebras over a field  $K$  the split extension classifier  $[A]$  of a Lie algebra  $A$  is given by the Lie algebra  $\mathbf{Der}(A)$  of derivations of  $A$ .

## 1.5 Centrality in the sense of Smith and of Huq

In this section, we define two types of centrality: the centrality in the sense of Smith and the centrality in the sense of Huq. We explain the link between these two notions and we recall the definition of the Huq commutator of two normal subobjects.

**Definition 1.5.1.** Let  $(R, r_0, r_1)$  and  $(S, s_0, s_1)$  be two equivalence relations over the same object  $X$ . We denote the pullback of  $s_0$  along  $r_1$  by

$$\begin{array}{ccc} R \times_X S & \xrightarrow{\pi_2} & S \\ \downarrow \pi_1 & & \downarrow s_0 \\ R & \xrightarrow{r_1} & X. \end{array}$$

A connector between  $R$  and  $S$  is an arrow  $\hat{p}: R \times_X S \rightarrow X$ , sending an element of the pullback  $xRySz$  to  $\hat{p}(x, y, z)$ , such that

- $xS\hat{p}(x, y, z)$  and  $zR\hat{p}(x, y, z)$ ,
- $\hat{p}(x, x, y) = y$  and  $\hat{p}(x, y, y) = x$ ,
- $\hat{p}(x, y, \hat{p}(y, u, v)) = \hat{p}(x, u, v)$  and  $\hat{p}(\hat{p}(x, y, u), u, v) = \hat{p}(x, y, v)$ .

When such an arrow exists, we say that the relations  $R$  and  $S$  centralize (in the sense of Smith).

**Definition 1.5.2.** One says that two subobjects  $x: X \rightarrow A$  and  $y: Y \rightarrow A$  of the same object  $A$  centralize (in the sense of Huq) [56], if and only if there exists an arrow  $p$  making the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{(1,0)} & X \times Y & \xleftarrow{(0,1)} & Y \\ & \searrow x & \downarrow p & \swarrow y & \\ & & A & & \end{array}$$

In the category of groups, let  $X$  and  $Y$  be two subgroups of a group  $A$ , then such an arrow  $p: X \times Y \rightarrow A$  exists if and only if  $xy = yx$  for any  $x \in X$  and  $y \in Y$ .

**Proposition 1.5.3.** [16] In a Mal'tsev category, the arrows  $\hat{p}$  and  $p$  in Definition 1.5.1 and Definition 1.5.2 are unique, whenever they exist.

Hence, in a Mal'tsev category, centralization becomes a property.

Note that these notions of centralization are not independent of each other. If two relations centralize to each other (in the sense of Smith) then it is true that their normalizations centralize in the sense of Huq [16].

More explicitly, let  $(S, s_0, s_1)$  and  $(R, r_0, r_1)$  be two equivalence relations over an object  $X$  that centralize, then their normalizations

$r_1 \cdot \ker(r_0): \text{Ker}(r_0) \rightarrow X$  and  $s_1 \cdot \ker(s_0): \text{Ker}(s_0) \rightarrow X$  centralize in the sense of Huq:

$$\begin{array}{ccccc} \text{Ker}(s_0) & \xrightarrow{(1,0)} & \text{Ker}(s_0) \times \text{Ker}(r_0) & \xleftarrow{(0,1)} & \text{Ker}(r_0), \\ & \searrow s_1 \cdot \ker(s_0) & \downarrow p & \swarrow r_1 \cdot \ker(r_0) & \\ & & X & & \end{array}$$

where  $\ker(r_0): \text{Ker}(r_0) \rightarrow R$  and  $\ker(s_0): \text{Ker}(s_0) \rightarrow S$  are the kernels of  $r_0$  and  $s_0$  respectively. We quickly recall the construction of the proof given in [16]. The arrow  $p$  is defined as  $p = \hat{p} \cdot \psi: \text{Ker}(s_0) \times \text{Ker}(r_0) \rightarrow X$ , where  $\hat{p}: R \times_X S \rightarrow X$  is the connector between  $R$  and  $S$  (see Definition 1.5.1) and  $\psi: \text{Ker}(s_0) \times \text{Ker}(r_0) \rightarrow R \times_X S$  is induced by the universal property of the pullback  $(R \times_X S, \pi_1, \pi_2)$ .

The converse is not true in general. If  $\mathcal{C}$  is a category such that any two relations always centralize to each other as soon as their normalizations centralize in the sense of Huq, will say that  $\mathcal{C}$  possesses the *Smith is Huq* property [16].

**Lemma 1.5.4.** *Any action representable category has the Smith is Huq property.*

This lemma is true more generally for any action accessible category [19].

To end this section we give the definition of the Huq commutator of two normal subobjects.

**Definition 1.5.5.** *In a semi-abelian category, the Huq commutator of two normal subobjects  $x: X \rightarrow A$  and  $y: Y \rightarrow A$ , denoted  $[X, Y]_{\text{Huq}}$ , is the smallest normal subobject  $K \rightarrow A$  such that its cokernel  $q: A \rightarrow A/K$  has the property that the images  $q(X)$  and  $q(Y)$  by  $q$  centralize (in the sense of Huq) in the quotient:*

$$\begin{array}{ccccc} & & q(X) \times q(Y) & & \\ & & \uparrow (1,0) \quad \downarrow (0,1) & & \\ & X & & Y & \\ & \searrow & & \swarrow & \\ K & \rightarrow & A & \xrightarrow{q} & A/K. \\ & & \downarrow q(X) & & \downarrow q(Y) \\ & & q(X) & & q(Y) \end{array}$$

As expected, two normal subobjects  $x: X \rightarrow A$  and  $y: Y \rightarrow A$  centralize in the sense of Huq, whenever  $[X, Y]_{\text{Huq}} = 0$ .

In the category of groups, the Huq commutator of two normal subgroups  $X, Y$  of  $A$  is given by  $[X, Y]_{\text{Huq}} = \{xyx^{-1}y^{-1} \mid x \in X, y \in Y\}$ .

See [56] for more details about the Huq commutator.

## 1.6 Internal structures

Let  $\mathcal{C}$  be a category with pullbacks, in this section we recall how to define a reflexive graph, a reflexive-multiplicative graph, and a groupoid internal to the category  $\mathcal{C}$  (see [27]).

The first internal construction that we recall is the category of points in  $\mathcal{C}$ , which is denoted by  $\text{Pt}(\mathcal{C})$ . The objects are the split epimorphisms, also called points, (see Definition 1.1.8) and the arrows are the morphisms of split epimorphisms, which are defined as follows.

**Definition 1.6.1.** *A morphism of split epimorphisms between the split epimorphisms  $A' \xleftarrow[\delta']{\iota'} B'$  and  $A \xleftarrow[\delta]{\iota} B$  is a pair of morphisms in  $\mathcal{C}$ ,  $(f: A' \rightarrow A, g: B' \rightarrow B)$  such that the two squares of corresponding arrows commute, i.e.  $g \cdot \delta' = \delta \cdot f$  and  $f \cdot \iota' = \iota \cdot g$*

$$\begin{array}{ccc} A' & \xleftarrow[\delta']{\iota'} & B' \\ \downarrow f & & \downarrow g \\ A & \xleftarrow[\delta]{\iota} & B. \end{array}$$

A point is an underlying structure in the definition of a reflexive graph.

**Definition 1.6.2.** *A reflexive graph  $\mathbb{A}$  in a category  $\mathcal{C}$  is a diagram in  $\mathcal{C}$  of the following form*

$$A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow[\gamma]{\iota} \\ \xrightarrow{\quad} \end{array} A_0, \quad (1.19)$$

where  $\delta$  is the “domain morphism”,  $\gamma$  the “codomain morphism”, and  $\iota$  the “identity morphism” satisfying  $\delta \cdot \iota = \gamma \cdot \iota = 1_{A_0}$ .

**Definition 1.6.3.** A morphism of reflexive graphs between the reflexive

graphs  $A'_1 \begin{array}{c} \xrightarrow{\delta'} \\ \xleftarrow{\iota'} \\ \xrightarrow{\gamma'} \end{array} A'_0$  and  $A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\iota} \\ \xrightarrow{\gamma} \end{array} A_0$  is given by a couple of

morphisms  $(f_1: A'_1 \rightarrow A_1, f_0: A'_0 \rightarrow A_0)$  such that the three squares of corresponding arrows of the following diagram commute

$$\begin{array}{ccc} A'_1 & \begin{array}{c} \xrightarrow{\delta'} \\ \xleftarrow{\iota'} \\ \xrightarrow{\gamma'} \end{array} & A'_0 \\ \downarrow f_1 & & \downarrow f_0 \\ A_1 & \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\iota} \\ \xrightarrow{\gamma} \end{array} & A_0. \end{array}$$

The category of reflexive graphs of a category  $\mathcal{C}$  is denoted by  $\mathbf{RGraph}(\mathcal{C})$ .

When we add a “multiplication” to a reflexive graph we obtain the notion of a reflexive-multiplicative graph.

**Definition 1.6.4.** In a category  $\mathcal{C}$  with pullbacks, a reflexive graph,

denoted by  $A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\iota} \\ \xrightarrow{\gamma} \end{array} A_0$ , together with a morphism  $m: A_1 \times_{A_0} A_1 \rightarrow A_1$

is called a reflexive-multiplicative graph

$$A_1 \times_{A_0} A_1 \xrightarrow{m} A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\iota} \\ \xrightarrow{\gamma} \end{array} A_0, \quad (1.20)$$

where  $A_1 \times_{A_0} A_1$  is the (object part of the) following pullback

$$\begin{array}{ccc} A_1 \times_{A_0} A_1 & \xrightarrow{p_2} & A_1 \\ p_1 \downarrow & & \downarrow \gamma \\ A_1 & \xrightarrow{\delta} & A_0 \end{array}$$

and  $m$  is a multiplication that is required to satisfy the identities

$$m \cdot (1_{A_1}, \iota \cdot \delta) = 1_{A_1} = m \cdot (\iota \cdot \gamma, 1_{A_1}), \quad (1.21)$$

where  $(1_{A_1}, \iota \cdot \delta): A_1 \rightarrow A_1 \times_{A_0} A_1$  and  $(\iota \cdot \gamma, 1_{A_1}): A_1 \rightarrow A_1 \times_{A_0} A_1$  are induced by the universal property of the pullback  $A_1 \times_{A_0} A_1$ .

**Definition 1.6.5.** A morphism of reflexive-multiplicative graphs is a morphism of reflexive graphs that preserve the multiplication, i.e. a couple  $(f_1, f_0)$  such that the four squares of corresponding arrows of the following diagram commute

$$\begin{array}{ccccc}
 A'_1 \times_{A'_0} A'_1 & \xrightarrow{m'} & A'_1 & \begin{array}{c} \xrightarrow{\delta'} \\ \xleftarrow{\iota'} \end{array} & A'_0 \\
 \downarrow f_1 \times f_1 & & \downarrow f_1 & \begin{array}{c} \xrightarrow{\gamma'} \\ \xleftarrow{\delta} \end{array} & \downarrow f_0 \\
 A_1 \times_{A_0} A_1 & \xrightarrow{m} & A_1 & \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\gamma} \end{array} & A_0
 \end{array}$$

**Definition 1.6.6.** A reflexive-multiplicative graph such that the multiplication  $m$  satisfies the additional identities

$$\delta \cdot m = \delta \cdot p_2,$$

$$\gamma \cdot m = \gamma \cdot p_1,$$

$$m \cdot (1_{A_1}, m) = m \cdot (m, 1_{A_1}),$$

is called an internal category.

An internal category in the category of sets is a (small) category.

**Definition 1.6.7.** An internal groupoid is an internal category such that there exists a morphism  $\varsigma: A_1 \rightarrow A_1$  (which plays the role of inverse) such that

$$\delta \cdot \varsigma = \gamma,$$

$$\gamma \cdot \varsigma = \delta,$$

$$m \cdot (\varsigma, 1_{A_1}) = \iota \cdot \delta,$$

$$m \cdot (1_{A_1}, \varsigma) = \iota \cdot \gamma.$$

We denote by  $\text{RMG}(\mathcal{C})$  the category of reflexive-multiplicative graphs with morphisms of reflexive-multiplicative graphs. The category of internal groupoids with the morphisms of reflexive multiplicative graphs will be

denoted by  $\mathbf{Grpd}(\mathcal{C})$ , and the category of internal categories with the morphisms of reflexive-multiplicative graphs will be denoted by  $\mathbf{Cat}(\mathcal{C})$ .

As shown in [27], when  $\mathcal{C}$  is a Mal'tsev category (see Definition 1.4.5), morphisms in  $\mathbf{RMG}(\mathcal{C})$ ,  $\mathbf{Cat}(\mathcal{C})$ , or  $\mathbf{Grpd}(\mathcal{C})$  are simply morphisms of reflexive graphs. In particular,  $\mathbf{Grpd}(\mathcal{C})$ ,  $\mathbf{Cat}(\mathcal{C})$ , and  $\mathbf{RMG}(\mathcal{C})$  become full subcategories of  $\mathbf{RGraph}(\mathcal{C})$ . The forgetful functors between these internal structures give rise to isomorphisms between the categories.

**Proposition 1.6.8.** *Let  $\mathcal{C}$  be a Mal'tsev category, then the following forgetful functors are isomorphisms of categories:*

$$\mathbf{Grpd}(\mathcal{C}) \rightarrow \mathbf{Cat}(\mathcal{C}) \rightarrow \mathbf{RMG}(\mathcal{C}).$$

Another interesting property of a Mal'tsev category is that the groupoid structure on a reflexive graph is unique whenever it exists. It becomes a property to be an internal groupoid for a reflexive graph. It is also equivalent for a reflexive graph  $A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\iota} \xrightarrow{\gamma} \\ \xrightarrow{\gamma} \end{array} A_0$ , to have an internal structure of groupoids or to have a connector (see Definition 1.5.1) between  $Eq(\delta)$  and  $Eq(\gamma)$  [27].

All of these observations imply that the following proposition holds for a pointed Mal'tsev category that satisfies the *Smith is Huq property*.

**Proposition 1.6.9.** *In any category which is a pointed Mal'tsev category and possesses the Smith is Huq property, the following statements are equivalent*

- (1.19) is a reflexive-multiplicative graph
- (1.19) is an internal category
- (1.19) is an internal groupoid
- the kernel pairs  $Eq(\delta)$  and  $Eq(\gamma)$  of (1.19) centralize to each other (in the sense of Smith)
- (1.19) satisfies the equality:  $[Ker(\delta), Ker(\gamma)]_{\mathbf{Huq}} = 0$ , where  $[\cdot, \cdot]_{\mathbf{Huq}}$  denotes the Huq commutator.

Note that, as explained in [\[74\]](#), for pointed protomodular categories, the equivalence of these conditions characterise the condition of *Smith is Huq*.

## Chapter 2

# The category of associative and cocommutative Hopf algebras over a field

In the first part of this thesis, we study the category of cocommutative and associative Hopf algebras in the symmetric monoidal category  $(\mathbf{Vect}_K, \otimes, K, \sigma)$  of vector spaces over any field  $K$ , where  $\sigma$  is the twist morphism defined by  $\sigma(x \otimes y) = y \otimes x$  for any  $x \otimes y \in X \otimes Y$ . In Chapter 2 and Chapter 3, we will use the classical Sweedler notation for calculations with the comultiplication, and we shall write  $\Delta(c) = c_1 \otimes c_2$  for any  $c \in C$  (with the usual summation convention, where  $c_1 \otimes c_2$  stands for  $\sum c_1 \otimes c_2$ ). The unit element of the Hopf algebra  $A$  will be denoted by  $u_A \in A$  and we recall that the notation  $1_A$  stands for the identity map on  $A$ . By a slight abuse of notation, in Chapter 2 and Chapter 3,  $u_A$  will denote the unit element (except for the field  $K$ , where the unit of  $K$  is written 1) but also the linear application  $u_A: K \rightarrow A$  which sends 1 to  $u_A$ .

During our investigation of the cocommutative and associative Hopf algebras in the symmetric monoidal category of vector spaces over any field, for a sake of readability, we will omit to specify that they are associative Hopf algebras in  $\mathbf{Vect}_K$ , where  $K$  is a field, and we will just call them cocommutative Hopf algebras.

In Sweedler notation a Hopf algebra  $(A, m, u_A, \Delta, \epsilon_A, S)$  is *cocommutative* if the following equation is satisfied for any  $a \in A$ :

$$a_1 \otimes a_2 = a_2 \otimes a_1. \quad (2.1)$$

We denote the category of cocommutative Hopf algebras in the monoidal category of vector spaces over the field  $K$ , by  $\mathbf{Hopf}_{K, \text{coc}}$ .

## 2.1 Constructions of limits and colimits

It is well-known that the category of Hopf algebras over any field is complete and cocomplete [88, 83]. In [1], the author describes the limits and colimits for general Hopf algebras. We would like to recall the description of some limits and colimits in the category of cocommutative Hopf algebras.

Let us note that the base field  $K$  is a Hopf algebra, where the comultiplication is given by the natural isomorphism  $\Delta: K \cong K \otimes K$ , the counit, the unit and the antipode are the identity maps. Moreover, for any Hopf algebra  $A$ , we have unique Hopf algebra morphisms  $u_A: K \rightarrow A$  and  $\epsilon_A: A \rightarrow K$ . The following proposition follows from these observations.

**Proposition 2.1.1.** *The base field  $K$  is the zero object in  $\mathbf{Hopf}_{K, \text{coc}}$ . The zero morphism between two Hopf algebras  $A$  and  $B$  is given by*

$$A \xrightarrow{\epsilon_A} K \xrightarrow{u_B} B$$

To prove that the category of cocommutative Hopf algebras over any field is semi-abelian, the fact that the categorical binary product is the tensor product is crucial. We will see that this is an essential difference between general Hopf algebras and cocommutative Hopf algebras. Since it is a key element, we give an explicit proof of this well-known proposition.

**Proposition 2.1.2.** *In the category  $\mathbf{Hopf}_{K, \text{coc}}$  the categorical product of two cocommutative Hopf algebras  $A$  and  $B$  is given by the tensor product  $(A \otimes B, \pi_A, \pi_B)$  where the projections  $\pi_A: A \otimes B \rightarrow A$  and  $\pi_B: A \otimes B \rightarrow B$  are defined by  $\pi_A(a \otimes b) = a\epsilon_B(b)$  and  $\pi_B(a \otimes b) = \epsilon_A(a)b$ , for any  $a \otimes b \in A \otimes B$ .*

*Proof.* It is clear that the projections  $\pi_A$  and  $\pi_B$  are Hopf algebra morphisms. Moreover, let  $f: C \rightarrow A$  and  $g: C \rightarrow B$  be two morphisms of Hopf algebras, we verify that the tensor product satisfies the universal property of the categorical product:

$$\begin{array}{ccccc}
 A & \xleftarrow{\pi_A} & A \otimes B & \xrightarrow{\pi_B} & B \\
 & \nwarrow f & \uparrow (f \otimes g) \cdot \Delta & \nearrow g & \\
 & & C & & 
 \end{array} \quad (2.2)$$

Note that the morphism  $(f \otimes g) \cdot \Delta$  is a coalgebra morphism thanks to the cocommutativity, as it is proved in the following equalities for any  $c$  in  $C$ ,

$$\begin{aligned}
 ((f \otimes g) \cdot \Delta) \otimes ((f \otimes g) \cdot \Delta) \cdot \Delta(c) &= (f \otimes g) \cdot \Delta(c_1) \otimes (f \otimes g) \cdot \Delta(c_2) \\
 &= f(c_{11}) \otimes g(c_{12}) \otimes f(c_{21}) \otimes g(c_{22}) \\
 &= f(c_1) \otimes g(c_2) \otimes f(c_3) \otimes g(c_4) \\
 &= f(c_1) \otimes g(c_3) \otimes f(c_2) \otimes g(c_4) \\
 &= f(c_{11}) \otimes g(c_{21}) \otimes f(c_{12}) \otimes g(c_{22}) \\
 &= f(c_1)_1 \otimes g(c_2)_1 \otimes f(c_1)_2 \otimes g(c_2)_2 \\
 &= \Delta_{A \otimes B}(f(c_1) \otimes g(c_2)) \\
 &= \Delta_{A \otimes B} \cdot (f \otimes g) \cdot \Delta(c).
 \end{aligned}$$

It is an algebra morphism since for any  $a \otimes b \in A \otimes B$ , we have

$$\begin{aligned}
 m_{A \otimes B} \left( ((f \otimes g) \cdot \Delta) \otimes ((f \otimes g) \cdot \Delta) \right) \cdot (a \otimes b) \\
 &= m_{A \otimes B} \cdot (f(a_1) \otimes g(a_2) \otimes f(b_1) \otimes g(b_2)) \\
 &= f(a_1)f(b_1) \otimes g(a_2)g(b_2) \\
 &= f(a_1b_1) \otimes g(a_2b_2) \\
 &= (f \otimes g) \cdot \Delta(ab).
 \end{aligned}$$

Moreover, the morphism  $(f \otimes g) \cdot \Delta: C \rightarrow A \otimes B$  is the unique morphism making the two above triangles commute (2.2). Let suppose there exists a morphism of Hopf algebras  $\psi: C \rightarrow A \otimes B$  such that  $\pi_A \cdot \psi = f$  and  $\pi_B \cdot \psi = g$ , then we have the following equalities

$$(f \otimes g) \cdot \Delta = (\pi_A \otimes \pi_B) \cdot (\psi \otimes \psi) \cdot \Delta$$

$$\begin{aligned}
 &= (\pi_A \otimes \pi_B) \cdot (1_A \otimes \sigma \otimes 1_B) \cdot (\Delta \otimes \Delta) \cdot \psi \\
 &= \psi,
 \end{aligned}$$

where the last equality follows from the definition of the projections  $\pi_A$ ,  $\pi_B$ , and the counit condition (1.5).  $\square$

An explicit description of the equalizers and the kernels in the category of (cocommutative) Hopf algebras was given in [5]. Note that the following constructions make sense for general Hopf algebras.

**Definition 2.1.3.** *Let  $f, g: A \rightarrow B$  be two Hopf algebra morphisms. We recall the three following constructions of subalgebras*

- $LEqual(f, g) = \{x \in A \mid f(x_1) \otimes x_2 = g(x_1) \otimes x_2\}$
- $REqual(f, g) = \{x \in A \mid x_1 \otimes f(x_2) = x_1 \otimes g(x_2)\}$
- $HEqual(f, g) = \{x \in A \mid x_1 \otimes f(x_2) \otimes x_3 = x_1 \otimes g(x_2) \otimes x_3\}$

It is clear that they are subalgebras. For general Hopf algebras, the subalgebra  $HEqual(f, g)$  is a Hopf subalgebra of  $A$ . This Hopf subalgebra with the inclusion morphism is the categorical equalizer in the category of arbitrary Hopf algebras.

**Proposition 2.1.4.** *In  $\text{Hopf}_{K, \text{coc}}$ , the equalizer of the arrows  $f: A \rightarrow B$  and  $g: A \rightarrow B$  is given by the inclusion of  $HEqual(f, g)$  in  $A$ . Moreover, the following equalities hold*

$$HEqual(f, g) = LEqual(f, g) = REqual(f, g).$$

The definition of *the kernel* in the category of cocommutative Hopf algebras follows directly.

**Proposition 2.1.5.** *The kernel of a morphism  $f: A \rightarrow B$  in  $\text{Hopf}_{K, \text{coc}}$  is given by the inclusion  $\text{HKer}(f) \rightarrow A$  of the following Hopf subalgebra of  $A$ ,*

$$\begin{aligned}
 \text{HKer}(f) &= \{a \in A \mid f(a_1) \otimes a_2 = u_B \otimes a\} \\
 &= \{a \in A \mid a_1 \otimes f(a_2) = a \otimes u_B\} \\
 &= \{a \in A \mid a_1 \otimes f(a_2) \otimes a_3 = a_1 \otimes u_B \otimes a_2\}.
 \end{aligned}$$

We give an explicit description of the pullback of cocommutative Hopf algebras.

**Proposition 2.1.6.** *The pullback of two morphisms  $f : A \rightarrow C$  and  $g : B \rightarrow C$  in  $\mathbf{Hopf}_{K, \text{coc}}$  is given by the following square*

$$\begin{array}{ccc} A \times_C B & \xrightarrow{p_B} & B \\ p_A \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C, \end{array}$$

where the object part is

$$A \times_C B = \{a \otimes b \in A \otimes B \mid a_1 \otimes f(a_2) \otimes b = a \otimes g(b_1) \otimes b_2\},$$

and the projections  $p_A$  and  $p_B$  are defined by  $p_A(a \otimes b) = a\epsilon_B(b)$  and  $p_B(a \otimes b) = \epsilon_A(a)b$ , for any  $a \otimes b \in A \times_C B$ .

We described the products, the equalizers, the kernels and the pullbacks in the category of cocommutative Hopf algebras, hence  $\mathbf{Hopf}_{K, \text{coc}}$  is a (finitely) complete category [88, 83].

Dually,  $\mathbf{Hopf}_{K, \text{coc}}$  is cocomplete [88, 83]. We give a description of the coequalizers and the cokernels, note that these constructions also hold for arbitrary Hopf algebras. First, we recall some basic properties of the quotients of algebras, coalgebras and Hopf algebras.

**Proposition 2.1.7.** *A two-sided ideal  $I$  in an algebra  $A$  is a linear subspace  $I \subset A$  such that  $aI$  and  $Ia$  belong to  $I$  for any  $a \in A$ . For any two-sided ideal  $I$ , the linear quotient  $A/I$  is an algebra and the canonical projection  $A \rightarrow A/I$  is an algebra morphism.*

*A two-sided coideal  $I$  in a coalgebra  $C$  is a linear subspace  $I \subset C$  such that  $\Delta(I) \subset I \otimes C + C \otimes I$  and  $\epsilon_C(I) = 0$ . For any two-sided coideal  $I$ , the linear quotient  $C/I$  is a coalgebra and the canonical projection  $C \rightarrow C/I$  is a coalgebra morphism.*

*A Hopf ideal  $I$  in a Hopf algebra  $H$  is a two-sided ideal and two-sided coideal that is stable under the antipode (i.e.  $S(I) \subseteq I$ ). For any Hopf ideal the linear quotient  $H/I$  is a Hopf algebra and the canonical projection is a Hopf algebra morphism.*

**Proposition 2.1.8.** *Let  $f, g: A \rightarrow B$  be two Hopf algebra morphisms. Then,  $BJB$ , the two-sided  $B$ -ideal generated by  $J := \{f(x) - g(x) | x \in A\}$ , is a Hopf ideal. Moreover, the coequalizer of  $f$  and  $g$  is given by  $HCoequal = B/BJB$  and  $\pi$  the usual projection.*

*Proof.* It is clear that  $BJB$  is a two-sided ideal in  $B$ . Moreover, it is a two-sided coideal since for any  $x \in A$ , we have

$$\begin{aligned} \Delta(f(x) - g(x)) &= \Delta(f(x)) - \Delta(g(x)) \\ &= f(x_1) \otimes f(x_2) - g(x_1) \otimes g(x_2) \\ &= f(x_1) \otimes f(x_2) - g(x_1) \otimes f(x_2) \\ &\quad + g(x_1) \otimes f(x_2) - g(x_1) \otimes g(x_2) \\ &= (f(x_1) - g(x_1)) \otimes f(x_2) + g(x_1) \otimes (f(x_2) - g(x_2)) \\ &\in BJB \otimes B + B \otimes BJB \end{aligned}$$

and  $\epsilon_B(BJB) = 0$ . Moreover, it is stable under the antipode since the following equality holds

$$S(b(f(x) - g(x))b') = S(b')(f(S(x)) - g(S(x)))S(b)$$

for any  $b, b' \in B$  and  $x \in A$ . Finally, the morphism of Hopf algebras  $\pi: B \rightarrow B/BJB$  has the universal property of a coequalizer thanks to the universal property of the linear quotient. Let  $\psi: B \rightarrow E$  be such that  $\psi \cdot f = \psi \cdot g$ , then  $BJB \subset \ker(\psi)$ , where  $\ker(\psi)$  is the linear kernel. Hence, there exists  $\bar{\psi}: B/BJB \rightarrow E$  such that  $\bar{\psi} \cdot \pi = \psi$  and we can check that  $\bar{\psi}$  is a morphism of Hopf algebras.

$$\begin{array}{ccccc} A & \xrightarrow[f]{g} & B & \xrightarrow{\pi} & B/BJB \\ & & \searrow \psi & & \downarrow \bar{\psi} \\ & & & & E. \end{array}$$

□

Given any coalgebra  $C$ , we shall write

$$C^+ = \{x \in C \mid \epsilon_C(x) = 0\}.$$

Remark that for any coalgebra morphism  $f : C \rightarrow D$ , we have that  $f(C^+) = f(C)^+$ . Indeed,  $y \in f(C)^+$  if and only if  $y = f(x)$  for some  $x \in C$  and  $0 = \epsilon_D(y) = \epsilon_D(f(x)) = \epsilon_C(x)$ . This is also equivalent to  $x \in C^+$  and  $y = f(x)$ . Note that if  $A$  is a bialgebra, then  $A^+$  is equal to the bialgebra  $\{a - \epsilon_A(a)u_A \mid a \in A\}$ .

Taking as “ $g$ ” the zero morphism in the construction of the coequalizer, we obtain the description of the cokernel.

**Proposition 2.1.9.** *The cokernel of an arrow  $f : A \rightarrow B$  in  $\text{Hopf}_{K, \text{coc}}$  is given by the canonical quotient*

$$q : B \rightarrow B/Bf(A)^+B,$$

where  $f(A) = \{f(a) \mid a \in A\}$  is the direct image of  $A$  along  $f$ .

To end this section about limits and colimits in the category of cocommutative Hopf algebras, we will give another explicit description of the pullback of an inclusion morphism along any morphism. This description is based on a construction given in [97].

As explained in [97] (Corollary 2) given a morphism  $p : A \rightarrow B$  in  $\text{Hopf}_{K, \text{coc}}$  and a Hopf subalgebra  $C \subseteq B$  of  $B$ , then the subset of  $A$  defined by

$$p^{-1}(C) = \{x \in A \mid (p \otimes 1_A)\Delta(x) \in C \otimes A\}$$

is a Hopf subalgebra of  $A$ . This subalgebra is called the  $h$ -inverse of  $C$  (along  $p$ ). The  $h$ -inverse and the direct image along  $p$  define an order preserving correspondence between the lattices of Hopf subalgebras of  $A$  and  $B$ .

**Lemma 2.1.10.** *Consider a morphism  $p : A \rightarrow B$  in  $\text{Hopf}_{K, \text{coc}}$ . Then:*

- (i) *for all Hopf subalgebras  $C$  of  $B$ ,  $p(p^{-1}(C)) \subseteq C$ ;*
- (ii) *for all Hopf subalgebras  $D$  of  $A$ ,  $D \subseteq p^{-1}(p(D))$ .*
- (iii) *for all Hopf subalgebras  $C \subseteq B$  and  $D \subseteq A$ ,*

$$D \subseteq p^{-1}(C) \Leftrightarrow p(D) \subseteq C;$$

(iv) for all Hopf subalgebras  $C \subseteq B$

$$C = p(p^{-1}(C)) \Leftrightarrow C = p(D), \text{ for some } D \subseteq A.$$

*Proof.* (i) If  $x \in p^{-1}(C)$ , then  $p(x_1) \otimes x_2 \in C \otimes A$ . By applying  $1_C \otimes \epsilon_A$  to this element we get  $p(x) \in C$ .

(ii) First remark that  $p(d) \in p(D)$ , for any  $d \in D$ . Then we have

$$p(d_1) \otimes d_2 \in p(D) \otimes A.$$

(iii) If  $p(D) \subseteq C$  then it follows by the previous observations that

$$D \subseteq p^{-1}(p(D)) \subseteq p^{-1}(C).$$

The other implication is proven in the same way.

(iv) In the case  $C = p(p^{-1}(C))$ , we can take  $D = p^{-1}(C)$ . On the other hand, if  $C = p(D)$  for some Hopf subalgebra  $D$  of  $A$  then by part (ii) we have  $D \subseteq p^{-1}(p(D)) = p^{-1}(C)$ . By applying  $p$ , one gets  $C = p(D) \subseteq p(p^{-1}(C))$ , and by part (i) we obtain the desired equality.  $\square$

Let us now observe that the  $h$ -inverse  $p^{-1}(C)$  is nothing but the “object part” of the pullback of the inclusion morphism  $i: C \rightarrow B$  along  $p: A \rightarrow B$ :

**Lemma 2.1.11.** *Given an inclusion of a Hopf subalgebra  $i: C \rightarrow B$  of a cocommutative Hopf algebra  $B$ , then the diagram*

$$\begin{array}{ccc} p^{-1}(C) & \xrightarrow{\hat{p}} & C \\ j \downarrow & & \downarrow i \\ A & \xrightarrow{p} & B. \end{array} \quad (2.3)$$

*is a pullback, where  $j$  is the inclusion of  $p^{-1}(C)$  in  $A$ , and  $\hat{p}$  the restriction of  $p$  to  $p^{-1}(C)$ .*

*Proof.* As it follows from Lemma 2.1.10 (i), for any  $x$  in  $p^{-1}(C)$  the element  $p(x)$  is in  $C$ , and the diagram (2.3) is commutative. To check the universal property, consider two morphisms  $\alpha: T \rightarrow A$  and  $\beta: T \rightarrow C$

such that  $p \cdot \alpha = i \cdot \beta$ . It suffices to show that  $\alpha(t) \in p^{-1}(C)$ , for any  $t \in T$ . We have the equalities:

$$\begin{aligned} (p \otimes 1_A)\Delta(\alpha(t)) &= (p \cdot \alpha \otimes \alpha)(t_1 \otimes t_2) \\ &= i \cdot \beta(t_1) \otimes \alpha(t_2) \end{aligned}$$

Since  $\beta(t) \in C$  for any  $t \in T$ , we obtain that  $\alpha(t) \in p^{-1}(C)$ , as desired.  $\square$

This other characterization of the pullback of an inclusion along any morphism will be useful to prove the regularity of the category of cocommutative Hopf algebras.

## 2.2 $\mathbf{Hopf}_{K, \text{coc}}$ is regular, semi-abelian, and action representable

In this section (based on [50]), we prove that the category of cocommutative Hopf algebras over any field is semi-abelian and action representable. Similar results were obtained in [48] for fields of characteristic zero. As explained in detail in the previous section,  $\mathbf{Hopf}_{K, \text{coc}}$  is complete and cocomplete [88, 83]. Moreover, it is pointed, with zero object the base field  $K$ , Proposition 2.1.1.

Our proof of the regularity of  $\mathbf{Hopf}_{K, \text{coc}}$  is based on a result due to Newman (Theorem 2.2.1). We start by establishing the framework of this result.

Let  $A$  be a Hopf algebra then for an  $A$ -module  $M$  (as defined in Definition 1.3.4 in the category  $\mathbf{Vect}_K$ ), we denote the action of an element  $a \in A$  on  $m \in M$  by

$$a \triangleright m.$$

It satisfies the following equations for any  $a, b \in A$  and any  $m \in M$ :

$$(b \triangleright (a \triangleright m)) = (ba \triangleright m),$$

$$u_A \triangleright m = m.$$

In Sweedler notation, an  $A$ -module coalgebra  $C$  satisfies the two following conditions

$$\Delta_C(a \triangleright c) = a_1 \triangleright c_1 \otimes a_2 \triangleright c_2, \quad \epsilon_C(a \triangleright c) = \epsilon_A(a)\epsilon_C(c)$$

for all  $a \in A$  and  $c \in C$ .

The following result, due to Newman [79], will play a central role in what follows. It states that when  $I$  is a left ideal of a Hopf algebra  $A$  that is also a (two-sided) coideal, the quotient  $A/I$  has the structure of an  $A$ -module coalgebra, and the canonical projection  $\pi : A \rightarrow A/I$  is a morphism of  $A$ -module coalgebras. Conversely, given any surjective morphism  $\pi : A \rightarrow B$  where  $A$  is a cocommutative Hopf algebra and  $B$  is a (cocommutative)  $A$ -module coalgebra, the vector space kernel of  $\pi$  naturally has the structure of a left ideal and (two-sided) coideal.

**Theorem 2.2.1.** [79] *Let  $A$  be a cocommutative Hopf algebra on a field  $K$ . Then there is a bijective correspondence between*

$$\mathcal{S} = \{D \subset A \mid D \text{ is a Hopf subalgebra of } A\}$$

and

$$\mathcal{I} = \{I \subset A \mid I \text{ is a left ideal and a two sided coideal of } A\}.$$

1. *The correspondence  $\Phi_A : \mathcal{S} \rightarrow \mathcal{I}$  associates, with a Hopf subalgebra  $D$  of  $A$ , the corresponding left ideal two-sided coideal given by  $\Phi_A(D) = AD^+$  (which is the  $A$ -module generated by  $D^+$ );*
2. *Conversely, given a left ideal two-sided coideal  $I$  in  $A$ , the inverse correspondence  $\Psi_A : \mathcal{I} \rightarrow \mathcal{S}$  sends  $I$  to the Hopf subalgebra of  $A$  given by*

$$\Psi_A(I) = \{x \in A \mid (1_A \otimes \pi)\Delta(x) = x \otimes \overline{u_A}\},$$

where  $\pi$  is the canonical quotient  $\pi : A \rightarrow A/I$  and  $\overline{u_A} = \pi(u_A)$  is the equivalence class of the unit  $u_A$  of the Hopf algebra  $A$ .

When  $A$  is a cocommutative Hopf algebra,  $B$  an  $A$ -module cocommutative coalgebra, and  $f : A \rightarrow B$  a morphism of  $A$ -module coalgebras we find that the vector space kernel  $\ker(f)$  is a left ideal two-sided coideal in  $A$ , and  $\ker(f) = A\Psi_A(\ker(f))^+$ .

Theorem 2.2.1 can be restricted to normal Hopf subalgebras. We recall that a Hopf subalgebra  $B \subset A$  of a cocommutative Hopf algebra  $A$  is called *normal* if for any  $b \in B$  and  $a \in A$ , one has that  $a_1 b S(a_2) \in B$ . The following corollary is well-known, but we include a proof for sake of clarity.

**Corollary 2.2.2.** *For a Hopf subalgebra  $B \subset A$  of a cocommutative Hopf algebra  $A$ , the following conditions are equivalent:*

1.  $B$  is a normal Hopf subalgebra;
2. the associated ideal  $\Phi_A(B)$  is a Hopf ideal (hence the quotient  $A/\Phi_A(B)$  is a Hopf algebra);
3. the inclusion morphism  $B \rightarrow A$  is a normal monomorphism, i.e. the kernel of some morphism in  $\mathbf{Hopf}_{K, \text{coc}}$ .

*In other words, Newman's correspondence can be restricted to a correspondence between normal Hopf subalgebras and Hopf ideals.*

*Proof.* (1)  $\Rightarrow$  (2). Let  $B$  be a normal Hopf subalgebra of  $A$ , and consider the associated left ideal, two-sided co-ideal  $\Phi_A(B) = AB^+$ . Let us show that it is also a right ideal. For any  $ab \in AB^+$  (sum implicitly understood) and any  $a' \in A$ , we find that

$$(ab)a' = a \epsilon_A(a'_1) b a'_2 = a a'_1 S(a'_2) b a'_3 \in AB^+.$$

To see that  $AB^+$  is stable under the antipode, remark that  $S(ab) = S(b)S(a)$ . Since  $b \in B^+$  and  $B$  is stable under the antipode,  $S(b) \in B^+ \subset AB^+$  and therefore  $S(b)S(a) \in AB^+$  since we have just shown this is a right ideal.

(2)  $\Rightarrow$  (3). If  $I = \Phi_A(B)$  is a Hopf ideal, then the quotient  $A/\Phi_A(B)$  is a Hopf algebra. By Newman's correspondence, we know that  $B = \Psi_A(I)$ . Observe that  $\Psi_A(I)$  is exactly the kernel  $\mathbf{HKer}(\pi)$  of the quotient  $\pi : A \rightarrow A/I$ .

(3)  $\Rightarrow$  (1). Suppose that  $B = \mathbf{HKer}(f)$  for some morphism  $f : A \rightarrow C$  in  $\mathbf{Hopf}_{K, \text{coc}}$ . Then we know that  $x \in \mathbf{HKer}(f)$  if and only if  $f(x_1) \otimes x_2 = u_C \otimes x$ . For any  $a \in A$ , we then find that

$$f(a_1 x_1 S(a_4)) \otimes a_2 x_2 S(a_3) = f(a_1 S(a_4)) \otimes a_2 x S(a_3)$$

$$\begin{aligned} &= f(a_1 S(a_2)) \otimes a_3 x S(a_4) \\ &= u_C \otimes a_1 x S(a_2), \end{aligned}$$

and therefore  $a_1 x S(a_2) \in \text{HKer}(f)$ .  $\square$

To prove the regularity of the category of cocommutative Hopf algebras, we show that any morphism of Hopf algebras has a regular epimorphism-monomorphism factorization. We also prove that for any regular epimorphism  $f : A \rightarrow B$ , the morphism  $1_E \otimes f : E \otimes A \rightarrow E \otimes B$  is a regular epimorphism. These two statements are the conditions (1) and (2) in Lemma 1.4.2. Note that in  $\text{Hopf}_{K, \text{coc}}$ , the regular epimorphisms are the surjective morphisms [30].

**Lemma 2.2.3.** *The category  $\text{Hopf}_{K, \text{coc}}$  of cocommutative Hopf algebras over a field  $K$  satisfies conditions (1) and (2) in Lemma 1.4.2.*

*Proof.* As remarked in [48], given a morphism  $f : A \rightarrow B$ , its regular epimorphism-monomorphism factorization  $i \cdot p$  in  $\text{Hopf}_{K, \text{coc}}$  is obtained by taking the cokernel  $p$  of the kernel  $k$  of  $f$ :

$$\begin{array}{ccccc} \text{HKer}(f) & \xrightarrow{k} & A & \xrightarrow{f} & B \\ & & \searrow p & & \nearrow i \\ & & \frac{A}{A(\text{HKer}(f))^+ A} & & \end{array}$$

Indeed, notice that the vector space kernel  $\ker(f)$  is a Hopf ideal when  $f : A \rightarrow B$  is a Hopf algebra morphism, then thanks to Newman's theorem and its Corollary 2.2.2, we know that  $\ker(f) = A(\text{HKer}(f))^+$ , where  $A(\text{HKer}(f))^+$  is equal to  $A(\text{HKer}(f))^+ A$  since  $\text{HKer}(f)$  is normal. Therefore, we have that

$$\frac{A}{A(\text{HKer}(f))^+ A} = \frac{A}{\ker(f)} \cong f(A),$$

and the regular epimorphism - monomorphism factorization in  $\text{Hopf}_{K, \text{coc}}$  is obtained as the usual one in vector spaces.

To see that condition (2) is satisfied, observe that in  $\text{Hopf}_{K, \text{coc}}$  the regular epimorphisms are the same as surjective morphisms [30], and products are tensor products:  $E \times A = E \otimes A$ . Accordingly, the induced arrow  $1_E \otimes f : E \otimes A \rightarrow E \otimes B$  is surjective whenever  $f$  is surjective.  $\square$

The point (i) in the following lemma was also observed in [97, 93], for instance.

**Lemma 2.2.4.** *Let  $p: A \rightarrow B$  be a surjective Hopf algebra morphism in  $\mathbf{Hopf}_{K, \text{coc}}$ . Then for any Hopf subalgebra  $D$  of  $A$ , we have that*

- (i) *if  $D$  is normal in  $A$ , its direct image  $p(D)$  is a normal Hopf subalgebra of  $B$ ,*
- (ii) *taking the direct image commutes with applying the correspondence  $\Phi$  from Theorem 2.2.1, i.e.*

$$\Phi_B \cdot p(D) = p \cdot \Phi_A(D).$$

*Proof.* (i) Assume that  $D$  is a normal Hopf subalgebra of  $A$ , so that  $a_1 d S(a_2) \in D$ , for any  $a \in A$  and any  $d \in D$ . Since  $p$  is surjective, for any  $b$  in  $B$  there exists  $a$  in  $A$  such that  $p(a) = b$  (and then  $b_1 \otimes b_2 = p(a_1) \otimes p(a_2)$ ). Hence, for any  $d$  in  $D$ ,

$$\begin{aligned} b_1 p(d) S(b_2) &= p(a_1) p(d) S(p(a_2)) \\ &= p(a_1 d S(a_2)) \in p(D), \end{aligned}$$

since  $a_1 d S(a_2) \in D$  (since  $D$  is normal in  $A$ ). Thus,  $p(D)$  is a normal Hopf subalgebra of  $B$ .

(ii) Using the surjectivity of  $p$  in the third equality, it follows that

$$p \cdot \Phi_A(D) = p(AD^+) = p(A)p(D^+) = Bp(D^+) = Bp(D)^+ = \Phi_B \cdot p(D).$$

□

The following result proves that the pullback of an epimorphism along a monomorphism in  $\mathbf{Hopf}_{K, \text{coc}}$  is again an epimorphism. To prove this, we use the observation from Lemma 2.1.11 that the pullback along a monomorphism in  $\mathbf{Hopf}_{K, \text{coc}}$  can be computed by means of the  $h$ -inverse.

**Proposition 2.2.5.** [50] *Consider a surjective morphism  $p: A \rightarrow B$  in  $\mathbf{Hopf}_{K, \text{coc}}$  and a Hopf subalgebra  $C$  of  $B$ , with inclusion  $i: C \rightarrow B$ . Then*

the morphism  $\hat{p}$  in the following pullback is also surjective.

$$\begin{array}{ccc} p^{-1}(C) & \xrightarrow{\hat{p}} & C \\ j \downarrow & & \downarrow i \\ A & \xrightarrow{p} & B. \end{array}$$

*Proof.* From Lemma 2.1.11, we know that  $\hat{p}$  is just given by the restriction of  $p$ . Hence,  $\hat{p}$  is surjective if and only if  $C = p(p^{-1}(C))$ . By Lemma 2.1.10(iv), it is equivalent to prove that  $C = p(D)$  for some Hopf subalgebra  $D$  of  $A$ . To construct this algebra  $D$ , consider the quotient  $B/BC^+$ , which is a left  $B$ -module coalgebra and thus a left  $A$ -module coalgebra by restriction of scalars via  $p$ . Consider now the following diagram in the category  $\mathbf{Vect}$  of vector spaces:

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ & \searrow \pi \cdot p & \downarrow \pi \\ & & B/BC^+. \end{array}$$

Since  $\pi$  and  $p$  are both surjective, we have

$$p(\ker(\pi \cdot p)) = \ker(\pi) = BC^+. \quad (2.4)$$

Furthermore,  $\pi \cdot p$  is an  $A$ -module coalgebra morphism, and therefore the (vector space) kernel of  $\pi \cdot p$  is a left ideal and a two-sided coideal in  $A$ . Using the correspondence of Theorem 2.2.1, we obtain therefore a Hopf subalgebra  $D := \Psi_A(\ker(\pi \cdot p))$  of  $A$ . To show that  $C = p(D)$ , let us observe that

$$\begin{aligned} \Phi_B(C) &= \ker(\pi) = p(\ker(\pi \cdot p)) = p \cdot \Phi_A \cdot \Psi_A(\ker(\pi \cdot p)) \\ &= p \cdot \Phi_A(D) = \Phi_B \cdot p(D) \end{aligned}$$

where we used (2.4) in the second equality, the Newman correspondence from Theorem 2.2.1 in the third equality, and Lemma 2.2.4(ii) in the last equality. Applying once more the Newman correspondence, we find that  $C = \Psi_B \cdot \Phi_B(C) = \Psi_B \cdot \Phi_B(p(D)) = p(D)$  and this completes the proof.  $\square$

The regularity of  $\mathbf{Hopf}_{K, \text{coc}}$  follows from Lemma 1.4.2, Lemma 2.2.3, and Proposition 2.2.5.

**Corollary 2.2.6.** [50] *The category  $\mathbf{Hopf}_{K, \text{coc}}$  is regular.*

*Proof.* In any pullback

$$\begin{array}{ccc} E \times_C B & \xrightarrow{p_2} & C \\ p_1 \downarrow & & \downarrow f \\ A & \xrightarrow{p} & B, \end{array}$$

where  $f: A \rightarrow B$  is a monomorphism, and  $p: E \rightarrow B$  a regular epimorphism, we know that  $p_2: E \times_C B \rightarrow C$  is surjective by Proposition 2.2.5. By Lemma 1.4.2 and Lemma 2.2.3 the proof is then complete.  $\square$

An important fact is that  $\mathbf{Hopf}_{K, \text{coc}}$  can be seen as the category of the internal groups in the category of cocommutative  $K$ -coalgebras. An internal group in a (finitely complete and pointed) category  $\mathcal{C}$  is given by a quadruple  $(G, m, e, i)$  where  $m: G \times G \rightarrow G$  is a morphism in  $\mathcal{C}$  which is associative (the multiplication),  $e: 0 \rightarrow G$  which is the unit for the multiplication  $m$  and the morphism  $i: G \rightarrow G$  such that

$$\begin{array}{ccccc} & & G \times G & \xrightleftharpoons[i \times 1_G]{1_G \times i} & G \times G \\ & \nearrow \Delta & & & \searrow m \\ G & \xrightarrow{\quad} & 0 & \xrightarrow{e} & G. \end{array}$$

It is then clear that the category of internal groups in  $\mathbf{Coalg}_{K, \text{coc}}$ , denoted by  $\mathbf{Grp}(\mathbf{Coalg}_{K, \text{coc}})$ , is  $\mathbf{Hopf}_{K, \text{coc}}$ .

Note that this observation is still true for the category of cocommutative Hopf algebras in any symmetric monoidal category.

The category  $\mathbf{Hopf}_{K, \text{coc}}$  is known to be protomodular [11, 48] since it can be seen as the category of internal groups in the category of cocommutative  $K$ -coalgebras, which is a finitely complete category (see [54, 83]).

**Proposition 2.2.7.** *The category  $\mathbf{Hopf}_{K, \text{coc}}$  is protomodular.*

Since  $\mathbf{Hopf}_{K, \text{coc}}$  is pointed, Proposition 2.2.7 implies that the Split Short Five Lemma holds: given the following diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X_1 & \xrightarrow{\kappa_1} & A_1 & \begin{array}{c} \xleftarrow{e_1} \\ \xrightarrow{\alpha_1} \end{array} & B_1 \longrightarrow 0 \\
 & & \downarrow v & & \downarrow p & & \downarrow g \\
 0 & \longrightarrow & X_2 & \xrightarrow{\kappa_2} & A_2 & \begin{array}{c} \xleftarrow{e_2} \\ \xrightarrow{\alpha_2} \end{array} & B_2 \longrightarrow 0,
 \end{array}$$

where  $\kappa_i$  is the kernel of  $\alpha_i$  and  $\alpha_i \cdot e_i = 1_{B_i}$  for any  $i \in \{1, 2\}$ , then  $p$  is an isomorphism whenever  $v$  and  $g$  are.

**Theorem 2.2.8.** [50] *The category  $\mathbf{Hopf}_{K, \text{coc}}$  is semi-abelian.*

*Proof.* The category  $\mathbf{Hopf}_{K, \text{coc}}$  is pointed, with zero object the base field  $K$ , and it is known to be protomodular [11] (see Proposition 2.2.7).  $\mathbf{Hopf}_{K, \text{coc}}$  is also (finitely) complete and cocomplete [83]. By Corollary 1.4.2 the category  $\mathbf{Hopf}_{K, \text{coc}}$  is regular, so thanks to Proposition 1.4.9 it will be semi-abelian provided that the direct image of a normal monomorphism is again a normal monomorphism. This property is true by Lemma 2.2.4 (i).  $\square$

From this theorem one can easily deduce Takeuchi's theorem concerning the category  $\mathbf{Hopf}_{K, \text{comm}, \text{coc}}$  of commutative and cocommutative Hopf algebras:

**Theorem 2.2.9.** [89] *The category  $\mathbf{Hopf}_{K, \text{comm}, \text{coc}}$  of commutative and cocommutative Hopf algebras over a field  $K$  is an abelian category.*

*Proof.* It is well-known that the category of abelian objects in a semi-abelian category is abelian (see Corollary 4.2 in [46], for instance). It turns out that the category  $\mathbf{Ab}(\mathbf{Hopf}_{K, \text{coc}})$  of abelian objects in  $\mathbf{Hopf}_{K, \text{coc}}$  is isomorphic to the category  $\mathbf{Hopf}_{K, \text{comm}, \text{coc}}$  of commutative and cocommutative Hopf algebras. Indeed, it follows from Proposition 9 in [12] that the abelian objects in any semi-abelian category  $\mathcal{C}$  are precisely the objects  $X$  with the property that the diagonal  $\Delta: X \rightarrow X \times X$  is a normal monomorphism (= a kernel of some morphism in  $\mathcal{C}$ ). Accordingly, as observed in [93], the abelian objects in  $\mathbf{Hopf}_{K, \text{coc}}$  are precisely the cocommutative Hopf algebras  $X$  such that the comultiplication  $\Delta: X \rightarrow X \otimes X$  is a normal monomorphism or, equivalently, such that  $X$  is commutative.  $\square$

We finish this section by an immediate corollary, that can be viewed as a semi-abelian extension of the result due to Grothendieck, which states the category of finite dimensional, commutative, and cocommutative Hopf algebras is abelian.

**Corollary 2.2.10.** *[50] The category  $\mathbf{Hopf}_{K,\text{coc}}^{\text{fd}}$  of finite dimensional cocommutative Hopf algebras over a field  $K$  is semi-abelian.*

*The opposite category  $(\mathbf{Hopf}_{K,\text{comm}}^{\text{fd}})^{\text{op}}$  of the category  $\mathbf{Hopf}_{K,\text{comm}}^{\text{fd}}$  of finite dimensional commutative Hopf algebras over a field  $K$  is semi-abelian.*

*Proof.* The first statement follows immediately from Theorem 2.2.8 and the observation that  $\mathbf{Hopf}_{K,\text{coc}}^{\text{fd}}$  is a full subcategory of  $\mathbf{Hopf}_{K,\text{coc}}$  which is closed under finite limits and regular quotients.

The second statement follows from the fact that the vector space dual induces an equivalence of categories between the categories  $\mathbf{Hopf}_{K,\text{coc}}^{\text{fd}}$  and  $(\mathbf{Hopf}_{K,\text{comm}}^{\text{fd}})^{\text{op}}$ .  $\square$

As it has been explained, the category of cocommutative Hopf algebras  $\mathbf{Hopf}_{K,\text{coc}}$  can be seen as the category of the internal groups of the category of cocommutative Hopf algebras  $\mathbf{Coalg}_{K,\text{coc}}$ . This category is a cartesian closed category, which means that it is a cartesian category (see Examples 1.2.3 (3)) and that for any cocommutative coalgebras  $C$ , the functors  $- \otimes C: \mathbf{Coalg}_{K,\text{coc}} \rightarrow \mathbf{Coalg}_{K,\text{coc}}$  and  $[C, -]: \mathbf{Coalg}_{K,\text{coc}} \rightarrow \mathbf{Coalg}_{K,\text{coc}}$ , which is the internal Hom functor, form a pair of adjoint functors.

**Proposition 2.2.11.** *[50] The category  $\mathbf{Hopf}_{K,\text{coc}}$  is an action representable category.*

*Proof.* By Theorem 4.4 in [10], one knows that the category of internal groups in a cartesian closed category is always action representable, provided it is semi-abelian. The result then follows from Theorem 2.2.8 and the observation that the category  $\mathbf{Hopf}_{K,\text{coc}}$  of cocommutative Hopf algebras is the category of internal groups in the cartesian closed category  $\mathbf{Coalg}_{K,\text{coc}}$  of cocommutative coalgebras (see [54] (2.3) for the fact that  $\mathbf{Coalg}_{K,\text{coc}}$  is a cartesian closed category).  $\square$

**Remark 2.2.12.** *The same argument as in the proof of the Proposition here above implies that  $\mathbf{Hopf}_{K,\text{coc}}$  is locally algebraically cartesian closed in the sense of [52], and then algebraically coherent in the sense of [31].*

**Remark 2.2.13.** *An explicit description of the split extension classifier  $[X]$  of a cocommutative Hopf algebra in  $\mathbf{Hopf}_{K,\text{coc}}$  was given in [49] in the special case when the characteristic of the base field  $K$  is zero, by using the canonical semi-direct product decomposition in the Milnor-Moore Theorem [75]. It would be interesting to give such a description in the case of a general field  $K$ .*

## 2.3 Application of the semi-abelianess of the category of cocommutative Hopf algebras over a field

In the previous section, we proved that the category of cocommutative Hopf algebras over any field is semi-abelian. We will use the theory of semi-abelian categories to investigate some categorical constructions and properties in the category of cocommutative Hopf algebras and deepen our understanding of this category.

### 2.3.1 Commutator of $\text{Huq}$ in $\mathbf{Hopf}_{K,\text{coc}}$

In this section we shall describe the commutator of  $\text{Huq}$  in the category of cocommutative Hopf algebras, [50]. We begin with the characterization of the centrality in the sense of  $\text{Huq}$  (see Definition 1.5.2) via the following lemma. A key element in this lemma is the fact that the categorical product  $X \times Y$  in  $\mathbf{Hopf}_{K,\text{coc}}$  is the tensor product  $X \otimes Y$ .

**Lemma 2.3.1.** [50] *In  $\mathbf{Hopf}_{K,\text{coc}}$ , for any two Hopf subalgebras  $x: X \rightarrow A$ , and  $y: Y \rightarrow A$  of a cocommutative Hopf algebra  $A$ , the following conditions are equivalent:*

(a) *there exists a unique morphism of Hopf algebras  $p : X \otimes Y \rightarrow A$  such that this diagram*

$$\begin{array}{ccccc}
 & & (1,0) & & (0,1) \\
 & & \longrightarrow & & \longleftarrow \\
 X & & X \otimes Y & & Y \\
 & \searrow x & \downarrow p & \swarrow y & \\
 & & A & & 
 \end{array} \tag{2.5}$$

*commutes;*

(b)  $ab = ba, \forall a \in X \text{ and } \forall b \in Y$ ;

(c)  $a_1 b_1 S(a_2) S(b_2) = \epsilon_A(a) \epsilon_A(b) u_A, \forall a \in X \text{ and } \forall b \in Y$ .

*Proof.* (a)  $\Leftrightarrow$  (b). First note that, in  $\mathbf{Hopf}_{K, coc}$ , if there is a  $p$  making diagram (2.5) commute then  $p$  has to be defined as  $p(a \otimes b) = ab$ : indeed,

$$\begin{aligned}
 p(a \otimes b) &= p((a \otimes u_A)(u_A \otimes b)) \\
 &= p(a \otimes u_A) p(u_A \otimes b) \\
 &= ab.
 \end{aligned}$$

Then note that, if (b) holds, then  $p$  is an algebra morphism (and it is always a coalgebra morphism by the cocommutativity assumption).

On the other hand, when (a) holds and  $p$  is an algebra morphism,

$$ab = p((a \otimes u_A)(u_A \otimes b)) = p((u_A \otimes b)(a \otimes u_A)) = ba.$$

(b)  $\Leftrightarrow$  (c) When condition (b) holds, then we have the following identities:

$$a_1 b_1 S(a_2) S(b_2) = a_1 S(a_2) b_1 S(b_2) = \epsilon_A(a) \epsilon_A(b).$$

Conversely,  $a_1 b_1 S(a_2) S(b_2) = \epsilon_A(a) \epsilon_A(b) u_A$  for all  $a \in X$  and  $b \in Y$  implies that

$$\begin{aligned}
 ab &= a_1 b \epsilon_A(a_2) = a_1 b S(a_2) a_3 = a_1 b_1 S(a_2) \epsilon_A(b_2) a_3 \\
 &= a_1 b_1 S(a_2) S(b_2) b_3 a_3 = \epsilon_A(a_1) \epsilon_A(b_1) b_2 a_2 = ba. \quad \square
 \end{aligned}$$

Thanks to Lemma 1.5.4 and Proposition 2.2.11, the Huq centrality implies the Smith centrality and vice-versa, which is express in the following remark.

**Remark 2.3.2.** *It is well-known that any action representable category has Smith is Huq property. It means that the notion of centrality of two equivalence relations  $R$  and  $S$  in the sense of Smith is equivalent to the notion of centrality of the corresponding normal subobjects in the sense of Huq [19]. Let us make it explicit, let  $(R, r_0, r_1)$  and  $(S, s_0, s_1)$  be two equivalence relations over  $X$  in  $\mathbf{Hopf}_{K, \text{coc}}$ , these relations centralize in the sense of Smith if and only if their normalizations  $r_1 \cdot hker(r_0): HKer(r_0) \rightarrow X$  and  $s_1 \cdot hker(s_0): HKer(s_0) \rightarrow X$  centralize in the sense of Huq:*

$$\begin{array}{ccc}
 HKer(s_0) & \xrightarrow{(1,0)} & HKer(s_0) \otimes HKer(r_0) \xleftarrow{(0,1)} HKer(r_0), \\
 & \searrow s_1 \cdot hker(s_0) & \downarrow \text{dotted arrow} \swarrow r_1 \cdot hker(r_0) \\
 & & X
 \end{array}$$

where  $hker(r_0): HKer(r_0) \rightarrow R$  and  $hker(s_0): HKer(s_0) \rightarrow S$  are the kernels of  $r_0$  and  $s_0$  respectively. Thanks to Lemma 2.3.1, it follows that two equivalence relations  $(R, r_0, r_1)$  and  $(S, s_0, s_1)$  of cocommutative Hopf algebras centralize if and only if  $r_1(k)s_1(x) = s_1(x)r_1(k)$  for any  $k \in HKer(r_0)$  and any  $x \in HKer(s_0)$ . Hence, the dotted arrow in the above diagram can be defined as  $p(x \otimes k) = s_1(x)r_1(k)$ , for any  $x \in HKer(s_0)$  and any  $k \in HKer(r_0)$ .

After the centrality in the sense of Huq, we specify now the Huq commutator (see Definition 1.5.5) in  $\mathbf{Hopf}_{K, \text{coc}}$ .

In particular, in the category  $\mathbf{Hopf}_{K, \text{coc}}$ , the Huq commutator of two normal Hopf subalgebras  $x: X \rightarrow A$  and  $y: Y \rightarrow A$ , denoted by  $[X, Y]_{\text{Huq}}$ , is defined as the smallest normal Hopf subalgebra in  $A$  such that, in the quotient  $A/A[X, Y]_{\text{Huq}}^+$ , the normal Hopf subalgebras  $q(X)$  and  $q(Y)$  commute in the sense of Huq, where  $q: A \rightarrow A/A[X, Y]_{\text{Huq}}^+$  is the canonical projection. By Lemma 2.3.1 this is also equivalent to the condition

$$\overline{ab} = \overline{ba}$$

$\forall a \in X, b \in Y$  or, equivalently, to

$$ab - ba \in A[X, Y]_{\text{Huq}}^+.$$

We shall now give an explicit description of the Huq commutator of two normal Hopf subalgebras  $X$  and  $Y$  of  $A$ , and show that it coincides with the commutator defined in [96]. We write  $[X, Y]$  for the subalgebra of  $A$  generated by all elements of the form

$$\{a, b\} = a_1 b_1 S(a_2) S(b_2)$$

for any  $a \in X$  and any  $b \in Y$ .

**Proposition 2.3.3.** *The algebra  $[X, Y]$  is a normal Hopf subalgebra of  $A$ .*

*Proof.* Thanks to the cocommutativity,  $[X, Y]$  is a subcoalgebra and hence a subbialgebra of  $A$ . Indeed,

$$\begin{aligned} \Delta(\{a, b\}) &= \Delta(a_1 b_1 S(a_2) S(b_2)) \\ &= a_1 b_1 S(a_4) S(b_4) \otimes a_2 b_2 S(a_3) S(b_3) \\ &= a_1 b_1 S(a_2) S(b_2) \otimes a_3 b_3 S(a_4) S(b_4) \in [X, Y] \otimes [X, Y]. \end{aligned}$$

Moreover, this subbialgebra is a Hopf subalgebra:

$$\begin{aligned} S(a_1 b_1 S(a_2) S(b_2)) &= b_2 a_2 S(b_1) S(a_1) \\ &= b_1 a_1 S(b_2) S(a_2) \in [X, Y], \end{aligned}$$

where we used that  $S^2 = 1_A$  since  $A$  is cocommutative (see e.g. [88, Proposition 4.0.1(6)]). Finally, given any  $c$  in  $A$ , we check that the element  $c_1 a_1 b_1 S(a_2) S(b_2) S(c_2)$  belongs to  $[X, Y]$  for any  $a \in X$  and any  $b \in Y$ . Indeed, this is the case, since  $X$  and  $Y$  are normal Hopf subalgebras of  $A$ :

$$\begin{aligned} &c_1 a_1 b_1 S(a_2) S(b_2) S(c_2) \\ &= c_1 a_1 S(c_2) c_3 b_1 S(c_4) c_5 S(a_2) S(c_6) c_7 S(b_2) S(c_8) \\ &= (c_1 a S(c_2))_1 (c_3 b S(c_4))_1 S((c_1 a S(c_2))_2) S((c_3 b S(c_4))_2). \quad \square \end{aligned}$$

**Proposition 2.3.4.** [50] *Let  $X, Y$  be two normal Hopf subalgebras of  $A$ . Then*

$$[X, Y] = [X, Y]_{\text{Huq}}.$$

*Proof.* If  $q: A \rightarrow A/A[X, Y]^+$  denotes the canonical quotient, as a first step we will prove that  $q(X)$  and  $q(Y)$  commute in  $A/A[X, Y]^+$ . As we have already observed, in  $\mathbf{Hopf}_{K, \text{coc}}$ , this is equivalent to the condition

$$ab - ba \in A[X, Y]^+,$$

for any  $a \in X$  and any  $b \in Y$ . Equivalently, this can be expressed by asking that

$$ab - ba = a_1 b_1 \left( \epsilon_A(a_2) \epsilon_A(b_2) u_A - S(b_2) S(a_2) b_3 a_3 \right) \in A[X, Y]^+,$$

that holds by definition of  $[X, Y]^+$ .

Next, we will show that  $[X, Y]$  is the smallest normal Hopf subalgebra which satisfies the definition of commutators. In other words, we are going to prove that  $[X, Y]$  factors through the categorical kernel of any surjective Hopf algebra morphism for which the images of  $X$  and  $Y$  commute. Let  $f: A \rightarrow B$  be any surjective Hopf algebra morphism such that  $f(X)$  and  $f(Y)$  commute in  $B$ .

For any generator  $\{a, b\}$  of  $[X, Y]$ , we have:

$$\begin{aligned} f(a_1 b_1 S(a_2) S(b_2)) \otimes a_3 b_3 S(a_4) S(b_4) \\ &= f(a_1) f(b_1) f(S(a_2)) f(S(b_2)) \otimes a_3 b_3 S(a_4) S(b_4) \\ &= f(a_1) f(S(a_2)) f(b_1) f(S(b_2)) \otimes a_3 b_3 S(a_4) S(b_4) \\ &= u_A \otimes a_1 b_1 S(a_2) S(b_2). \end{aligned}$$

In conclusion, the Huq commutator  $[X, Y]_{\text{Huq}}$  of two normal Hopf subalgebras  $X$  and  $Y$  of  $A$  is  $[X, Y]$ .  $\square$

Hence, the categorical definition of the Huq commutator in  $\mathbf{Hopf}_{K, \text{coc}}$  corresponds to the existing notion of the commutator of cocommutative Hopf algebras given by Yanagihara in [96].

### 2.3.2 The Zassenhaus Lemma

We will use that the category of cocommutative Hopf algebras is semi-abelian to obtain a new result in this category: the Zassenhaus Lemma for cocommutative Hopf algebras.

We will apply to  $\mathbf{Hopf}_{K, \text{coc}}$ , the results developed in [95]. In [95], O. Wyler investigated a large class of pointed categories in which the Zassenhaus Lemma, well-known in group theory, remains valid. These categories are very close to the *ideal determined categories* introduced by G. Janelidze, L. Márki, W. Tholen and A. Ursini in [59]. We recommend the article [81] for the description of the Zassenhaus Lemma in the much more general framework of *star-regular categories*, which unified the results obtained in the pointed context [95] and the non-pointed context [91].

We recall the framework and the tools used in [95].

A *normal category* is a pointed regular category in which every regular epimorphism is a normal epimorphism. [60]

An *ideal determined category* is a normal category (with binary coproducts) in which the normal image of a normal monomorphism is again a normal monomorphism [59].

In particular, any semi-abelian category is normal and ideal determined.

We recall the definition of the asymmetric join [80].

**Definition 2.3.5.** *Let  $\mathcal{C}$  be a normal category. For any kernel  $\kappa: K \rightarrow A$  with cokernel  $f: A \rightarrow A/K$ , and any monomorphism  $m: M \rightarrow A$ , we define their asymmetric join  $K \mathbb{W}_A M$  as the subobject  $\kappa \mathbb{W}_A m: K \mathbb{W}_A M \rightarrow A$  of  $A$  in the following pullback*

$$\begin{array}{ccccc}
 M & & & & \\
 \searrow & & \swarrow & & \\
 & K \mathbb{W}_A M & \xrightarrow{\quad} & f(M) & \\
 \downarrow m & \downarrow \kappa \mathbb{W}_A m & \lrcorner & \downarrow f(m) & \\
 K & \xrightarrow{\quad} & A & \xrightarrow{\quad f \quad} & A/K.
 \end{array}$$

The asymmetric join  $K \mathbb{W}_A M$  is a subobject of  $A$  containing both  $K$  and  $M$ .

We recall that when it exists, the supremum  $K \vee M$  of two subobjects  $K$  and  $M$  of the same object  $A$  is the smallest subobject of  $A$  containing  $K$  and  $M$ . In other words, for any object  $L$  such that  $K$  and  $M$  factor

through  $L$ , the supremum also factors through it. In particular, if the category has binary coproducts and any arrow has a factorization as a strong epimorphism followed by a monomorphism, the supremum always exists.

The following lemma gives another characterization of the asymmetric join in a semi-abelian category.

**Lemma 2.3.6.** *If  $\mathcal{C}$  is a semi-abelian category,  $K \vee_A M$  is exactly the supremum of  $K$  and  $M$  (as subobjects of  $A$ ).*

*Proof.* Let  $L$  be another subobject of  $A$  containing both  $K$  and  $M$  and consider the following diagram

$$\begin{array}{ccccc}
 & & & \curvearrowright & \\
 K \vee_A M & \xleftarrow{\quad} & M & \xrightarrow{\quad} & f(M) \\
 & \nearrow \text{dotted} & \downarrow & & \downarrow \text{dashed} \\
 K & \xrightarrow{n} & L & \xrightarrow{p} & f(L) \\
 \parallel & & \downarrow l & & \downarrow \\
 K & \xrightarrow{\kappa} & A & \xrightarrow{f} & A/K
 \end{array}$$

where the vertical dashed arrow is induced by the fact that regular epimorphisms are strong epimorphisms (in any category with binary product). One can verify that the monomorphism  $n$  is in fact the kernel of the normal epimorphism  $p$ , which is clear since  $\kappa$  is the kernel of  $f$  and  $l$  is a monomorphism. The protomodularity of  $\mathcal{C}$  allows us to conclude that the bottom right-hand square is a pullback (see Proposition 1.4.6). This implies the existence of the diagonal dotted arrow.

□

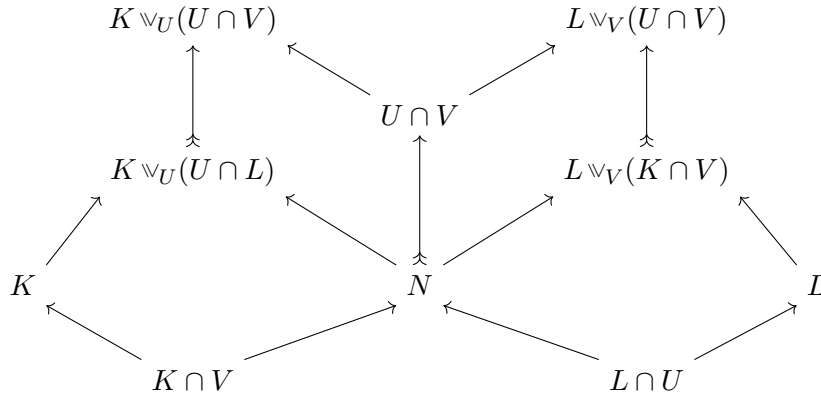
In the particular case of groups,  $K \vee_A M$  is exactly  $KM = MK = \{km \mid k \in K, m \in M\}$ .

Thanks to this notion of the asymmetric join, O. Wyler stated the Zassenhaus Lemma as follows (see also [81]).

**Corollary 2.3.7.** *Let  $\mathcal{C}$  be an ideal determined category. Let  $K \rightarrow U, L \rightarrow V$  be two kernels and  $U \rightarrow A, V \rightarrow A$  be two monomorphisms, then we have the isomorphisms*

$$\frac{K \mathbb{W}_U(U \cap V)}{K \mathbb{W}_U(U \cap L)} \cong \frac{U \cap V}{(K \cap V) \mathbb{W}_{U \cap V}(L \cap U)} \cong \frac{L \mathbb{W}_V(U \cap V)}{L \mathbb{W}_V(K \cap V)}.$$

This result is summarized in the following diagram



where  $N = (K \cap V) \mathbb{W}_{U \cap V}(L \cap U)$  and the three vertical kernels have isomorphic quotients.

In  $\mathbf{Grp}$ , let  $H_1, H_2, N_1, N_2$  be four subgroups of  $G$  such that  $N_1$  is normal in  $H_1$  and  $N_2$  is normal in  $H_2$ . Then Corollary 2.3.7 becomes the well-known Zassenhaus Lemma in group theory

$$\frac{N_1(H_1 \cap H_2)}{N_1(N_2 \cap H_1)} \cong \frac{H_1 \cap H_2}{(N_1 \cap H_2)(H_1 \cap N_2)} \cong \frac{N_2(H_1 \cap H_2)}{N_2(N_1 \cap H_2)}.$$

We describe the asymmetric join and the Zassenhaus Lemma in the category of cocommutative Hopf algebras  $\mathbf{Hopf}_{\mathbf{K}, \text{coc}}$ .

**Lemma 2.3.8.** *Let  $M$  and  $K$  be two Hopf subalgebras of a Hopf algebra  $A$ , with  $K$  normal in  $A$ . Then the vector spaces  $KM$  and  $MK$  are equal, where  $KM$  and  $MK$  are the vector spaces generated by the sums of elements in  $\{km \mid k \in K, m \in M\}$  and  $\{mk \mid k \in K, m \in M\}$  respectively.*

*Proof.* Let  $k$  be an element of  $K$  and  $m$  be an element of  $M$ , then

$$km = m_1(S(m_2)km_3) \in MK$$

and

$$mk = (m_1 k S(m_2)) m_3 \in KM$$

since  $K$  is normal in  $A$ .  $\square$

**Lemma 2.3.9.** *Let  $M$  and  $K$  be two Hopf subalgebras of a cocommutative Hopf algebra  $A$ , with  $K$  normal in  $A$ . Then the vector space  $KM$  is a cocommutative Hopf algebra.*

*Proof.* Consider  $k, h \in K$  and  $n, m \in M$ , thanks to the previous lemma,  $KM$  is a subalgebra:

$$knhm \in KMKM = KKMM = KM.$$

$KM$  is a subcoalgebra:

$$\Delta(km) = k_1 m_1 \otimes k_2 m_2 \in KM \otimes KM.$$

$KM$  is Hopf subalgebra:

$$S(km) = S(m)S(k) \in MK = KM.$$

The cocommutativity is clear.  $\square$

**Proposition 2.3.10.** *Let  $M$  and  $K$  be two Hopf subalgebras of  $A$  with  $K$  normal in  $A$ . Then  $KM$  is the supremum of  $K$  and  $M$  as Hopf subalgebras.*

*Proof.* It is clear that if there exists a Hopf subalgebra  $L$  containing  $K$  and  $M$ , then  $L$  contains also  $KM$  (since a Hopf subalgebra is closed under products and sums).  $\square$

In conclusion, in the specific context of Hopf algebras, for any kernel  $k: K \rightarrow A$  and any monomorphism  $m: M \rightarrow A$ ,

$$K \mathbb{W}_A M = KM.$$

Thanks to the above descriptions and Theorem 2.2.8 we can apply the results obtained in the framework of ideal determined categories to cocommutative Hopf algebras.

**Theorem 2.3.11.** *[81][Zassenhaus Lemma] In  $\mathbf{Hopf}_{K, \text{coc}}$ , let  $K$  be a normal Hopf subalgebra of  $U$ ,  $L$  a normal Hopf subalgebra of  $V$ , and  $U, V$  two Hopf subalgebras of  $A$ . There is the following isomorphism*

$$\begin{aligned} \frac{K(U \cap V)}{K(U \cap V)(K(L \cap U))^+} &\cong \frac{U \cap V}{(U \cap V)((K \cap V)(L \cap U))^+} \\ &\cong \frac{L(U \cap V)}{L(U \cap V)(L(K \cap V))^+}. \end{aligned}$$

This theorem is the analogue, for cocommutative Hopf algebras, of Theorem 3.10 of S. Natale [78], who proved a similar result for finite dimensional Hopf algebras. Thanks to the above result we now have a larger class of Hopf algebras in which the Zassenhaus Lemma is satisfied.

Other applications of the semi-abelianess of the category of cocommutative Hopf algebras are explained and exhibited in the next chapter.



## Chapter 3

# Crossed modules and crossed squares of associative and cocommutative Hopf algebras

This chapter is dedicated to other applications of the semi-abelianess of the category of associative and cocommutative Hopf algebras. Explicitly we will describe the internal crossed modules and the “double” internal crossed modules in the category of cocommutative Hopf algebras. In this chapter, as in the preceding one, we use the classical Sweedler notation for calculations with the comultiplication, and we shall write  $\Delta(c) = c_1 \otimes c_2$  for any  $c \in C$  (with the usual summation convention, where  $c_1 \otimes c_2$  stands for  $\sum c_1 \otimes c_2$ ). We also recall that the unit element of the Hopf algebra  $A$  will be denoted by  $u_A \in A$  and  $1_A$  stands for the identity map over  $A$ . By a slight abuse of notation, in this chapter,  $u_A$  will denote the element (except for the field  $K$ , where the unit of  $K$  is written 1) but also the linear application  $u_A: K \rightarrow A$  which sends 1 to  $u_A$ .

### 3.1 Actions and semi-direct products of cocommutative Hopf algebras

In this section, we recall an important relation between actions and split extensions of cocommutative Hopf algebras. The module Hopf algebras (see Definition 3.1.1) is an important notion in Hopf algebra theory. Independently, the concept of internal action was defined in category theory [10, 18]. The fact that  $\mathbf{Hopf}_{K, \text{coc}}$  is semi-abelian (Theorem 2.2.8), directly implies an equivalence between actions and split extensions of cocommutative Hopf algebras. However, long before, in [76], an equivalence between split extensions and module Hopf algebras was proven in an “ad hoc” way for cocommutative Hopf algebras. It implies that the modules Hopf algebras of cocommutative Hopf algebras coincide with the internal actions in the category of cocommutative Hopf algebras. We recall the definitions and constructions involved in the proof of this equivalence in detail. In particular, one important notion used is the “semi-direct product”. Similarly, this notion was introduced in both Hopf algebras (called smash-product) [76] and category theory [18]. This semi-direct product will offer us a way of decomposing any cocommutative Hopf algebra over an algebraically closed field of characteristic zero into a *group-like* part and a *primitive* part.

Recall that for a cocommutative Hopf algebra  $B$ , the category of  $B$ -modules is a symmetric monoidal category (see Definition 1.3.4 and Proposition 1.3.6, where the condition (1.15) becomes trivial). Hence, one can consider a Hopf algebra in the category of  $B$ -modules, which is then called a  $B$ -module Hopf algebra. Explicitly, we have the following definition:

**Definition 3.1.1.** *A (cocommutative)  $B$ -module Hopf algebra  $X$  is a (cocommutative) Hopf algebra  $X$  endowed with a linear map  $\triangleright: B \otimes X \rightarrow X, b \otimes x \rightarrow b \triangleright x$ , satisfying the following identities*

$$(bb' \triangleright x) = b \triangleright (b' \triangleright x), \quad (3.1)$$

$$u_B \triangleright x = x, \quad (3.2)$$

$$(b \triangleright xy) = (b_1 \triangleright x)(b_2 \triangleright y), \quad (3.3)$$

$$b \triangleright u_X = \epsilon_B(b)u_X, \quad (3.4)$$

$$(b \triangleright x)_1 \otimes (b \triangleright x)_2 = b_1 \triangleright x_1 \otimes b_2 \triangleright x_2, \quad (3.5)$$

$$\epsilon_X(b \triangleright x) = \epsilon_B(b)\epsilon_X(x), \quad (3.6)$$

for any  $b, b' \in B$ , and  $x, y \in X$ .

A morphism between  $X$  a  $B$ -module and  $Y$  a  $C$ -module is given by a pair of morphisms of Hopf algebras  $(\alpha: B \rightarrow C, \beta: X \rightarrow Y)$  such that  $\beta(b \triangleright x) = \alpha(b) \triangleright \beta(x)$  for any  $b \in B, x \in X$ .

They form the category of cocommutative module Hopf algebras, this category is denoted by  $\mathbf{Act}(\mathbf{Hopf}_{K, \text{coc}})$ .

When  $A$  is a  $B$ -module Hopf algebra, one can define the semi-direct product  $A \rtimes B$  (which is, in the Hopf algebra context, usually called the *smash product* and is denoted by  $A \# B$  [76]) as the vector space  $A \otimes B$  with the following structure maps:

$$\begin{aligned} (a \otimes b)(a' \otimes b') &= a(b_1 \triangleright a') \otimes b_2 b', \\ u_{A \rtimes B} &= u_A \otimes u_B, \\ \Delta_{A \rtimes B}(a \otimes b) &= a_1 \otimes b_1 \otimes a_2 \otimes b_2, \\ \epsilon_{A \rtimes B}(a \otimes b) &= \epsilon_A(a)\epsilon_B(b), \\ S_{A \rtimes B}(a \otimes b) &= S_B(b_1) \triangleright S_A(a) \otimes S_B(b_2), \end{aligned}$$

for any  $a, a' \in A$  and any  $b, b' \in B$ .

The semi-direct product of cocommutative Hopf algebras is a useful tool allowing us to decompose any cocommutative Hopf algebras (over an algebraically closed field) into a “group part” and a “Lie algebra part”. More precisely, we recall how we can decompose a Hopf algebra in a *group-like* part and a *primitive* part, thanks to the following functors,

$$\begin{aligned} \mathcal{G}: \mathbf{Hopf}_{K, \text{coc}} &\rightarrow \mathbf{Grp} \\ H &\rightarrow G_H = \{x \in H \mid \Delta(x) = x \otimes x, \epsilon_H(x) = 1\}, \\ \mathbf{K}[-]: \mathbf{Grp} &\rightarrow \mathbf{Hopf}_{K, \text{coc}} \\ G &\rightarrow \mathbf{K}[G], \\ \mathcal{P}: \mathbf{Hopf}_{K, \text{coc}} &\rightarrow \mathbf{LieAlg}_K \\ H &\rightarrow L_H = \{x \in H \mid \Delta(x) = u_H \otimes x + x \otimes u_H\}, \end{aligned}$$

$$\begin{aligned} \mathbf{U}: \mathbf{LieAlg}_K &\rightarrow \mathbf{Hopf}_{K, \text{coc}} \\ L &\rightarrow \mathbf{U}(L), \end{aligned}$$

where  $K[G]$  denotes the group Hopf algebra of the group  $G$ , the Hopf algebra structure is given by  $\Delta(g) = g \otimes g$ ,  $\epsilon_G(g) = 1$ , and  $S(g) = g^{-1}$  for any  $g \in G$ , and  $\mathbf{U}(L)$  is the universal enveloping algebra of the Lie algebra  $L$  with a Hopf algebra structure given by  $\Delta(x) = u_H \otimes x + x \otimes u_H$ ,  $\epsilon_H(x) = 0$ , and  $S(x) = -x$  for any  $x \in L$ . The above functors appear in the following adjunctions.

$$\mathbf{Grp} \begin{array}{c} \xrightarrow{K[-]} \\ \xleftarrow[\mathcal{G}]{\perp} \end{array} \mathbf{Hopf}_{K, \text{coc}} \begin{array}{c} \xleftarrow{\mathbf{U}} \\ \xrightarrow[\mathcal{P}]{\perp} \end{array} \mathbf{LieAlg}_K. \quad (3.7)$$

Moreover,  $\mathbf{U}(L_H)$  can be seen as a  $K[G_H]$ -module thanks to the following linear map

$$g \triangleright x = gxS(g),$$

for any  $x \in L_H$  and  $g \in G_H$ .

The following Theorem gives us a way to decompose any cocommutative Hopf algebra over an algebraically closed field  $K$  of characteristic zero in a *group-like* part and a *primitive* part [75].

**Theorem 3.1.2** (Cartier-Gabriel-Konstant-Milnor-Moore). *Let  $H$  be a cocommutative Hopf algebra over an algebraically closed field  $K$  of characteristic zero. We have the following isomorphism*

$$U(L_H) \rtimes K[G_H] \rightarrow H: x \otimes k \mapsto xk.$$

**Remark 3.1.3.** *Note that the particular case of Theorem 2.2.8 where the field was assumed to have zero characteristic [48] depended on Theorem 3.1.2*

The semi-direct product induces a well-known correspondence between split epimorphisms and module Hopf algebras (see [76]), as in the category of groups Proposition 0.0.1. For the reader's convenience, we recall this correspondence in detail.

**Proposition 3.1.4.** *Let  $A \xrightleftharpoons[i]{\delta} B$  be a split epimorphism of Hopf algebras, i.e.  $\delta \cdot i = 1_B$ . Then,  $\mathbf{HKer}(\delta)$  is a  $B$ -module Hopf algebra for*

the action  $\xi: B \otimes \mathrm{HKer}(\delta) \rightarrow \mathrm{HKer}(\delta)$  defined by

$$\xi(b \otimes k) = i(b_1)ki(S(b_2)).$$

Moreover, there exists a natural isomorphism

$$\varphi: \mathrm{HKer}(\delta) \rtimes B \rightarrow A: k \otimes b \rightarrow ki(b). \quad (3.8)$$

*Proof.* Using the explicit description of the kernel of a morphism in  $\mathrm{Hopf}_{\mathbf{K}, \mathrm{coc}}$ , one can check that  $\xi(b \otimes k) \in \mathrm{HKer}(\delta)$  for all  $b \otimes k \in B \otimes \mathrm{HKer}(\delta)$ .

$$\begin{aligned} & (i(b_1)ki(S(b_2)))_1 \otimes \delta((i(b_1)ki(S(b_2)))_2) \\ &= i(b_1)k_1i(S(b_4)) \otimes b_2\delta(k_2)S(b_3) \\ &= i(b_1)ki(S(b_4)) \otimes b_2u_BS(b_3) \\ &= i(b_1)ki(S(b_2)) \otimes u_B. \end{aligned}$$

Furthermore, it is easy to verify that  $\xi$  defines a  $B$ -module Hopf algebra structure on  $\mathrm{HKer}(\delta)$ . Moreover, one can check that the maps  $\varphi: \mathrm{HKer}(\delta) \rtimes B \rightarrow A$  and  $\psi: A \rightarrow \mathrm{HKer}(\delta) \rtimes B$  defined by  $\varphi(k \otimes b) = ki(b)$  and  $\psi(a) = a_1(i \cdot \delta)(S(a_2)) \otimes \delta(a_3)$  are well-defined, and they are Hopf algebra morphisms. Moreover, these morphisms are mutual inverses.  $\square$

This isomorphism gives rise to an equivalence of categories between the category  $\mathrm{Act}(\mathrm{Hopf}_{\mathbf{K}, \mathrm{coc}})$  and the category  $\mathrm{Pt}(\mathrm{Hopf}_{\mathbf{K}, \mathrm{coc}})$ , see [76], [50], [7], [93]

**Proposition 3.1.5.** *There is an equivalence of categories between the category  $\mathrm{Act}(\mathrm{Hopf}_{\mathbf{K}, \mathrm{coc}})$  and the category  $\mathrm{Pt}(\mathrm{Hopf}_{\mathbf{K}, \mathrm{coc}})$ .*

*Proof.* For a sake of clarity we explicit the functors of this equivalence.

$$A: \mathrm{Pt}(\mathrm{Hopf}_{\mathbf{K}, \mathrm{coc}}) \rightarrow \mathrm{Act}(\mathrm{Hopf}_{\mathbf{K}, \mathrm{coc}}),$$

$$A \left( \begin{array}{ccc} A & \xrightleftharpoons[\quad i \quad]{\quad \delta \quad} & B \\ f_1 \downarrow & & \downarrow f_0 \\ A' & \xrightleftharpoons[\quad i' \quad]{\quad \delta' \quad} & B' \end{array} \right) = \begin{array}{ccc} B \otimes \mathrm{HKer}(\delta) & \xrightarrow{\quad \xi \quad} & \mathrm{HKer}(\delta) \\ \downarrow f_0 \otimes \tilde{f}_1 & & \downarrow \tilde{f}_1 \\ B' \otimes \mathrm{HKer}(\delta') & \xrightarrow{\quad \xi \quad} & \mathrm{HKer}(\delta'), \end{array}$$

where  $\tilde{f}_1$  is the morphism induces by the kernel's universal property and  $\xi$  is defined as in Proposition 3.1.4.

$$B : \text{Act}(\text{Hopf}_{K, \text{coc}}) \rightarrow \text{Pt}(\text{Hopf}_{K, \text{coc}})$$

$$B \left( \begin{array}{ccc} B \otimes X & \xrightarrow{\triangleright} & X \\ \downarrow \beta \otimes \alpha & & \downarrow \alpha \\ B' \otimes X' & \xrightarrow{\triangleright'} & X' \end{array} \right) = \begin{array}{ccc} X \rtimes B & \xrightleftharpoons[e]{p_2} & B \\ \downarrow \alpha \otimes \beta & & \downarrow \beta \\ X' \rtimes B' & \xrightleftharpoons[e]{p_2} & B' \end{array}$$

where  $p_2$  is the projection and  $e$  is the inclusion defined by  $p_2(x \otimes b) = \epsilon_X(x)b$  and  $e(b) = u_X \otimes b$ , respectively.

The pair of morphisms  $(\varphi, 1_B)$  is an isomorphism between the split epimorphisms  $A \xrightleftharpoons[i]{\delta} B$  and  $H\text{Ker}(\delta) \rtimes B \xrightleftharpoons[e]{p_2} B$ . Moreover, since  $X \rightarrow X \rtimes B$  is the kernel of  $p_2$ , we can conclude that these functors form an equivalence of categories.  $\square$

This categorical equivalence between actions and points in  $\text{Hopf}_{K, \text{coc}}$  will be a building block for the equivalence between crossed modules and internal groupoids of cocommutative Hopf algebras, which we will prove in Proposition 3.2.4.

## 3.2 Internal crossed modules of cocommutative Hopf algebras

The abstract notion of internal crossed modules was defined and investigated in the general context of semi-abelian categories by G. Janelidze [57]. Internal crossed modules in a semi-abelian category  $\mathcal{C}$  form a category, denoted  $\text{XMod}(\mathcal{C})$  that is equivalent to the category of internal groupoids in  $\mathcal{C}$ .

In the context of Hopf algebras, the notion of *Hopf crossed module* was defined independently by S. Majid [70] and by J.M. Fernández Vilaboa, M.P. López López, and E. Villanueva Novoa [43] (see also [42]). We will show here that in the cocommutative case, the category  $\text{HXMod}_{K, \text{coc}}$  of Hopf crossed modules is equivalent to the category of internal groupoids

in  $\mathbf{Hopf}_{K, \text{coc}}$  and is consequently also equivalent to the category of internal crossed modules in  $\mathbf{Hopf}_{K, \text{coc}}$ . Moreover, from this definition, we recovered the notions of crossed modules of groups and Lie algebras.

We recall Majid's definition of Hopf crossed module of cocommutative Hopf algebras which, in this case, also coincides with the definition given in [43].

**Definition 3.2.1.** [70] *In  $\mathbf{Hopf}_{K, \text{coc}}$ , a Hopf crossed module is given by a triple  $(B, X, d)$  where  $B$  is a cocommutative Hopf algebra,  $X$  is a cocommutative  $B$ -module Hopf algebra and  $d : X \rightarrow B$  is a Hopf algebra morphism satisfying*

$$d(b \triangleright x) = b_1 d(x) S(b_2) \quad (3.9)$$

$$d(y) \triangleright x = y_1 x S(y_2) \quad (3.10)$$

for any  $x, y \in X$  and  $b \in B$ .

The axiom (3.10) is usually called the *Peiffer identity*. Let  $\mathbf{HXMod}_{K, \text{coc}}$  be the category of Hopf crossed modules, where a morphism

$$(\alpha, \beta) : (B, X, d) \rightarrow (B', X', d')$$

is a pair of Hopf algebra morphisms  $\alpha : X \rightarrow X'$  and  $\beta : B \rightarrow B'$  such that  $d' \cdot \alpha = \beta \cdot d$  and  $\alpha(b \triangleright x) = \beta(b) \triangleright \alpha(x)$ .

We give some examples of the definition of Hopf crossed modules.

**Examples 3.2.2.** 1) *The inclusion of a normal Hopf subalgebra  $N$  of  $M$  is a crossed module of Hopf algebras where the action of  $M$  on  $N$  is given by  $m \triangleright n = m_1 n S(m_2)$ .*

2) *Let  $f : A \rightarrow B$  be a morphism of Hopf algebras where  $A$  and  $B$  are commutative, i.e.  $m \cdot \sigma = m$ . The morphism  $f$  endowed with trivial action,  $b \triangleright a := \epsilon_B(b)a$  is a Hopf crossed module since we have for any  $b \in B$  and any  $a, a' \in A$ ,*

$$f(b \triangleright a) = f(\epsilon_B(b)a) = \epsilon_B(b)f(a) = b_1 S(b_2)f(a) = b_1 f(a) S(b_2),$$

$$f(a') \triangleright a = \epsilon_B(f(a'))a = \epsilon_A(a')a = a'_1 S(a'_2)a = a'_1 a S(a'_2).$$

3) *Let  $g : A \rightarrow B$  be a morphism of Hopf algebras such that  $H\text{Ker}(g) \subseteq Z(A) = \{x \in A \mid xa = ax \ \forall a \in A\}$  and there exists a section of*

coalgebras, i.e. there exists  $s : B \rightarrow A$  a morphism of coalgebras such that  $g \cdot s = 1_B$ .

The Hopf algebra  $A$  is a  $B$ -module Hopf algebra via  $\triangleright : B \otimes A \rightarrow A : b \otimes a \rightarrow s(b_1)aS(s(b_2))$

The first condition to be a crossed module is trivially verified

$$g(b \triangleright a) = g(s(b_1)aS(s(b_2))) = b_1g(a)S(b_2).$$

To prove the second condition, also called Peiffer identity, we remark that

$$a'_1 \otimes S(a'_2)s(g(a'_3))a \otimes S(s(g(a'_4))) = a'_1 \otimes aS(a'_2)s(g(a'_3)) \otimes S(s(g(a'_4)))$$

since  $S(x_1)s(g(x_2))$  belongs to  $HKer(g)$  for any  $x \in A$ ,

$$\begin{aligned} & g((S(x_1)s(g(x_2)))_1) \otimes (S(x_1)s(g(x_2)))_2 \\ &= g(S(x_2)s(g(x_3))) \otimes S(x_1)s(g(x_4)) \\ &= S(g(x_2))g(x_3) \otimes S(x_1)s(g(x_4)) \\ &= u_B \otimes S(x_1)s(g(x_2)). \end{aligned}$$

Hence, by multiplying, we obtain

$$g(a') \triangleright a = s(g(a'_1))aS(s(g(a'_2))) = a'_1aS(a'_2).$$

Before proving the equivalence between crossed modules of cocommutative Hopf algebras and internal groupoids in  $\mathbf{Hopf}_{K, \text{coc}}$ , we identify the internal groupoids with some other internal structures in the category of cocommutative Hopf algebras.

In particular, we recall that a  $cat^1$ -Hopf algebra, in the sense of [43], is a reflexive graph

$$A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{i} \\ \xrightarrow{\gamma} \end{array} A_0 \quad (3.11)$$

such that

$$ab = ba, \quad (3.12)$$

$\forall a \in HKer(\delta)$  and  $\forall b \in HKer(\gamma)$ .

We write  $\mathbf{cat}^1(\mathbf{Hopf}_{K, \text{coc}})$  for the category whose objects are  $cat^1$ -Hopf algebras and whose morphisms are the ones of reflexive graphs.

In the context of cocommutative Hopf algebras, Proposition 1.6.9 leads us to the following proposition:

**Proposition 3.2.3.** [50] *In  $\text{Hopf}_{K,\text{coc}}$ , for a reflexive graph (3.11) the following conditions are equivalent:*

- (a) (3.11) is a reflexive-multiplicative graph;
- (b) (3.11) is an internal groupoid;
- (c) (3.11) satisfies  $[\text{HKer}(\delta), \text{HKer}(\gamma)]_{\text{Huq}} = 0$ ;
- (d) (3.11) is a  $\text{cat}^1$ -Hopf algebra;
- (e) (3.11) satisfies  $[\text{HKer}(\delta), \text{HKer}(\gamma)] = 0$ .

*This implies that the categories  $\text{RMG}(\text{Hopf}_{K,\text{coc}})$ ,  $\text{Grpd}(\text{Hopf}_{K,\text{coc}})$ , and  $\text{cat}^1(\text{Hopf}_{K,\text{coc}})$  are isomorphic.*

*Proof.* Since  $\mathcal{C} = \text{Hopf}_{K,\text{coc}}$  is a semi-abelian category (Theorem 2.2.8), thus, in particular, a Mal'tsev category, this proposition follows from Proposition 1.6.9, Lemma 1.5.4, Proposition 2.2.11 and the description of the Huq commutator given in Proposition 2.3.4.  $\square$

We will now show that the categorical notion of crossed modules in the category of cocommutative Hopf algebras corresponds exactly with the notion of Hopf crossed modules defined by Majid (Definition 3.2.1), by using the fact that both categories are equivalent to  $\text{Grpd}(\text{Hopf}_{K,\text{coc}})$ . As mentioned in [50], the next theorem could be derived from Theorem 14 in [43] or Proposition 1.5 in [7], but we prefer to give a sketch of the proof for sake of completeness.

**Proposition 3.2.4.** *The categories  $\text{HXMod}_{K,\text{coc}}$  and  $\text{Grpd}(\text{Hopf}_{K,\text{coc}})$  are equivalent.*

*Proof.* Thanks to Proposition 3.2.3, it will suffice to prove that the categories  $\text{RMG}(\text{Hopf}_{K,\text{coc}})$  and  $\text{HXMod}_{K,\text{coc}}$  are equivalent. For clarity, we recall the definition of the functors on the objects, yielding this equivalence.

With a reflexive multiplicative graph

$$A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{i} \xrightarrow{\gamma} \end{array} A_0$$

one associates the Hopf algebra morphism

$$d = \gamma \cdot hker(\delta) : \mathbf{HKer}(\delta) \rightarrow A_0$$

(where  $hker(\delta) : \mathbf{HKer}(\delta) \rightarrow A_1$  is the kernel of  $\delta$ ), equipped with the action  $A_0 \otimes \mathbf{HKer}(\delta) \rightarrow \mathbf{HKer}(\delta)$  defined by

$$a \triangleright k = i(a_1)ki(S(a_2)),$$

for any  $a \in A_0$  and  $k \in \mathbf{HKer}(\delta)$ . Thanks to Proposition 3.1.4,  $\mathbf{HKer}(\delta)$  is an  $A_0$ -module Hopf algebra. The morphism  $d = \gamma \cdot hker(\delta)$  is a Hopf algebra morphism by construction, so it remains to see that the two axioms in the Definition 3.2.1 of Hopf crossed module are satisfied. The first axiom holds, since

$$\begin{aligned} d(a \triangleright k) &= [\gamma \cdot hker(\delta)](i(a_1)ki(S(a_2))) \\ &= (\gamma \cdot i)(a_1)(\gamma \cdot hker(\delta))(k)(\gamma \cdot i)(S(a_2)) \\ &= a_1 d(k) S(a_2). \end{aligned}$$

By Proposition 3.2.3, the elements of  $\mathbf{HKer}(\delta)$  and  $\mathbf{HKer}(\gamma)$  commute, hence for all  $b, k \in \mathbf{HKer}(\delta)$ ,

$$(S(k_1)(i \cdot \gamma)(k_2))b = b((S(k_1)(i \cdot \gamma)(k_2))) \quad (3.13)$$

since  $(S(k_1)(i \cdot \gamma)(k_2)) \in \mathbf{HKer}(\gamma)$ , as it follows from

$$\begin{aligned} \gamma(S(k_2)(i \cdot \gamma)(k_3)) \otimes S(k_1)(i \cdot \gamma)(k_4) &= \gamma(S(k_2))\gamma(k_3) \otimes S(k_1)(i \cdot \gamma)(k_4) \\ &= u_B \otimes S(k_1)(i \cdot \gamma)(k_2). \end{aligned}$$

Accordingly, the equality (3.13) implies

$$\begin{aligned} d(k) \triangleright b &= (i \cdot \gamma)(k_1)b(i \cdot \gamma)(S(k_2)) \\ &= k_1 S(k_2)(i \cdot \gamma)(k_3)b(i \cdot \gamma)(S(k_4)) \\ &= k_1 b S(k_2)(i \cdot \gamma)(k_3)(i \cdot \gamma)(S(k_4)) \\ &= k_1 b S(k_2) \end{aligned}$$

showing that the *Peiffer identity* holds and  $d: \mathbf{HKer}(\delta) \rightarrow A_0$  is indeed a Hopf crossed module in Definition 3.2.1.

Conversely, given a Hopf crossed module  $d: X \rightarrow B$ , we define the reflexive graph

$$X \rtimes B \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow[e]{p_1} \end{array} B, \quad (3.14)$$

The morphism  $p_2$  is the second projection  $p_2(x \otimes b) = \epsilon_X(x)b$ , the morphisms  $p_1$  and  $e$  are defined by  $p_1(x \otimes b) = d(x)b$  and  $e(b) = u_X \otimes b$ . This reflexive graph is equipped with a groupoid multiplication by setting

$$m(x \otimes b \otimes x' \otimes b') = xx' \otimes \epsilon_B(b)b',$$

for any  $(x \otimes b \otimes x' \otimes b')$  in the pullback  $P$  defined by

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & X \rtimes B \\ \pi_1 \downarrow & & \downarrow p_1 \\ X \rtimes B & \xrightarrow{p_2} & B. \end{array}$$

We observe that the Peiffer identity is essential to prove that the map  $m: P \rightarrow X \rtimes B$  is an algebra morphism, and is then a morphism in  $\mathbf{Hopf}_{K, \text{coc}}$ . Let  $(x \otimes b \otimes x' \otimes b')$  be an element in  $P$ , in particular, this element satisfies the equation

$$x \otimes b_1 \otimes b_2 \otimes x' \otimes b' = x \otimes b \otimes d(x'_1)b'_1 \otimes x'_2 \otimes b'_2,$$

by definition of the pullback (Proposition 2.1.6). By applying the counit and the comultiplication the equation

$$x \otimes b \otimes x' \otimes b'_1 \otimes b'_2 = x \otimes d(x'_1)b'_1 \otimes x'_2 \otimes b'_2 \otimes \epsilon_B(b)b'_3 \quad (3.15)$$

also holds. We denote the multiplication in  $P$  and in  $X \rtimes B$  by juxtaposition, we have for any  $(x \otimes b \otimes x' \otimes b'), (y \otimes c \otimes y' \otimes c')$  in  $P$ ,

$$\begin{aligned} & m((x \otimes b \otimes x' \otimes b')(y \otimes c \otimes y' \otimes c')) \\ &= m((x \otimes b)(y \otimes c) \otimes (x' \otimes b')(y' \otimes c')) \\ &= m(x(b_1 \triangleright y) \otimes b_2 c \otimes x'(b'_1 \triangleright y') \otimes b'_2 c') \\ &= x(b \triangleright y)x'(b'_1 \triangleright y') \otimes \epsilon_B(c)b'_2 c' \end{aligned}$$

$$\begin{aligned}
& \stackrel{(3.15)}{=} x(d(x'_1)b'_1 \triangleright y)x'_2(b'_2 \triangleright y') \otimes \epsilon_B(c)\epsilon_B(b)b'_3c' \\
& \stackrel{(3.10)}{=} xx'_1(b'_1 \triangleright y)S(x'_2)x'_3(b'_2 \triangleright y') \otimes \epsilon_B(cb)b'_3c' \\
& = xx'(b'_1 \triangleright y)(b'_2 \triangleright y') \otimes \epsilon_B(cb)b'_3c' \\
& = xx'(b'_1 \triangleright yy') \otimes \epsilon_B(cb)b'_2c' \\
& = (xx' \otimes \epsilon_B(b)b')(yy' \otimes \epsilon_B(c)c') \\
& = m(x \otimes b \otimes x' \otimes b')m(y \otimes c \otimes y' \otimes c').
\end{aligned}$$

The correspondence described above naturally extends to morphisms, yielding two functors

$$F : \text{RMG}(\text{Hopf}_{K, \text{coc}}) \rightarrow \text{HXMod}_{K, \text{coc}},$$

$$F \left( \begin{array}{ccc} A_1 & \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{i} \\ \xrightarrow{\gamma} \end{array} & A_0 \\ f_1 \downarrow & & \downarrow f_0 \\ A'_1 & \begin{array}{c} \xrightarrow{\delta'} \\ \xleftarrow{i'} \\ \xrightarrow{\gamma'} \end{array} & A'_0 \end{array} \right) = \begin{array}{ccc} \text{HKer}(\delta) & \xrightarrow{d} & A_0 \\ \tilde{f}_1 \downarrow & & \downarrow f_0 \\ \text{HKer}(\delta') & \xrightarrow{d'} & A'_0, \end{array}$$

where  $\tilde{f}_1$  is the morphism induces by the kernel's universal property.

$$G : \text{HXMod}_{K, \text{coc}} \rightarrow \text{RMG}(\text{Hopf}_{K, \text{coc}})$$

$$G \left( \begin{array}{ccc} X & \xrightarrow{d} & B \\ \alpha \downarrow & & \downarrow \beta \\ X' & \xrightarrow{d'} & B' \end{array} \right) = \begin{array}{ccc} X \rtimes B & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e} \\ \xrightarrow{p_1} \end{array} & B \\ \alpha \otimes \beta \downarrow & & \downarrow \beta \\ X' \rtimes B' & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e} \\ \xrightarrow{p_1} \end{array} & B'. \end{array}$$

These functors give rise to an equivalence of categories thanks to the isomorphism of Proposition 3.1.4, which provides with the identity a morphism of reflexive graphs. On the one hand, we have

$$F \cdot G \left( X \xrightarrow{d} B \right) = F \left( X \rtimes B \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e} \\ \xrightarrow{p_1} \end{array} B \right) = \text{HKer}(p_2) \xrightarrow{p_1 \cdot hker(p_2)} B,$$

which is isomorphic to  $d: X \rightarrow B$  by definition of  $p_1$  and  $p_2$ .

On the other hand, we have

$$\begin{aligned} G \cdot F \left( A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\iota} \\ \xrightarrow{\gamma} \end{array} A_0 \right) &= G \left( \text{HKer}(\delta) \xrightarrow{\gamma \cdot hker(\delta)} A_0 \right) \\ &= \text{HKer}(\delta) \rtimes A_0 \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{e} \\ \xrightarrow{p_2} \end{array} A_0, \end{aligned}$$

this multiplicative reflexive graph is isomorphic to  $A_1 \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\iota} \\ \xrightarrow{\gamma} \end{array} A_0$  due to the isomorphism (3.8)

$$\varphi: \text{HKer}(\delta) \rtimes A_0 \rightarrow A_1: k \otimes a \rightarrow k\iota(a).$$

Indeed, in the following diagram

$$\begin{array}{ccc} \text{HKer}(\delta) \rtimes A_0 & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e} \\ \xrightarrow{p_1} \end{array} & A_0 \\ \varphi \downarrow & & \parallel \\ A_1 & \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\iota} \\ \xrightarrow{\gamma} \end{array} & A_0, \end{array}$$

the three rectangles commute.  $\square$

**Remark 3.2.5.** *The previous result can also be deduced from the recent results of Böhm (see Proposition 3.13 in [7], where a Hopf monoid in  $\text{Vect}_K$  is precisely a Hopf algebra).*

Let us write  $\text{XMod}(\text{Hopf}_{K, \text{coc}})$  for the category of internal crossed modules in  $\text{Hopf}_{K, \text{coc}}$  in the sense of Janelidze [57].

**Corollary 3.2.6.** [50] *There is an equivalence of categories between  $\text{HXMod}_{K, \text{coc}}$  and  $\text{XMod}(\text{Hopf}_{K, \text{coc}})$ .*

*Proof.* In any semi-abelian category, the equivalence between internal groupoids and internal crossed modules was established in [57]. The result then follows from Theorem 2.2.8 and Proposition 3.2.4, since both the categories  $\text{HXMod}_{K, \text{coc}}$  and  $\text{XMod}(\text{Hopf}_{K, \text{coc}})$  are equivalent to the category  $\text{Grpd}(\text{Hopf}_{K, \text{coc}})$ .  $\square$

**Corollary 3.2.7.** [50] *The category  $\mathbf{XMod}(\mathbf{Hopf}_{K, \text{coc}})$  and the category  $\mathbf{HXMod}_{K, \text{coc}}$  are semi-abelian.*

*Proof.* By Theorem 2.2.8 we know that  $\mathbf{Hopf}_{K, \text{coc}}$  is semi-abelian, and this implies that the category  $\mathbf{Grpd}(\mathbf{Hopf}_{K, \text{coc}})$  of internal groupoids in  $\mathbf{Hopf}_{K, \text{coc}}$  is itself semi-abelian (see [46], and Lemma 4.1 in [15]). The result then follows from Proposition 3.2.4 and Corollary 3.2.6.  $\square$

To end this section about crossed modules of cocommutative Hopf algebras, we show that we recover the definitions of crossed modules of groups and Lie algebras when we restrict the crossed modules of cocommutative Hopf algebras to group-like elements and primitive elements, respectively.

We recall the definitions of the crossed modules of groups and Lie algebras

**Definition 3.2.8.** [94] *A crossed module of groups is given by a morphism of groups  $d: X \rightarrow B$  endowed with an action of groups of  $B$  on  $X$  such that for any  $b$  in  $B$  and any  $x, y$  in  $X$ ,*

$$d(b \triangleright x) = bd(x)b^{-1}, \quad (3.16)$$

$$d(x) \triangleright y = xyx^{-1}. \quad (3.17)$$

**Definition 3.2.9.** [62] *A crossed module of Lie algebras is given by a morphism of Lie algebras  $d: \mathfrak{a} \rightarrow \mathfrak{b}$  endowed with an action of Lie algebras of  $\mathfrak{b}$  on  $\mathfrak{a}$  such that for any  $b$  in  $\mathfrak{b}$  and any  $x, y$  in  $\mathfrak{a}$ ,*

$$d(b \triangleright x) = [b, d(x)], \quad (3.18)$$

$$d(x) \triangleright y = [x, y]. \quad (3.19)$$

The morphisms of these two algebraic structures are defined as expected, and they respectively form the category of crossed modules of groups,  $\mathbf{XMod}$ , and the category of crossed modules of Lie algebras,  $\mathbf{XLieAlg}_K$ . In the following result, we exhibit a lifting of the adjunctions (3.7) to the “crossed” level.

**Lemma 3.2.10.** *The adjunctions (3.7) are lifted to the respective categories of crossed modules.*

$$\begin{array}{ccccc}
 \mathbf{XMod} & \xrightleftharpoons[\mathcal{G}]{K[-]} & \mathbf{HMod}_{K, \text{coc}} & \xrightleftharpoons[\mathcal{P}]{U} & \mathbf{XLieAlg}_K \\
 \uparrow D & & \uparrow D & & \uparrow D \\
 \mathbf{Grp} & \xrightleftharpoons[\mathcal{G}]{K[-]} & \mathbf{Hopf}_{K, \text{coc}} & \xrightleftharpoons[\mathcal{P}]{U} & \mathbf{LieAlg}_K
 \end{array}$$

where  $K$  is the ground field and  $D$  are the discrete functors. More precisely the functor  $D: \mathbf{Hopf}_{K, \text{coc}} \rightarrow \mathbf{HMod}_{K, \text{coc}}$  is defined as follows  $D(H) = (H, K, u_H)$ .

*Proof.* First, it is clear that  $K[-]$  sends a crossed module of groups to a Hopf crossed module and  $\mathcal{G}$  does the opposite. We give some details of how the functor  $\mathcal{P}$  sends a Hopf crossed module to a crossed module of Lie algebras. We recall that  $\mathcal{P}(H) = \{x \in H \mid \Delta(x) = x \otimes u_H + u_H \otimes x\}$  is a Lie algebra where the bracket is given by  $[x, y] = xy - yx$ . Let  $(B, X, d)$  be a Hopf crossed module, then by restricting this definition to the primitive elements, we obtain a morphism of Lie algebras, such that

$$\begin{aligned}
 d(b \triangleright x) &= b_1 d(x) S(b_2) = bd(x) - d(x)b = [b, d(x)] \\
 d(x) \triangleright x' &= x_1 x' S(x_2) = xx' - x'x = [x, x'].
 \end{aligned}$$

This is the definition of a crossed module of Lie algebras (Definition 3.2.9). The adjunction structure is also lifted to the “crossed structure”. Indeed, we know that for  $G$  a group and  $H$  a Hopf algebra we have the isomorphism

$$\mathbf{Hom}_{\mathbf{Hopf}_{K, \text{coc}}} (K[G], H) \cong \mathbf{Hom}_{\mathbf{Grp}} (G, \mathcal{G}(H)) \quad (3.20)$$

Let  $(B, X, d)$  be a crossed module of groups and  $(H, Y, \mu)$  a crossed module of Hopf algebras. Then by using (3.20) we can prove that there is a bijection between  $\mathbf{Hom}_{\mathbf{HMod}_{K, \text{coc}}} (K[(B, X, d)], (H, Y, \mu))$  and  $\mathbf{Hom}_{\mathbf{XMod}} ((B, X, d), \mathcal{G}(H, Y, \mu))$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 K[X] & \xrightarrow{K[d]} & K[B] \\
 \downarrow f_1 & & \downarrow f_2 \\
 Y & \xrightarrow{\mu} & H
 \end{array} & \mapsto & \begin{array}{ccc}
 X & \xrightarrow{d} & B \\
 \downarrow \tilde{f}_1 & & \downarrow \tilde{f}_2 \\
 \mathcal{G}(Y) & \xrightarrow{\mathcal{G}(\mu)} & \mathcal{G}(H)
 \end{array}
 \end{array}$$

where  $\tilde{f}_i$  for  $i = 1, 2$  is the restriction of  $f_i$  to the elements of  $B$  and  $X$  respectively. The same computations work for the other adjunction.  $\square$

To sum up, in this section we proved that the internal crossed modules of cocommutative Hopf algebras (in the sense of G. Janelidze) are the ones defined in [43] and [70] in the cocommutative case. Moreover, from this definition, we recovered the notions of crossed modules of groups and Lie algebras.

### 3.3 Crossed squares of cocommutative Hopf algebras

Thanks to Corollary 3.2.7, we can construct internal crossed modules in the category of crossed modules of cocommutative Hopf algebras and one obtains a “2-dimensional” version of Proposition 3.2.4, providing an equivalence between internal “double crossed modules of cocommutative Hopf algebras” and internal double groupoids of cocommutative Hopf algebras.

In this section, we investigate this notion of “double crossed modules”. More precisely, we define the notion of Hopf crossed squares. We prove that the category of Hopf crossed squares is equivalent to the category of internal double groupoids of cocommutative Hopf algebras. We also prove that the notion Hopf crossed squares generalizes the notions of crossed squares of groups and Lie algebras (Proposition 3.3.13). This section is based on the article [86]. We point out to the interested reader that in [86] moreover a notion of Hopf 2-actions is defined and shown to be equivalent to 2-fold split epimorphisms ( $\text{Pt}(\text{Pt}(\text{Hopf}_{K, \text{coc}}))$ ).

**Lemma 3.3.1.** *Since  $\text{HXMod}_{K, \text{coc}}$  is semi-abelian, we have the following equivalence of categories,*

$$\begin{aligned} \text{XMod}(\text{HXMod}_{K, \text{coc}}) &\cong \text{Grpd}(\text{HXMod}_{K, \text{coc}}) \\ &\cong \text{Grpd}(\text{Grpd}(\text{Hopf}_{K, \text{coc}})) \\ &= \text{Grpd}^2(\text{Hopf}_{K, \text{coc}}). \end{aligned}$$

Here  $\text{Grpd}^2(\text{Hopf}_{K, \text{coc}})$  denotes the category of double internal groupoids in  $\text{Hopf}_{K, \text{coc}}$ . In this section, we will describe the category of internal

crossed modules in the category of crossed modules of cocommutative Hopf algebras, denoted by  $\mathbf{XMod}(\mathbf{HMod}_{\mathbf{K}, \text{coc}})$ . But first, let us describe  $\mathbf{Grpd}^2(\mathbf{Hopf}_{\mathbf{K}, \text{coc}})$  the category of double internal groupoids.

**Definition 3.3.2.** *A double internal groupoid of Hopf algebras is an internal groupoid in  $\mathbf{Grpd}(\mathbf{Hopf}_{\mathbf{K}, \text{coc}})$ , i.e. it is defined by a commutative square*

$$\begin{array}{ccc}
 & \xrightarrow{\gamma_2} & \\
 H_{\{1,2\}} & \xleftarrow{\iota_2} \xrightarrow{\delta_2} & H_2 \\
 \delta_1 \downarrow \uparrow \gamma_1 & & \delta'_1 \downarrow \uparrow \gamma'_1 \\
 H_1 & \xleftarrow{\iota'_2} \xrightarrow{\delta'_2} & H_\emptyset
 \end{array} \quad (3.21)$$

where the  $(H_{\{1,2\}}, H_1, \gamma_1, \delta_1, \iota_1)$ ,  $(H_{\{1,2\}}, H_2, \gamma_2, \delta_2, \iota_2)$ ,  $(H_2, H_\emptyset, \gamma'_1, \delta'_1, \iota'_1)$  and  $(H_1, H_\emptyset, \gamma'_2, \delta'_2, \iota'_2)$  are not only reflexive graphs, but also internal groupoids.

Thanks to the isomorphism of Proposition 3.2.3 we can prove that the notion of double internal groupoids is equivalent to the notion of  $\text{cat}^2$ -Hopf algebra that is defined as follows.

**Definition 3.3.3.** *A  $\text{cat}^2$ -Hopf algebra is a 7-tuple  $(H, N, M, s_N, t_N, s_M, t_M)$ , where  $H \xleftarrow{s_N} \xrightarrow{i_N} N$  and  $H \xleftarrow{s_M} \xrightarrow{i_M} M$  are  $\text{cat}^1$ -Hopf algebras, where the injective morphisms  $i_N: N \rightarrow H$  and  $i_M: M \rightarrow H$  are from now on regarded as inclusions and which will be omitted, satisfying the following four conditions of compatibility*

- (2C1)  $s_N \cdot s_M = s_M \cdot s_N$ ,
- (2C2)  $t_N \cdot t_M = t_M \cdot t_N$ ,
- (2C3)  $s_N \cdot t_M = t_M \cdot s_N$ ,
- (2C4)  $t_N \cdot s_M = s_M \cdot t_N$ .

**Definition 3.3.4.** A morphism of  $\text{cat}^2$ -Hopf algebras *between the  $\text{cat}^2$ -Hopf algebras*  $(H, N, M, s_N, t_N, s_M, t_M)$  and  $(H', N', M', s'_{N'}, t'_{N'}, s'_{M'}, t'_{M'})$  is a morphism of Hopf algebras  $f: H \rightarrow H'$  such that the image of  $f|_N := f \cdot i_N$  ( $f|_M := f \cdot i_M$ ) lies in  $N'$  ( $M'$ ) and the four following diagrams commute

$$\begin{array}{ccc} H & \xrightleftharpoons[t_N]{s_N} & N \\ \downarrow f & & \downarrow f|_N \\ H' & \xrightleftharpoons[t'_{N'}]{s'_{N'}} & N' \end{array} \quad \begin{array}{ccc} H & \xrightleftharpoons[t_M]{s_M} & M \\ \downarrow f & & \downarrow f|_M \\ H' & \xrightleftharpoons[t'_{M'}]{s'_{M'}} & M'. \end{array}$$

In other words, a morphism  $f: H \rightarrow H'$  is a morphism of  $\text{cat}^2$ -Hopf algebras if and only if it is a morphism of  $\text{cat}^1$ -Hopf algebras for the two underlying  $\text{cat}^1$ -Hopf algebras.

The category of  $\text{cat}^2$ -Hopf algebras and  $\text{cat}^2$ -Hopf algebra morphisms is denoted by  $\text{cat}^2(\text{Hopf}_{\text{K}, \text{coc}})$ . Proposition 3.2.3 implies a correspondence between the  $\text{cat}^2$ -Hopf algebras and the double groupoids of Hopf algebras. For the sake of clarity, we describe explicitly the following isomorphism.

**Proposition 3.3.5.** *There is an isomorphism between  $\text{cat}^2(\text{Hopf}_{\text{K}, \text{coc}})$  and  $\text{Grpd}^2(\text{Hopf}_{\text{K}, \text{coc}})$ .*

*Proof.* From the  $\text{cat}^2$ -Hopf algebra  $(H, N, M, s_N, t_N, s_M, t_M)$ , we construct the following double groupoid

$$\begin{array}{ccc} & \xrightleftharpoons[t_M]{s_M} & \\ H & \xrightleftharpoons[t_M]{i_M} & M \\ \uparrow t_N & & \uparrow s_N \\ & \xrightleftharpoons[t_M \cdot i_N]{s_M \cdot i_N} & \\ N & \xrightleftharpoons[t_M \cdot i_N]{s_M \cdot i_N} & N \cap M, \end{array}$$

where the maps  $i$  are the inclusion. The Hopf algebra  $N \cap M$  is the intersection of  $N$  and  $M$  seen as Hopf subalgebras of  $H$ , it is equivalent to say that it is the pullback of  $i_M$  along  $i_N$ . All the squares in this diagram commute thanks to the definition of  $\text{cat}^2$ -Hopf algebras.

Moreover, by using the correspondence between  $\text{cat}^1$ -Hopf algebras and internal groupoids, the above diagram is a double groupoid of cocommutative Hopf algebras since the kernels and their restrictions centralize,

$$\begin{aligned} [HKer(s_N), HKer(t_N)] &= 0, \\ [HKer(s_M), HKer(t_M)] &= 0, \\ [HKer(s_N \cdot i_M), HKer(t_N \cdot i_M)] &= 0, \\ [HKer(s_M \cdot i_N), HKer(t_M \cdot i_N)] &= 0. \end{aligned}$$

On the other way around, if we have a double groupoid (3.21), the 7-tuple  $(H_{\{1,2\}}, H_1, H_2, \iota_1 \cdot \gamma_1, \iota_1 \cdot \delta_1, \iota_2 \cdot \gamma_2, \iota_2 \cdot \delta_2)$  is a  $\text{cat}^2$ -Hopf algebra. Indeed, the kernels have trivial commutators since the top reflexive graph and the left one are internal groupoids. Moreover, we have all the conditions of compatibility between the four maps thanks to the commutativity of (3.21). For example, the condition (2C1) holds as we can see in the following equalities,

$$\begin{aligned} (\iota_1 \cdot \gamma_1) \cdot (\iota_2 \cdot \gamma_2) &= \iota_1 \cdot \iota'_2 \cdot \gamma'_1 \cdot \gamma_2 \\ &= \iota_2 \cdot \iota'_1 \cdot \gamma'_2 \cdot \gamma_1 \\ &= (\iota_2 \cdot \gamma_2) \cdot (\iota_1 \cdot \gamma_1). \end{aligned}$$

□

### 3.3.1 Definition of a Hopf crossed square

We define the notion of Hopf crossed squares in the context of cocommutative Hopf algebras. Let us notice that the definition makes sense also in the non-cocommutative case. The following definition is inspired by the definitions in groups and of Lie algebras given by Ellis in his thesis [37].

**Definition 3.3.6.** [86] *In  $\text{Hopf}_{K, \text{coc}}$ , a Hopf crossed square  $(L, M, N, P, h, \lambda, \lambda', \mu, \nu)$  is a commutative square in  $\text{Hopf}_{K, \text{coc}}$ ,*

$$\begin{array}{ccc}
L & \xrightarrow{\lambda} & M \\
\lambda' \downarrow & & \downarrow \mu \\
N & \xrightarrow{\nu} & P,
\end{array}$$

where  $L, M, N$  are  $P$ -modules Hopf algebras and  $h: M \otimes N \rightarrow L$  a coalgebra map such that for any  $l \in L, n, n' \in N, m, m' \in M$  and  $p \in P$ ,

(CS1)  $(P, M, \mu), (P, N, \nu)$  and  $(P, L, \kappa = \mu \cdot \lambda = \nu \cdot \lambda')$  are Hopf crossed modules,

(CS2)  $\lambda$  and  $\lambda'$  are  $P$ -linear,

(CS3)  $h$  is  $P$ -linear,

(CS4)  $\lambda \cdot h(m \otimes n) = m_1(\nu(n) \triangleright S(m_2)),$   
 $\lambda' \cdot h(m \otimes n) = (\mu(m) \triangleright n_1)S(n_2),$

(CS5)  $h(\lambda(l) \otimes n) = l_1(\nu(n) \triangleright S(l_2)),$   
 $h(m \otimes \lambda'(l)) = (\mu(m) \triangleright l_1)S(l_2),$

(CS6)  $h(m \otimes nn') = h(m_1 \otimes n_1)(\nu(n_2) \triangleright h(m_2 \otimes n')),$   
 $h(mm' \otimes n) = (\mu(m_1) \triangleright h(m' \otimes n_1))h(m_2 \otimes n_2).$

**Definition 3.3.7.** A morphism of Hopf crossed squares between the crossed squares  $(L, M, N, P, h, \lambda, \lambda', \mu, \nu)$  and  $(\hat{L}, \hat{M}, \hat{N}, \hat{P}, \hat{h}, \hat{\lambda}, \hat{\lambda}', \hat{\mu}, \hat{\nu})$  is given by 4 Hopf algebra morphisms  $\alpha: L \rightarrow \hat{L}, \beta: M \rightarrow \hat{M}, \gamma: N \rightarrow \hat{N}$ , and  $\delta: P \rightarrow \hat{P}$ , such that each face of the following cube commutes

$$\begin{array}{ccccc}
& & \hat{L} & \xrightarrow{\hat{\lambda}} & \hat{M} \\
& \nearrow \alpha & \downarrow \lambda & & \nearrow \beta \\
L & \xrightarrow{\lambda} & M & & \\
\downarrow \lambda' & & \downarrow \hat{\lambda}' & & \downarrow \hat{\mu} \\
& \nearrow \gamma & \hat{N} & \xrightarrow{\hat{\mu}} & \hat{P} \\
& & \downarrow \mu & & \nearrow \delta \\
N & \xrightarrow{\nu} & P & & 
\end{array}$$

and  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  are morphisms of modules i.e.

$$\alpha(p \triangleright l) = \delta(p) \triangleright \alpha(l),$$

$$\beta(p \triangleright m) = \delta(p) \triangleright \beta(m),$$

$$\gamma(p \triangleright n) = \delta(p) \triangleright \gamma(n).$$

Moreover, they are compatible with the coalgebra morphisms  $h$  and  $\hat{h}$  i.e. we have the following identity

$$\alpha \cdot h = \hat{h} \cdot (\beta \otimes \gamma).$$

The Hopf crossed squares and the morphisms of Hopf crossed squares form the category of Hopf crossed squares,  $\mathbf{X}^2(\mathbf{Hopf}_{K, \text{coc}})$ .

**Remark 3.3.8.** We observe the following facts about the notion of Hopf crossed squares. The following observations will be useful later.

- (1) Let us consider a Hopf crossed square  $(L, M, N, P, h, \lambda, \lambda', \mu, \nu)$ . Thanks to the morphisms  $\mu$  and  $\nu$ , we can define the linear map

$$M \otimes L \rightarrow L: m \otimes l \rightarrow \mu(m) \triangleright l,$$

which turns  $L$  into a  $M$ -module Hopf algebra and the linear map

$$N \otimes L \rightarrow L: n \otimes l \rightarrow \nu(n) \triangleright l,$$

which turns  $L$  into a  $N$ -module Hopf algebra.

- (2) Let  $(L, M, N, P, \lambda, \lambda', \mu, \nu)$  be a Hopf crossed square, then  $(M, L, \lambda)$  (where  $M$  acts on  $L$  as  $M \otimes L \rightarrow L: m \otimes l \rightarrow \mu(m) \triangleright l$ ) and  $(N, L, \lambda')$  (where  $N$  acts on  $L$  as  $N \otimes L \rightarrow L: n \otimes l \rightarrow \nu(n) \triangleright l$ ) are crossed modules. We use that  $\lambda$  is  $P$ -linear, and  $\kappa = \mu \cdot \lambda$  and  $\mu$  are crossed modules to prove the conditions (3.9) and (3.10) for  $(M, L, \lambda)$ .

$$\lambda(\mu(m) \triangleright l) = \mu(m) \triangleright \lambda(l) = m_1 \lambda(l) S(m_2),$$

$$(\mu \cdot \lambda(l)) \triangleright l' = \kappa(l) \triangleright l' = l_1 l' S(l_2).$$

The same computations work for  $(N, L, \lambda')$ .

- (3)  $(\lambda, 1_P)$  is a morphism of crossed modules between  $(P, L, \kappa)$  and  $(P, M, \mu)$

$$\begin{array}{ccc}
 L & \xrightarrow{\lambda} & M \\
 \downarrow \kappa & & \downarrow \mu \\
 P & \xlongequal{\quad} & P
 \end{array}$$

since  $\lambda$  is  $P$ -linear.

$(1_L, \mu)$  is a morphism of crossed modules between  $(M, L, \lambda)$  and  $(P, L, \kappa)$

$$\begin{array}{ccc}
 L & \xlongequal{\quad} & L \\
 \downarrow \lambda & & \downarrow \kappa \\
 M & \xrightarrow{\mu} & P
 \end{array}$$

by the construction of  $L$  as an  $M$ -module.

Similarly, we can prove that  $(\lambda', 1_P)$  and  $(1_L, \nu)$  are morphisms of crossed modules.

(4) In  $\mathbf{Hopf}_{K, \text{coc}}$ , if  $(L, M, N, P, h, \lambda, \lambda', \mu, \nu)$  is a crossed square,

$$\begin{array}{ccc}
 L & \xrightarrow{\lambda} & M \\
 \downarrow \lambda' & & \downarrow \mu \\
 N & \xrightarrow{\nu} & P,
 \end{array}$$

then  $(L, N, M, P, S \cdot h \cdot \sigma, \lambda', \lambda, \nu, \mu)$  is also a Hopf crossed square

$$\begin{array}{ccc}
 L & \xrightarrow{\lambda'} & N \\
 \downarrow \lambda & & \downarrow \nu \\
 M & \xrightarrow{\mu} & P.
 \end{array}$$

Indeed, we can easily check that all the conditions are satisfied, for example, we obtain  $(CS_4)$  for any  $n \in N$  and  $m \in M$ ,

$$\lambda' \cdot S \cdot h \cdot \sigma(n \otimes m) = S(\lambda' \cdot h(m \otimes n))$$

$$\begin{aligned} & \stackrel{(CS4)}{=} S((\mu(m) \triangleright n_1)S(n_2)) \\ & \stackrel{(2.1)}{=} n_1(\mu(m) \triangleright S(n_2)). \end{aligned}$$

We prove that a Hopf crossed square satisfies some equalities, which will turn out to be useful later on.

**Proposition 3.3.9.** *Let the 9-tuple  $(L, M, N, P, h, \lambda, \lambda', \mu, \nu)$  be a Hopf crossed square, then the following equalities hold for any  $m \in M$ ,  $l \in L$ ,  $n \in N$ ,  $p \in P$*

$$\mu(p_1 \triangleright m) \triangleright (p_2 \triangleright l) = p \triangleright (\mu(m) \triangleright l), \quad (3.22)$$

$$\nu(p_1 \triangleright n) \triangleright (p_2 \triangleright l) = p \triangleright (\nu(n) \triangleright l), \quad (3.23)$$

$$h(u_M \otimes n) = \epsilon_N(n)u_L, \quad (3.24)$$

$$h(m \otimes u_N) = \epsilon_M(m)u_L, \quad (3.25)$$

$$(\mu(m_1) \triangleright (\nu(n_1) \triangleright l))h(m_2 \otimes n_2) = h(m_1 \otimes n_1)(\nu(n_2) \triangleright (\mu(m_2) \triangleright l)). \quad (3.26)$$

*Proof.* We use the equivariance of crossed modules (3.9), the property (3.1), and the property of the antipode to obtain (3.22) and (3.23) for any  $m \in M$ ,  $n \in N$ ,  $p \in P$

$$\begin{aligned} \mu(p_1 \triangleright m) \triangleright (p_2 \triangleright l) & \stackrel{(3.9)}{=} (p_1 \mu(m) S(p_2)) \triangleright (p_3 \triangleright l) \\ & \stackrel{(3.1)}{=} (p_1 \mu(m) S(p_2) p_3) \triangleright l \\ & \stackrel{(3.1)}{=} p \triangleright (\mu(m) \triangleright l), \end{aligned}$$

$$\begin{aligned} \nu(p_1 \triangleright n) \triangleright (p_2 \triangleright l) & \stackrel{(3.9)}{=} (p_1 \nu(n) S(p_2)) \triangleright (p_3 \triangleright l) \\ & \stackrel{(3.1)}{=} (p_1 \nu(n) S(p_2) p_3) \triangleright l \\ & \stackrel{(3.1)}{=} p \triangleright (\nu(n) \triangleright l). \end{aligned}$$

Thanks to the condition (CS5) and (3.4), we prove that  $h$  satisfies (3.24) and (3.25), for any  $m \in M$  and any  $n \in N$ ,

$$\begin{aligned} h(u_M \otimes n) &= h(\lambda(u_L) \otimes n) \stackrel{(CS5)}{=} u_L(\nu(n) \triangleright S(u_L)) \stackrel{(3.4)}{=} \epsilon_N(n)u_L \\ h(m \otimes u_N) &= h(m \otimes \lambda'(u_L)) \stackrel{(CS5)}{=} (\mu(m) \triangleright u_L) S(u_L) \stackrel{(3.4)}{=} \epsilon_M(m)u_L. \end{aligned}$$

The equality (3.26) is a consequence of the conditions (CS4) and (CS5). Indeed, on the one hand, we have for any  $m' \otimes n' \otimes l'$  in  $M \otimes N \otimes L$ ,

$$\begin{aligned} h(\lambda(h(m' \otimes n')) \otimes \lambda'(l')) &\stackrel{(CS5)}{=} (\mu \cdot \lambda(h(m' \otimes n')) \triangleright l'_1) S(l'_2) \\ &\stackrel{(CS4)}{=} (\mu(m'_1(\nu(n') \triangleright S(m'_2))) \triangleright l'_1) S(l'_2) \\ &= (\mu(m'_1) \mu(\nu(n') \triangleright S(m'_2)) \triangleright l'_1) S(l'_2) \\ &\stackrel{(3.9)}{=} (\mu(m'_1) \nu(n'_1) \mu(S(m'_2)) S(\nu(n'_2)) \triangleright l'_1) S(l'_2) \end{aligned}$$

and on the other hand

$$\begin{aligned} h(\lambda(h(m' \otimes n')) \otimes \lambda'(l')) &\stackrel{(CS5)}{=} h(m' \otimes n')_1 (\nu \cdot \lambda'(l') \triangleright S(h(m' \otimes n')_2)) \\ &= h(m'_1 \otimes n'_1) (\nu \cdot \lambda'(l') \triangleright S(h(m'_2 \otimes n'_2))) \\ &\stackrel{(3.10)}{=} h(m'_1 \otimes n'_1) l'_1 S(h(m'_2 \otimes n'_2)) S(l'_2). \end{aligned}$$

By combining these two results, we have the following identity

$$\begin{aligned} &(\mu(m'_1) \nu(n'_1) \mu(S(m'_2)) S(\nu(n'_2)) \triangleright l'_1) S(l'_2) \otimes l'_3 \otimes h(m'_3 \otimes n'_3) \\ &= h(m'_1 \otimes n'_1) l'_1 S(h(m'_2 \otimes n'_2)) S(l'_2) \otimes l'_3 \otimes h(m'_3 \otimes n'_3). \end{aligned}$$

By multiplying the four components of the tensor products together, we obtain

$$(\mu(m'_1) \nu(n'_1) \mu(S(m'_2)) S(\nu(n'_2)) \triangleright l') h(m'_3 \otimes n'_3) = h(m' \otimes n') l'.$$

For the particular element  $m' \otimes n' \otimes l' = m_1 \otimes n_1 \otimes (\nu(n_2) \mu(m_2) \triangleright l)$ , in  $M \otimes N \otimes L$ , with any  $m \in M$ , any  $n \in N$ , and any  $l \in L$ , we recover the above equality (thanks to (3.1)),

$$(\mu(m_1) \triangleright (\nu(n_1) \triangleright l)) h(m_2 \otimes n_2) = h(m_1 \otimes n_1) (\nu(n_2) \triangleright (\mu(m_2) \triangleright l)).$$

□

We illustrate the notion of Hopf crossed square with some examples:

**Examples 3.3.10.** (1) Let  $N$  and  $M$  be two normal Hopf subalgebras of  $P$  a cocommutative Hopf algebra, we construct the following square of inclusions

$$\begin{array}{ccc}
 N \cap M & \longrightarrow & M \\
 \downarrow & & \downarrow \\
 N & \longrightarrow & P
 \end{array}$$

with  $h: M \otimes N \rightarrow N \cap M: m \otimes n \mapsto m_1 n_1 S(m_2) S(n_2)$ , where the modules are given by conjugation. This construction is a Hopf crossed square. In particular the unit square

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \parallel & & \parallel \\
 A & \xlongequal{\quad} & A
 \end{array}$$

is then a Hopf crossed square with  $A \otimes A \rightarrow A: a \otimes a' \mapsto a_1 a' S(a_2)$  and  $h(a \otimes a') = a_1 a'_1 S(a_2) S(a'_2)$ . Note that  $A = A$  is the normalization of the largest equivalent relation on  $A$ .

- (2) Any commutative diagram of commutative (and cocommutative) Hopf algebras,

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow g & & \downarrow k \\
 D & \xrightarrow{l} & C
 \end{array}$$

can be seen as a Hopf crossed square where the modules are the trivial ones (i.e.  $C \otimes B \rightarrow B: c \otimes b \mapsto b \epsilon_C(c)$ ), and the coalgebra map  $h: B \otimes D \rightarrow A$  is chosen as  $h(b \otimes d) = \epsilon_B(b) \epsilon_D(d) u_A$ . Note that this example follows from the fact the category of commutative and cocommutative Hopf algebras is abelian (see Theorem 2.2.9).

- (3) Thanks to Theorem 3.1.2, we know that we can re-express any cocommutative Hopf algebra over an algebraically closed field of characteristic zero as

$$X \cong U(L_X) \rtimes K[G_X].$$

Let  $B$  and  $X$  be two commutative and cocommutative Hopf algebras over an algebraically closed field of characteristic zero,  $d: X \rightarrow B$  be a crossed module, then the following square

$$\begin{array}{ccc} U(L_X) & \xrightarrow{d} & U(L_B) \\ \downarrow u_{K[G_X]} \cdot \epsilon_{U(L_X)} & & \downarrow u_{K[G_B]} \cdot \epsilon_{U(L_B)} \\ K[G_X] & \xrightarrow{d} & K[G_B] \end{array}$$

is a Hopf crossed square where  $h: U(L_B) \otimes K[G_X] \rightarrow U(L_X)$  is the trivial map ( $h: U(L_B) \otimes K[G_X] \rightarrow U(L_X): k \otimes g \mapsto 0$ ). Note that the vertical arrows correspond to the normalization of a totally disconnected groupoid, i.e. a groupoid with  $\delta = \gamma$  (see [9] for details).

- (4) Another example of Hopf crossed square is given by the following square

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \parallel & & \downarrow d \\ X & \xrightarrow{d} & B, \end{array}$$

where  $d: X \rightarrow B$  is a Hopf crossed module and  $h: X \otimes X \rightarrow X$  is given by  $h(x \otimes y) = x_1 y_1 S(x_2) S(y_2)$ .

- (5) Let  $d: X \rightarrow B$  be a crossed module, then the commutative square

$$\begin{array}{ccc} K & \xlongequal{\quad} & K \\ \downarrow u_X & & \downarrow u_B \\ X & \xrightarrow{d} & B \end{array}$$

is a Hopf crossed square with  $h := 1_K \otimes \epsilon_X: K \otimes X \rightarrow K$ . This is the definition of the discrete functor denoted  $D': \mathbf{HMod}_{K, \text{coc}} \rightarrow \mathbf{X}^2(\mathbf{Hopf}_{K, \text{coc}})$  on the objects.

An important property that we want for our definition of Hopf crossed squares is to recover the definitions of crossed squares of groups and Lie algebras by restricting the elements to the group-like elements and primitive elements, respectively. To verify this, we recall the definition of crossed squares of groups and Lie algebras.

The definition of crossed squares of groups given below is equivalent to the one introduced by Loday in [64]. Note that the observations (2) and (3) in Remark 3.3.8 are the analogues for cocommutative Hopf algebras, of the (redundant) conditions of the original definition of crossed squares of groups given by Loday.

**Definition 3.3.11.** *A crossed square of groups is a commutative square of groups*

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \downarrow \lambda' & & \downarrow \mu \\ M' & \xrightarrow{\mu'} & P, \end{array}$$

together with actions of  $P$  on  $L$ ,  $M$  and  $M'$ , and with a function  $h: M \times M' \rightarrow L$  satisfying the following axioms

- (i) the homomorphisms  $\mu, \mu'$  and  $\kappa = \mu \cdot \lambda = \mu' \cdot \lambda'$  are crossed modules.
- (ii)  $\lambda \cdot h(m, n) = m(\mu'(n) \triangleright m^{-1})$ ,  
 $\lambda' \cdot h(m, n) = (\mu(m) \triangleright n)n^{-1}$ ,
- (iii)  $h(\lambda(l), n) = l(\mu'(n) \triangleright l^{-1})$ ,  
 $h(m, \lambda'(l)) = (\mu(m) \triangleright l)l^{-1}$ ,
- (iv)  $h(m, nn') = h(m, n)(\mu'(n) \triangleright h(m, n'))$ ,  
 $h(mm', n) = (\mu(m) \triangleright h(m', n))h(m, n)$ ,
- (v)  ${}^p h(m, n) = h({}^p m, {}^p n)$ ,

for all  $m, m'$  in  $M$ ,  $n, n'$  in  $M'$ ,  $l$  in  $L$  and  $p$  in  $P$ .

The following definition is equivalent to the one given by Ellis in his thesis [37] and recalled in [28].

**Definition 3.3.12.** A crossed square of Lie  $K$ -algebras is a commutative square of Lie  $K$ -algebras

$$\begin{array}{ccc} \mathfrak{l} & \xrightarrow{\lambda} & \mathfrak{m} \\ \downarrow \lambda' & & \downarrow \mu \\ \mathfrak{n} & \xrightarrow{\nu} & \mathfrak{p} \end{array}$$

together with an action of  $\mathfrak{p}$  on  $\mathfrak{l}$ ,  $\mathfrak{m}$ , and  $\mathfrak{n}$  and with a function  $h: \mathfrak{m} \times \mathfrak{n} \rightarrow \mathfrak{l}$  satisfying the following axioms

- (i)  $\lambda$  and  $\lambda'$  preserve the action of  $\mathfrak{p}$ ,
- (ii) the homomorphisms  $\mu, \mu'$  and  $\kappa = \mu \cdot \lambda = \mu' \cdot \lambda'$  are crossed modules,
- (iii)  $\alpha h(m, n) = h(\alpha m, n) = h(m, \alpha n)$ ,
- (iv)  $h(m + m', n) = h(m, n) + h(m', n)$ ,  
 $h(m, n + n') = h(m, n) + h(m, n')$ ,
- (v)  $h([m, m'], n) = (\mu(m) \triangleright h(m', n)) - (\mu(m') \triangleright h(m, n))$ ,  
 $h(m, [n, n']) = (\nu(n) \triangleright h(m, n')) - (\nu(n') \triangleright h(m, n))$ ,
- (vi)  $p \triangleright h(m, n) = h(p \triangleright m, n) + h(m, p \triangleright n)$ ,
- (vii)  $\lambda \cdot h(m, n) = -(\nu(n) \triangleright m)$ ,  
 $\lambda' \cdot h(m, n) = (\mu(m) \triangleright n)$ ,
- (viii)  $h(\lambda(l), n) = -(\nu(n) \triangleright l)$ ,  
 $h(m, \lambda'(l)) = (\mu(m) \triangleright l)$ ,

for all  $m, m'$  in  $\mathfrak{m}$ ,  $n, n'$  in  $\mathfrak{n}$ ,  $l$  in  $\mathfrak{l}$ ,  $p$  in  $\mathfrak{p}$  and  $\alpha \in K$ .

The morphisms of these structures are defined as expected, and they respectively form the category of crossed squares of groups  $\mathbf{X}^2\mathbf{Mod}$  and the category of crossed squares of Lie algebras,  $\mathbf{X}^2\mathbf{LieAlg}_K$ . Similarly to Lemma 3.2.10, the following result exhibits a lifting of the adjunctions (3.7).

**Proposition 3.3.13.** *The adjunctions are lifted to the respective categories of crossed squares*

$$\begin{array}{ccccc}
 \text{X}^2\text{Mod} & \xrightleftharpoons[\mathcal{G}]{K[-]} & \text{X}^2(\text{Hopf}_{K,\text{coc}}) & \xrightleftharpoons[\mathcal{P}]{U} & \text{X}^2\text{LieAlg}_K \\
 \uparrow D' & & \uparrow D' & & \uparrow D' \\
 \text{XMod} & \xrightleftharpoons[\mathcal{G}]{K[-]} & \text{HXMod}_{K,\text{coc}} & \xrightleftharpoons[\mathcal{P}]{U} & \text{XLieAlg}_K \\
 \uparrow D & & \uparrow D & & \uparrow D \\
 \text{Grp} & \xrightleftharpoons[\mathcal{G}]{K[-]} & \text{Hopf}_{K,\text{coc}} & \xrightleftharpoons[\mathcal{P}]{U} & \text{LieAlg}_K
 \end{array}$$

where  $K$  is the ground field and  $D$  and  $D'$  are the discrete functors (see Examples 3.3.10 (5) and Lemma 3.2.10 for an explicit description of  $D'$  and  $D$ , respectively).

*Proof.* Similar computations as in the Lemma 3.2.10 work for crossed squares. Let  $(L, M, N, P, h, \lambda, \lambda', \mu, \nu)$  be a Hopf crossed square. By applying the functor  $\mathcal{P}: \text{Hopf}_{K,\text{coc}} \rightarrow \text{Lie}_K$ , we obtain the definition of a crossed square of Lie algebras. To give an idea of how it works, we prove that the condition (CS6) implies the condition (v) of Definition 3.3.12. We use the explicit form of the Hopf algebra structure of primitive elements (i.e.  $\Delta(x) = x \otimes u_H + u_H \otimes x$  for any  $x \in \mathcal{P}(H)$ , with  $H$  a cocommutative Hopf algebra). For any  $m, m' \in M$  and  $n \in N$ , we have

$$\begin{aligned}
 h([m, m'] \otimes n) &= h(mm' - m'm \otimes n) \\
 &= h(mm' \otimes n) - h(m'm \otimes n) \\
 &\stackrel{(CS6)}{=} (\mu(m_1) \triangleright h(m' \otimes n_1))h(m_2 \otimes n_2) - (\mu(m'_1) \triangleright h(m \otimes n_1))h(m'_2 \otimes n_2) \\
 &= (\mu(m) \triangleright h(m' \otimes n_1))h(u_M \otimes n_2) + (\mu(u_M) \triangleright h(m' \otimes n_1))h(m \otimes n_2) \\
 &\quad - (\mu(m') \triangleright h(m \otimes n_1))h(u_M \otimes n_2) - (\mu(u_M) \triangleright h(m \otimes n_1))h(m' \otimes n_2) \\
 &= (\mu(m) \triangleright h(m' \otimes n))h(u_M \otimes u_N) + (\mu(u_M) \triangleright h(m' \otimes n))h(m \otimes u_N) \\
 &\quad + \mu(m) \triangleright h(m' \otimes u_N)h(u_M \otimes n) + (\mu(u_M) \triangleright h(m' \otimes u_N))h(m \otimes n) \\
 &\quad - (\mu(m') \triangleright h(m \otimes n))h(u_M \otimes u_N) - (\mu(u_M) \triangleright h(m \otimes n))h(m' \otimes u_N) \\
 &\quad - (\mu(m') \triangleright h(m \otimes u_N))h(u_M \otimes n) - (\mu(u_M) \triangleright h(m \otimes u_N))h(m' \otimes n)
 \end{aligned}$$

$$\stackrel{(3.24)+(3.25)}{=} (\mu(m) \triangleright h(m' \otimes n)) - (\mu(m') \triangleright h(m \otimes n)).$$

□

Definition 3.3.6 of a Hopf crossed square allows us to recover the definitions of crossed squares of groups and Lie algebras, similarly to the definition of Hopf crossed modules generalizing the definitions of crossed modules of groups and Lie algebras. This result gives us confidence that Definition 3.3.6 is adequate.

Furthermore, this definition is equivalent to the internal crossed modules in the category of Hopf crossed modules, since the category  $X^2(\text{Hopf}_{K, \text{coc}})$  of Hopf crossed squares and the category  $\text{cat}^2(\text{Hopf}_{K, \text{coc}})$  of  $\text{cat}^2$ -Hopf algebras are equivalent. The next section is devoted to the proof of Theorem 3.3.14.

**Theorem 3.3.14.** [86] *There is an equivalence of categories between the category  $X^2(\text{Hopf}_{K, \text{coc}})$  and the category  $\text{cat}^2(\text{Hopf}_{K, \text{coc}})$ .*

Thanks to this theorem we have totally described the notion of internal crossed modules (in the sense of G. Janelidze) in the category of crossed modules of cocommutative Hopf algebras, which are what we called Hopf crossed squares. This remark is clarified in the following proposition.

**Corollary 3.3.15.** [86] *The category  $X\text{Mod}(\text{HXMod}_{K, \text{coc}})$  is equivalent to the category  $X^2(\text{Hopf}_{K, \text{coc}})$ .*

*Proof.* This equivalence is clear thanks to the Lemma 3.3.1 and the Theorem 3.3.14 □

### 3.3.2 Detailed proof of Theorem 3.3.14

In this section, we give a detailed proof of Theorem 3.3.14. This section is quite technical and tedious. There are no new results or definitions in this section, if the reader is not interested in the details of this proof, he can skip this section.

We construct two functors

$$\hat{F}: \text{cat}^2(\text{Hopf}_{K, \text{coc}}) \rightarrow X^2(\text{Hopf}_{K, \text{coc}})$$

$$\hat{G}: \mathbf{X}^2(\mathbf{Hopf}_{K, \text{coc}}) \rightarrow \mathbf{cat}^2(\mathbf{Hopf}_{K, \text{coc}})$$

and show that they form an equivalence of categories.

**Proposition 3.3.16.** *There is a functor*

$$\hat{F}: \mathbf{cat}^2(\mathbf{Hopf}_{K, \text{coc}}) \rightarrow \mathbf{X}^2(\mathbf{Hopf}_{K, \text{coc}}).$$

*Proof.* Let us consider a  $\mathbf{cat}^2$ -Hopf algebra  $(H, N, M, s_N, t_N, s_M, t_M)$ . We construct the following square

$$\begin{array}{ccc} HKer(s_N) \cap HKer(s_M) & \xrightarrow{t_N} & N \cap HKer(s_M) \\ \downarrow t_M & & \downarrow t_M \\ HKer(s_N) \cap M & \xrightarrow{t_N} & N \cap M, \end{array} \quad (3.27)$$

where the modules are given by the action of conjugation

$$\begin{aligned} (N \cap M) \otimes (N \cap HKer(s_M)) &\rightarrow N \cap HKer(s_M) \\ x \otimes k &\rightarrow x_1 k S(x_2), \\ (N \cap M) \otimes (HKer(s_N) \cap M) &\rightarrow HKer(s_N) \cap M \\ x \otimes k &\rightarrow x_1 k S(x_2), \\ (N \cap M) \otimes (HKer(s_N) \cap HKer(s_M)) &\rightarrow HKer(s_N) \cap HKer(s_M) \\ x \otimes k &\rightarrow x_1 k S(x_2), \end{aligned}$$

Thanks to these applications we can check that the intersections  $N \cap HKer(s_M)$ ,  $HKer(s_N) \cap M$ , and  $HKer(s_N) \cap HKer(s_M)$  are  $(N \cap M)$ -module Hopf algebras.

We prove that  $t_N(HKer(s_N) \cap HKer(s_M))$  is included in the Hopf algebra  $N \cap HKer(s_M)$ . Let  $k$  be an element of  $HKer(s_N) \cap HKer(s_M)$ , then  $t_N(k)$  is in  $N$  and  $t_N(k)$  belongs to the Hopf subalgebra  $HKer(s_M)$  since we have

$$\begin{aligned} (s_M \cdot t_N)(k_1) \otimes t_N(k_2) &\stackrel{(2C4)}{=} (t_N \cdot s_M)(k_1) \otimes t_N(k_2) \\ &= u_M \otimes t_N(k). \end{aligned}$$

The morphism  $t_N: HKer(s_N) \cap M \rightarrow N \cap M$  is well-defined since for any element  $m$  in  $M$ ,

$$t_N(m) \stackrel{t_M \cdot i_M = 1_M}{=} t_N \cdot t_M(m) \stackrel{(2C2)}{=} t_M \cdot t_N(m) \in M.$$

Same computations work for the morphisms  $t_M: N \cap HKer(s_M) \rightarrow N \cap M$  and  $t_M: HKer(s_N) \cap HKer(s_M) \rightarrow HKer(s_N) \cap M$  by using (2C2), (2C3), and  $t_N \cdot i_N = 1_N$ .

Moreover, the square (3.27) commutes thanks to the compatibility condition  $t_M \cdot t_N = t_N \cdot t_M$  (2C2).

Thanks to Theorem 3.2.4, this square is composed of crossed modules, which gives the condition (CS1).

The morphisms  $t_N: HKer(s_N) \cap HKer(s_M) \rightarrow N \cap HKer(s_M)$  and  $t_M: HKer(s_N) \cap HKer(s_M) \rightarrow HKer(s_N) \cap M$  are  $(N \cap M)$ -module morphisms (CS2). For any  $p \in N \cap M$  and  $l \in HKer(s_N) \cap HKer(s_M)$ , we have

$$\begin{aligned} t_N(p \triangleright l) &= t_N(p_1 l S(p_2)) \\ &= t_N(p_1) t_N(l) t_N(S(p_2)) \\ &= p_1 t_N(l) S(p_2) \\ &= p \triangleright t_N(l), \end{aligned}$$

since  $t_N \cdot i_N = 1_N$ , and a similar computation holds for  $t_M$ .

We define  $h: (N \cap HKer(s_M)) \otimes (HKer(s_N) \cap M) \rightarrow HKer(s_N) \cap HKer(s_M)$  as

$$h(n \otimes m) = n_1 m_1 S(n_2) S(m_2).$$

We verify that  $h$  is well-defined by using the fact that  $s_N$  is a morphism of Hopf algebras, that  $m \in HKer(s_N)$  and the cocommutativity. Indeed, we can see that for  $n \in N \cap HKer(s_M)$  and  $m \in HKer(s_N) \cap M$ ,  $n_1 m_1 S(n_2) S(m_2)$  belongs to  $HKer(s_N) \cap HKer(s_M)$  thanks to the following identities,

$$\begin{aligned} & s_N((n_1 m_1 S(n_2) S(m_2))_1) \otimes (n_1 m_1 S(n_2) S(m_2))_2 \\ &= s_N(n_1) s_N(m_1) s_N(S(n_4)) s_N(S(m_4)) \otimes n_2 m_2 S(n_3) S(m_3) \\ &= s_N(n_1) s_N(S(n_4)) \otimes n_1 m_2 S(n_2) S(m_3) \\ &= u_N \otimes n_1 m_1 S(n_2) S(m_2). \end{aligned}$$

Similar computations work for  $HKer(s_M)$ .

Since we are in  $\mathbf{Hopf}_{K, \text{coc}}$ ,  $h$  is a coalgebra morphism. Let us check that it is also an  $(N \cap M)$ -module morphism (i.e. condition (CS3)). With

$p \in N \cap M$ ,  $n \in N \cap HKer(s_M)$  and  $m \in HKer(s_N) \cap M$ , we obtain

$$\begin{aligned}
 h(p_1 \triangleright n \otimes p_2 \triangleright m) &= (p_1 \triangleright n)_1 (p_2 \triangleright m)_1 S((p_1 \triangleright n)_2) S((p_2 \triangleright m)_2) \\
 &= (p_{1_1} n S(p_{1_2}))_1 (p_{2_1} m S(p_{2_2}))_1 S((p_{1_1} n S(p_{1_2}))_2) S((p_{2_1} m S(p_{2_2}))_2) \\
 &= p_1 n_1 S(p_2) p_3 m_1 S(p_4) S(p_5 n_2 S(p_6)) S(p_7 m_2 S(p_8)) \\
 &= p_1 n_1 m_1 S(n_2) S(m_2) S(p_2) \\
 &= p \triangleright h(n \otimes m)
 \end{aligned}$$

The condition (CS4) holds for  $t_M$  (and similarly for  $t_N$ ) thanks to  $t_M \cdot i_M = 1_M$  since, for  $n$  an element of  $N \cap HKer(s_M)$  and  $m$  an element of  $HKer(s_N) \cap M$ , we have the following equalities,

$$\begin{aligned}
 t_M \cdot h(n \otimes m) &= t_M(n_1 m_1 S(n_2) S(m_2)) \\
 &= t_M(n_1) m_1 t_M(S(n_2)) S(m_2) \\
 &= (t_M(n) \triangleright m_1) S(m_2).
 \end{aligned}$$

To obtain the conditions (CS5) and (CS6), we show two intermediate results thanks to the equalities  $[HKer(s_M), HKer(t_M)] = 0$  and  $[HKer(s_N), HKer(t_N)] = 0$  in the definition of  $\text{cat}^2$ -Hopf algebras,

$$t_M(n) \triangleright k = n \triangleright k, \quad (3.28)$$

$$t_N(m) \triangleright k = m \triangleright k, \quad (3.29)$$

for any  $n \in N \cap HKer(s_M)$ ,  $m \in HKer(s_N) \cap M$  and  $k \in HKer(s_N) \cap HKer(s_M)$ .

First, we can check that  $t_M(S(n_1))n_2$  belongs to  $HKer(t_M)$  for any  $n \in N \cap HKer(s_M)$ . Indeed, since  $t_M$  is split ( $t_M \cdot i_M = 1_M$ ) it implies that  $t_M \cdot t_M = t_M$ , and we have the following equalities

$$\begin{aligned}
 &t_M(t_M(S(n_1))n_2)_1 \otimes (t_M(S(n_1))n_2)_2 \\
 &= t_M(t_M(S(n_2))n_3) \otimes t_M(S(n_1))n_4 \\
 &= (t_M \cdot t_M)(S(n_2))t_M(n_3) \otimes t_M(S(n_1))n_4 \\
 &= u_M \otimes t_M(S(n_1))n_2.
 \end{aligned}$$

Thanks to this observation, we use  $[HKer(s_M), HKer(t_M)] = 0$  to obtain the equality (3.28) for  $n \in N \cap HKer(s_M)$  and  $k \in HKer(s_N) \cap HKer(s_M)$ ,

$$t_M(n) \triangleright k = t_M(n_1) k t_M(S(n_2))$$

$$\begin{aligned}
&= t_M(n_1)k(t_M(S(n_2))n_3)S(n_4) \\
&= t_M(n_1)(t_M(S(n_2))n_3)kS(n_4) \\
&= n_1kS(n_2) \\
&= n \triangleright k.
\end{aligned}$$

Similarly, for  $m \in HKer(s_N) \cap M$  and  $k \in HKer(s_N) \cap HKer(s_M)$ , we can show (3.29).

Thanks to (3.28), (3.29), and  $[HKer(s_N), HKer(t_N)] = 0$  we obtain the condition (CS5), for  $l \in HKer(s_N) \cap HKer(s_M)$  and  $m \in HKers(s_N) \cap M$ , we have

$$\begin{aligned}
h(t_N(l) \otimes m) &= t_N(l_1)m_1t_N(S(l_2))S(m_2) \\
&= t_N(l_1)m_1t_N(S(l_2))l_3S(l_4)S(m_2) \\
&= t_N(l_1)(t_N(S(l_2))l_3)m_1S(l_4)S(m_2) \\
&= l_1m_1S(l_2)S(m_2) \\
&= l_1(m \triangleright S(l_2)) \\
&\stackrel{(3.29)}{=} l_1(t_N(m) \triangleright S(l_2))
\end{aligned}$$

Similarly, for  $l \in HKer(s_N) \cap HKer(s_M)$  and  $n \in N \cap HKers(s_M)$ , by using the fact that  $[HKer(s_M), HKer(t_M)] = 0$ , we can show that  $(t_M(n) \triangleright l_1)S(l_2) = h(n \otimes t_M(l))$ .

The property (CS6) of the definition of Hopf crossed square is easily checked by definition of the modules and of  $h$ , (3.28), and (3.29) as it is shown in the following equalities, where  $p \in N \cap M$ ,  $n \in N \cap HKer(s_M)$ , and  $m, m' \in HKer(s_N) \cap M$ ,

$$\begin{aligned}
h(n \otimes mm') &= n_1m_1m'_1S(n_2)S(m'_2)S(m_2) \\
&= n_1m_1S(n_2)S(m_2)m_3n_3m'_1S(n_4)S(m'_2)S(m_4) \\
&= h(n_1 \otimes m_1)(m_2 \triangleright h(n_2 \otimes m')) \\
&\stackrel{(3.29)}{=} h(n_1 \otimes m_1)(t_N(m_2) \triangleright h(n_2 \otimes m'))
\end{aligned}$$

The second equality also holds with a similar computation as above.

This construction extends to the morphisms. Let  $f$  be a morphism of  $\text{cat}^2$ -Hopf algebras

$$\begin{array}{ccc}
 H & \xrightarrow{s_N} & N \\
 \downarrow f & \searrow t_N & \downarrow f|_N \\
 H' & \xrightarrow{s'_{N'}} & N' \\
 & \searrow t'_{N'} & \\
 H & \xrightarrow{s_M} & M \\
 \downarrow f & \searrow t_M & \downarrow f|_M \\
 H' & \xrightarrow{s'_{M'}} & M' \\
 & \searrow t'_{M'} & 
 \end{array}$$

We construct the morphisms of the associate Hopf crossed squares, defined by the restrictions of  $f$ . For any  $p \in N \cap M$ ,  $l \in HKer(s_N) \cap HKer(s_M)$ ,  $n \in N \cap HKer(s_M)$ ,  $m \in HKer(s_N) \cap M$ , we have

$$\begin{aligned}
 f(p \triangleright n) &= f(p_1)f(n)f(S(p_2)) = f(p) \triangleright f(n), \\
 f(p \triangleright l) &= f(p_1)f(l)f(S(p_2)) = f(p) \triangleright f(l) \\
 f(p \triangleright m) &= f(p_1)f(m)f(S(p_2)) = f(p) \triangleright f(m), \\
 f \cdot h(n \otimes m) &= f(n_1m_1S(n_2)S(m_2)) = h'(f(n) \otimes f(m)).
 \end{aligned}$$

where the modules are defined thanks to the action by conjugation and  $h: (N \cap HKer(s_M)) \otimes (HKer(s_N) \cap M) \rightarrow HKer(s_N) \cap HKer(s_M)$  is given by  $h(n \otimes m) = n_1m_1S(n_2)S(m_2)$ .

Moreover, the following cube commutes since  $f \cdot i_N \cdot t_N = t_{N'} \cdot f$ ,  $f \cdot i_M \cdot t_M = t_{M'} \cdot f$

$$\begin{array}{ccccc}
 & & HKer(s'_{N'}) \cap HKer(s'_{M'}) & \xrightarrow{t'_{N'}} & N' \cap HKer(s'_{M'}) \\
 & \nearrow f_{HKer(s_N) \cap HKer(s_M)} & \downarrow & \nearrow f_{N \cap HKer(s_M)} & \downarrow t'_{M'} \\
 HKer(s_N) \cap HKer(s_M) & \xrightarrow{t_N} & N \cap HKer(s_M) & & \\
 \downarrow t_M & & \downarrow t'_{M'} & & \\
 HKer(s_N) \cap M & \xrightarrow{t_N} & N \cap M & & \\
 \downarrow f_{HKer(s_N) \cap M} & & \downarrow f_{N \cap M} & & \\
 HKer(s_N) \cap M & \xrightarrow{t_N} & N \cap M & & \\
 & \nearrow f_{HKer(s_N) \cap M} & \downarrow & \nearrow f_{N \cap M} & \downarrow t'_{M'} \\
 & & HKer(s'_{N'}) \cap M' & \xrightarrow{t'_{N'}} & N' \cap M'
 \end{array}$$

□

**Proposition 3.3.17.** *There is a functor*

$$\hat{G}: X^2(\text{Hopf}_{K, \text{coc}}) \rightarrow \text{cat}^2(\text{Hopf}_{K, \text{coc}}).$$

*Proof.* Let  $(L, M, N, P, h, \lambda, \lambda', \mu, \nu)$  be a Hopf crossed square. Since  $M$  is a  $P$ -module Hopf algebra and  $L$  is an  $N$ -module Hopf algebra, via the linear application  $N \otimes L \rightarrow L: n \otimes l \mapsto \nu(n) \triangleright m$  (as explained in Remark 3.3.8 (1)), we can construct the two following Hopf algebras:  $M \rtimes P$  and  $L \rtimes N$ . We verify in details that  $L \rtimes N$  can be seen as an  $(M \rtimes P)$ -module Hopf algebra thanks to the following linear map

$$(m \otimes p) \triangleright (l \otimes n) = (\mu(m_1) \triangleright (p_1 \triangleright l))h(m_2 \otimes p_2 \triangleright n_1) \otimes p_3 \triangleright n_2. \quad (3.30)$$

We verify the condition (3.1) for (3.30) in the following computations for any  $l \in L$ ,  $m, m' \in M$ ,  $n \in N$ , and  $p, p' \in P$ ,

$$\begin{aligned} & (m \otimes p) \triangleright ((m' \otimes p') \triangleright (l \otimes n)) \\ &= (m \otimes p) \triangleright ((\mu(m'_1) \triangleright (p'_1 \triangleright l))h(m'_2 \otimes p'_2 \triangleright n_1) \otimes p'_3 \triangleright n_2) \\ &= \left( \mu(m_1) \triangleright (p_1 \triangleright (\mu(m'_1) \triangleright (p'_1 \triangleright l)))h(m'_2 \otimes p'_2 \triangleright n_1) \right) \\ & \quad h(m_2 \otimes p_2 \triangleright (p'_3 \triangleright n_2)_1) \otimes p_3 \triangleright (p'_3 \triangleright n_2)_2 \\ & \stackrel{(3.3)(3.1)}{=} (\mu(m_1) \triangleright (p_1 \triangleright (\mu(m'_1) \triangleright (p'_1 \triangleright l)))) (\mu(m_2) \triangleright (p_2 \triangleright h(m'_2 \otimes p'_2 \triangleright n_1))) \\ & \quad h(m_3 \otimes (p_3 p'_3) \triangleright n_2) \otimes (p_4 p'_4) \triangleright n_3 \\ & \stackrel{(CS3)}{=} (\mu(m_1) \triangleright (p_1 \triangleright (\mu(m'_1) \triangleright (p'_1 \triangleright l)))) \\ & \quad (\mu(m_2) \triangleright h(p_{2_1} \triangleright m'_2 \otimes (p_{2_2} \triangleright (p'_2 \triangleright n_1)))) h(m_3 \otimes (p_3 p'_3) \triangleright n_2) \\ & \quad \otimes (p_4 p'_4) \triangleright n_3 \\ & \stackrel{(3.1)}{=} (\mu(m_1) \triangleright (p_1 \triangleright (\mu(m'_1) \triangleright (p'_1 \triangleright l)))) \\ & \quad (\mu(m_{2_1}) \triangleright h(p_2 \triangleright m'_2 \otimes ((p_3 p'_2) \triangleright n_1)_1)) \\ & \quad h(m_{2_2} \otimes ((p_3 p'_2) \triangleright n_1)_2) \otimes (p_4 p'_3) \triangleright n_2 \\ & \stackrel{(CS6)}{=} (\mu(m_1) \triangleright (p_1 \triangleright (\mu(m'_1) \triangleright (p'_1 \triangleright l)))) h(m_2(p_2 \triangleright m'_2) \otimes (p_3 p'_2) \triangleright n_1) \\ & \quad \otimes (p_4 p'_3) \triangleright n_2 \\ & \stackrel{(3.22)}{=} (\mu(m_1) \triangleright ((\mu(p_{1_1} \triangleright m'_1)) \triangleright (p_{1_2} \triangleright (p'_1 \triangleright l)))) \\ & \quad h(m_2(p_2 \triangleright m'_2) \otimes (p_3 p'_2) \triangleright n_1) \otimes (p_4 p'_3) \triangleright n_2 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(2.1)(3.1)}{=} ((\mu((m(p_1 \triangleright m'))_1) \triangleright ((p_2 p')_1 \triangleright l)) \\
 & \quad h((m(p_1 \triangleright m'))_2 \otimes (p_2 p')_2 \triangleright n_1) \otimes (p_2 p')_3 \triangleright n_2 \\
 & = (m(p_1 \triangleright m') \otimes p_2 p') \triangleright (l \otimes n) \\
 & = ((m \otimes p)(m' \otimes p')) \triangleright (l \otimes n).
 \end{aligned}$$

The cocommutativity of the Hopf algebras  $M$  and  $P$ , the property (3.5) and the fact that  $h$  is a morphism of coalgebras imply the condition (3.5) for the linear map defined by (3.30) for any  $l \in L$ ,  $m \in M$ ,  $n \in N$ , and  $p \in P$ .

$$\begin{aligned}
 & ((m \otimes p) \triangleright (l \otimes n))_1 \otimes ((m \otimes p) \triangleright (l \otimes n))_2 \\
 & = ((\mu(m_1) \triangleright (p_1 \triangleright l))h(m_2 \otimes p_2 \triangleright n_1) \otimes p_3 \triangleright n_2)_1 \\
 & \quad \otimes ((\mu(m_1) \triangleright (p_1 \triangleright l))h(m_2 \otimes p_2 \triangleright n_1) \otimes p_3 \triangleright n_2)_2 \\
 & \stackrel{(3.5)}{=} (\mu(m_{11}) \triangleright (p_{11} \triangleright l_1))h(m_2 \otimes p_2 \triangleright n_1)_1 \otimes p_{31} \triangleright n_{21} \\
 & \quad \otimes (\mu(m_{12}) \triangleright (p_{12} \triangleright l_2))h(m_2 \otimes p_2 \triangleright n_1)_2 \otimes p_{32} \triangleright n_{22} \\
 & = (\mu(m_1) \triangleright (p_1 \triangleright l_1))h(m_3 \otimes p_3 \triangleright n_1) \otimes p_5 \triangleright n_3 \\
 & \quad \otimes (\mu(m_2) \triangleright (p_2 \triangleright l_2))h(m_4 \otimes p_4 \triangleright n_2) \otimes p_6 \triangleright n_4 \\
 & \stackrel{(2.1)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l_1))h(m_2 \otimes p_2 \triangleright n_1) \otimes p_3 \triangleright n_2 \\
 & \quad \otimes (\mu(m_3) \triangleright (p_4 \triangleright l_2))h(m_4 \otimes p_5 \triangleright n_3) \otimes p_6 \triangleright n_4 \\
 & = (m_1 \otimes p_1) \triangleright (l_1 \otimes n_1) \otimes (m_2 \otimes p_2) \triangleright (l_2 \otimes n_2).
 \end{aligned}$$

Via the following equalities, for any  $l \in L$ ,  $m \in M$ ,  $n \in N$ , and  $p \in P$ ,

$$\begin{aligned}
 & (\epsilon_L \otimes \epsilon_N)((m \otimes p) \triangleright (l \otimes n)) \\
 & = (\epsilon_L \otimes \epsilon_N)((\mu(m_1) \triangleright (p_1 \triangleright l))h(m_2 \otimes p_2 \triangleright n_1) \otimes p_3 \triangleright n_2) \\
 & = \epsilon_L(\mu(m_1) \triangleright (p_1 \triangleright l))\epsilon_L(h(m_2 \otimes p_2 \triangleright n_1)) \otimes \epsilon_N(p_3 \triangleright n_2) \\
 & = \epsilon_L(l)\epsilon_L(h(m \otimes p \triangleright n)) \\
 & = \epsilon_L(l)\epsilon_M(m)\epsilon_P(p)\epsilon_N(n),
 \end{aligned}$$

the linear map (3.30) satisfies (3.6) thanks to the properties of  $\epsilon$ .

The conditions (3.2) and (3.4) for (3.30) are satisfied thanks to (3.24) and (3.25) for any  $l \in L$ ,  $m \in M$ ,  $n \in N$ , and  $p \in P$ :

$$(u_M \otimes u_P) \triangleright (l \otimes n) = (\mu(u_M) \triangleright (u_P \triangleright l))h(u_M \otimes u_P \triangleright n_1) \otimes u_P \triangleright n_2$$

$$\begin{aligned} & \stackrel{(3.24)}{=} l\epsilon_N(n_1) \otimes n_2 \\ & = l \otimes n, \end{aligned}$$

$$\begin{aligned} (m \otimes p) \triangleright (u_L \otimes u_N) &= (\mu(m_1) \triangleright (p_1 \triangleright u_L)) h(m_2 \otimes p_2 \triangleright u_N) \otimes p_3 \triangleright u_N \\ &= h(m\epsilon_P(p) \otimes u_N) \otimes u_N \\ & \stackrel{(3.25)}{=} \epsilon_M(m)\epsilon_P(p)(u_L \otimes u_N). \end{aligned}$$

We verify the condition (3.3) for (3.30) for any  $l, l' \in L$ ,  $m \in M$ ,  $n, n' \in N$ , and  $p \in P$ . It holds thanks to several properties of the definition of Hopf crossed square and the modules, as we can see in the following identities.

$$\begin{aligned} (m \otimes p) \triangleright ((l \otimes n)(l' \otimes n')) &= (m \otimes p) \triangleright (l(\nu(n_1) \triangleright l') \otimes n_2 n'_1) \\ &= (\mu(m_1) \triangleright (p_1 \triangleright l(\nu(n_1) \triangleright l')))) h(m_2 \otimes p_2 \triangleright (n_2 n'_1)) \otimes p_3 \triangleright (n_3 n'_2) \\ & \stackrel{(3.3)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l)) (\mu(m_2) \triangleright (p_2 \triangleright (\nu(n_1) \triangleright l'))) \\ & \quad h(m_3 \otimes p_3 \triangleright (n_2 n'_1)) \otimes p_4 \triangleright (n_3 n'_2) \\ & \stackrel{(3.23)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l)) (\mu(m_2) \triangleright (\nu(p_2 \triangleright n_1) \triangleright (p_2 \triangleright l'))) \\ & \quad h(m_3 \otimes p_3 \triangleright (n_2 n'_1)) \otimes p_4 \triangleright (n_3 n'_2) \\ & \stackrel{(3.3)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l)) (\mu(m_2) \triangleright (\nu(p_2 \triangleright n_1) \triangleright (p_3 \triangleright l'))) \\ & \quad h(m_3 \otimes (p_4 \triangleright n_2)(p_5 \triangleright n'_1)) \otimes p_6 \triangleright (n_3 n'_2) \\ & \stackrel{(CS6)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l)) (\mu(m_2) \triangleright (\nu(p_2 \triangleright n_1) \triangleright (p_3 \triangleright l'))) \\ & \quad h(m_3 \otimes (p_4 \triangleright n_2)_1 ((\nu(p_4 \triangleright n_2)_2) \triangleright h(m_4 \otimes p_5 \triangleright n'_1))) \otimes p_6 \triangleright (n_3 n'_2) \\ & \stackrel{(3.5)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l)) (\mu(m_2) \triangleright (\nu(p_2 \triangleright n_1) \triangleright (p_3 \triangleright l'))) h(m_3 \otimes p_4 \triangleright n_2) \\ & \quad ((\nu(p_5 \triangleright n_3)) \triangleright h(m_4 \otimes p_6 \triangleright n'_1)) \otimes p_7 \triangleright (n_4 n'_2) \\ & \stackrel{(2.1)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l)) (\mu(m_2) \triangleright (\nu((p_2 \triangleright n_1)_1) \triangleright (p_3 \triangleright l'))) \\ & \quad h(m_3 \otimes (p_2 \triangleright n_1)_2 (\nu(p_4 \triangleright n_2) \triangleright h(m_4 \otimes p_5 \triangleright n'_1))) \otimes p_6 \triangleright (n_3 n'_2) \\ & \stackrel{(3.26)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l)) h(m_2 \otimes (p_2 \triangleright n_1)_1) \\ & \quad (\nu((p_2 \triangleright n_1)_2) \triangleright (\mu(m_3) \triangleright (p_3 \triangleright l'))) \\ & \quad ((\nu(p_4 \triangleright n_2)) \triangleright h(m_4 \otimes p_5 \triangleright n'_1)) \otimes p_6 \triangleright (n_3 n'_2) \\ & \stackrel{(3.3)}{=} (\mu(m_1) \triangleright (p_1 \triangleright l)) h(m_2 \otimes (p_2 \triangleright n_1)) \end{aligned}$$

$$\begin{aligned}
 & (\nu(p_3 \triangleright n_2) \triangleright ((\mu(m_3) \triangleright (p_4 \triangleright l'))h(m_4 \otimes p_5 \triangleright n'_1))) \otimes p_6 \triangleright (n_3 n'_2)) \\
 & \stackrel{(3.3)}{=} \left( (\mu(m_1) \triangleright (p_1 \triangleright l))h(m_2 \otimes p_2 \triangleright n_1) \otimes p_3 \triangleright n_2 \right) \\
 & \quad \left( (\mu(m_3) \triangleright (p_4 \triangleright l'))h(m_4 \otimes p_5 \triangleright n'_1) \otimes p_6 \triangleright n'_2 \right) \\
 & = ((m_1 \otimes p_1) \triangleright (l \otimes n))((m_2 \otimes p_2) \triangleright (l' \otimes n')).
 \end{aligned}$$

Hence,  $L \rtimes N$  is an  $(M \rtimes P)$ -module Hopf algebra via the linear application defined in (3.30).

Then, we can define the semi-direct product  $(L \rtimes N) \rtimes (M \rtimes P)$ , where the algebra structure is given as follows

$$\begin{aligned}
 (l \otimes n \otimes m \otimes p)(l' \otimes n' \otimes m' \otimes p') &= l \left( \nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l')) \right) \\
 & \quad (\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1)) \otimes n_3(p_3 \triangleright n'_2) \otimes m_3(p_4 \triangleright m') \otimes p_5 p',
 \end{aligned}$$

Thanks to (4) of Remark 3.3.8, we know that  $(L, N, M, P, S \cdot h \cdot \sigma)$  satisfies all the conditions of the definition of Hopf crossed square where the role of  $M$  and  $N$  are inverted. Hence, we can construct similarly to above, the semi-direct product  $(L \rtimes M) \rtimes (N \rtimes P)$ , with  $(L \rtimes M)$  seen as an  $(N \rtimes P)$ -module, thanks to the following linear map

$$(n \otimes p) \triangleright (l \otimes m) = (\nu(n_1) \triangleright (p_1 \triangleright l))S(h(p_2 \triangleright m_1 \otimes n_2)) \otimes p_3 \triangleright m_2.$$

The multiplication in  $(L \rtimes M) \rtimes (N \rtimes P)$ , is given by the following linear map

$$\begin{aligned}
 (l \otimes m \otimes n \otimes p)(l' \otimes m' \otimes n' \otimes p') &= l \left( \mu(m_1) \triangleright (\nu(n_1) \triangleright (p_1 \triangleright l')) \right) \\
 & \quad (\mu(m_2) \triangleright S(h(p_2 \triangleright m'_1 \otimes n_2))) \otimes m_3(p_3 \triangleright m'_2) \otimes n_3(p_4 \triangleright n') \otimes p_5 p'.
 \end{aligned}$$

Note that the semi-direct products  $(L \rtimes N) \rtimes (M \rtimes P)$  and  $(L \rtimes M) \rtimes (N \rtimes P)$  are isomorphic, the isomorphisms are given by

$$\begin{aligned}
 \psi: (L \rtimes N) \rtimes (M \rtimes P) &\rightarrow (L \rtimes M) \rtimes (N \rtimes P) \\
 (l \otimes n \otimes m \otimes p) &\rightarrow lS(h(m_1 \otimes n_1)) \otimes m_2 \otimes n_2 \otimes p
 \end{aligned} \tag{3.31}$$

$$\begin{aligned}
 \psi^{-1}: (L \rtimes M) \rtimes (N \rtimes P) &\rightarrow (L \rtimes N) \rtimes (M \rtimes P) \\
 (l \otimes m \otimes n \otimes p) &\mapsto lh(m_1 \otimes n_1) \otimes n_2 \otimes m_2 \otimes p.
 \end{aligned}$$

We verify that these maps are morphisms of Hopf algebras. On one hand, we obtain, for  $l, l' \in L$ ,  $m, m' \in M$ ,  $n, n' \in N$ , and  $p, p' \in P$

$$\begin{aligned}
& \psi((l \otimes n \otimes m \otimes p)(l' \otimes n' \otimes m' \otimes p')) \\
&= \psi(l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l'))))(\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1)) \\
&\quad \otimes n_3(p_3 \triangleright n'_2) \otimes m_3(p_4 \triangleright m') \otimes p_5 p') \\
&= l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l')))(\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1)) \\
&\quad S(h((m_3(p_4 \triangleright m'))_1 \otimes (n_3(p_3 \triangleright n'_2))_1) \otimes (m_3(p_4 \triangleright m'))_2 \\
&\quad \otimes (n_3(p_3 \triangleright n'_2))_2 \otimes p_5 p') \\
&\stackrel{(3.5)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l')))(\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1)) \\
&\quad S(h(m_3(p_5 \triangleright m'_1) \otimes n_3(p_3 \triangleright n'_2))) \otimes m_4(p_6 \triangleright m'_2) \otimes n_4(p_4 \triangleright n'_3) \otimes p_7 p') \\
&\stackrel{(CS6)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l')))(\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1)) \\
&\quad S((\mu(m_3) \triangleright h(p_6 \triangleright m'_1 \otimes n_3(p_3 \triangleright n'_2)))h(m_4 \otimes n_4(p_4 \triangleright n'_3))) \\
&\quad \otimes m_5(p_7 \triangleright m'_2) \otimes n_5(p_5 \triangleright n'_4) \otimes p_8 p') \\
&\stackrel{(CS6)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l')))(\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1)) \\
&\quad S((\mu(m_3) \triangleright h(p_6 \triangleright m'_1 \otimes n_3(p_3 \triangleright n'_2)))h(m_4 \otimes n_4) \\
&\quad (\nu(n_5) \triangleright h(m_5 \otimes p_4 \triangleright n'_3))) \otimes m_6(p_7 \triangleright m'_2) \otimes n_6(p_5 \triangleright n'_4) \otimes p_8 p') \\
&\stackrel{(2.1)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l')))(\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1)) \\
&\quad (\nu(n_3) \triangleright S(h(m_3 \otimes p_3 \triangleright n'_2)))S(\mu(m_4) \triangleright h(p_5 \triangleright m'_1 \otimes n_6(p_4 \triangleright n'_3)) \\
&\quad h(m_5 \otimes n_4)) \otimes m_6(p_6 \triangleright m'_2) \otimes n_5(p_7 \triangleright n'_4) \otimes p_8 p') \\
&\stackrel{(3.3)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l')))) \\
&\quad (\nu(n_2) \triangleright (h(m_2 \otimes p_2 \triangleright n'_1)S(h(m_3 \otimes p_3 \triangleright n'_2)))) \\
&\quad S(\mu(m_4) \triangleright h(p_5 \triangleright m'_1 \otimes n_5(p_4 \triangleright n'_3))h(m_5 \otimes n_3)) \\
&\quad \otimes m_6(p_6 \triangleright m'_2) \otimes n_4(p_7 \triangleright n'_4) \otimes p_8 p') \\
&\stackrel{(1.11)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l'))))S(\mu(m_2) \triangleright (h(p_3 \triangleright m'_1 \otimes n_4(p_2 \triangleright n'_1)) \\
&\quad h(m_3 \otimes n_2))) \otimes m_4(p_4 \triangleright m'_2) \otimes n_3(p_5 \triangleright n'_2) \otimes p_6 p') \\
&\stackrel{(2.1)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l'))))S(h(m_2 \otimes n_2))S((\mu(m_3) \triangleright h(p_2 \triangleright m'_1 \\
&\quad \otimes n_3(p_3 \triangleright n'_1)))) \otimes m_4(p_4 \triangleright m'_2) \otimes n_4(p_5 \triangleright n'_2) \otimes p_6 p')
\end{aligned}$$

and on the other hand, by using the properties of  $(L, N, M, P, S \cdot h \cdot \sigma)$  we have

$$\begin{aligned}
& \psi(l \otimes n \otimes m \otimes p) \psi(l' \otimes n' \otimes m' \otimes p') \\
&= (lS(h(m_1 \otimes n_1)) \otimes m_2 \otimes n_2 \otimes p)(l'S(h(m'_1 \otimes n'_1)) \otimes m'_2 \otimes n'_2 \otimes p') \\
&= lS(h(m_1 \otimes n_1))(\mu(m_2) \triangleright (\nu(n_2) \triangleright (p_1 \triangleright l'S(h(m'_1 \otimes n'_1))))) \\
&\quad (\mu(m_3) \triangleright S(h(p_2 \triangleright m'_2 \otimes n_3))) \otimes m_4(p_3 \triangleright m'_3) \otimes n_4(p_4 \triangleright n'_2) \otimes p_5 p' \\
&\stackrel{(3.3)}{=} lS(h(m_1 \otimes n_1))(\mu(m_2) \triangleright (\nu(n_2) \triangleright (p_1 \triangleright l'))) \\
&\quad (\mu(m_3) \triangleright (\nu(n_3) \triangleright (p_2 \triangleright S(h(m'_1 \otimes n'_1))))) (\mu(m_4) \triangleright S(h(p_3 \triangleright m'_2 \otimes n_4))) \\
&\quad \otimes m_5(p_4 \triangleright m'_3) \otimes n_5(p_5 \triangleright n'_2) \otimes p_6 p' \\
&\stackrel{(3.26)}{=} {}^{+(CS3)} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l'))) S(h(m_2 \otimes n_2)) \\
&\quad (\mu(m_3) \triangleright (\nu(n_3) \triangleright (S(h(p_2 \triangleright m'_1 \otimes p_3 \triangleright n'_1))))) \\
&\quad (\mu(m_4) \triangleright S(h(p_4 \triangleright m'_2 \otimes n_4))) \otimes m_5(p_5 \triangleright m'_3) \otimes n_5(p_6 \triangleright n'_2) \otimes p_7 p' \\
&\stackrel{(3.3)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l'))) S(h(m_2 \otimes n_2)) \\
&\quad \left( (\mu(m_3) \triangleright (\nu(n_3) \triangleright S(h(p_2 \triangleright m'_1 \otimes p_3 \triangleright n'_1))) S(h(p_4 \triangleright m'_2 \otimes n_4))) \right) \\
&\quad \otimes m_4(p_5 \triangleright m'_3) \otimes n_5(p_6 \triangleright n'_2) \otimes p_7 p' \\
&\stackrel{(CS6)}{=} l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l'))) S(h(m_2 \otimes n_2)) \\
&\quad (\mu(m_3) \triangleright S(h(p_2 \triangleright m'_1 \otimes n_3(p_3 \triangleright n'_1)))) \otimes m_4(p_4 \triangleright m'_2) \\
&\quad \otimes n_4(p_5 \triangleright n'_2) \otimes p_6 p'.
\end{aligned}$$

Thus, the linear map  $\psi$  is a morphism of Hopf algebras between  $(L \rtimes N) \rtimes (M \rtimes P)$  and  $(L \rtimes M) \rtimes (N \rtimes P)$ , since it is a coalgebra morphism thanks to the cocommutativity. Moreover, it is an isomorphism as we can check as follows

$$\begin{aligned}
\psi \cdot \psi^{-1}(l \otimes m \otimes n \otimes p) &= \psi(lh(m_1 \otimes n_1) \otimes n_2 \otimes m_2 \otimes p) \\
&= lh(m_1 \otimes n_1) S(h(m_{2_1} \otimes n_{2_1})) \otimes m_{2_2} \otimes n_{2_2} \otimes p \\
&= lh(m_1 \otimes n_1)_1 S(h(m_1 \otimes n_1)_2) \otimes m_2 \otimes n_2 \otimes p \\
&= l \otimes m \otimes n \otimes p,
\end{aligned}$$

and, thanks to similar computations,  $\psi^{-1} \cdot \psi = 1_{(L \rtimes N) \rtimes (M \rtimes P)}$  also holds.

Remark that  $d: L \rtimes N \rightarrow M \rtimes P$  and  $d': L \rtimes M \rightarrow N \rtimes P$  defined as

$$d(l \otimes n) = \lambda(l) \otimes \nu(n),$$

$$d'(l \otimes m) = \lambda'(l) \otimes \mu(m),$$

are crossed modules.

First, we check that  $d$  is a morphism of algebras thanks to the fact that  $\lambda$  is  $P$ -linear for  $l, l'$  elements of  $L$ , and  $n, n'$  elements of  $N$ , we have the following equalities

$$\begin{aligned} d((l \otimes n)(l' \otimes n')) &= d(l(\nu(n_1) \triangleright l') \otimes n_2 n') \\ &= \lambda(l) \lambda(\nu(n_1) \triangleright l') \otimes \nu(n_2 n') \\ &= \lambda(l) (\nu(n_1) \triangleright \lambda(l')) \otimes \nu(n_2) \nu(n') \\ &= (\lambda(l) \otimes \nu(n)) (\lambda(l') \otimes \nu(n')) \\ &= d(l \otimes n) d(l' \otimes n'). \end{aligned}$$

Moreover, this morphism satisfies the two properties to be a crossed module, since for any  $m \in M$ ,  $p \in P$ ,  $l, l' \in L$ , and  $n, n' \in N$ , we have

$$\begin{aligned} &d((m \otimes p) \triangleright (l \otimes n)) \\ &= d((\mu(m_1) p_1 \triangleright l) h(m_2 \otimes p_2 \triangleright n_1) \otimes p_3 \triangleright n_2) \\ &= \lambda((\mu(m_1) p_1 \triangleright l) h(m_2 \otimes p_2 \triangleright n_1)) \otimes \nu(p_3 \triangleright n_2) \\ &\stackrel{(3.1)(CS2)}{=} (\mu(m_1) \triangleright (p_1 \triangleright \lambda(l))) \lambda(h(m_2 \otimes p_2 \triangleright n_1)) \otimes \nu(p_3 \triangleright n_2) \\ &\stackrel{(3.10)}{=} m_1(p_1 \triangleright \lambda(l)) S(m_2) \lambda(h(m_3 \otimes p_2 \triangleright n_1)) \otimes \nu(p_3 \triangleright n_2) \\ &\stackrel{(CS4)}{=} m_1(p_1 \triangleright \lambda(l)) S(m_2) m_3(\nu(p_2 \triangleright n_1) \triangleright S(m_4)) \otimes \nu(p_3 \triangleright n_2) \\ &\stackrel{(3.9)}{=} m_1(p_1 \triangleright \lambda(l)) (p_2 \nu(n_1) S(p_3) \triangleright S(m_2)) \otimes p_4 \nu(n_2) S(p_5) \\ &\stackrel{(3.1)}{=} (m_1(p_1 \triangleright \lambda(l)) \otimes p_2 \nu(n)) (S(p_3) \triangleright S(m_2) \otimes S(p_4)) \\ &= (m_1 \otimes p_1) (\lambda(l) \otimes \nu(n)) (S(p_2) \triangleright S(m_2) \otimes S(p_3)) \\ &= (m_1 \otimes p_1) d(l \otimes n) S(m_2 \otimes p_2), \end{aligned}$$

$$\begin{aligned} &d(l \otimes n) \triangleright (l' \otimes n') = (\lambda(l) \otimes \nu(n)) \triangleright (l' \otimes n') \\ &= (\mu \cdot \lambda(l_1) \triangleright (\nu(n_1) \triangleright l')) h(\lambda(l_2) \otimes \nu(n_2) \triangleright n'_1) \otimes \nu(n_3) \triangleright n'_2 \\ &\stackrel{(CS5)}{=} (\mu \cdot \lambda(l_1) \triangleright (\nu(n_1) \triangleright l')) l_2(\nu(\nu(n_2) \triangleright n'_1) \triangleright S(l_3)) \otimes \nu(n_3) \triangleright n'_2 \\ &\stackrel{(3.10)}{=} l_1(\nu(n_1) \triangleright l') S(l_2) l_3(\nu(\nu(n_2) \triangleright n'_1) \triangleright S(l_4)) \otimes \nu(n_3) \triangleright n'_2 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(3.9)}{=} l_1(\nu(n_1) \triangleright l)'(\nu(n_2)\nu(n'_1)S(\nu(n_3)) \triangleright S(l_2)) \otimes n_4 n'_2 S(n_5) \\
 & \stackrel{(3.1)}{=} l_1(\nu(n_1) \triangleright l)'(\nu(n_2 n'_1) \triangleright (S(\nu(n_3) \triangleright S(l_2))) \otimes n_4 n'_2 S(n_5) \\
 & = (l_1(\nu(n_1) \triangleright l') \otimes n_2 n') (S(\nu(n_3) \triangleright S(l_2)) \otimes S(n_4)) \\
 & = (l_1 \otimes n_1)(l' \otimes n') (S(\nu(n_2) \triangleright S(l_2)) \otimes S(n_3)) \\
 & = (l_1 \otimes n_1)(l' \otimes n') S(l_2 \otimes n_2).
 \end{aligned}$$

We obtain the same conclusion for  $d'$  with similar computations. Hence, we have two Hopf crossed modules.

Thus, we apply the equivalence in one dimension of Proposition 3.2.4, to obtain two  $\text{cat}^1$ -Hopf algebras where the source and the target maps are defined as

$$\begin{aligned}
 s_1 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (M \rtimes P) \\
 & \quad l \otimes n \otimes m \otimes p \rightarrow \epsilon_L(l) \epsilon_N(n) (m \otimes p), \\
 t_1 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (M \rtimes P) \\
 & \quad l \otimes n \otimes m \otimes p \rightarrow \lambda(l) (\nu(n_1) \triangleright m) \otimes \nu(n_2) p, \\
 s'_1 &: (L \rtimes M) \rtimes (N \rtimes P) \rightarrow (N \rtimes P) \\
 & \quad l \otimes m \otimes n \otimes p \rightarrow \epsilon_L(l) \epsilon_M(m) (n \otimes p), \\
 t'_1 &: (L \rtimes M) \rtimes (N \rtimes P) \rightarrow (N \rtimes P) \\
 & \quad l \otimes m \otimes n \otimes p \rightarrow \lambda'(l) (\mu(m_1) \triangleright n) \otimes \mu(m_2) p,
 \end{aligned}$$

the section is given by  $e_1(m \otimes p) = u_L \otimes u_N \otimes m \otimes p$ ,  $e'_1(n \otimes p) = u_L \otimes u_M \otimes n \otimes p$  and they have trivial commutators.

By using the isomorphism  $\psi$  as defined in (3.31) we obtain the following two  $\text{cat}^1$ -Hopf algebras

$$\begin{aligned}
 s_1 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (M \rtimes P) \\
 & \quad l \otimes n \otimes m \otimes p \rightarrow \epsilon_L(l) \epsilon_N(n) (m \otimes p), \\
 t_1 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (M \rtimes P) \\
 & \quad l \otimes n \otimes m \otimes p \rightarrow \lambda(l) (\nu(n_1) \triangleright m) \otimes \nu(n_2) p, \\
 s_2 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (N \rtimes P) \\
 & \quad l \otimes n \otimes m \otimes p \rightarrow \epsilon_L(l) \epsilon_M(m) (n \otimes p), \\
 t_2 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (N \rtimes P)
 \end{aligned}$$

$$l \otimes n \otimes m \otimes p \rightarrow \lambda'(l)n \otimes \mu(m)p,$$

where  $s_2 = s'_1 \cdot \psi$  and  $t_2 = t'_1 \cdot \psi$ .

We note that  $[HKer(s_2), HKer(t_2)] = 0$  since  $[HKer(s'_1), HKer(t'_1)] = 0$ . Indeed, let  $a$  be an element of  $HKer(s_2)$  (i.e.  $s_2(a_1) \otimes a_2 = u_{N \rtimes P} \otimes a$ ) and  $b$  an element of  $HKer(t_2)$ , then  $\psi(a)$  belongs to  $HKer(s'_1)$  ( $s'_1 \cdot \psi(a_1) \otimes \psi(a_2) = u_{N \rtimes P} \otimes \psi(a)$ ) and  $\psi(b)$  belongs to  $HKer(t'_1)$ . Hence,  $ab = \psi^{-1}(\psi(a)\psi(b)) = \psi^{-1}(\psi(b)\psi(a)) = ba$ .

We easily check the conditions of compatibility for being a  $\text{cat}^2$ -Hopf algebra.

Let  $(l \otimes n \otimes m \otimes p)$  be an element in  $(L \rtimes N) \rtimes (M \rtimes P)$ , we have the following identities

$$\begin{aligned} t_1 \cdot t_2(l \otimes n \otimes m \otimes p) &= t_1(u_L \otimes \lambda'(l)n \otimes u_M \otimes \mu(m)p) \\ &= u_L \otimes u_N \otimes \lambda(u_L)(\nu((\lambda'(l)n)_1) \triangleright u_M) \otimes \nu((\lambda'(l)n)_2)\mu(m)p \\ &\stackrel{(3.4)}{=} u_L \otimes u_N \otimes u_M \otimes \nu \cdot \lambda'(l)\nu(n)\mu(m)p \\ &= u_L \otimes u_N \otimes u_M \otimes \mu \cdot \lambda(l)\nu(n)\mu(m)p \\ &\stackrel{(3.10)}{=} u_L \otimes \lambda'(u_L)u_N \otimes u_M \otimes \mu(\lambda(l)(\nu(n_1) \triangleright m))\nu(n_2)p \\ &= t_2(u_L \otimes u_N \otimes \lambda(l)(\nu(n_1) \triangleright m) \otimes \nu(n_2)p) \\ &= t_2 \cdot t_1(l \otimes n \otimes m \otimes p) \end{aligned}$$

$$\begin{aligned} s_1 \cdot t_2(l \otimes n \otimes m \otimes p) &= s_1(u_L \otimes \lambda'(l)n \otimes u_M \otimes \mu(m)p) \\ &= u_L \otimes u_N \otimes \epsilon_L(l)\epsilon_N(n)u_M \otimes \mu(m)p \\ &= t_2(\epsilon_L(l)\epsilon_N(n)u_L \otimes u_N \otimes m \otimes p) \\ &= t_2 \cdot s_1(l \otimes n \otimes m \otimes p) \end{aligned}$$

$$\begin{aligned} t_1 \cdot s_2(l \otimes n \otimes m \otimes p) &= t_1(\epsilon_L(l)\epsilon_M(m)u_L \otimes n \otimes u_M \otimes p) \\ &= \epsilon_L(l)\epsilon_M(m)u_L \otimes u_N \otimes (\nu(n_1) \triangleright u_M) \otimes \nu(n_2)p \\ &\stackrel{(3.4)}{=} \epsilon_L(l)\epsilon_M(m)u_L \otimes u_N \otimes u_M \otimes \nu(n)p \\ &= s_2(u_L \otimes u_N \otimes \lambda(l)(\nu(n_1) \triangleright m) \otimes \nu(n_2)p) \end{aligned}$$

$$= s_2 \cdot t_1(l \otimes n \otimes m \otimes p).$$

Hence, the 7-tuple  $((L \rtimes N) \rtimes (M \rtimes P), M \rtimes P, N \rtimes P, s_1, t_1, s_2, t_2)$  is a  $\text{cat}^2$ -Hopf algebra.

if we consider a morphism of Hopf crossed squares  $(\alpha, \beta, \gamma, \delta)$

$$\begin{array}{ccccc}
 & & \hat{L} & \xrightarrow{\hat{\lambda}} & \hat{M} \\
 & \nearrow \alpha & \downarrow \lambda & \nearrow \beta & \downarrow \hat{\mu} \\
 L & \xrightarrow{\hat{\lambda}'} & M & & \\
 \downarrow \lambda' & \nearrow \gamma & \downarrow \mu & \nearrow \delta & \\
 N & \xrightarrow{\nu} & P & & 
 \end{array}$$

First, let us check that  $\alpha \otimes \gamma \otimes \beta \otimes \delta: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (L' \rtimes N') \rtimes (M' \rtimes P')$  is a morphism of Hopf algebras (the only non-trivial part is the respect of the multiplication), thanks to the axioms of the definition of a morphism of Hopf crossed square (Definition 3.3.7):

$$\begin{aligned}
 & (\alpha \otimes \gamma \otimes \beta \otimes \delta)((l \otimes n \otimes m \otimes p)(l' \otimes n' \otimes m' \otimes p')) \\
 &= (\alpha \otimes \gamma \otimes \beta \otimes \delta)(l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l')))) \\
 & \quad (\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1)) \otimes n_3(p_3 \triangleright n'_2) \otimes m_3(p_4 \triangleright m') \otimes p_5 p') \\
 &= \alpha(l(\nu(n_1) \triangleright (\mu(m_1) \triangleright (p_1 \triangleright l'))))(\nu(n_2) \triangleright h(m_2 \otimes p_2 \triangleright n'_1))) \\
 & \quad \otimes \gamma(n_3(p_3 \triangleright n'_2)) \otimes \beta(m_3(p_4 \triangleright m')) \otimes \delta(p_5 p') \\
 &= \alpha(l(\gamma \cdot \nu(n_1) \triangleright (\beta \cdot \mu(m_1) \triangleright (\delta(p_1) \triangleright \alpha(l'))))) \\
 & \quad (\gamma \cdot \nu(n_2) \triangleright \alpha \cdot h(m_2 \otimes p_2 \triangleright n'_1)) \\
 & \quad \otimes \gamma(n_3)(\delta(p_3) \triangleright \gamma(n'_2)) \otimes \beta(m_3)(\delta(p_4) \triangleright \beta(m')) \otimes \delta(p_5) \delta(p')) \\
 &= \alpha(l(\hat{\nu} \cdot \gamma(n_1) \triangleright (\hat{\mu} \cdot \beta(m_1) \triangleright (\delta(p_1) \triangleright \alpha(l'))))) \\
 & \quad (\hat{\nu} \cdot \gamma(n_2) \triangleright h(\beta(m_2) \otimes \delta(p_2) \triangleright \gamma(n'_1))) \otimes \gamma(n_3)(\delta(p_3) \triangleright \gamma(n'_2)) \\
 & \quad \otimes \beta(m_3)(\delta(p_4) \triangleright \beta(m')) \otimes \delta(p_5) \delta(p'))
 \end{aligned}$$

$$= (\alpha(l) \otimes \gamma(n) \otimes \beta(m) \otimes \delta(p))(\alpha(l') \otimes \gamma(n') \otimes \beta(m') \otimes \delta(p')).$$

This morphism is, in particular, a morphism of  $\text{cat}^2$ -Hopf algebras

$$\begin{array}{ccc} (L \rtimes N) \rtimes (M \rtimes P) & \xrightarrow[t_1]{s_1} & M \rtimes P \\ \downarrow \alpha \otimes \gamma \otimes \beta \otimes \delta & & \downarrow \beta \otimes \delta \\ (L' \rtimes N') \rtimes (M' \rtimes P') & \xrightarrow[t'_1]{s'_1} & M' \rtimes P' \\ \\ (L \rtimes N) \rtimes (M \rtimes P) & \xrightarrow[t_2]{s_2} & N \rtimes P \\ \downarrow \alpha \otimes \gamma \otimes \beta \otimes \delta & & \downarrow \gamma \otimes \delta \\ (L' \rtimes N') \rtimes (M' \rtimes P') & \xrightarrow[t'_2]{s'_2} & N' \rtimes P'. \end{array}$$

Indeed, the four rectangles commute since for any  $l \otimes m \otimes n \otimes p \in (L \rtimes N) \rtimes (M \rtimes P)$  we have

$$\begin{aligned} (\beta \otimes \delta) \cdot s_1(l \otimes n \otimes m \otimes p) &= (\beta \otimes \delta)(\epsilon_L(l)\epsilon_N(n)m \otimes p) \\ &= \epsilon_L(l)\epsilon_N(n)\beta(m) \otimes \delta(p) \\ &= s'_1(\alpha(l) \otimes \gamma(n) \otimes \beta(m) \otimes \delta(p)) \end{aligned}$$

$$\begin{aligned} (\gamma \otimes \delta) \cdot s_2(l \otimes n \otimes m \otimes p) &= (\gamma \otimes \delta)(\epsilon_L(l)\epsilon_M(m)n \otimes p) \\ &= \epsilon_L(l)\epsilon_M(m)\gamma(n) \otimes \delta(p) \\ &= s'_2(\alpha(l) \otimes \gamma(n) \otimes \beta(m) \otimes \delta(p)). \end{aligned}$$

$$\begin{aligned} (\beta \otimes \delta) \cdot t_1(l \otimes n \otimes m \otimes p) &= (\beta \otimes \delta)(\lambda(l)(\nu(n_1) \triangleright m) \otimes \nu(n_2)p) \\ &= \beta(\lambda(l)(\nu(n_1) \triangleright m)) \otimes \delta(\nu(n_2)p) \\ &= \beta \cdot \lambda(l)\beta(\nu(n_1) \triangleright m) \otimes \delta(\nu(n_2)p) \\ &= \hat{\lambda} \cdot \alpha(l)(\hat{\nu} \cdot \gamma(n_1) \triangleright \beta(m)) \otimes \hat{\nu} \cdot \gamma(n_2)\delta(p) \\ &= t'_1 \cdot (\alpha \otimes \gamma \otimes \beta \otimes \delta)(l \otimes n \otimes m \otimes p), \end{aligned}$$

$$(\gamma \otimes \delta) \cdot t_2(l \otimes n \otimes m \otimes p) = (\gamma \otimes \delta)(\lambda'(l)n \otimes \mu(m)p)$$

$$\begin{aligned}
&= \gamma \cdot \lambda'(l)\gamma(n) \otimes \delta \cdot \mu(m)\delta(p) \\
&= \hat{\lambda}' \cdot \alpha(l)\gamma(n) \otimes \hat{\mu} \cdot \beta(m)\delta(p) \\
&= t'_2 \cdot (\alpha \otimes \gamma \otimes \beta \otimes \delta)(l \otimes n \otimes m \otimes p).
\end{aligned}$$

Hence, we can conclude that this is a morphism of  $\text{cat}^2$ -Hopf algebras.  $\square$

**Theorem 3.3.18.** *There is an equivalence of categories between the category  $X^2(\text{Hopf}_{K,\text{coc}})$  and the category  $\text{cat}^2(\text{Hopf}_{K,\text{coc}})$ .*

*Proof.* Thanks to Proposition 3.3.16 and Proposition 3.3.17, we have the following 2 functors

$$\begin{aligned}
\hat{F}: \text{cat}^2(\text{Hopf}_{K,\text{coc}}) &\rightarrow X^2(\text{Hopf}_{K,\text{coc}}), \\
\hat{G}: X^2(\text{Hopf}_{K,\text{coc}}) &\rightarrow \text{cat}^2(\text{Hopf}_{K,\text{coc}}).
\end{aligned}$$

We show that they describe an equivalence of categories between Hopf crossed squares and  $\text{cat}^2$ -Hopf algebras.

On one hand, let us consider a  $\text{cat}^2$ -Hopf algebra  $(H, M, N, s_N, t_N, s_M, t_M)$ , by applying the functors  $\hat{F}$  and  $\hat{G}$ , we obtain the following  $\text{cat}^2$ -Hopf algebra,

$$\begin{aligned}
H' &= (HKer(s_N) \cap HKer(s_M)) \rtimes (HKer(s_N) \cap M) \\
&\quad \rtimes (N \cap HKer(s_M)) \rtimes (N \cap M), \\
M' &= (HKer(s_N) \cap M) \rtimes (N \cap M), \\
N' &= (N \cap HKer(s_M)) \rtimes (N \cap M) \\
s_1: H' &\rightarrow N': s_1(l \otimes n \otimes m \otimes p) = \epsilon_L(l)\epsilon_N(n)(m \otimes p) \\
t_1: H' &\rightarrow N': t_1(l \otimes n \otimes m \otimes p) = t_N(l)t_N(n)mt_N(S(n_2)) \otimes t_N(n_3)p \\
s_2: H' &\rightarrow M': s_2(l \otimes n \otimes m \otimes p) = \epsilon_L(l)\epsilon_M(m)(n \otimes p) \\
t_2: H' &\rightarrow M': t_2(l \otimes n \otimes m \otimes p) = t_M(l)n \otimes t_M(m)p.
\end{aligned}$$

Thanks to the equivalence between modules Hopf algebras and split epimorphisms that was recalled in Lemma 3.1.4 we have the following isomorphisms,

$$\begin{aligned}
N' &= (N \cap HKer(s_M)) \rtimes (N \cap M) \cong N \cap H = N \\
M' &= (HKer(s_N) \cap M) \rtimes (N \cap M) \cong H \cap M = M
\end{aligned}$$

$$\begin{aligned}
 H' &= (HKer(s_N) \cap HKer(s_M)) \rtimes (HKer(s_N) \cap M) \\
 &\rtimes (N \cap HKer(s_M)) \rtimes (N \cap M) \\
 &\cong HKer(s_N) \rtimes N \cong H,
 \end{aligned}$$

given by the multiplication in  $H$ . Moreover, the isomorphism

$$\varphi: H' \rightarrow H: (l \otimes n \otimes m \otimes p) \rightarrow lnmp, \quad (3.32)$$

is a morphism of Hopf crossed square, since it makes the two following diagram commute

$$\begin{array}{ccc}
 H' & \xrightarrow{s_1} & N' \\
 \downarrow \varphi & \searrow t_1 & \downarrow \varphi|_{N'} \\
 H & \xrightarrow{s_N} & N
 \end{array}
 \quad
 \begin{array}{ccc}
 H' & \xrightarrow{s_2} & M' \\
 \downarrow \varphi & \searrow t_2 & \downarrow \varphi|_{M'} \\
 H & \xrightarrow{s_M} & M
 \end{array}$$

Indeed, for any  $l \in (HKer(s_N) \cap HKer(s_M))$ ,  $n \in (HKer(s_N) \cap M)$ ,  $m \in (N \cap HKer(s_M))$ , and  $p \in (N \cap M)$  we have, thanks to the property of  $HKer(s_N)$ , the following equalities

$$\begin{aligned}
 \varphi|_{N'} \cdot s_1(l \otimes n \otimes m \otimes p) &= \varphi|_{N'}(\epsilon_L(l)\epsilon_N(n)m \otimes p) \\
 &= \epsilon_L(l)\epsilon_N(n)mp \\
 &= s_N(lnmp) \\
 &= s_N \cdot \varphi(l \otimes n \otimes m \otimes p)
 \end{aligned}$$

$$\begin{aligned}
 \varphi|_{N'} \cdot t_1(l \otimes n \otimes m \otimes p) &= \varphi|_{N'}(t_N(l)t_N(n_1)mt_N(S(n_2)) \otimes t_N(n_3)p) \\
 &= t_N(l)t_N(n_1)mt_N(S(n_2))t_N(n_3)p \\
 &= t_N(l)t_N(n)mp \\
 &\stackrel{t_N \cdot i_N = 1_N}{=} t_N(lnmp) \\
 &= t_N \cdot \varphi(l \otimes n \otimes m \otimes p).
 \end{aligned}$$

Same computation as above makes the right diagram commutes.

By applying  $\hat{F} \cdot \hat{G}$  on the Hopf crossed square  $(L, M, N, P, h, \lambda, \lambda', \mu, \nu)$ , we obtain the following Hopf crossed square

$$\begin{array}{ccc}
 HKer(s_1) \cap HKer(s_2) & \xrightarrow{t_1} & (M \rtimes P) \cap HKer(s_2) \\
 \downarrow t_2 & & \downarrow t_2 \\
 HKer(s_1) \cap (N \rtimes P) & \xrightarrow{t_1} & (N \rtimes P) \cap (M \rtimes P),
 \end{array}$$

where

$$\begin{aligned}
 s_1 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (M \rtimes P) \\
 &\quad l \otimes n \otimes m \otimes p \mapsto \epsilon_L(l) \epsilon_N(n) (m \otimes p), \\
 s_2 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (N \rtimes P) \\
 &\quad l \otimes n \otimes m \otimes p \mapsto \epsilon_L(l) \epsilon_M(m) (n \otimes p), \\
 t_1 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (M \rtimes P) \\
 &\quad l \otimes n \otimes m \otimes p \mapsto \lambda(l) (\nu(n_1) \triangleright m) \otimes \nu(n_2) p, \\
 t_2 &: (L \rtimes N) \rtimes (M \rtimes P) \rightarrow (N \rtimes P) \\
 &\quad l \otimes n \otimes m \otimes p \mapsto \lambda'(l) n \otimes \mu(m) p.
 \end{aligned}$$

Since  $s_1$  and  $s_2$  are just projections, it is obvious that

$$HKer(s_1) \cong L \rtimes N$$

$$HKer(s_2) \cong L \rtimes M.$$

The expression  $(N \rtimes P) \cap (M \rtimes P)$  stands for the intersection of the vector spaces  $K \otimes N \otimes K \otimes P$  and  $K \otimes K \otimes M \otimes P$ , where the Hopf algebra structures on these two vector spaces are the underlying ones of  $(L \rtimes N) \rtimes (M \rtimes P)$ .

More precisely,

$$(N \rtimes P) \cap (M \rtimes P) = \{l \otimes n \otimes m \otimes p \in (L \rtimes N) \rtimes (M \rtimes P) \mid l \otimes n \otimes m \otimes p \in (K \otimes N \otimes K \otimes P) \cap (K \otimes K \otimes M \otimes P)\}.$$

It follows that  $(N \rtimes P) \cap (M \rtimes P) = K \otimes K \otimes K \otimes P$ , see for example Lemma 1.4.5 of [33].

The canonical morphism between  $K \otimes K \otimes K \otimes P$  and  $P$  allows us to say that  $(N \rtimes P) \cap (M \rtimes P) \cong P$ .

Similar arguments imply that  $HKer(s_1) \cap HKer(s_2) \cong L$ ,  $(M \rtimes P) \cap HKer(s_2) \cong M$ , and  $HKer(s_1) \cap (N \rtimes P) \cong N$ , via the canonical isomorphisms. The canonical isomorphisms form a morphism of Hopf crossed squares.

Hence, this isomorphism implies that  $\hat{F} \cdot \hat{G} = 1_{X^2(\text{Hopf}_{K, \text{coc}})}$ , and we conclude that  $X^2(\text{Hopf}_{K, \text{coc}})$  and  $\text{cat}^2(\text{Hopf}_{K, \text{coc}})$  are equivalent.  $\square$

## Chapter 4

# Split extensions and actions of bialgebras

In Chapter 2 and Chapter 3, we studied in detail the category of cocommutative Hopf algebras. In particular, we recalled a well-known equivalence between the actions of cocommutative Hopf algebras and the split extensions of cocommutative Hopf algebras (Proposition 3.1.5). Note that, in any semi-abelian category, [58], there is an equivalence between (internal) actions and split extensions ([10] and [19]). Unfortunately, this property does not hold in more general categories: for example, there is not such an equivalence between split extensions and monoid actions in the category of monoids. Nevertheless, in [20, 72], the authors prove a restricted equivalence for the “Schreier split epimorphisms”. The terminology “Schreier split epimorphisms” came from a paper of Patchkoria [82], who worked on a notion of a Schreier internal category in the category of monoids and proved that the category of Schreier internal categories in the category of monoids is equivalent to the category of crossed semimodules. A step of generalization of this result was done in [47]. The authors defined a suitable notion of split extensions of magmas and showed they correspond to actions. Another example is given in [73], where the authors studied the split extensions in the full subcategory of preordered monoids whose preorder is determined by the corresponding positive cone.

Inspired by these results, we want to define a good notion of split extensions for (non-)associative bialgebras. Note that the case of cocommutative (non-)associative bialgebras over a symmetric monoidal category  $\mathcal{C}$  follows from the internalization of the results of [47] (see section 4.6 in [47]). Indeed, (non-)associative bialgebras can be seen as internal monoids (or magmas) in the category of cocommutative coalgebras over  $\mathcal{C}$ .

But what is an adequate notion of split extension of non-cocommutative bialgebras? What properties do they have?

This chapter will answer these questions. We define split extensions of non-associative bialgebras that are equivalent to the “actions” of non-associative bialgebras in any symmetric monoidal category. A split extension of bialgebras always forms an exact sequence. We also observe several properties of these split extensions, for example, the fact that Split Short Five Lemma holds for these split extensions. The associative case follows from this study of split extensions of non-associative bialgebras (Corollary 4.3.3). They give a generalization of the concept of a Schreier split epimorphism from the case of monoids to associative bialgebras in a symmetric monoidal category. In the next chapter, we will obtain similar results for the Hopf algebras. Chapter 4 and Chapter 5 are based on the article [87].

**Convention:** For the sake of simplicity, “bialgebras” will mean “non-associative bialgebras” (unless the associativity is explicitly mentioned). For the monoidal product of  $n$  copies  $A \otimes \cdots \otimes A$ , the notation  $A^n$  will be used. The same convention will be used for the morphisms, for example, we denote  $\alpha \otimes \alpha: A \otimes A \rightarrow B \otimes B$  by  $\alpha^2: A^2 \rightarrow B^2$ . For the diagrams appearing in the proof, we will also omit “ $\otimes$ ” and just use the juxtaposition,  $\alpha \otimes \beta: A \otimes B \rightarrow C \otimes D$  will be expressed as  $\alpha\beta: AB \rightarrow CD$ .

## 4.1 Definitions of the actions and the split extensions of bialgebras

This section is devoted to the definition of the split extensions of general bialgebras. The axioms of this definition are inspired by the properties

that we observe for the split extension provided via the semi-direct product (see Lemma 4.1.2).

In the preliminaries, we define a symmetric monoidal category of non-associative  $B$ -modules,  ${}_B\overline{\text{Mod}}$ , for  $B$  a bialgebra (see Remark 1.3.7). In this chapter, we investigate unital coalgebras in this category, which is what we called actions of bialgebras.

**Definition 4.1.1.** *Let  $X$  and  $B$  be bialgebras in a symmetric monoidal category  $(\mathcal{C}, \otimes, I, \sigma)$ . An action of  $B$  on  $X$  is a morphism  $\triangleright: B \otimes X \rightarrow X$  in  $\mathcal{C}$ , such that the following diagrams commute in  $\mathcal{C}$*

$$\triangleright \cdot (u_B \otimes 1_X) = 1_X, \quad (4.1)$$

$$\begin{array}{ccc} X & \xrightarrow{u_B \otimes 1_X} & B \otimes X \\ & \searrow & \downarrow \triangleright \\ & & X \end{array}$$

$$(1_B \otimes \triangleright) \cdot (\Delta \otimes 1_X) = (1_B \otimes \triangleright) \cdot (\sigma \otimes 1_X) \cdot (\Delta \otimes 1_X), \quad (4.2)$$

$$\begin{array}{ccccc} B \otimes X & \xrightarrow{\Delta \otimes 1_X} & B \otimes B \otimes X & & \\ \downarrow \Delta \otimes 1_X & & \downarrow 1_B \otimes \triangleright & & \\ B \otimes B \otimes X & \xrightarrow{\sigma \otimes 1_X} B \otimes B \otimes X \xrightarrow{1_B \otimes \triangleright} & B \otimes X & & \end{array}$$

$$\triangleright \cdot (1_B \otimes u_X) = u_X \cdot \epsilon_B, \quad (4.3)$$

$$\begin{array}{ccc} B & \xrightarrow{1_B \otimes u_X} & B \otimes X \\ & \searrow u_X \cdot \epsilon_B & \downarrow \triangleright \\ & & X \end{array}$$

$$\epsilon_X \cdot \triangleright = \epsilon_B \otimes \epsilon_X, \quad (4.4)$$

$$\begin{array}{ccc} B \otimes X & \xrightarrow{\triangleright} & X \\ & \searrow \epsilon_B \otimes \epsilon_X & \downarrow \epsilon_X \\ & & I \end{array}$$

$$\Delta \cdot \triangleright = (\triangleright \otimes \triangleright) \cdot (1_B \otimes \sigma \otimes 1_X) \cdot (\Delta \otimes \Delta). \quad (4.5)$$

$$\begin{array}{ccc} B \otimes X & \xrightarrow{\quad \triangleright \quad} & X \\ \downarrow \Delta \otimes \Delta & & \downarrow \Delta \\ B \otimes B \otimes X \otimes X & \xrightarrow{1_B \otimes \sigma \otimes 1_X} B \otimes X \otimes B \otimes X \xrightarrow{\quad \triangleright \otimes \triangleright \quad} & X \otimes X \end{array}$$

Let us note that (4.5) and (4.4) mean that  $\epsilon_X$  and  $\Delta$  are morphisms of  $B$ -modules. The axiom (4.3) means that  $u_X$  is a morphism of  $B$ -modules. The axiom (4.2) is inspired by the condition (1) that Majid used in [70] to define Hopf algebra crossed modules. This condition also appears in the definition of the Yetter-Drinfeld modules (see Remark 1.3.7).

This definition of action is what we need to define a bialgebra via the semi-direct product construction. In particular, we are interested in diagrams of the form

$$X \xleftarrow[\quad i_1 \quad]{\quad \pi_1 \quad} X \rtimes B \xleftarrow[\quad \pi_2 \quad]{\quad i_2 \quad} B \quad (4.6)$$

in  $\mathcal{C}$ , where  $i_1 = 1_X \otimes u_B$ ,  $i_2 = u_X \otimes 1_B$ ,  $\pi_1 = 1_X \otimes \epsilon_B$ ,  $\pi_2 = \epsilon_X \otimes 1_B$ , and  $X \rtimes B$  is the object  $X \otimes B$ , where the bialgebra structure is given by the following morphisms in  $\mathcal{C}$ ,

$$\begin{aligned} m_{X \rtimes B} &= (m \otimes m) \cdot (1_X \otimes \triangleright \otimes 1_B \otimes 1_B) \cdot (1_X \otimes 1_B \otimes \sigma \otimes 1_B) \\ &\quad \cdot (1_X \otimes \Delta \otimes 1_X \otimes 1_B) \\ u_{X \rtimes B} &= u_X \otimes u_B, \\ \Delta_{X \rtimes B} &= (1_X \otimes \sigma \otimes 1_B) \cdot (\Delta \otimes \Delta), \\ \epsilon_{X \rtimes B} &= \epsilon_X \otimes \epsilon_B. \end{aligned}$$

By combining Figure 4.1 and Figure 4.2, we check that this definition provides a bialgebra structure on  $X \otimes B$  making the morphisms  $i_1$ ,  $i_2$  and  $\pi_2$  bialgebra morphisms, and  $\pi_1$  a coalgebra morphism. Note that we can check that  $u_X \otimes u_B$  is the neutral element for the multiplication thanks to the first two axioms (4.1) and (4.3) of the definition of action. Note that the coassociativity of this structure is obvious.

We investigate this construction in detail in the case of associative bialgebras in Section 4.3.2.

**Figure 4.1:** The semi-direct product is a bialgebra: part 1

[illegible]

[illegible]

Let us make some observations about the graph (4.6), which are analogous to the ones made in [47],

**Lemma 4.1.2.** *The graph*

$$X \begin{array}{c} \xleftarrow{\pi_1} \\ \xrightarrow{i_1} \end{array} X \rtimes B \begin{array}{c} \xleftarrow{i_2} \\ \xrightarrow{\pi_2} \end{array} B ,$$

as defined in (4.6), where  $i_1, i_2, \pi_2$  are morphisms of bialgebras, satisfies the following properties

- (1)  $\pi_1 \cdot i_1 = 1_X, \pi_2 \cdot i_2 = 1_B$
- (2)  $\pi_1 \cdot i_2 = u_X \cdot \epsilon_B, \pi_2 \cdot i_1 = u_B \cdot \epsilon_X$
- (3)  $m_{X \rtimes B} \cdot (i_1 \cdot \pi_1 \otimes i_2 \cdot \pi_2)(1_X \otimes \sigma \otimes 1_B) \cdot (\Delta \otimes \Delta) = 1_X \otimes 1_B$
- (4)  $\pi_1 \cdot m_{X \rtimes B} \cdot (i_1 \otimes i_2) = 1_X \otimes \epsilon_B$
- (5)  $(1_B \otimes \pi_1) \cdot (1_B \otimes m_{X \rtimes B}) \cdot (1_B \otimes i_2 \otimes i_1) \cdot (\Delta \otimes 1_X) = (1_B \otimes \pi_1) \cdot (1_B \otimes m_{X \rtimes B}) \cdot (1_B \otimes i_2 \otimes i_1) \cdot (\sigma \otimes 1_X) \cdot (\Delta \otimes 1_X)$
- (6)  $m_{X \rtimes B} \cdot (1_X \otimes 1_B \otimes m_{X \rtimes B}) \cdot (i_1 \otimes i_2 \otimes 1_X \otimes 1_B) = m_{X \rtimes B} \cdot (m_{X \rtimes B} \otimes 1_X \otimes 1_B) \cdot (i_1 \otimes i_2 \otimes 1_X \otimes 1_B)$
- (7)  $m_{X \rtimes B} \cdot (1_X \otimes 1_B \otimes m_{X \rtimes B}) \cdot (i_1 \otimes 1_X \otimes 1_B \otimes i_2) = m_{X \rtimes B} \cdot (m_{X \rtimes B} \otimes 1_X \otimes 1_B) \cdot (i_1 \otimes 1_X \otimes 1_B \otimes i_2)$
- (8)  $m_{X \rtimes B} \cdot (1_X \otimes 1_B \otimes m_{X \rtimes B}) \cdot (1_X \otimes 1_B \otimes i_1 \otimes i_2) = m_{X \rtimes B} \cdot (m_{X \rtimes B} \otimes 1_X \otimes 1_B) \cdot (1_X \otimes 1_B \otimes i_1 \otimes i_2)$
- (9)  $\pi_1$  is a morphism of coalgebras and  $\pi_1$  preserves the unit.

*Proof.* The properties (1) and (2) are trivial. The condition (3) is proven via the commutativity of the following diagram

$$\begin{array}{ccccc}
 XB & \xrightarrow{\Delta\Delta} & X^2B^2 & \xrightarrow{1_X\sigma 1_B} & (XB)^2 \\
 \parallel & & \searrow (1.5) & \searrow 1_X\epsilon_X\epsilon_B 1_B & \downarrow 1_X\epsilon_B\epsilon_X 1_B \\
 XB & \xlongequal{\quad\quad\quad} & & & XB \\
 \parallel & & & & \downarrow 1_Xu_Bu_X 1_B \\
 & (1.8) & & & (XB)^2 \\
 XB & \xrightarrow{1_Xu_Bu_Bu_X 1_B} & & & \downarrow 1_X\Delta 1_X 1_B \\
 & & & & XB^2XB \\
 \parallel & & & & \downarrow 1_X 1_B\sigma 1_B \\
 XB & \xrightarrow{1_Xu_Bu_Xu_B 1_B} & & & XBXB^2 \\
 \parallel & & & & \downarrow 1_X \triangleright 1_B 1_B \\
 XB & \xrightarrow{1_Xu_Xu_B 1_B} & & & X^2B^2 \\
 \parallel & & & & \downarrow mm \\
 XB & \xlongequal{\quad\quad\quad} & & & XB.
 \end{array}$$

$(i_1 \cdot \pi_1)(i_2 \cdot \pi_2)$   
 $m_{X \rtimes B}$

The equality (4) holds since this diagram commutes

$$\begin{array}{ccc}
 XB & \xrightarrow{1_Xu_Bu_X 1_B} & (XB)^2 \\
 \parallel & \searrow (1.8) & \downarrow 1_X\Delta 1_X 1_B \\
 & 1_Xu_Bu_Xu_B 1_B & XB^2XB \\
 & \searrow (4.1) & \downarrow 1_X 1_B\sigma 1_B \\
 & & XBXB^2 \\
 & & \downarrow 1_X \triangleright (1_B)^2 \\
 & & X^2B^2 \\
 & \xrightarrow{1_Xu_Xu_B 1_B} & \downarrow mm \\
 XB & \xlongequal{\quad\quad\quad} & XB. \\
 \downarrow 1_X\epsilon_B & \swarrow (1.3) & \\
 X & \xleftarrow{1_X\epsilon_B} & 
 \end{array}$$

$m_{X \rtimes B}$

The property (5) is due to (4.2) and the commutativity of the squares denoted (A)

$$\begin{array}{ccccc}
 BX & \xrightarrow{\Delta 1_X} & BBX & \xrightarrow{1_B i_2 i_1} & B(XB)^2 \\
 \downarrow \Delta 1_X & & \downarrow 1_B \triangleright & & \downarrow 1_B m_{X \rtimes B} \\
 BBX & \xrightarrow{\sigma 1_X} & BBX & \xrightarrow{1_B \triangleright} & BX \\
 \downarrow 1_B i_2 i_1 & & \downarrow 1_B 1_X \epsilon_B & & \downarrow 1_B 1_X \epsilon_B \\
 B(XB)^2 & \xrightarrow{1_B m_{X \rtimes B}} & BXB & \xleftarrow{1_B 1_X \epsilon_B} & BXB
 \end{array}
 \quad (A)$$

We observe that the squares (A) commute thanks to the unitality of the multiplication and the counitality of the comultiplication, as explained in the following diagram

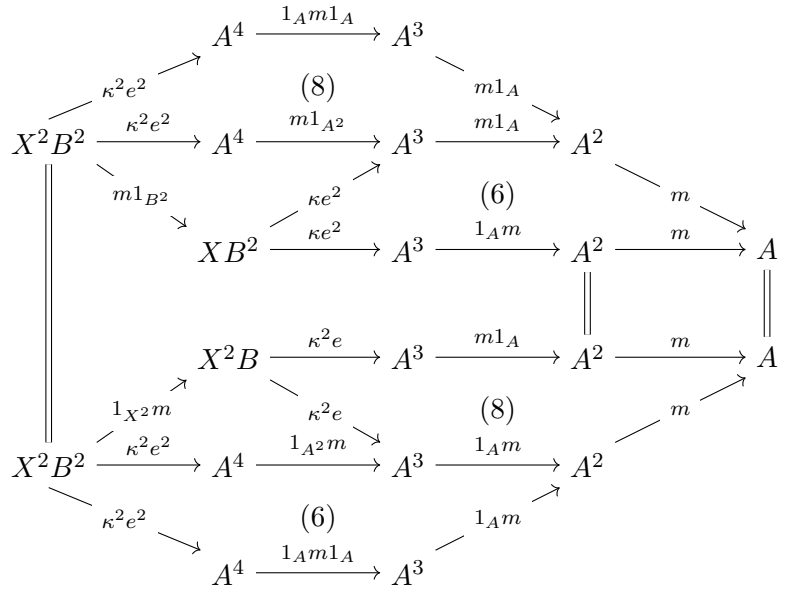
$$\begin{array}{ccccc}
 B^2 X & \xrightarrow{1_B u_X 1_B 1_X u_B} & B(XB)^2 & \xrightarrow{1_B 1_X \Delta 1_X 1_B} & BXB^2 XB \\
 \parallel & & & & \downarrow 1_B 1_X 1_B \sigma 1_B \\
 B^2 X & \xrightarrow{1_B \Delta 1_X} & B^3 X & & (BX)^2 B^2 \\
 \downarrow 1_B \triangleright & & \downarrow 1_B 1_B \sigma & & \downarrow 1_B 1_X \triangleright 1_B 1_B \\
 BXB & \xrightarrow{1_B \triangleright 1_B} & BXB & \xrightarrow{1_B u_X 1_X 1_B u_B} & BXB^2 \\
 \downarrow 1_B \triangleright & & \downarrow 1_B 1_X \epsilon_B & & \downarrow 1_B m \\
 BX & \xleftarrow{1_B 1_X \epsilon_B} & BXB & \xleftarrow{1_B 1_X \epsilon_B} & BXB
 \end{array}
 \quad (1.3)$$

We prove the “partial associativity” condition (6) thanks to the following diagram



- (3)  $m \cdot ((\kappa \cdot \lambda) \otimes (e \cdot \alpha)) \cdot \Delta = 1_A$
- (4)  $\lambda \cdot m \cdot (\kappa \otimes e) = 1_X \otimes \epsilon_B$
- (5)  $(1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\Delta \otimes 1_X) = (1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\sigma \otimes 1_X) \cdot (\Delta \otimes 1_X)$
- (6)  $m \cdot (m \otimes 1_A) \cdot (\kappa \otimes e \otimes 1_A) = m \cdot (1_A \otimes m) \cdot (\kappa \otimes e \otimes 1_A)$
- (7)  $m \cdot (m \otimes 1_A) \cdot (\kappa \otimes 1_A \otimes e) = m \cdot (1_A \otimes m) \cdot (\kappa \otimes 1_A \otimes e)$
- (8)  $m \cdot (m \otimes 1_A) \cdot (1_A \otimes \kappa \otimes e) = m \cdot (1_A \otimes m) \cdot (1_A \otimes \kappa \otimes e)$
- (9)  $\lambda$  is a morphism of coalgebras preserving the unit.

**Remark 4.1.4.** We notice that the conditions  $\lambda \cdot \kappa = 1_X$ ,  $\lambda \cdot e = u_X \cdot \epsilon_B$ , and  $\lambda$  preserving the unit are consequences of the axiom (4). The condition (7) follows from (3), (6), and (8), as we show in the following diagrams, where we use that  $e$  and  $\kappa$  are (bi)algebra morphisms.



Hence, by pre-composing with  $(1_X \otimes \lambda \otimes \alpha \otimes 1_B) \cdot (1_X \otimes \Delta \otimes 1_B)$  and using the condition (3), we obtain the following diagram

$$\begin{array}{ccccc}
 XAB & \xlongequal{\hspace{1.5cm}} & & & XAB \\
 \downarrow 1_X \Delta 1_B & & & & \downarrow \kappa 1_A e \\
 & XA^2B & & & \\
 & \downarrow 1_X \lambda \alpha 1_B & & & \downarrow 1_X m 1_B \\
 & X^2B^2 & \xrightarrow{1_X \kappa e 1_B} & XA^2B & \\
 & \downarrow 1_X \kappa e 1_B & & & \downarrow \kappa 1_A e \\
 & XA^2B & & & A^3 \\
 & \downarrow 1_X m 1_B & & & \downarrow m 1_A \\
 & & & & A^2 \\
 & & & & \downarrow m \\
 XAB & \xrightarrow{\kappa 1_A e} & A^3 & \xrightarrow{1_A m} & A^2 & \xrightarrow{m} & A
 \end{array}$$

(3)

Then, we conclude that condition (7) holds.

**Proposition 4.1.5.** *If there exists a morphism  $\lambda$  satisfying the conditions of Definition 4.1.3, it has to be unique.*

*Proof.* Let us suppose that there exist two morphisms  $\lambda$  and  $\lambda'$  satisfying the conditions of Definition 4.1.3. Then the commutativity of the following diagram shows that  $\lambda'$  has to be equal to  $\lambda$ .

$$\begin{array}{ccccccc}
 A & \xrightarrow{\Delta} & A^2 & \xrightarrow{\lambda' \alpha} & XB & \xrightarrow{\kappa e} & A^2 & \xrightarrow{m} & A \\
 \parallel & & & & \searrow 1_X \epsilon_B & & & & \downarrow \lambda \\
 A & \xrightarrow{\lambda'} & & & & & & & X
 \end{array}$$

(1.5)

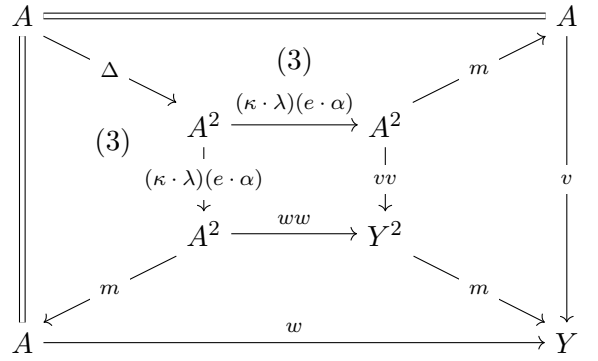
□

In the associative case, the conditions (6), (7) and (8) trivially become redundant. Note that these “partial associativity conditions” are versions of the conditions considered in the article [47].

Another important property of the split extensions of bialgebras is given by the following proposition:

**Proposition 4.1.6.** *Let  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  be a split extension of bialgebras, then  $\kappa$  and  $e$  are jointly epimorphic in the category of bialgebras (and in the category of algebras)*

*Proof.* Let  $v, w: A \rightarrow Y$  be two morphisms of bialgebras such that  $v \cdot \kappa = w \cdot \kappa$  and  $v \cdot e = w \cdot e$ .



The above diagram allows us to conclude that  $v$  and  $w$  are equal. Hence,  $\kappa$  and  $e$  are jointly epimorphic. Note that we only use that  $v$  and  $w$  are algebra morphisms (we do not need them to be coalgebra morphisms).  $\square$

Note that any point (split extension) in a pointed protomodular category satisfies this property. Hence, even if the category of bialgebras is not protomodular, Proposition 4.1.6 is interesting and seems to indicate that this definition of split extensions of bialgebras is adequate.

**Definition 4.1.7.** *A morphism of split extensions of bialgebras from  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  to  $X' \xleftarrow[\kappa']{\lambda'} A' \xleftarrow[\alpha']{e'} B'$  is given by three morphisms of bialgebras  $g: B \rightarrow B'$ ,  $v: X \rightarrow X'$  and  $p: A \rightarrow A'$  such that the following diagram commutes*

$$\begin{array}{ccccc}
X & \xleftarrow[\kappa]{\lambda} & A & \xleftarrow[\alpha]{e} & B \\
\downarrow v & & \downarrow p & & \downarrow g \\
X' & \xleftarrow[\kappa']{\lambda'} & A' & \xleftarrow[\alpha']{e'} & B'
\end{array} . \quad (4.8)$$

We do not need to ask the commutativity of all the squares thanks to this corollary:

**Corollary 4.1.8.** *Let  $g: B \rightarrow B'$ ,  $v: X \rightarrow X'$ ,  $p: A \rightarrow A'$  be morphisms of bialgebras*

$$\begin{array}{ccccc}
X & \xleftarrow[\kappa]{\lambda} & A & \xleftarrow[\alpha]{e} & B \\
\downarrow v & & \downarrow p & & \downarrow g \\
X' & \xleftarrow[\kappa']{\lambda'} & A' & \xleftarrow[\alpha']{e'} & B'
\end{array} .$$

*If  $p \cdot \kappa = \kappa' \cdot v$  and  $p \cdot e = e' \cdot g$  hold, then the identities  $\lambda' \cdot p = v \cdot \lambda$  and  $\alpha' \cdot p = g \cdot \alpha$  follow, and conversely.*

*Proof.* Let us suppose that  $p \cdot \kappa = \kappa' \cdot v$  and  $p \cdot e = e' \cdot g$ , then by Proposition 4.1.6, we can prove that  $\lambda' \cdot p = v \cdot \lambda$  by checking that  $(\lambda' \cdot p) \cdot \kappa = (v \cdot \lambda) \cdot \kappa$  and  $(\lambda' \cdot p) \cdot e = (v \cdot \lambda) \cdot e$ , which is done in the following identities:

$$(\lambda' \cdot p) \cdot \kappa = \lambda' \cdot \kappa' \cdot v = v = (v \cdot \lambda) \cdot \kappa,$$

$$(\lambda' \cdot p) \cdot e = \lambda' \cdot e' \cdot g = \epsilon_{X'} = (v \cdot \lambda) \cdot e.$$

Similarly, one can check that  $\alpha' \cdot p = g \cdot \alpha$ . □

**Proposition 4.1.9.** *Given a split extension  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  of bialgebras, the following properties hold:*

a)  $\kappa$  is the kernel of  $\alpha$  in the category of bialgebras;

- b)  $\alpha$  is the cokernel of  $\kappa$  in the category of bialgebras;
- c)  $e$  is the kernel of  $\lambda$  in the category of pointed coalgebras.

Hence, a split extension of bialgebras is a short exact sequence (as defined in Definition 1.1.11), so the term “extension” is appropriate.

*Proof.* a) Let  $\omega : D \rightarrow A$  be a morphism of bialgebras such that  $\alpha \cdot \omega = u_B \cdot \epsilon_D$ . We build the morphism  $\hat{\omega} : D \rightarrow X$  by setting

$$\hat{\omega} = \lambda \cdot \omega.$$

First, we verify that  $\kappa \cdot \hat{\omega} = w$  thanks to the commutativity of the following diagram

$$\begin{array}{ccccc}
 D & \xlongequal{\quad} & D & & \\
 \downarrow \omega & \searrow \Delta & & & \downarrow \kappa \cdot \lambda \cdot \omega \\
 & D^2 & & & \\
 & \downarrow \omega\omega & \searrow (\kappa \cdot \lambda \cdot \omega)(u_A \cdot \epsilon_D) & & \\
 & A^2 & \xrightarrow{(\kappa \cdot \lambda)(e \cdot \alpha)} & A^2 & \\
 \downarrow \omega & \nearrow \Delta & \xrightarrow{(3)} & \searrow m & \downarrow \kappa \cdot \lambda \cdot \omega \\
 A & \xlongequal{\quad} & A & & 
 \end{array} \tag{4.9}$$

Moreover,  $\hat{\omega}$  is a coalgebra morphism by construction and an algebra morphism since the following equalities hold thanks to (1) and the fact that  $\kappa \cdot \hat{\omega} = \omega$ :

$$m \cdot \hat{\omega}^2 = \lambda \cdot \kappa \cdot m \cdot \hat{\omega}^2 = \lambda \cdot m \cdot \kappa^2 \cdot \hat{\omega}^2 = \lambda \cdot m \cdot \omega^2 = \lambda \cdot \omega \cdot m = \lambda \cdot \kappa \cdot \hat{\omega} \cdot m = \hat{\omega} \cdot m.$$

Finally, if there exists another morphism  $\omega'$  such that  $\kappa \cdot \omega' = w$ , then by (1)

$$\omega' = \lambda \cdot \kappa \cdot \omega' = \lambda \cdot \omega = \lambda \cdot \kappa \cdot \hat{\omega} = \hat{\omega}.$$

Hence,  $\kappa$  is the kernel of  $\alpha$ .

b) Let  $\beta : A \rightarrow C$  be a bialgebra morphism such that  $\beta \cdot \kappa = u_C \cdot \epsilon_X$ . We define  $\tilde{\beta} : B \rightarrow C$  by

$$\tilde{\beta} = \beta \cdot e.$$

This morphism is a bialgebra morphism, and thanks to Proposition 4.1.6, it is enough to remark that  $\tilde{\beta} \cdot \alpha \cdot \kappa = u_C \cdot \epsilon_X = \beta \cdot \kappa$  and  $\tilde{\beta} \cdot \alpha \cdot e = \beta \cdot e$ , to conclude that  $\tilde{\beta} \cdot \alpha = \beta$ . Moreover, if there exists another  $\beta'$  such that  $\beta' \cdot \alpha = \beta$  then, thanks to (1) we have

$$\beta' = \beta' \cdot \alpha \cdot e = \beta \cdot e = \tilde{\beta} \cdot \alpha \cdot e = \tilde{\beta}$$

and  $\alpha$  is the cokernel of  $\kappa$ .

c) Let  $\gamma: Y \rightarrow A$  be a morphism of coalgebras such that  $\lambda \cdot \gamma = u_X \cdot \epsilon_Y$ . We can verify that the equality  $e \cdot \hat{\gamma} = \gamma$  holds for  $\hat{\gamma} := \alpha \cdot \gamma: Y \rightarrow B$ , using the condition (3) in Definition 4.1.3 as in diagram (4.9). Moreover, it is clear that  $\hat{\gamma}$  is a coalgebra morphism and it is the unique morphism such that  $e \cdot \hat{\gamma} = \gamma$ .  $\square$

## 4.2 Equivalence of categories between the split extensions and actions of bialgebras

To prove the equivalence between actions of bialgebras and split extensions of bialgebras, we need the following proposition. The morphism of coalgebras  $\lambda$ , in Definition 4.1.3, is not an algebra morphism but it satisfies the following property.

**Proposition 4.2.1.** *Let  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  be a split extension of bialgebras, then the following diagram commutes*

$$\begin{array}{ccccc}
 A^2 & \xrightarrow{\Delta \otimes 1_A} & A^3 & \xrightarrow{1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda)} & A^3 & \xrightarrow{1_A \otimes m} & A^2 \\
 \downarrow m & & & & & & \downarrow \lambda \otimes \lambda \\
 & & & & & & X^2 \\
 & & & & & & \downarrow m \\
 A & \xrightarrow{\lambda} & & & & & X.
 \end{array} \quad (4.10)$$

*Proof.* In order to prove the proposition, we make the three diagrams (A), (B) and (C) commute. Then we compose them to obtain the conclusion. The diagram (A) is commutative since  $e$  and  $\alpha$  are morphisms of (bi)algebras.

$$\begin{array}{ccccc}
 (A) & A^5 & \xrightarrow{1_A(e \cdot \alpha)(\kappa \cdot \lambda)m} & & A^4 \\
 & \parallel & & & \downarrow 1_A m 1_A \\
 & A^5 & \xrightarrow{1_A(e \cdot \alpha)(\kappa \cdot \lambda)(e \cdot \alpha)(e \cdot \alpha)} & A^5 & \xrightarrow{1_A m^2} A^3 \xleftarrow{1_{A^2}(e \cdot \alpha)} A^3 \\
 & \downarrow 1_A(e \cdot \alpha)(\kappa \cdot \lambda)(e \cdot \alpha)^2 & & \downarrow 1_A m 1_{A^2} & \downarrow (\kappa \cdot \lambda)^2(e \cdot \alpha) \\
 & A^5 & & A^4 & \downarrow (\kappa \cdot \lambda)^2 1_A \\
 & \downarrow 1_A m 1_{A^2} & & \downarrow (\kappa \cdot \lambda)^2 1_{A^2} & \downarrow 1_{A^2} m \\
 & A^4 & & A^4 & \xrightarrow{1_{A^2} m} A^3 \\
 & \downarrow (\kappa \cdot \lambda)^2 1_{A^2} & & \downarrow 1_A m & \downarrow m \\
 & A^4 & \xrightarrow{1_A m 1_A} & A^3 & \xrightarrow{1_A m} A^2 \xrightarrow{m} A. \\
 & & & & \downarrow m \\
 & & & & A.
 \end{array}
 \quad (8)$$

The following diagram (B) commutes thanks to the counitality of the comultiplication, the unitality of the multiplication, and the fact that  $(e \cdot \alpha)$  is a morphism of (bi)algebras.

$$\begin{array}{ccccccc}
 (B) & A^2 & \xrightarrow{\Delta^2} & A^4 & \xrightarrow{1_A \Delta 1_{A^2}} & A^5 & \xrightarrow{1_{A^2} \sigma 1_A} A^5 \\
 & \downarrow \Delta^2 a & & & & & \downarrow ((\kappa \cdot \lambda)(e \cdot \alpha))^2(e \cdot \alpha) \\
 & A^4 & & & & & A^5 \\
 & \downarrow 1_A \Delta^2 1_A & & & & & \downarrow 1_A m 1_{A^2} \\
 & A^6 & & & & & A^4 \\
 & \downarrow 1_{A^2} \sigma 1_{A^2} & & & & & \downarrow 1_A(\kappa \cdot \lambda) 1_{A^2} \\
 & A^6 & \xrightarrow{((\kappa \cdot \lambda)(e \cdot \alpha))^2(u_A \cdot \epsilon_A)(e \cdot \alpha)} & A^6 & \xrightarrow{1_A m 1_{A^3}} & A^5 & \xrightarrow{1_A(\kappa \cdot \lambda) m 1_A} A^4 \\
 & \downarrow ((\kappa \cdot \lambda)(e \cdot \alpha))^3 & & & & & \downarrow \\
 & A^6 & \xrightarrow{1_A m(e \cdot \alpha)^2 1_A} & A^5 & \xrightarrow{1_A(\kappa \cdot \lambda) m 1_A} & & A^4 \\
 & \parallel & & & & & \parallel \\
 & A^6 & \xrightarrow{1_A m^2 1_A} & A^4 & \xrightarrow{1_A(\kappa \cdot \lambda)(e \cdot \alpha) 1_A} & & A^4.
 \end{array}
 \quad (1.3) + (1.5)$$

The diagram (C) is the following, where we are using the fact that we work with coalgebra morphisms.

$$\begin{array}{c}
(C) \\
\begin{array}{ccccccc}
A^2 & \xlongequal{\quad} & A^2 & \xrightarrow{\Delta^2} & A^4 & \xlongequal{\quad} & A^4 & \xrightarrow{1_A \Delta^2 1_A} & A^6 & \xrightarrow{1_{A^2} \sigma 1_{A^2}} & A^6 \\
\parallel & & \downarrow \Delta^2 & & \downarrow ((\kappa \cdot \lambda)(e \cdot \alpha))^2 & & \downarrow ((\kappa \cdot \lambda)(e \cdot \alpha))^2 & & \downarrow ((\kappa \cdot \lambda)(e \cdot \alpha))^2 & & \downarrow ((\kappa \cdot \lambda)(e \cdot \alpha))^3 \\
& & A^4 & & A^4 & & A^4 & \xrightarrow{1_A \Delta^2 1_A} & A^6 & \xrightarrow{1_{A^2} \sigma 1_{A^2}} & A^6 \\
& & \downarrow ((\kappa \cdot \lambda)(e \cdot \alpha))^2 & & \downarrow 1_{A^2} m & & \downarrow 1_A m 1_A & & \downarrow 1_A m^2 1_A & & \downarrow 1_A m^2 1_A \\
& & A^4 & & A^3 & & A^3 & \xrightarrow{1_A \Delta 1_A} & A^4 & & A^4 \\
& & \downarrow 1_{A^2} m & & \downarrow 1_A m & & \downarrow 1_A m & & \downarrow 1_A m & & \downarrow 1_A m \\
& & A^3 & & A^3 & & A^3 & \xrightarrow{1_A m} & A^2 & & A^2 \\
& & \downarrow m 1_A & & \downarrow m & & \downarrow m & & \downarrow m & & \downarrow m \\
& & A^2 & \xrightarrow{\quad m \quad} & A^2 & \xrightarrow{\quad m \quad} & A^2 & \xrightarrow{\quad m \quad} & A^2 & \xrightarrow{\quad m \quad} & A^2 \\
& & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
& & A^2 & \xrightarrow{\quad m \quad} & A^2 & \xrightarrow{\quad m \quad} & A^2 & \xrightarrow{\quad m \quad} & A^2 & \xrightarrow{\quad m \quad} & A^2
\end{array}
\end{array}$$

Finally, we combine the three above diagrams in Figure 4.3, and we obtain the following equality

$$\begin{aligned}
m &= m \cdot (\kappa \otimes e) \cdot (m \otimes 1_B) \cdot (\lambda \otimes \lambda \otimes \alpha) \cdot (1_A \otimes m \otimes 1_A) \\
&\quad \cdot (1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda) \otimes m) \cdot (1_A \otimes 1_A \otimes \sigma \otimes 1_A) \\
&\quad \cdot (1_A \otimes \Delta \otimes 1_A \otimes 1_A) \cdot (\Delta \otimes \Delta).
\end{aligned}$$

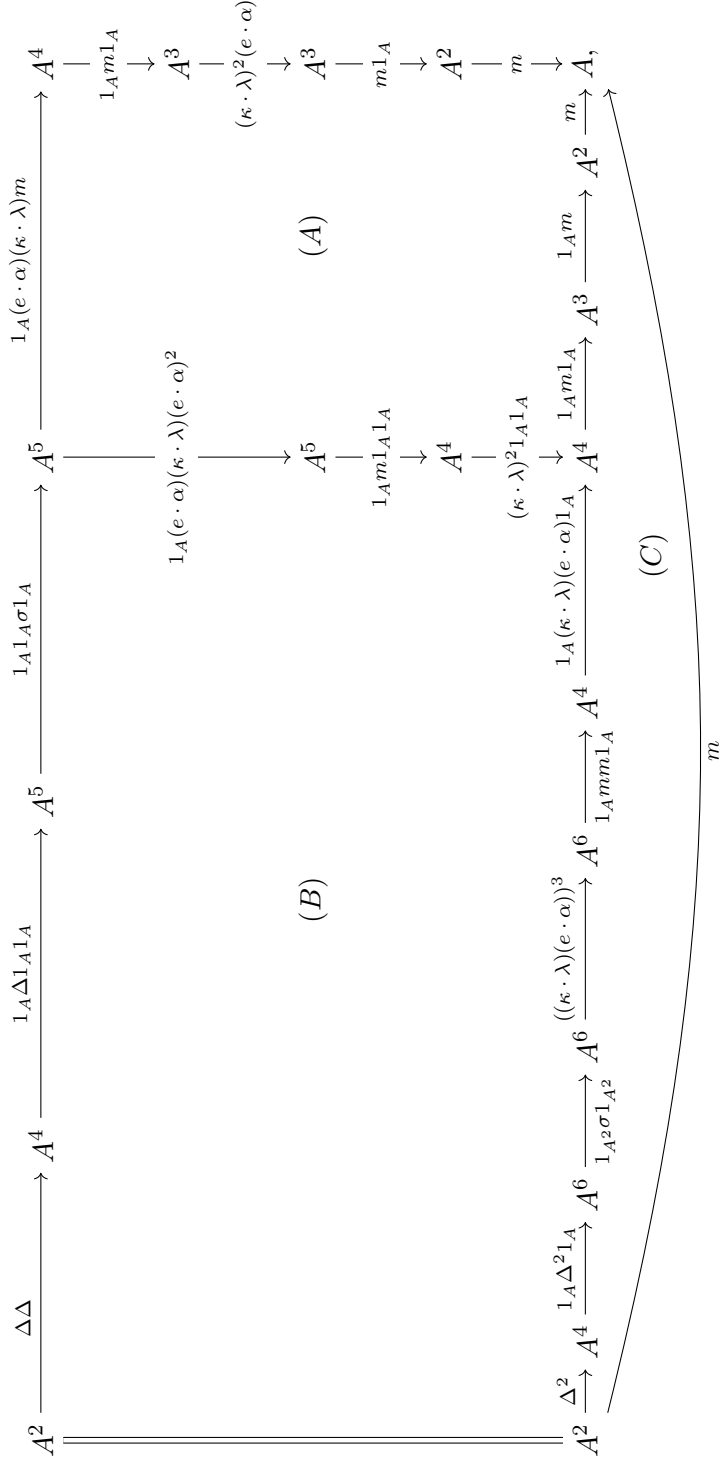
By composing with  $\lambda$  and using the condition (4), we can conclude that

$$\lambda \cdot m = m \cdot (\lambda \otimes \lambda) \cdot (1_A \otimes m) \cdot (1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda)) \cdot (\Delta \otimes 1_A).$$

□

The construction in Lemma 4.1.2 taught us how to build a split extension of bialgebras from an action of bialgebras. The following Proposition constructs an action from a split extension of bialgebras.

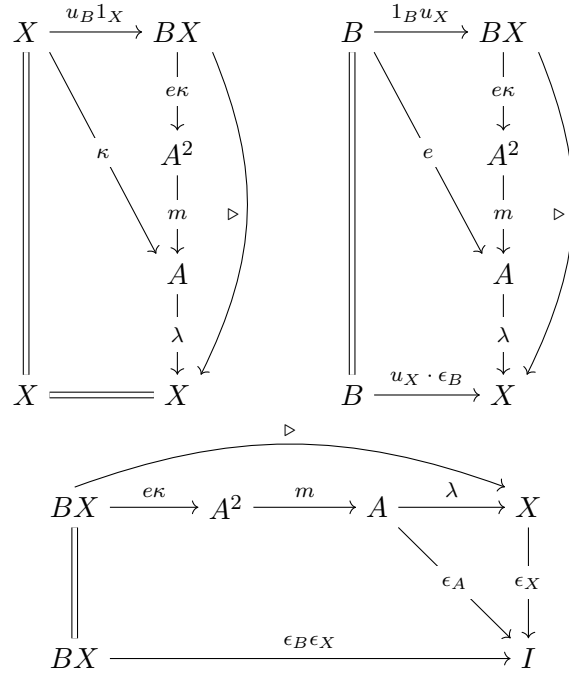
Figure 4.3: Combination of the diagrams (A), (B) and (C)



**Proposition 4.2.2.** *Given a split extension  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  of bialgebras, we can construct an action of bialgebras,  $\triangleright: B \otimes X \rightarrow X$  defined by*

$$\triangleright = \lambda \cdot m \cdot (e \otimes \kappa).$$

*Proof.* We check all the axioms of the definition of actions of bialgebras (Definition 4.1.1).



$$\begin{array}{ccccccc}
 & & \triangleright & & & & \\
 BX & \xrightarrow{ek} & A^2 & \xrightarrow{m} & A & \xrightarrow{\lambda} & X \\
 \downarrow \Delta\Delta & & \downarrow \Delta\Delta & & \downarrow \Delta & & \downarrow \Delta \\
 B^2X^2 & \xrightarrow{eekk} & A^4 & & & & \\
 \downarrow 1_B\sigma 1_X & & \downarrow 1_A\sigma 1_A & & & & \\
 (BX)^2 & \xrightarrow{ekek} & A^4 & \xrightarrow{mm} & A^2 & \xrightarrow{\lambda\lambda} & X^2.
 \end{array}
 \quad (1.7)$$

$\triangleright\triangleright$

Moreover, the condition (4.2) for the particular action  $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$  is exactly the condition (5) in the definition of split extensions of bialgebras (Definition 4.1.3).  $\square$

Consider the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\lambda} & A & \xleftarrow{e} & B \\
 \parallel & \swarrow \kappa & \uparrow \psi & \searrow \alpha & \parallel \\
 X & \xleftarrow{\pi_1} & X \rtimes B & \xleftarrow{i_2} & B, \\
 & \swarrow i_1 & & \searrow \pi_2 & 
 \end{array}
 \quad (4.11)$$

where the two rows are split extensions, the top row by definition and the bottom one by Lemma 4.1.2, where the action of bialgebras is given by Proposition 4.2.2. The morphisms  $\varphi$  and  $\psi$  are defined by

$$\psi = (\lambda \otimes \alpha) \cdot \Delta,$$

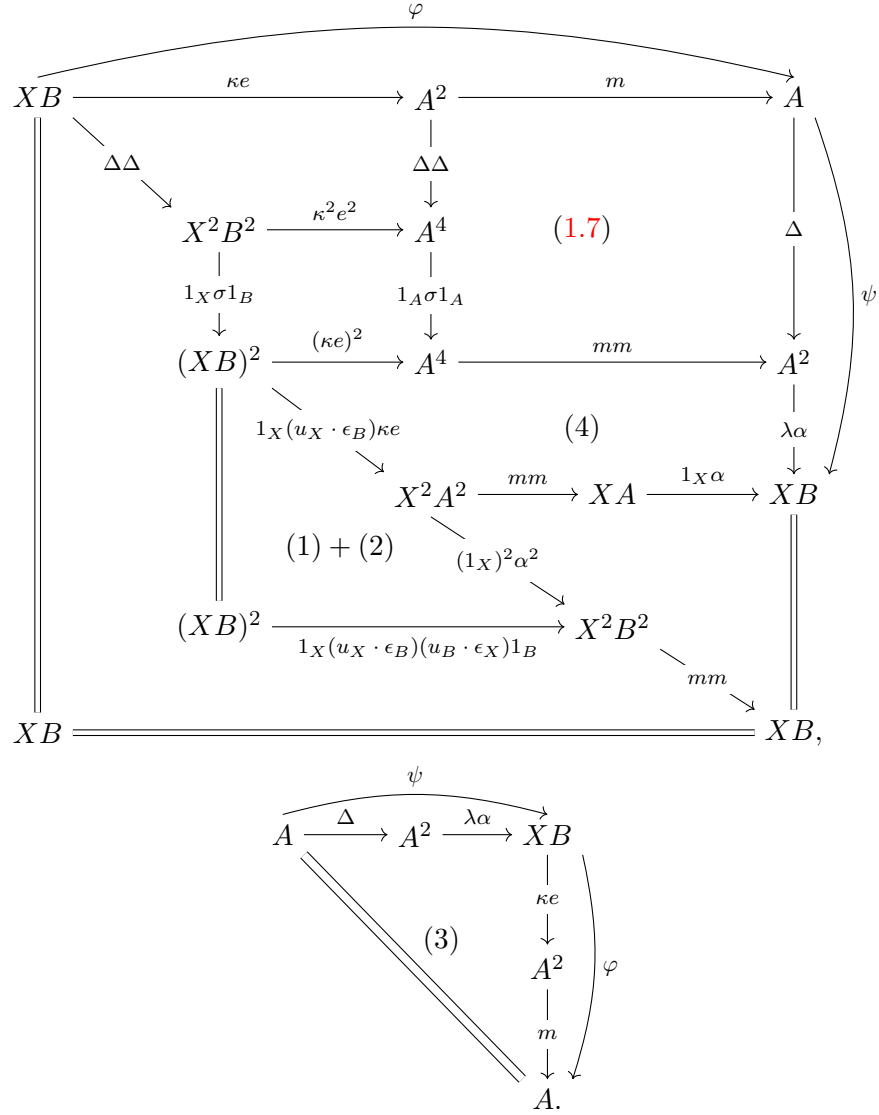
and

$$\varphi = m \cdot (\kappa \otimes e).$$

In the following lemmas, we prove step by step that  $\psi$  and  $\varphi$  are isomorphisms of split extensions. First, we prove that they are inverse to each other.

**Lemma 4.2.3.** *The morphisms  $\varphi$  and  $\psi$ , in  $\mathcal{C}$ , are inverse to each other.*

*Proof.* We prove in the following two diagrams that  $\psi \cdot \varphi = 1_X \otimes 1_B$  and  $\varphi \cdot \psi = 1_A$  by using the properties of the split extensions.



□

To prove that  $\psi$  and  $\varphi$  of (4.11) are morphisms of bialgebras, we need the following technical Lemma. Notice that this Lemma will also be

convenient to express the action  $\triangleright := \lambda \cdot m \cdot (e \otimes \kappa)$  in the particular case of Hopf algebras (See Remark 5.1.6).

**Lemma 4.2.4.** *Given a split extension  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  of bialgebras, we have*

$$m \cdot (e \otimes \kappa) = m \cdot (\kappa \otimes e) \cdot (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X), \quad (4.12)$$

where  $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$ .

*Proof.* The equality of the lemma is proven thanks to the commutativity of the following diagram where we use that  $\alpha$ ,  $\kappa$  and  $e$  are morphisms of bialgebras

$$\begin{array}{ccccccc}
 BX & \xlongequal{\quad} & BX & \xrightarrow{e\kappa} & A^2 & \xrightarrow{m} & A \\
 \Delta 1_X \downarrow & & (1_B \sigma 1_X) \cdot (\Delta \Delta) \downarrow & & (1_B \sigma 1_X) \cdot (\Delta \Delta) \downarrow & (1.7) & \Delta \downarrow \\
 B^2 X & & BXB X & \xrightarrow{(e\kappa)^2} & A^4 & \xrightarrow{mm} & A^2 \\
 1_B \sigma \downarrow & & e\kappa 1_{BX} \downarrow & & m\alpha\alpha \downarrow & & \lambda\alpha \downarrow (3) \\
 BXB & & A^2 BX & \xrightarrow{m 1_B(u_B \cdot \epsilon_X)} & AB^2 & \xrightarrow{\lambda m} & XB \\
 e\kappa 1_B \downarrow & & & & & & \kappa e \downarrow \\
 A^2 B & & (1.3) + (1.5) & & & & A^2 \\
 m 1_B \downarrow & & & & & & m \downarrow \\
 AB & \xrightarrow{\lambda 1_B} & XB & \xrightarrow{\kappa e} & A^2 & \xrightarrow{m} & A
 \end{array}$$

□

**Lemma 4.2.5.** *The morphism  $\varphi := m \cdot (\kappa \otimes e): X \otimes B \rightarrow A$  is a morphism of bialgebras.*

*Proof.* First, we establish two diagrams denoted by (A) and (B). We will need them to conclude that  $\varphi$  is a morphism of algebras.

(A) We use the partial associativity of the split extensions to check that the following diagram commutes

$$\begin{array}{ccccccc}
 XBXB & \xrightarrow{(\kappa e)^2} & A^4 & \xrightarrow{1_A m 1_A} & A^3 & \xrightarrow{m 1_A} & A^2 \\
 \parallel & & & & & & \downarrow m \\
 XBXB & \xrightarrow{(\kappa e)^2} & A^4 & \xrightarrow{m 1_A 1_A} & A^3 & \xrightarrow{m 1_A} & A^2 \xrightarrow{m} A \\
 \parallel & & & & & & \parallel \\
 XBXB & \xrightarrow{(\kappa e)^2} & A^4 & \xrightarrow{m 1_A 1_A} & A^3 & \xrightarrow{1_A m} & A^2 \xrightarrow{m} A
 \end{array}
 \quad (6)$$

(8)

(B) The diagram commutes since  $e$  and  $\kappa$  are (bi)algebra morphisms

$$\begin{array}{ccccccc}
 X^2 B^2 & \xrightarrow{\kappa^2 e^2} & A^4 & \xrightarrow{1_A m 1_A} & A^3 & \xrightarrow{m 1_A} & A^2 \\
 \parallel & & & & & \nearrow m 1_A & \parallel \\
 X^2 B^2 & \xrightarrow{\kappa^2 e^2} & A^4 & \xrightarrow{m 1_A 1_A} & A^3 & & \\
 \downarrow m 1_B 1_A & & \downarrow m 1_A 1_A & & & & \downarrow m \\
 X B^2 & \xrightarrow{\kappa e^2} & A^3 & \xrightarrow{m 1_A} & A^2 & & \\
 \parallel & & & & & & \parallel \\
 X B^2 & \xrightarrow{\kappa e^2} & A^3 & & & & A^2 \\
 \downarrow 1_X m & & \downarrow 1_A m & & & & \downarrow m \\
 X B & \xrightarrow{\kappa e} & A^2 & \xrightarrow{m} & A & & A
 \end{array}
 \quad (7)$$

(6)

Thanks to these two diagrams and Lemma 4.2.4 we have the following diagram showing that  $\varphi := m \cdot (\kappa \otimes e)$  is a morphism of algebras

$$\begin{array}{ccccccc}
 (XB)^2 & \xrightarrow{1_X \Delta 1_{XB}} & XB^2 XB & \xrightarrow{1_{XB} \sigma 1_B} & XBXB^2 & \xrightarrow{1_X \triangleright 1_{B^2}} & X^2 B^2 \\
 \downarrow (\kappa e)^2 & \searrow (\kappa e)^2 & & & & \swarrow \kappa^2 e^2 & \downarrow m^2 \\
 & & A^4 & & A^4 & & XB \\
 & & \searrow 1_A m 1_A & & \swarrow 1_A m 1_A & & \downarrow \kappa e \\
 & & & A^3 & \xrightarrow{m 1_A} & A^2 & \downarrow m \\
 & & (A) & & & & A \\
 & & & & & \searrow m & \\
 A^4 & \xrightarrow{mm} & A^2 & \xrightarrow{m} & A
 \end{array}
 \quad (4.12)$$

Finally,  $\varphi$  is a morphism of bialgebras since it is also a morphism of coalgebras:

$$\begin{array}{ccccc}
 XB & \xrightarrow{\kappa e} & A^2 & \xrightarrow{m} & A \\
 \downarrow \Delta \Delta & & \downarrow \Delta \Delta & & \downarrow \Delta \\
 XXBB & \xrightarrow{\kappa^2 e^2} & A^4 & & \Delta \\
 \downarrow 1_X \sigma 1_B & & \downarrow 1_A \sigma 1_A & & \downarrow \\
 XBXB & \xrightarrow{(\kappa e)^2} & A^4 & \xrightarrow{mm} & A^2
 \end{array}
 \quad (1.7)$$

The above diagram commutes since  $e$  and  $\kappa$  are coalgebra morphisms.  $\square$

From this Lemma and Lemma 4.2.3, it is straightforward that  $\psi$  is also a morphism of bialgebras, and it brings us to the following useful proposition.

**Proposition 4.2.6.** *The diagram (4.11)*

$$\begin{array}{ccccc}
 X & \xleftarrow{\lambda} & A & \xleftarrow{e} & B \\
 \parallel & \searrow \kappa & \uparrow \mathbb{R}\xi & \searrow \alpha & \parallel \\
 X & \xleftarrow{\pi_1} & X \rtimes B & \xleftarrow{i_2} & B \\
 & \searrow i_1 & & \searrow \pi_2 & \\
 & & & & 
 \end{array}$$

in the category of bialgebras in  $\mathcal{C}$  commutes. Accordingly,  $(1_B, 1_X, \psi)$  and  $(1_B, 1_X, \varphi)$  are isomorphisms of split extensions of bialgebras.

*Proof.* Thanks to Proposition 4.1.8 and Lemma 4.2.3, it is enough to check that the two following diagrams of (4.11) commute to conclude that  $\varphi$  and  $\psi$  are morphisms of split extensions of bialgebras.

$$\begin{array}{ccc}
 B & \xrightarrow{u_X 1_B} & XB \\
 \parallel & \searrow u_A e & \downarrow \kappa e \\
 & & A^2 \\
 & & \downarrow m \\
 B & \xrightarrow{e} & A
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{1_X u_B} & XB \\
 \parallel & \searrow \kappa u_A & \downarrow \kappa e \\
 & & A^2 \\
 & & \downarrow m \\
 X & \xrightarrow{\kappa} & A
 \end{array}$$

□

**Lemma 4.2.7.** *Let  $(g, v, p)$  be a morphism of split extensions as in Definition 4.1.7, then*

$$v \cdot \triangleright = \triangleright \cdot (g \otimes v),$$

where the actions are induced by the split extensions.

*Proof.* This follows from the fact that  $(g, v, p)$  is a morphism of split extensions and  $p$  is a morphism of bialgebras, as we can see in the following diagram

$$\begin{array}{ccccccc}
 & & \triangleright & & & & \\
 & \nearrow^{e\kappa} & & \xrightarrow{m} & \xrightarrow{\lambda} & & \\
 BX & \xrightarrow{e\kappa} & A^2 & \xrightarrow{m} & A & \xrightarrow{\lambda} & X \\
 \downarrow gv & & \downarrow pp & & \downarrow p & & \downarrow v \\
 B'X' & \xrightarrow{e'\kappa'} & A'^2 & \xrightarrow{m'} & A' & \xrightarrow{\lambda'} & X' \\
 & \searrow & & & & \searrow & \\
 & & \triangleright & & & & 
 \end{array}$$

□

**Lemma 4.2.8.** *Let  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  and  $X' \xleftarrow[\kappa']{\lambda'} A' \xleftarrow[\alpha']{e'} B'$  be two split extensions of bialgebras, and  $g: B \rightarrow B'$  and  $v: X \rightarrow X'$  two morphisms of bialgebras. Then the following conditions are equivalent:*

- 1) *there exists  $p: A \rightarrow A'$  such that  $(g, v, p)$  is a morphism of split extensions;*
- 2) *there exists a unique  $p: A \rightarrow A'$  such that  $(g, v, p)$  is a morphism of split extensions;*
- 3)  $v \cdot \triangleright = \triangleright \cdot (g \otimes v).$

*Proof.* Thanks to Proposition 4.1.6 and Lemma 4.2.7, we just need to check that 3)  $\Rightarrow$  1). Let us define  $\tilde{p}: X \rtimes B \rightarrow X' \rtimes B'$  as

$$\tilde{p} = v \otimes g.$$

It is clear that this morphism is a morphism of coalgebras. Moreover,  $\tilde{p}$  is a morphism of algebras as we can see in the following diagram

$$\begin{array}{ccccccc}
 & & m_{X \rtimes B} & & & & \\
 & \swarrow & & \searrow & & & \\
 (XB)^2 & \xrightarrow{1_X \Delta 1_{XB}} & XB^2 XB & \xrightarrow{1_{XB} \sigma 1_B} & (XB)^2 B & \xrightarrow{1_X \triangleright 1_{B^2}} & X^2 B^2 \xrightarrow{mm} XB \\
 \downarrow (vg)^2 & & \downarrow vg^2 vg & & \downarrow (vg)^2 g & & \downarrow v^2 g^2 \\
 (X'B')^2 & \xrightarrow{1_{X'} \Delta 1_{X'B'}} & X'B'^2 X'B' & \xrightarrow{1_{X'B'} \sigma 1_{B'}} & (X'B')^2 B' & \xrightarrow{1_{X'} \triangleright 1_{B'^2}} & X'^2 B'^2 \xrightarrow{mm} X'B' \\
 & \swarrow & & \searrow & & & \\
 & & m_{X' \rtimes B'} & & & & 
 \end{array}$$

Then we can define a morphism  $p: A \rightarrow A'$  as  $\varphi_{A'} \cdot \tilde{p} \cdot \psi_A$  where  $\varphi$  and  $\psi$  are the isomorphisms in Proposition 4.2.6. In particular, that gives us

$$p = m \cdot (\kappa' \otimes e') \cdot (v \otimes g) \cdot (\lambda \otimes \alpha) \cdot \Delta.$$

Finally,  $(g, v, p)$  is a morphism of split extensions of bialgebras,

$$\begin{array}{ccc}
 \begin{array}{c}
 X \xrightarrow{\kappa} A \\
 \searrow \Delta \downarrow \\
 X^2 \xrightarrow{\kappa\kappa} A^2 \\
 \downarrow 1_X(u_B \cdot \epsilon_X) \downarrow \lambda\alpha \\
 XB \xrightarrow{vg} X'B' \\
 \downarrow \kappa'e' \\
 A'^2 \xrightarrow{m} A' \\
 \downarrow \kappa' \\
 X' \xrightarrow{\kappa'} A'
 \end{array}
 &
 \begin{array}{c}
 B \xrightarrow{e} A \\
 \searrow \Delta \downarrow \\
 B^2 \xrightarrow{ee} A^2 \\
 \downarrow (u_X \cdot \epsilon_B)1_B \downarrow \lambda\alpha \\
 XB \xrightarrow{vg} X'B' \\
 \downarrow \kappa'e' \\
 A'^2 \xrightarrow{m} A' \\
 \downarrow e' \\
 B' \xrightarrow{e'} A'
 \end{array}
 &
 \begin{array}{c}
 p \\
 p \\
 p
 \end{array}
 \end{array}$$

Thanks to Corollary 4.1.8 the commutativity of these two diagrams is enough to conclude that diagram (4.8) commutes.  $\square$

**Definition 4.2.9.** Let  $\triangleright: B \otimes X \rightarrow X$  and  $\triangleright': B' \otimes X' \rightarrow X'$  be two actions of bialgebras. A morphism between them is defined as a pair of

morphisms of bialgebras  $g: B \rightarrow B'$  and  $v: X \rightarrow X'$  such that

$$v \cdot \triangleright = \triangleright' \cdot (g \otimes v).$$

The split extensions of bialgebras (Definition 4.1.3) endowed with the morphisms of split extensions of bialgebras (Definition 4.1.7) form the category of split extensions of bialgebras in the symmetric monoidal category  $\mathcal{C}$  denoted by  $\text{SplitExt}(\text{BiAlg}_{\mathcal{C}})$ . The actions of bialgebras (Definition 4.1.1) with the morphisms of actions (Definition 4.2.9) form the category of actions of bialgebras in  $\mathcal{C}$ , denoted by  $\text{Act}(\text{BiAlg}_{\mathcal{C}})$ .

**Theorem 4.2.10.** [87] *The category  $\text{SplitExt}(\text{BiAlg}_{\mathcal{C}})$  of split extensions of bialgebras in  $\mathcal{C}$  and the category  $\text{Act}(\text{BiAlg}_{\mathcal{C}})$  of actions of bialgebras in  $\mathcal{C}$  are equivalent.*

*Proof.* The functor  $F: \text{SplitExt}(\text{BiAlg}_{\mathcal{C}}) \rightarrow \text{Act}(\text{BiAlg}_{\mathcal{C}})$  is defined as

$$F \left( \begin{array}{ccccc} X & \xleftarrow[\kappa]{\lambda} & A & \xleftarrow[\alpha]{e} & B \\ v \downarrow & & p \downarrow & & g \downarrow \\ X' & \xleftarrow[\kappa']{\lambda'} & A' & \xleftarrow[\alpha']{e'} & B' \end{array} \right) = \begin{array}{ccc} B \otimes X & \xrightarrow{\triangleright} & X \\ g \otimes v \downarrow & & v \downarrow \\ B' \otimes X' & \xrightarrow{\triangleright'} & X' \end{array}$$

where  $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$  as in Proposition 4.2.2, and we have a morphism of actions thanks to Lemma 4.2.7. The functor  $G: \text{Act}(\text{BiAlg}_{\mathcal{C}}) \rightarrow \text{SplitExt}(\text{BiAlg}_{\mathcal{C}})$  is defined as

$$G \left( \begin{array}{ccc} B \otimes X & \xrightarrow{\triangleright} & X \\ g \otimes v \downarrow & & v \downarrow \\ B' \otimes X' & \xrightarrow{\triangleright'} & X' \end{array} \right) = \begin{array}{ccccc} X & \xleftarrow[\pi_1]{i_1} & X \rtimes B & \xleftarrow[\pi_2]{i_2} & B \\ v \downarrow & & p \downarrow & & g \downarrow \\ X' & \xleftarrow[\pi'_1]{i'_1} & X' \rtimes B' & \xleftarrow[\pi'_2]{i'_2} & B' \end{array}$$

where  $p := v \otimes g$  is given by Lemma 4.2.8 and the bialgebra structures of the semi-direct products  $X \rtimes B$  and  $X' \rtimes B'$  are defined as in (4.6).

We observe that

$$F \cdot G(\triangleright) = \pi_1 \cdot m_{X \rtimes B} \cdot (i_2 \otimes i_1) = \triangleright, \quad (4.13)$$

where the last equality holds thanks to the commutativity of the following diagram

$$\begin{array}{ccccccc}
 BX & \xrightarrow{u_X 1_B 1_X u_B} & (XB)^2 & \xrightarrow{1_X \Delta 1_{XB}} & XB^2 XB & \xrightarrow{1_{XB} \sigma 1_B} & XBXB^2 \\
 \parallel & & & & & & \downarrow 1_X \triangleright 1_{B^2} \\
 BX & \xrightarrow{\Delta 1_X} & B^2 X & \xrightarrow{1_B \sigma} & BXB & \xrightarrow{\triangleright 1_B} & XB \xrightarrow{u_X 1_X 1_B u_B} X^2 B^2 \\
 \parallel & & & & & & \downarrow m^2 \\
 & & & & & & XB \\
 & & & & & & \downarrow 1_X \epsilon_B \\
 BX & \xrightarrow{\quad \triangleright \quad} & & & & & X.
 \end{array}$$

(1.5)

(1.3)

Thanks to this observation and the isomorphisms  $\varphi$  and  $\psi$  of (4.11), the functors  $F$  and  $G$  give rise to an equivalence of categories.  $\square$

**Remark 4.2.11.** *Note that the idea to work in the “non-associative” setting follow from the paper [47]. We also remark that we directly obtain dual results for non-coassociative bialgebras, by considering non-associative bialgebras in the symmetric monoidal category  $\mathcal{C}^{op}$ , the dual of  $\mathcal{C}$ . The notion of action becomes coaction which satisfies the dual conditions of Definition 4.1.1. In particular, (4.2) becomes the compatibility condition of a Yetter-Drinfeld module with a trivial action. These coactions are equivalent to the split extensions of bialgebras in  $\mathcal{C}^{op}$ . In particular, these extensions are split monomorphisms such that the cokernel is split in the category of algebras in  $\mathcal{C}$  (the category of coalgebras in  $\mathcal{C}^{op}$ ).*

To end this section, we prove that a variation of the Split Short Five Lemma holds in  $\text{BiAlg}_{\mathcal{C}}$ .

**Theorem 4.2.12.** *Let  $(g, v, p)$  be a morphism of split extensions of bialgebras*

$$\begin{array}{ccccc}
 X & \xleftarrow{\lambda} & A & \xleftarrow{e} & B \\
 \downarrow v & & \downarrow p & & \downarrow g \\
 X' & \xleftarrow{\lambda'} & A' & \xleftarrow{e'} & B'
 \end{array}
 \quad \begin{array}{c}
 \xrightarrow{\kappa} \\
 \xrightarrow{\alpha} \\
 \xrightarrow{\kappa'} \\
 \xrightarrow{\alpha'}
 \end{array}
 \quad (4.14)$$

then  $p$  is an isomorphism whenever  $v$  and  $g$  are.

*Proof.* Thanks to Theorem 4.2.10, the diagram (4.14) is canonically isomorphic to

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_1} & X \rtimes B & \xleftarrow{i_2} & B \\
 \downarrow v & & \downarrow v \otimes g & & \downarrow g \\
 X' & \xleftarrow{\pi'_1} & X' \rtimes B' & \xleftarrow{i'_2} & B'
 \end{array}
 \quad \begin{array}{c}
 \xrightarrow{i_1} \\
 \xrightarrow{\pi_2} \\
 \xrightarrow{i'_1} \\
 \xrightarrow{\pi'_2}
 \end{array}$$

It follows that  $v \otimes g$  is an isomorphism whenever  $v$  and  $g$  are. □

### 4.3 The cocommutative and associative case

In this section, we give two explicit restrictions of our results: the cases of cocommutative bialgebras and of associative bialgebras.

#### 4.3.1 Cocommutative bialgebras in a symmetric monoidal category

If we consider cocommutative bialgebras, then the category  $\mathbf{BiAlg}_{\mathcal{C}, \text{coc}}$  of cocommutative bialgebras in  $\mathcal{C}$  can be seen as the category of internal magmas in the category of cocommutative coalgebras. Indeed, the categorical product of cocommutative coalgebras is given by the tensor product.

In this particular case, the condition (4.2) in Definition 4.1.1 becomes trivial and the definition can be reformulated explicitly as:

**Definition 4.3.1.** *Let  $X$  and  $B$  be cocommutative bialgebras. An action of  $B$  on  $X$  is a morphism of coalgebras  $\triangleright: B \otimes X \rightarrow X$  such that*

$$\triangleright \cdot (u_B \otimes 1_X) = 1_X,$$

$$\triangleright \cdot (1_B \otimes u_X) = \epsilon_B \cdot u_X.$$

Similarly, thanks to the cocommutativity we can drop the condition (5) in the definition of split extensions of bialgebras (Definition 4.1.3), and this turns out to be exactly the internal version, in the category of coalgebras, of Definition 1.4 in [47]. Then, in the case of cocommutative bialgebras, the above theorem reduces to the results in Section 4.6 in [47]. Indeed, if  $\mathcal{C}$  is a category with finite limits, it is, in particular, a cartesian monoidal category. Hence, any magma in such a category can be seen as a non-associative cocommutative bialgebra in this category (where the unique comultiplication is the diagonal). Accordingly, the results in [47] become a particular case of our theorem.

### 4.3.2 Associative bialgebras in a symmetric monoidal category

In this subsection, we restrict our results to the associative bialgebras in a symmetric monoidal category.

First, it is interesting to note that the multiplication  $m_{X \rtimes B}: (X \otimes B) \otimes (X \otimes B) \rightarrow X \otimes B$ , defined in (4.6), is not associative in general. Even if  $X$  and  $B$  are associative bialgebras this structure is not associative. Let us illustrate this observation with a simple example. Let  $\mathcal{C}$  be the category of sets, and we consider the monoids  $(\mathbb{N}, +, 0)$  and  $(\mathbb{N}_0, \cdot, 1)$ . In particular we can construct an action of  $(\mathbb{N}_0, \cdot, 1)$  on  $(\mathbb{N}, +, 0)$  via the function  $\triangleright: \mathbb{N}_0 \times \mathbb{N} \rightarrow \mathbb{N}$  sending  $(b, x)$  to  $x$  to the power of  $b$ ,  $x^b$ . We can observe that this function satisfies the conditions (4.1) and (4.3) of Definition 4.1.1 (the other ones being trivial in  $\mathcal{C} = \mathbf{Set}$ ):  $x^1 = x$ ,  $0^b = 0$  for any  $x \in \mathbb{N}, b \in \mathbb{N}_0$ . By defining, via this action, the bialgebra structure on  $\mathbb{N} \rtimes \mathbb{N}_0$  as  $m((x, b), (y, c)) = (x + y^b, b \cdot c)$ , this multiplication is non-associative:

$$m(m((0, 2), (1, 1)), (1, 1)) = m((1, 2), (1, 1)) = (1 + 1^2, 2 \cdot 1) = (2, 2)$$

$$m((0, 2), m((1, 1), (1, 1))) = m((0, 2), (2, 1)) = (0 + 2^2, 2.1) = (4, 2).$$

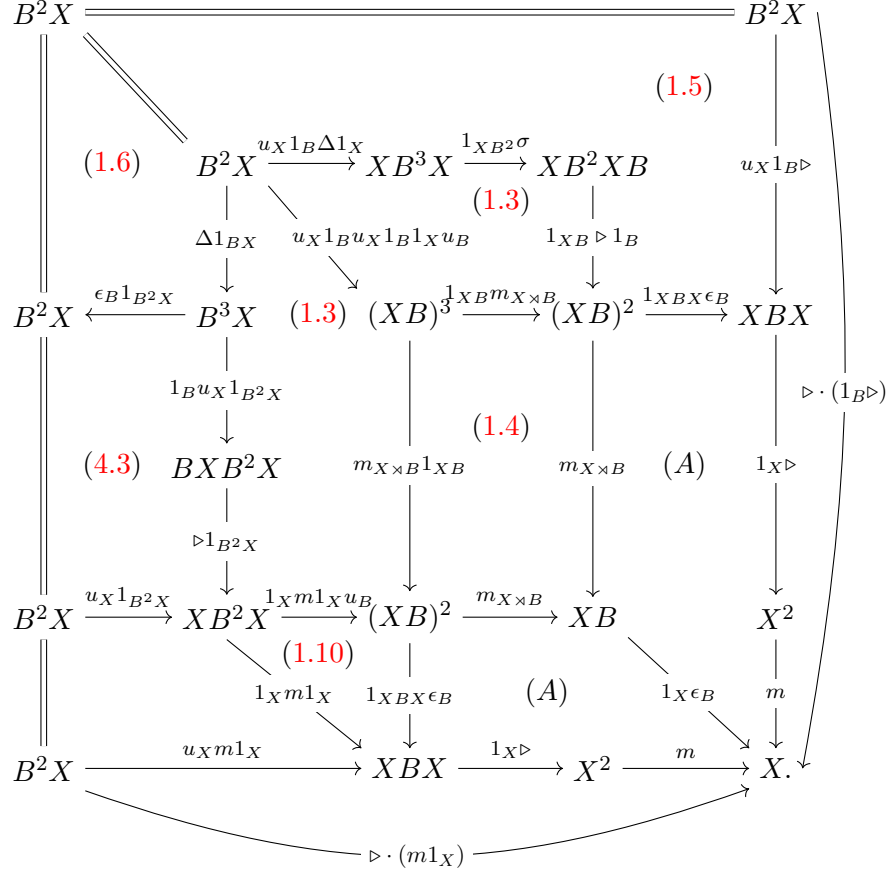
We need additional conditions on the actions of two associative bialgebras  $B$  and  $X$  to obtain an associative bialgebra structure on  $X \rtimes B$ . In particular, we have the following Lemma.

**Lemma 4.3.2.** *The multiplication  $m_{X \rtimes B} : (X \otimes B) \otimes (X \otimes B) \rightarrow X \otimes B$ , defined in (4.6), is not associative, in general. If  $X$  and  $B$  are associative, then  $m_{X \rtimes B}$  is associative as well if and only if*

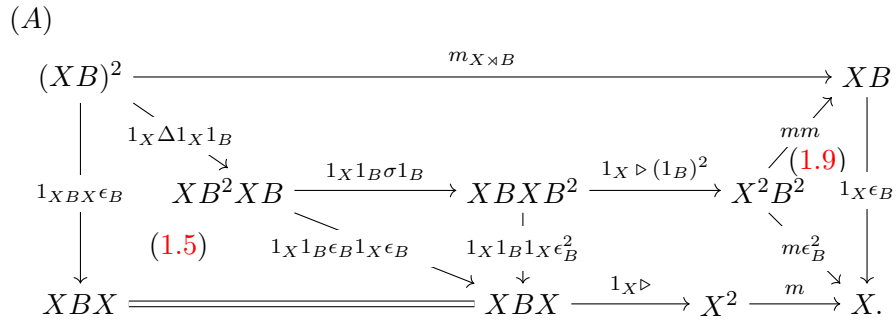
$$\triangleright \cdot (m \otimes 1_X) = \triangleright \cdot (1_B \otimes \triangleright), \quad (4.15)$$

$$\triangleright \cdot (1_B \otimes m) = m \cdot (\triangleright \otimes \triangleright) \cdot (1_B \otimes \sigma \otimes 1_X) \cdot (\Delta \otimes 1_X \otimes 1_X). \quad (4.16)$$

*Proof.* Via the following diagram, we show that if  $m_{X \rtimes B}$  is associative then (4.15) is immediately satisfied. With similar computations, we can show that associativity also gives condition (4.16). The other implication is given by the commutativity of the diagram in Figure 4.4 (it is a direct computation using the conditions (4.15) and (4.16)).



The trapezes (A) commute thanks to (1.5) and (1.9), as is shown in the following diagram



□

Figure 4.4: The semi-direct product is associative

[illegible]

We note that if  $X$  and  $B$  are associative bialgebras, then an action of  $B$  on  $X$  satisfying (4.15) and (4.16) is a bialgebra in  ${}_B\overline{\mathbf{Mod}}$  (see Proposition 1.3.6).

Let us consider associative bialgebras in  $\mathcal{C}$ . We define two categories:  $\mathbf{Act}(\mathbf{AssBiAlg}_{\mathcal{C}})$  and  $\mathbf{SplitExt}(\mathbf{AssBiAlg}_{\mathcal{C}})$ . An object in  $\mathbf{Act}(\mathbf{AssBiAlg}_{\mathcal{C}})$ , is an action of associative bialgebras (Definition 4.1.1) satisfying (4.15) and (4.16), the morphisms are the morphisms of  $\mathbf{Act}(\mathbf{BiAlg}_{\mathcal{C}})$ . The category  $\mathbf{SplitExt}(\mathbf{AssBiAlg}_{\mathcal{C}})$  is a full subcategory of  $\mathbf{SplitExt}(\mathbf{BiAlg}_{\mathcal{C}})$  since the conditions (6), (7) and (8) become redundant.

The definition of a split extension of associative bialgebras generalizes the concept of “Schreier split epimorphism” introduced and developed in [72]. In particular, let us describe the associative version of Definition 4.1.3 in the symmetric monoidal category of sets. A split extension of monoids is given by a diagram

$$X \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\kappa} \end{array} A \begin{array}{c} \xleftarrow{e} \\ \xrightarrow{\alpha} \end{array} B, \quad (4.17)$$

where  $X, A, B$  are monoids (the multiplications are expressed by juxtaposition and the unit is denoted by 1),  $\kappa, \alpha, e$  are morphisms of monoids,  $\lambda$  is a function such that for any  $a \in A, b \in B, x \in X$ , the non-trivial conditions are

- (1)  $\alpha \cdot e(b) = b$
- (2)  $\alpha \cdot \kappa(x) = 1$
- (3)  $(\kappa \cdot \lambda)(a)(e \cdot \alpha)(a) = a$
- (4)  $\lambda(\kappa(x)e(b)) = x$

This is the definition of a Scheier split epimorphism introduced in [20].

**Corollary 4.3.3.** [87] *There is an equivalence of categories between the category  $\mathbf{SplitExt}(\mathbf{AssBiAlg}_{\mathcal{C}})$  of split extensions of associative bialgebras in  $\mathcal{C}$  and the category  $\mathbf{Act}(\mathbf{AssBiAlg}_{\mathcal{C}})$  of actions of associative bialgebras in  $\mathcal{C}$ .*

*Proof.* It is clear by applying Lemma 4.3.2 and Theorem 4.2.10. □

To conclude, in this chapter, we defined split extensions of bialgebras in a symmetric monoidal category. These split extensions form exact sequences, satisfy the Split Short Five Lemma and are equivalent to actions. Moreover, the kernel and the splitting,  $\kappa$  and  $e$ , of a split extension  $X \overset{\lambda}{\underset{\kappa}{\dashrightarrow}} A \overset{e}{\underset{\alpha}{\dashrightarrow}} B$  are jointly epimorphic. Our results can be restricted to the cocommutative and associative bialgebras and they generalize the ones obtained for monoids [20] and for magmas [47], respectively.



## Chapter 5

# Split extensions and actions of general Hopf algebras

In this Chapter, we work with non-associative Hopf algebras. Similarly to the previous chapter, we give a definition of a split extension of Hopf algebras. We prove an equivalence between the category of split extensions of non-associative Hopf algebras and the category of actions of non-associative Hopf algebras. The definition of a split extension of non-associative Hopf algebras has the expected properties and the associative case is obtained as a corollary. We also give some examples of this notion of split extensions of Hopf algebras, and we compare our definition of split extensions with the definition of an exact cleft sequence given by N. Andruskiewitsch and J. Devoto [5] and with a definition given by Böhm [7].

For the sake of simplicity, in this section “Hopf algebra” will mean “non-associative Hopf algebra” (unless the associativity is explicitly mentioned).

## 5.1 Definition and equivalence of split extensions and actions of Hopf algebras

**Definition 5.1.1.** [87] *A split extension of Hopf algebras is a split extension of bialgebras*

$$X \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\kappa} \end{array} A \begin{array}{c} \xleftarrow{e} \\ \xrightarrow{\alpha} \end{array} B, \quad (5.1)$$

such that  $X, A, B$  are Hopf algebras and  $\kappa, \alpha, e$  are morphisms of Hopf algebras, with an additional condition of associativity (condition (9')) and an additional condition about the antipode (conditions (10')). More precisely, the split extension (5.1) satisfies

$$(1') \quad \lambda \cdot \kappa = 1_X, \quad \alpha \cdot e = 1_B$$

$$(2') \quad \lambda \cdot e = u_X \cdot \epsilon_B, \quad \alpha \cdot \kappa = u_B \cdot \epsilon_X$$

$$(3') \quad m \cdot ((\kappa \cdot \lambda) \otimes (e \cdot \alpha)) \cdot \Delta = 1_A$$

$$(4') \quad \lambda \cdot m \cdot (\kappa \otimes e) = 1_X \otimes \epsilon_B$$

$$(5') \quad (1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\Delta \otimes 1_X) = (1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\sigma \otimes 1_X) \cdot (\Delta \otimes 1_X)$$

$$(6') \quad m \cdot (m \otimes 1_A) \cdot (\kappa \otimes e \otimes 1_A) = m \cdot (1_A \otimes m) \cdot (\kappa \otimes e \otimes 1_A)$$

$$(7') \quad m \cdot (m \otimes 1_A) \cdot (\kappa \otimes 1_A \otimes e) = m \cdot (1_A \otimes m) \cdot (\kappa \otimes 1_A \otimes e)$$

$$(8') \quad m \cdot (m \otimes 1_A) \cdot (1_A \otimes \kappa \otimes e) = m \cdot (1_A \otimes m) \cdot (1_A \otimes \kappa \otimes e)$$

$$(9') \quad m \cdot (m \otimes 1_A) \cdot (e \otimes 1_A \otimes \kappa) = m \cdot (1_A \otimes m) \cdot (e \otimes 1_A \otimes \kappa)$$

$$(10') \quad S \cdot \lambda \cdot m \cdot (e \otimes \kappa) = \lambda \cdot m \cdot (e \otimes \kappa) \cdot (1_B \otimes S),$$

$$(11') \quad \lambda \text{ is a morphism of coalgebras preserving the unit.}$$

Similarly to Proposition 4.1.9, a split extension of Hopf algebras is a short exact sequence of Hopf algebras.

**Proposition 5.1.2.** *Given a split extension  $X \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\kappa} \end{array} A \begin{array}{c} \xleftarrow{e} \\ \xrightarrow{\alpha} \end{array} B$  of Hopf algebras, the following properties hold:*

- a)  $\kappa$  is the kernel of  $\alpha$  in the category of Hopf algebras;
- b)  $\alpha$  is the cokernel of  $\kappa$  in the category of Hopf algebras.

Hence, a split extension of Hopf algebras is a short exact sequence, so the term extension is appropriate.

*Proof.* Let  $\omega : D \rightarrow A$  be a morphism of Hopf algebras such that  $\alpha \cdot \omega = u_B \cdot \epsilon_D$ . In Proposition 4.1.9, we checked that the morphism  $\hat{\omega} := \lambda \cdot \omega : D \rightarrow X$  is a morphism of bialgebras and that  $\kappa \cdot \hat{\omega} = \omega$ . Moreover, it preserves the antipode since we have the following equalities

$$S \cdot \tilde{\omega} = \lambda \cdot \kappa \cdot S \cdot \tilde{\omega} = \lambda \cdot S \cdot \kappa \cdot \tilde{\omega} = \lambda \cdot S \cdot \omega = \lambda \cdot \omega \cdot S = \tilde{\omega} \cdot S.$$

Hence, we can conclude that a) and b) hold thanks to Proposition 4.1.9.  $\square$

We define an action of Hopf algebras as a Hopf algebra in the symmetric monoidal category  ${}_B\overline{\mathbf{Mod}}$  (see Proposition 1.3.6).

**Definition 5.1.3.** Let  $X$  and  $B$  be Hopf algebras,  $\triangleright : B \otimes X \rightarrow X$  is an action of Hopf algebras if it is an action of bialgebras such that the following additional conditions are satisfied

$$\triangleright \cdot (1_B \otimes \triangleright) = \triangleright \cdot (m \otimes 1_X), \quad (5.2)$$

$$\triangleright \cdot (1_B \otimes m) = m \cdot (\triangleright \otimes \triangleright) \cdot (1_B \otimes \sigma \otimes 1_X) \cdot (\Delta \otimes 1_X \otimes 1_X), \quad (5.3)$$

$$\triangleright \cdot (1_B \otimes S) = S \cdot \triangleright, \quad (5.4)$$

These conditions can be expressed by the commutativity of the following diagrams

$$\begin{array}{ccccc}
 B \otimes B \otimes X & \xrightarrow{1_B \otimes \triangleright} & B \otimes X & & \\
 \downarrow m \otimes 1_X & (5.2) & \downarrow \triangleright & & \\
 B \otimes X & \xrightarrow{\triangleright} & X & & \\
 \\
 B \otimes X \otimes X & \xrightarrow{\Delta \otimes 1_X \otimes 1_X} & B^2 \otimes X^2 & \xrightarrow{1_B \otimes \sigma \otimes 1_X} & (B \otimes X)^2 & \xrightarrow{\triangleright \otimes \triangleright} & X^2 \\
 \downarrow 1_B \otimes m & (5.3) & & & \downarrow m & & \\
 B \otimes X & \xrightarrow{\triangleright} & & & X & & \\
 \\
 B \otimes X & \xrightarrow{\triangleright} & X & & \\
 \downarrow 1_B \otimes S & (5.4) & \downarrow S & & \\
 B \otimes X & \xrightarrow{\triangleright} & X & & 
 \end{array}$$

Note that the condition (5.3) means that  $m$  is a morphism of  $B$ -modules and the condition (5.4) means that  $S$  is a morphism of  $B$ -modules.

We notice that, when we consider associative Hopf algebras, condition (5.4) is trivially satisfied thanks to the uniqueness of the antipode.

We define the map  $\Theta: B \otimes X \rightarrow X \otimes B$  as the composition

$$\Theta := (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X).$$

We will use this map to obtain shorter computations. We re-formulate the conditions (5.3), (4.1), (5.2), and (4.3) in terms of  $\Theta$ . These new conditions will help us to prove that the semi-direct product is a Hopf algebra when we construct it with an action as defined above (Definition 5.1.3).

**Lemma 5.1.4.** *Let  $\triangleright: B \otimes X \rightarrow X$  be an action of Hopf algebras, the morphism  $\Theta := (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X)$  is a coalgebra morphism and satisfies the following conditions*

$$(m \otimes 1_B) \cdot (1_X \otimes \Theta) \cdot (\Theta \otimes 1_X) = \Theta \cdot (1_B \otimes m), \quad (5.5)$$

$$\Theta \cdot (u_B \otimes 1_X) = 1_X \otimes u_B, \quad (5.6)$$

$$(1_X \otimes m) \cdot (\Theta \otimes 1_B) \cdot (1_B \otimes \Theta) = \Theta \cdot (m \otimes 1_X), \quad (5.7)$$

$$\Theta \cdot (1_B \otimes u_X) = u_X \otimes 1_B, \quad (5.8)$$

$$\begin{array}{ccc}
 B \otimes X \otimes X & \xrightarrow{\Theta \otimes 1_X} & X \otimes B \otimes X \xrightarrow{1_X \otimes \Theta} X \otimes X \otimes B \\
 \downarrow 1_B \otimes m & & \downarrow m \otimes 1_B \\
 B \otimes X & \xrightarrow{\Theta} & X \otimes B \\
 & & \begin{array}{ccc}
 X & \xrightarrow{u_B \otimes 1_X} & B \otimes X \\
 \searrow 1_X \otimes u_B & & \downarrow \Theta \\
 & & X \otimes B
 \end{array}
 \end{array}$$
  

$$\begin{array}{ccc}
 B \otimes B \otimes X & \xrightarrow{1_B \otimes \Theta} B \otimes X \otimes B \xrightarrow{\Theta \otimes 1_B} X \otimes B \otimes B \\
 \downarrow m \otimes 1_X & & \downarrow 1_X \otimes m \\
 B \otimes X & \xrightarrow{\Theta} & X \otimes B
 \end{array}
 \qquad
 \begin{array}{ccc}
 B & \xrightarrow{1_B \otimes u_X} & B \otimes X \\
 \searrow u_X \otimes 1_B & & \downarrow \Theta \\
 & & X \otimes B.
 \end{array}$$

*Proof.* First, we prove that this morphism is a morphism of coalgebras in Figure 5.1

We only show the two first equalities since the computations are similar. We prove (5.5) via the following diagram, where the key part is given by (5.3).

(1.6)

$$(5.3)$$

$$(5.6)$$

(1.8)

$$(4.1)$$



[illegible]

Starting from an action of Hopf algebras, we can define a split extension of Hopf algebras

$$X \xrightleftharpoons[i_1]{\pi_1} X \rtimes B \xrightleftharpoons[\pi_2]{i_2} B \quad (5.9)$$

where the structure of  $X \rtimes B$  is given by

$$\begin{aligned} m_{X \rtimes B} &= (m \otimes m) \cdot (1_X \otimes \Theta \otimes 1_B) \\ u_{X \rtimes B} &= u_X \otimes u_B, \\ \Delta_{X \rtimes B} &= (1_X \otimes \sigma \otimes 1_B) \cdot (\Delta \otimes \Delta), \\ \epsilon_{X \rtimes B} &= \epsilon_X \otimes \epsilon_B, \\ S_{X \rtimes B} &= \Theta \cdot (S \otimes S) \cdot \sigma. \end{aligned}$$

Thanks to Lemma 4.1.2, we already know that (5.9) is a split extension of bialgebras. It is also easy to check that  $i_1$ ,  $i_2$ , and  $\pi_2$  are morphisms of Hopf algebras as it is shown in the following diagrams,

$$\begin{array}{ccc} B \xrightarrow{u_X 1_B} XB \xrightarrow{\sigma} BX & & X \xrightarrow{1_X u_B} XB \xrightarrow{\sigma} BX \\ \downarrow S & \downarrow SS & \downarrow S \\ B \xrightarrow{1_B u_X} BX & & X \xrightarrow{u_B 1_X} BX \\ \parallel & \text{(5.8)} & \parallel \\ B \xrightarrow{u_X 1_B} XB & & X \xrightarrow{1_X u_B} XB \end{array}$$
  

$$\begin{array}{ccccc} XB & \xrightarrow{\sigma} & BX & \xrightarrow{SS} & BX \\ \downarrow \epsilon_X 1_B & & & \searrow & \downarrow \Delta 1_X \\ B & & & & B^2 X \\ \downarrow S & \swarrow 1_B \epsilon_X & & & \downarrow 1_B \sigma \\ B & \xleftarrow{\epsilon_B \epsilon_X 1_B} & BXB & & \\ \parallel & \text{(4.4)} & \downarrow \triangleright 1_B & & \\ B & \xleftarrow{\epsilon_X 1_B} & XB. & & \end{array}$$

Furthermore, since  $\Theta$  is a morphism of coalgebras, we can show that  $S_{X \rtimes B}$  is an anti-homomorphism of coalgebras as we can see in the following diagram

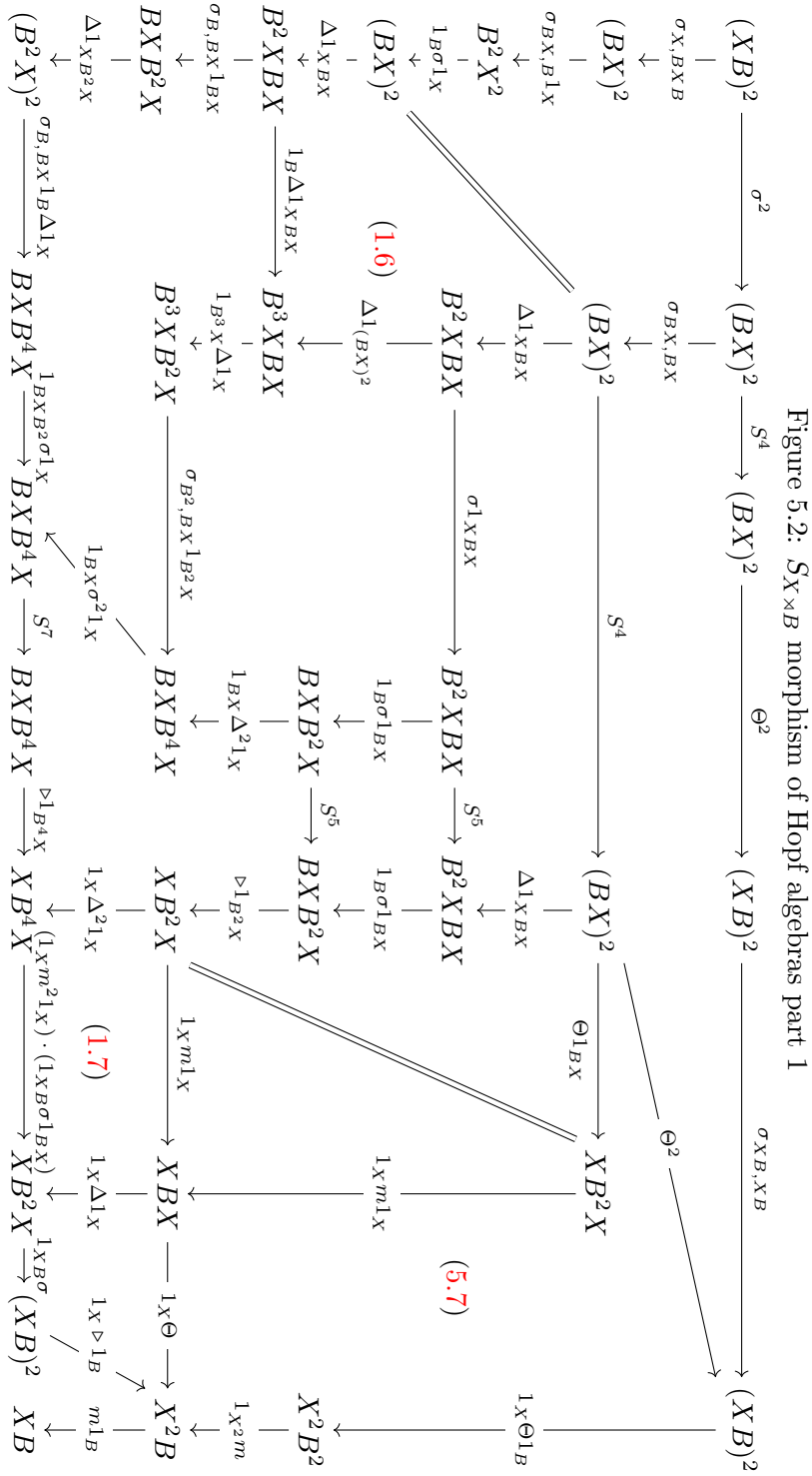
(1.2)

Moreover, we check that the construction (5.9) satisfies the antipode condition (1.11) thanks to the following two diagrams

(1.11)

(5.6)

(5.7)



$$\begin{array}{c}
\begin{array}{c}
(XB) \xrightarrow{(1_X \triangleright 1_B)} XB \\
\downarrow \scriptstyle (1_X \sigma_B, BX \cdot 1_B) \cdot (1_X \Delta 1_{XB}) \\
XB \xrightarrow{(4.2)} XB \\
\downarrow \scriptstyle (1_X \sigma_B, BX \cdot 1_B) \cdot (1_X \Delta 1_{XB}) \\
XB \xrightarrow{(5.4)} XB \\
\downarrow \scriptstyle (1_X \sigma_B, BX \cdot 1_B) \cdot (1_X \Delta 1_{XB}) \\
XB \xrightarrow{(5.2)} XB \\
\downarrow \scriptstyle (1_X \sigma_B, BX \cdot 1_B) \cdot (1_X \Delta 1_{XB}) \\
XB \xrightarrow{(5.3)} XB
\end{array}
\end{array}$$



[illegible]



$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & \Delta_{X \rtimes B} & & S_{X \rtimes B} 1_{XB} & & \\
 & \nearrow & & \searrow & & \nearrow & \\
 XB & \xrightarrow{\Delta^2} & X^2 B^2 & \xrightarrow{1_X \sigma 1_B} & (XB)^2 & \xrightarrow{\sigma 1_{XB}} & BX^2 B \xrightarrow{S^2 1_{XB}} BX^2 B \xrightarrow{\Theta 1_{XB}} (XB)^2 \\
 & \parallel & \parallel & & & & \\
 & & X^2 B^2 & \xrightarrow{S 1_X S 1_B} & X^2 B^2 & \xrightarrow{m 1_{B^2}} & XB^2 \xrightarrow{\sigma 1_B} BXB \\
 & & (1.11) & & & & (5.8) \\
 & & & & 1_B u_X 1_B & \nearrow & \\
 & & & & u_X 1_{B^2} & \nearrow & \\
 XB & \xrightarrow{1_X \Delta} & XB^2 & \xrightarrow{\epsilon_X S 1_B} & B^2 & \xrightarrow{u_X 1_{B^2}} & XB^2 \xrightarrow{1_X m} XB. \\
 & & (1.11) & & & & \\
 & & & & (u_X u_B) \cdot (\epsilon_X \epsilon_B) & \searrow & 
 \end{array}
 \end{array}$$

Furthermore, the conditions (5.7) and (5.5) imply that the split extension

$$X \xleftarrow[\pi_1]{i_1} X \rtimes B \xleftarrow[\pi_2]{i_2} B \text{ satisfies the condition (9') of the Definition 5.1.1}$$

as it is shown in the following diagram

$$\begin{array}{c}
 \begin{array}{ccccccc}
 (BX)^2 & \xrightarrow{u_X 1_{BXBX} u_B} & (XB)^3 & \xrightarrow{1_{XB} \Theta 1_B} & XBX^2 B^2 & \xrightarrow{1_{XB} m^2} & (XB)^2 \\
 \parallel & & (1.3) & & & & \downarrow 1_X \Theta 1_B \\
 (BX)^2 & \xrightarrow{1_{BX} \Theta} & BX^2 B & \xrightarrow{1_B m 1_B} & BXB & \xrightarrow{\Theta 1_B} & XB^2 \\
 & & \searrow \Theta 1_{XB} & \nearrow m(1_B)^2 & & & \downarrow \\
 & & (XB)^2 & \xrightarrow{1_X \Theta 1_B} & X^2 B^2 & & \downarrow m^2 \\
 \parallel & & & & & & \downarrow \\
 (BX)^2 & \xrightarrow{\Theta^2} & (XB)^2 & \xrightarrow{1_X \Theta 1_B} & X^2 B^2 & \xrightarrow{m^2} & XB \\
 \parallel & & \nearrow 1_{XB} \Theta & \nearrow (1_X)^2 m & \nearrow m 1_B & \nearrow m^2 & \uparrow \\
 (BX)^2 & \xrightarrow{\Theta 1_{BX}} & XB^2 X & \xrightarrow{1_X m 1_X} & XBX & \xrightarrow{1_X \Theta} & X^2 B \\
 & & (1.3) & & & & \uparrow 1_X \Theta 1_B \\
 (BX)^2 & \xrightarrow{u_X 1_{BXBX} u_B} & (XB)^3 & \xrightarrow{1_X \Theta 1_{BXB}} & X^2 B^2 XB & \xrightarrow{m^2 1_{XB}} & (XB)^2.
 \end{array}
 \end{array}$$

Finally, the condition (10') holds thanks to (5.4) and (4.13), and we can conclude that (5.9) is a split extension of Hopf algebras as defined in Definition 5.1.1.

Conversely, if we have a split extension of Hopf algebras, we can define an action of Hopf algebras. Thanks to Proposition 4.2.1 and condition (9'), we can prove two identities which are crucial properties, for our purposes, of a split extension of Hopf algebras.

**Lemma 5.1.5.** *Let  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  be a split extension of Hopf algebras, we have*

$$\lambda \cdot m \cdot (e \otimes \kappa) \cdot (m \otimes 1_X) = \lambda \cdot m \cdot (e \otimes (\kappa \cdot \lambda)) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa), \quad (5.10)$$

$$\begin{aligned} \lambda \cdot m \cdot (e \otimes \kappa) \cdot (1_B \otimes m) &= m \cdot (\lambda \otimes \lambda) \cdot (m \otimes m) \cdot (e \otimes \kappa \otimes e \otimes \kappa) \\ &\quad \cdot (1_B \otimes \sigma \otimes 1_X) \cdot (\Delta \otimes 1_X \otimes 1_X), \end{aligned} \quad (5.11)$$

$$\begin{array}{ccccc} B^2 \otimes X & \xrightarrow{1_B \otimes (\lambda \cdot m \cdot (e \otimes \kappa))} & B \otimes X & & \\ \downarrow m \otimes 1_X & & \downarrow \lambda \cdot m \cdot (e \otimes \kappa) & & \\ B \otimes X & \xrightarrow{\lambda \cdot m \cdot (e \otimes \kappa)} & X & & \\ \downarrow 1_B \otimes m & & \downarrow m & & \\ B \otimes X & \xrightarrow{\lambda \cdot m \cdot (e \otimes \kappa)} & X & & \end{array}$$

$$\begin{array}{ccccc} B \otimes X \otimes X & \xrightarrow{\Delta \otimes 1_X \otimes 1_X} & B^2 \otimes X^2 & \xrightarrow{1_B \otimes \sigma \otimes 1_X} & (B \otimes X)^2 \\ \downarrow 1_B \otimes m & & \downarrow \lambda \cdot m \cdot (e \otimes \kappa) & & \downarrow (\lambda \cdot m \cdot (e \otimes \kappa))^2 \\ B \otimes X & \xrightarrow{\lambda \cdot m \cdot (e \otimes \kappa)} & X & & X \otimes X \\ & & & & \downarrow m \\ & & & & X \end{array}$$

(5.11)

*Proof.* Thanks to Proposition 4.2.1, the result follows as we can check by means of Figure 5.7 and Figure 5.8, where we use that  $e, \alpha, \kappa$  are morphisms of bialgebras.  $\square$

This lemma implies that the action of bialgebras defined by (4.2.2) ( $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$ ) satisfies the conditions (5.2) and (5.3). Hence, this

Figure 5.7: Proof of (5.10)

$$\begin{array}{ccccccc}
 B^2X & \xrightarrow{1_{Be\kappa}} & BA^2 & \xrightarrow{1_Bm} & BA & \xrightarrow{1_B\lambda} & BX \xrightarrow{e\kappa} A^2 \\
 & \searrow 1_{Be\kappa} & & & & & \nearrow (e \cdot \alpha)(\kappa \cdot \lambda) \\
 & & BA^2 & \xrightarrow{1_Bm} & BA & \xrightarrow{e1_A} & A^2 \\
 & & & & & & \downarrow m \\
 & & & & & & A \\
 & & & & & & \downarrow \lambda \\
 & & & & & & X \\
 & & & & & & \downarrow m \\
 & & & & & & X^2 \\
 & & & & & & \downarrow \lambda \\
 & & & & & & A \\
 & & & & & & \downarrow m \\
 & & & & & & A \\
 & & & & & & \downarrow m \\
 & & & & & & A
 \end{array}$$

(1')

(1.3) + (1.6)

$$\begin{array}{ccccccc}
 B^2X & \xrightarrow{\Delta e\kappa} & B^2A^2 & \xrightarrow{(1_B)^2m} & B^2A & \xrightarrow{(u_X \cdot e_B)e1_A} & XA^2 \\
 & & & & \downarrow e^21_A & & \downarrow 1_A(e \cdot \alpha)(\kappa \cdot \lambda) \\
 & & & & A^3 & \xrightarrow{1_A(e \cdot \alpha)(\kappa \cdot \lambda)} & A^3 \\
 & & & & \uparrow \Delta 1_A & & \uparrow \lambda^2 \\
 & & & & & & X^2 \\
 & & & & & & \downarrow m \\
 & & & & & & X \\
 & & & & & & \downarrow \lambda \\
 & & & & & & A \\
 & & & & & & \downarrow m \\
 & & & & & & A
 \end{array}$$

(1') + (2')

(4.10)

$$\begin{array}{ccccccc}
 B^2X & \xrightarrow{e^2\kappa} & A^3 & \xrightarrow{1_Am} & A^2 & \xrightarrow{m} & A \\
 & \searrow e^2\kappa & & & & & \downarrow m \\
 & & A^3 & \xrightarrow{m1_A} & A^2 & \xrightarrow{m} & A \\
 & & & & & & \downarrow m \\
 & & & & & & A
 \end{array}$$

(9')

$$\begin{array}{ccccccc}
 B^2X & \xrightarrow{m1_X} & BX & \xrightarrow{e\kappa} & A^2 & \xrightarrow{m} & A
 \end{array}$$

Figure 5.8: Proof of (5.11)

[illegible]

action becomes an action of Hopf algebras since the axiom (5.4) is given by the condition (10').

**Remark 5.1.6.** *The construction of the action of Hopf algebras given by  $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$  can be reformulated without  $\lambda$ , when we compose it by  $\kappa$ . Indeed, by pre-composing by  $(1_B \otimes 1_X \otimes e) \cdot (1_B \otimes 1_X \otimes S) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X)$  and post-composing by  $m$ , the two components of the equality (4.12), we obtain the following equality*

$$\kappa \cdot \triangleright = m \cdot (m \otimes 1_A) \cdot (e \otimes \kappa \otimes e) \cdot (1_B \otimes 1_X \otimes S) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X). \quad (5.12)$$

We notice that thanks to the condition (8'), this is equivalent to

$$\kappa \cdot \triangleright = m \cdot (1_A \otimes m) \cdot (e \otimes \kappa \otimes e) \cdot (1_B \otimes 1_X \otimes S) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X).$$

When the symmetric monoidal category  $\mathcal{C}$  is  $\mathbf{Vect}_K$ ,  $\kappa$  can be viewed as an inclusion and (5.12) give us a way to construct the action without  $\lambda$ .

For the sake of clarity, we give an explicit description of the morphisms of split extensions and actions of Hopf algebras in  $\mathcal{C}$ .

**Definition 5.1.7.** *A morphism of split extensions of Hopf algebras from  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  to  $X' \xleftarrow[\kappa']{\lambda'} A' \xleftarrow[\alpha']{e'} B'$  is given by three morphisms of Hopf algebras  $g: B \rightarrow B'$ ,  $v: X \rightarrow X'$  and  $p: A \rightarrow A'$  such that the following diagram commutes*

$$\begin{array}{ccccc} X & \xleftarrow[\kappa]{\lambda} & A & \xleftarrow[\alpha]{e} & B \\ v \downarrow & & p \downarrow & & \downarrow g \\ X' & \xleftarrow[\kappa']{\lambda'} & A' & \xleftarrow[\alpha']{e'} & B' \end{array} \quad (5.13)$$

We recall that, by using Corollary 4.1.8, it is enough that the two following equalities hold  $p \cdot \kappa = \kappa' \cdot v$  and  $p \cdot e = e' \cdot g$ , to imply that all the four squares of (5.13) commute.

**Definition 5.1.8.** *Let  $\triangleright: B \otimes X \rightarrow X$  and  $\triangleright': B' \otimes X' \rightarrow X'$  be two actions of Hopf algebras. A morphism between them is defined as a pair of morphisms of Hopf algebras  $g: B \rightarrow B'$  and  $v: X \rightarrow X'$  such that*

$$v \cdot \triangleright = \triangleright' \cdot (g \otimes v).$$

The actions of Hopf algebras (Definition 5.1.3) endowed with the morphisms of actions of Hopf algebras (Definition 5.1.8) form the category  $\text{Act}(\text{Hopf}_{\mathcal{C}})$  of actions of Hopf algebras in  $\mathcal{C}$ . The split extensions of Hopf algebras with the morphisms given by Definition 5.1.7 form the category  $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$  of split extensions of Hopf algebras in  $\mathcal{C}$ .

**Theorem 5.1.9.** [87] *There is an equivalence of categories between the category  $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$  of split extensions of Hopf algebras in the symmetric monoidal category  $\mathcal{C}$  and the category  $\text{Act}(\text{Hopf}_{\mathcal{C}})$  of actions of Hopf algebras in the symmetric monoidal category  $\mathcal{C}$ .*

*Proof.* Let  $(g, v, p)$  be a morphism in  $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$ , then it is clear that  $(v, g)$  is a morphism in  $\text{Act}(\text{Hopf}_{\mathcal{C}})$ . On the other hand, if  $(v, g)$  is a morphism of actions of Hopf algebras, the triple  $(g, v, v \otimes g)$  is a morphism of split extensions of Hopf algebras since  $v \otimes g$  preserves the antipode. Moreover, the isomorphisms  $\varphi := m \cdot (\kappa \otimes e)$  and  $\psi := (\lambda \otimes \alpha) \cdot \Delta$  (in (4.2.6)) form an isomorphism in  $\text{SplitExt}(\text{Hopf})$  since they are morphisms of Hopf algebras, as we can see in the following diagram

$$\begin{array}{ccccccc}
 XB & \xrightarrow{\sigma} & BX & \xrightarrow{SS} & BX & \xrightarrow{\Delta 1_X} & B^2X \xrightarrow{1_B \sigma} BXB \xrightarrow{\triangleright 1_B} XB \\
 \downarrow \kappa e & & & & \downarrow e\kappa & & \downarrow \kappa e \\
 & & & & A^2 & & A^2 \\
 & & & & \downarrow m & & \downarrow m \\
 A^2 & \xrightarrow{m} & A & \xrightarrow{S} & A & \xlongequal{\quad\quad\quad} & A,
 \end{array} \tag{4.12}$$

where the left square commutes since  $S$  is an anti-homomorphism of algebras. In conclusion, we obtain our statement thanks to the observations about the split extension (5.9) and Lemma 5.1.5.  $\square$

In the case of associative Hopf algebras, we can define  $\text{SplitExt}(\text{AssHopf}_{\mathcal{C}})$  and  $\text{Act}(\text{AssHopf}_{\mathcal{C}})$ . The actions of associative Hopf algebras are actions of Hopf algebras where the condition (5.4) always holds thanks to the uniqueness of the antipode. A split extension of associative Hopf algebras is the same as in  $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$  where the conditions (6'), (7'), (8'), and (9') become trivial. Moreover, the condition (10') is not required, it becomes a property that any split extension of associative Hopf algebras has.

**Corollary 5.1.10.** [87] *There is an equivalence of categories between the category  $\text{SplitExt}(\text{AssHopf}_{\mathcal{C}})$  of split extensions of associative Hopf algebras in  $\mathcal{C}$  and the category  $\text{Act}(\text{AssHopf}_{\mathcal{C}})$  of actions of associative Hopf algebras in  $\mathcal{C}$ .*

**Remark 5.1.11.** *Let us notice that, the Split Short Five Lemma (as proved in Theorem 4.2.12) also holds in the categories:  $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$ ,  $\text{SplitExt}(\text{AssHopf}_{\mathcal{C}})$ , and  $\text{SplitExt}(\text{AssBiAlg}_{\mathcal{C}})$ .*

## 5.2 Examples of split extensions of Hopf algebras

We investigate the two main symmetric monoidal categories of interest:  $\text{Set}$  and  $\text{Vect}_{\mathbb{K}}$ . On the one hand, we specify our results in  $\text{Set}$ .

### 5.2.1 Split extensions of Hopf algebras in the category of sets

- 1) Any split extension of groups

$$0 \longrightarrow K \xrightarrow{k} G \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{f} \end{array} H \longrightarrow 0$$

is a split extension of associative Hopf algebras when the symmetric monoidal category is  $\text{Set}$ . In particular, Corollary 5.1.10 becomes the well-known equivalence of categories between split extensions of groups and group actions Proposition 0.0.1.

- 2) In  $\text{Set}$ , non-associative Hopf algebras will be structures given by a set  $G$ , with a non-associative multiplication, a neutral element  $u_G$ , an inverse such that

$$g^{-1}g = u_G = gg^{-1}. \quad (5.14)$$

In particular, the non-zero octonions [29, 51] are equipped with a non-associative multiplication satisfying (5.14).

We can describe what split extensions of this algebraic structure should be to be equivalent to actions of such algebraic structures.

Indeed, a split extension should be a split morphism of these algebraic structures  $X \xrightarrow{\kappa} A \xleftarrow[\alpha]{e} B$  such that the following conditions are satisfied for any  $a \in A, b \in B, x \in X$

$$(3') \quad (a(e \cdot \alpha)(a^{-1}))(e \cdot \alpha)(a) = a$$

$$(6') \quad (\kappa(x)e(b))a = \kappa(x)(e(b)a)$$

$$(7') \quad (\kappa(x)a)e(b) = \kappa(x)(ae(b))$$

$$(8') \quad a(\kappa(x)e(b)) = (a\kappa(x))e(b)$$

$$(9') \quad (e(b)a)\kappa(x) = e(b)(a\kappa(x))$$

$$(10') \quad e(b^{-1})^{-1} = e(b)$$

The other conditions are trivially satisfied with

$$\lambda(a) = a(e \cdot \alpha)(a^{-1}).$$

### 5.2.2 Split extensions of Hopf algebras in the category of vector spaces

We give a particular example of associative split extensions of Hopf algebras, in the symmetric monoidal category  $\mathbf{Vect}_K$  of vector spaces over a field  $K$ .

We consider a split epimorphism  $\alpha$  of associative Hopf  $K$ -algebras,

$$HKer(\alpha) \xrightarrow{\kappa_\alpha} A \xleftarrow[\alpha]{e} B, \quad (5.15)$$

where  $HKer(\alpha)$  is the kernel of  $\alpha$  in the category  $\mathbf{Hopf}_K$  of Hopf  $K$ -algebras,  $\kappa_\alpha$  stands for the equalizer in  $\mathbf{Vect}_K$  of  $(1_A \otimes u_B \otimes 1_A) \cdot \Delta$  and  $(1_A \otimes \alpha \otimes 1_A) \cdot (\Delta \otimes 1_A) \cdot \Delta$ ,

$$HKer(\alpha) \xrightarrow{\kappa_\alpha} A \xrightarrow[(1_A \otimes \alpha \otimes 1_A) \cdot (\Delta \otimes 1_A) \cdot \Delta]{(1_A \otimes u_B \otimes 1_A) \cdot \Delta} A \otimes B \otimes A. \quad (5.16)$$

We recall that the equalizer in  $\mathbf{Vect}_K$  of  $f, g: A \rightarrow B$  is given by  $\{a \in A \mid f(a) = g(a)\}$ . We also define the following equalizers

$$LKer(\alpha) \xrightarrow{\kappa_{\alpha,L}} A \xrightarrow[(\alpha \otimes 1_A) \cdot \Delta]{u_B \otimes 1_A} B \otimes A,$$

$$RKer(\alpha) \xrightarrow{\kappa_{\alpha,R}} A \xrightleftharpoons[(1_A \otimes \alpha) \cdot \Delta]{1_A \otimes u_B} A \otimes B .$$

**Proposition 5.2.1.** *A split epimorphism (5.15) such that*

$$HKer(\alpha) = LKer(\alpha)$$

*is an extension of associative Hopf algebras (Definition 5.1.1) in the symmetric monoidal category  $\mathbf{Vect}_K$ .*

*Proof.* First, we recall that for any morphism  $\alpha$  in  $\mathbf{AssHopf}_K$  it is well-known that the following conditions are equivalent (see [4])

- $HKer(\alpha) = LKer(\alpha)$ ,
- $HKer(\alpha) = RKer(\alpha)$ ,
- $LKer(\alpha) = RKer(\alpha)$ ,
- $LKer(\alpha)$  is an associative Hopf algebra,
- $RKer(\alpha)$  is an associative Hopf algebra.

Let  $A \xrightleftharpoons[\alpha]{e} B$ , be a split epimorphism of associative Hopf algebras satisfying the condition  $HKer(\alpha) = LKer(\alpha)$ , since  $A$  is an associative Hopf algebra we can define the following section of  $\kappa_\alpha: HKer(\alpha) \rightarrow A$

$$\lambda = m \cdot (1_A \otimes (S \cdot e \cdot \alpha)) \cdot \Delta.$$

First, we use the condition  $HKer(\alpha) = RKer(\alpha)$  on the kernel to prove that  $\lambda$  factors through  $HKer(\alpha)$ ,

$$\begin{array}{ccccccc}
 A & \xrightarrow{\Delta} & A^2 & \xrightarrow{1_A(S \cdot e \cdot \alpha)} & A^2 & \xrightarrow{m} & A \\
 & \searrow \Delta & \downarrow \Delta^2 & & \downarrow \Delta\Delta & & \downarrow \Delta \\
 & & A^2 & (1.6) & A^4 & (1.7) & \\
 & & \downarrow 1_A\Delta & & \downarrow 1_{A^2}\sigma & & \\
 & & A^3 & & A^4 & & \\
 & & \downarrow 1_A\sigma & & \downarrow 1_A\sigma 1_A & & \\
 & & A^3 & \xrightarrow{1_{A^2}\Delta} & A^4 & \xrightarrow{(1_A(S \cdot e \cdot \alpha))^2} & A^4 \xrightarrow{m^2} A^2 \\
 & & \downarrow 1_A(S \cdot e \cdot \alpha)\epsilon_A & & \downarrow 1_A(S \cdot e \cdot \alpha)\alpha(S \cdot \alpha) & & \downarrow 1_A\alpha \\
 & & A^2 & \xrightarrow{1_A(S \cdot e \cdot \alpha)} & A^2 & \xrightarrow{mu_B} & A^2B^2 \xrightarrow{m^2} AB \\
 & & & & & & \downarrow 1_A\alpha \\
 & & & & & & AB
 \end{array}$$

By using that  $\lambda$  factors through  $RKer(\alpha) = HKer(\alpha)$ , we prove that  $\Delta \cdot \lambda = (\lambda \otimes \lambda) \cdot \Delta$ . The commutativity of part (A) is clarified in the second diagram

$$\begin{array}{ccccccc}
 A & \xrightarrow{\lambda} & A & & & & A \\
 \parallel & \searrow \Delta & \downarrow \Delta & \downarrow \Delta & \downarrow \Delta & & \downarrow \Delta \\
 A & \xrightarrow{\Delta} & A^2 & \xrightarrow{1_A(S \cdot e \cdot \alpha)} & A^2 & \xrightarrow{m} & A \\
 & \downarrow \Delta & \downarrow \Delta & \downarrow 1_A(S \cdot e \cdot \alpha) & \downarrow \Delta & & \downarrow \Delta \\
 & & A^2 & \downarrow m & A^2 & \xrightarrow{1_A u_B 1_A} & A^2 \\
 & & & \downarrow \Delta & A & \xrightarrow{\Delta 1_A} & A^3 \xrightarrow{1_A \alpha 1_A} & ABA \quad (1.3) \\
 & & & & & \downarrow 1_A(S \cdot e) 1_A & \downarrow m 1_A \\
 & & & & & & A^3 \xrightarrow{m 1_A} A^2 \\
 & & & & & & \parallel \\
 A^2 & \xrightarrow{\lambda^2} & A^2 & & & & A^2
 \end{array}$$

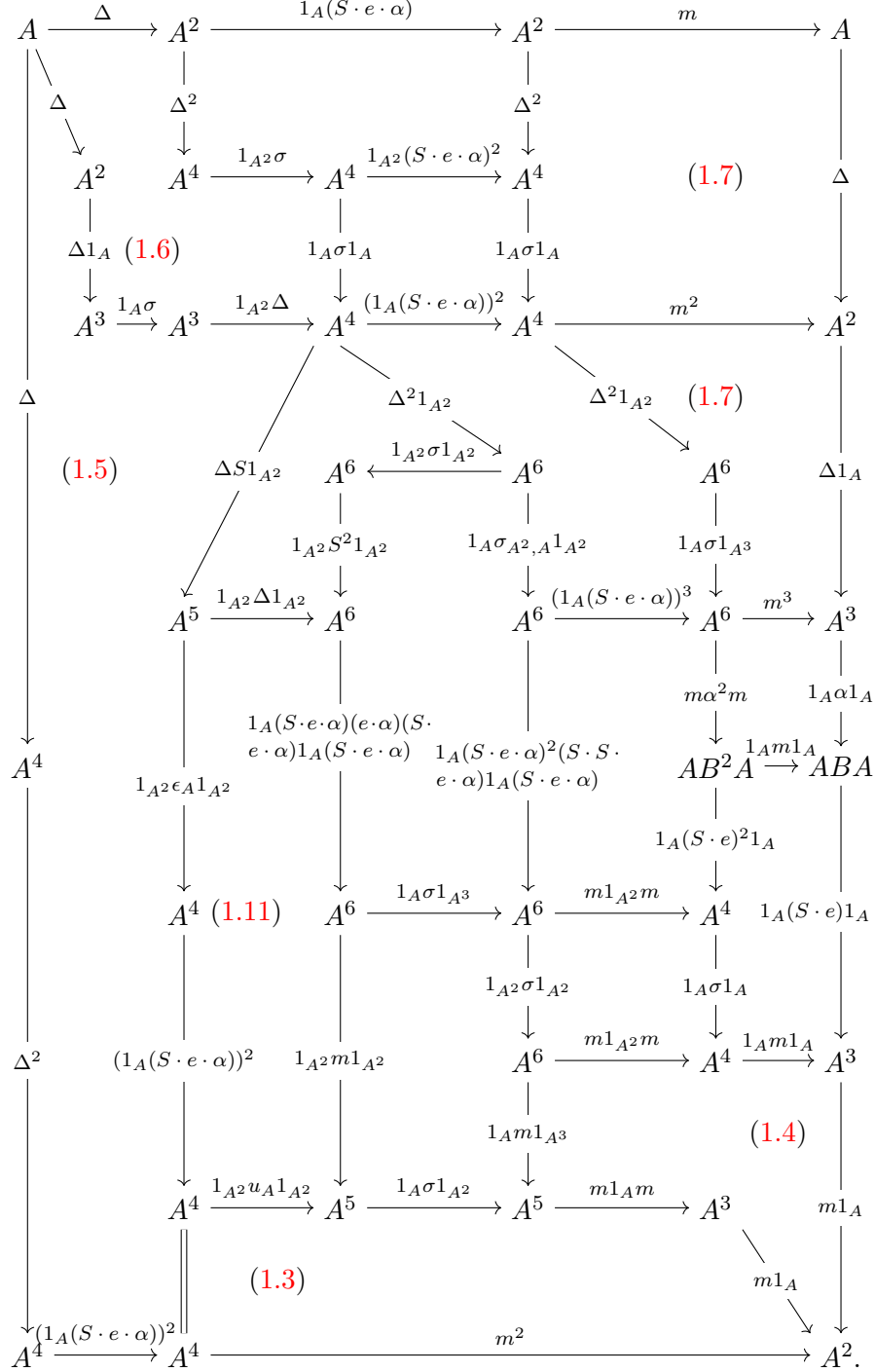
(A)

The condition (4') is also satisfied by this definition of  $\lambda$  thanks to the commutativity of the following diagram, where we use that  $HKer(\alpha) = RKer(\alpha)$  (denoted by  $(\star)$ ),

$$\begin{array}{c}
\begin{array}{ccccc}
HKer(\alpha)B & \xrightarrow{\kappa_\alpha e} & A^2 & \xrightarrow{m} & A \\
\downarrow \kappa_\alpha 1_B & & \downarrow \Delta\Delta & & \downarrow \Delta \\
AB & \xrightarrow{\Delta^2} & A^2 B^2 & \xrightarrow{1_A \sigma 1_A} & A^4 \xrightarrow{m^2} A^2 \\
\downarrow \Delta 1_B & & \downarrow 1_{AB} \sigma & & \downarrow 1_{A^2} \sigma \\
A^2 B & \xrightarrow{1_A \sigma} & ABA & \xrightarrow{1_A \Delta 1_A} & AB^2 A \xrightarrow{1_A e^2 1_A} A^4 \\
\downarrow 1_A \epsilon_B 1_A & & \downarrow 1_A \epsilon_B 1_A & & \downarrow 1_A m 1_A \\
A & \xrightarrow{\Delta} & A^2 & \xrightarrow{1_A u_A (S \cdot e \cdot \alpha)} & A^3 \xrightarrow{1_A m} A^2 \xrightarrow{m} A \\
\downarrow 1_A \alpha & & \downarrow 1_A \alpha & & \downarrow 1_A \alpha \\
AB & \xrightarrow{1_A (S \cdot e)} & AB & \xrightarrow{1_A (S \cdot e)} & A^2 \xrightarrow{m} A \\
\downarrow 1_A u_B & & \downarrow 1_A u_B & & \downarrow 1_A u_B \\
HKer(\alpha)B & \xrightarrow{\kappa_\alpha \epsilon_B} & A & \xrightarrow{\kappa_\alpha \epsilon_B} & A
\end{array}
\end{array}$$

$$\lambda \cdot m \cdot (e \otimes \kappa_\alpha) = m \cdot (1_A \otimes m) \cdot (1_A \otimes 1_A \otimes S) \cdot (1_A \otimes \sigma) \cdot (\Delta \otimes 1_A) \cdot (e \otimes \kappa_\alpha). \quad (5.17)$$

Figure 5.9: (A)



$$\begin{array}{ccccccc}
(A') & & & & & & \\
BHKer(\alpha) & \xrightarrow{e\kappa_\alpha} & A^2 & \xrightarrow{m} & A & \xrightarrow{\lambda} & A \\
\downarrow e\kappa_\alpha & \searrow \Delta^2 & \downarrow \Delta^2 & & \downarrow \Delta & & \downarrow \Delta \\
& B^2HKer(\alpha)^2 & A^4 & (1.7) & & & \\
& \downarrow 1_B\sigma 1_{HKer(\alpha)} & \downarrow 1_A\sigma 1_A & & & & \\
A^2 & \xrightarrow{(e\kappa_\alpha)^2} & A^4 & \xrightarrow{m^2} & A^2 & \xrightarrow{\lambda^2} & A^2 \\
\downarrow m & & \downarrow \epsilon_A\Delta & & \downarrow \Delta^2 & & \downarrow \Delta^2 \\
& & A^2 & & A^4 & \xrightarrow{(1_A(S \cdot e \cdot \alpha))^2} & A^4 \xrightarrow{m^2} A^2 \\
& & \downarrow 1_A(S \cdot e \cdot \alpha) & & \downarrow \alpha(S \cdot \alpha) 1_A(S \cdot e \cdot \alpha) & & \downarrow \alpha 1_A \\
& & A^2 & & B^2A^2 & & \\
& & \downarrow u_B m & & \downarrow m^2 & & \\
A & \xrightarrow{u_B \lambda} & BA & \xlongequal{\quad} & BA & & BA
\end{array}$$

Besides this observation, we establish two diagrams (B) and (C) in Figure 5.10 and 5.11 that we use in Figure 5.12 to prove that the condition (5') holds. In these figures,  $HKer(\alpha)$  is denoted by  $H$ .

To conclude, a split epimorphism  $HKer(\alpha) \xrightarrow{\kappa_\alpha} A \xleftarrow[\alpha]{e} B$  such that  $HKer(\alpha) = RKer(\alpha)$ , is a split extension of associative Hopf algebras.

□

### 5.3 Comparison with other definitions

In this section, we compare the definition of split extensions of associative Hopf algebras in  $\mathbf{Vect}_K$  with the definition given by Andruskiewitsch and Devoto *exact cleft sequences*. We also compare the Proposition 1.5 of a paper of Böhm, [7], with Corollary 5.1.10.



Figure 5.11: Diagram (C)

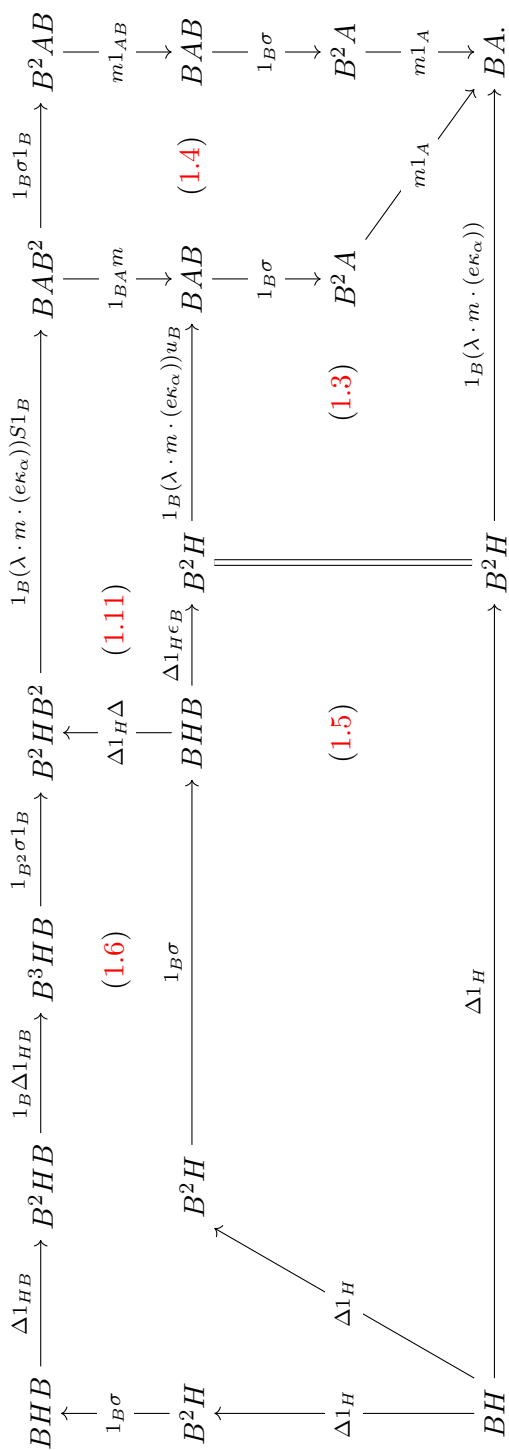
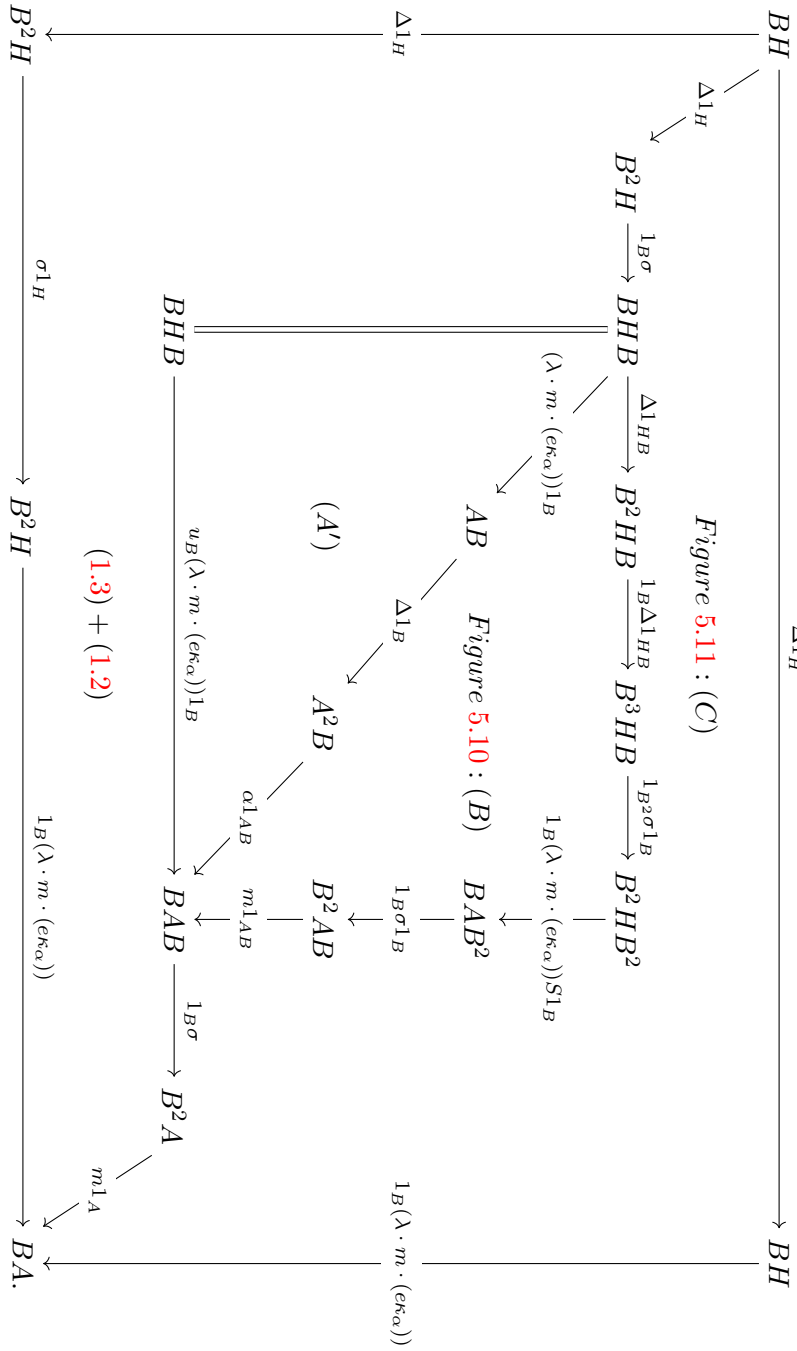


Figure 5.12: Proof of the condition (5')



### 5.3.1 Definition of an exact cleft sequence

In some sense, Theorem 4.2.12, for the associative Hopf algebras (see Remark 5.1.11), is similar to a property obtained for the *exact cleft sequences* of associative Hopf  $K$ -algebras (with bijective antipodes) investigated by [5] (see Lemma 3.2.19). We would like to emphasize the differences and shared properties between the definition of an exact cleft sequence of associative Hopf algebras and the definition of split extension of associative Hopf algebras (Definition 5.1.1), in the symmetric monoidal category  $\mathbf{Vect}_K$  of vector spaces. First, we recall the definition of an exact cleft sequence of associative Hopf algebras [4].

**Definition 5.3.1.** *A sequence of morphisms of associative Hopf algebras*

$$A' \xrightarrow{\iota} C' \xrightarrow{\pi} B' , \quad (5.18)$$

*is exact in the sense of [4] if*

- 1)  $\iota$  is injective,
- 2)  $\pi$  is surjective,
- 3)  $\ker(\pi) = C' \iota(A')^+ \text{ (} \ker(\pi) \text{ is the kernel in } \mathbf{Vect} \text{ and } \iota(A')^+ = \{x \in \iota(A') \mid \epsilon_{\iota(A')}(x) = 0\} \text{) ,}$
- 4)  $\iota(A') = LKer(\pi) = \{x \in C' \mid (\pi \otimes 1_{C'}) \cdot \Delta(x) = u_{B'} \otimes x\}.$

**Definition 5.3.2.** *Let (5.18) be an exact sequence of associative Hopf algebras, then the sequence*

$$A' \xrightleftharpoons[\iota]{\xi} C' \xrightleftharpoons[\pi]{\chi} B' , \quad (5.19)$$

*is cleft if and only if there exist a morphism of  $A'$ -modules  $\xi : C' \rightarrow A'$  (i.e. the equality  $\xi \cdot m \cdot (\iota \otimes 1_{C'}) = m \cdot (1_{A'} \otimes \xi)$  holds) and a morphism of  $B'$ -comodules  $\chi : B' \rightarrow C'$  (i.e. the equality  $(\pi \otimes 1_{C'}) \cdot \Delta \cdot \chi = (1_{B'} \otimes \chi) \cdot \Delta$  is satisfied) such that the following two equations hold*

$$\xi \cdot \chi = u_{A'} \cdot \epsilon_{B'} , \quad (5.20)$$

$$m \cdot ((\iota \cdot \xi) \otimes (\chi \cdot \pi)) \cdot \Delta = 1_{C'} . \quad (5.21)$$

Remark that in [4, 5] the above definition is not the definition of an exact cleft sequence, but it is equivalent to it by Lemma 3.1.14 in [4]. It is straightforward to observe that the conditions (5.20) and (5.21) of the sequence (5.19) are the same as the conditions (2') and (3') in Definition 5.1.1. Moreover, let  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  be a split extension of associative Hopf algebras, then  $\lambda$  is an  $X$ -module morphism (thanks to Proposition 4.2.1) and  $e$  is a  $B$ -comodule morphism.

However, there are major differences. Indeed, a split extension of associative Hopf algebras (Definition 5.1.1) is (in general) not exact in the sense of [4]. Conversely, the Hopf algebra morphism  $\pi$  in the exact cleft sequence (5.19) is not a split epimorphism of Hopf algebras, since  $\chi$  is neither a morphism of algebras nor a morphism of coalgebras (see [5] for such an example). Then, it is clear that one definition does not imply the other and vice versa. Nevertheless, there are sequences of associative Hopf algebras that are exact cleft sequences and split extensions of associative Hopf algebras. For example, any exact sequence of associative Hopf algebras (5.18) such that  $\pi$  is a split epimorphism is an example of both definitions (see Proposition 5.2.1).

### 5.3.2 A split epimorphism in the sense of [7]

In [7], G. Böhm studied a similar equivalence of categories as Corollary 5.1.10 in a more general context. The author used a monoidal admissible class of spans to define pullbacks and internal categories. As a particular case of her theory, she establishes an equivalence between two categories. The first one is the category of split epimorphisms

$$A \xleftarrow[\alpha]{e} B, \text{ where } B \text{ is a Hopf algebra and } A \text{ is a bialgebra, having the}$$

following cocommutativity conditions

$$(\alpha \otimes 1_A) \cdot \sigma \cdot \Delta = (\alpha \otimes 1_A) \cdot \Delta. \quad (5.22)$$

The second one is the category whose objects consist of a cocommutative Hopf algebra  $B$  and a bialgebra  $X$ , such that  $X$  is a  $B$ -module bialgebra. Proposition 1.5 in [7] shows an equivalence between these two categories.

Note that Corollary 5.1.10 does not follow from this interesting result of [7].

Indeed, there are examples of split extensions of associative Hopf algebras which do not satisfy the condition (5.22).

The identity over a non-cocommutative associative Hopf algebra

$$A \begin{smallmatrix} \xleftarrow{1_A} \\ \xrightarrow{1_A} \end{smallmatrix} A ,$$

is an example of a split extension of associative Hopf algebras, as defined in Definition 5.1.1, that does not verify the condition (5.22).

A split epimorphism of Hopf algebras satisfying (5.22) is an example of split epimorphisms described in Proposition 5.15.

The results in Chapter 5 and the results in [7] coincide for cocommutative associative Hopf algebras, but they are both different generalizations of the cocommutative case.



## Chapter 6

# S-reflexive graphs of general Hopf algebras

In the previous chapter, we established an equivalence of categories between actions and split epimorphisms of (non-associative) Hopf algebras: Theorem 5.1.9. This interesting result is a first step in the equivalence between internal and crossed structures. Indeed, actions are an essential ingredient in the definition of “crossed structures”. Similarly, split extensions are a fundamental component in the definition of internal structure. In this chapter, we use the actions and split extensions of Hopf algebras to define the  $S$ -reflexive graphs and the precrossed modules of Hopf algebras. On the one hand, we define the notion of  $S$ -reflexive graph as a reflexive graph with one of its “legs” being a split extension of Hopf algebras. This definition is natural in the sense of [20] and [82]. On the other hand, we define the notion of precrossed modules for general Hopf algebras via the actions of Hopf algebras. We show an equivalence between precrossed modules and the “ $S$ -reflexive graphs” of general Hopf algebras.

In this chapter, we still work with general Hopf algebras in a symmetric monoidal category, “Hopf algebras” still means “non-associative Hopf algebras”.

We use the results of Chapter 5 to define the following notions: *precrossed modules of Hopf algebras* and  *$S$ -reflexive graphs of Hopf algebras*.

**Definition 6.0.1.** A precrossed module of Hopf algebras is given by a pair  $(\mu: X \rightarrow B, \triangleright: B \otimes X \rightarrow X)$  where  $\mu$  is a morphism of Hopf algebras and  $\triangleright$  is an action of Hopf algebras such that

$$m \cdot (\mu \otimes 1_B) \cdot (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X) = m \cdot (1_B \otimes \mu), \quad (6.1)$$

$$m \cdot (1_B \otimes m) \cdot (\mu \otimes 1_B \otimes 1_B) = m \cdot (m \otimes 1_B) \cdot (\mu \otimes 1_B \otimes 1_B), \quad (6.2)$$

$$m \cdot (1_B \otimes m) \cdot (1_B \otimes \mu \otimes 1_B) = m \cdot (m \otimes 1_B) \cdot (1_B \otimes \mu \otimes 1_B). \quad (6.3)$$

In the category of Hopf algebras, the condition (6.1) can be rewritten thanks to (6.2), (1.11), and (1.6) as

$$\mu \cdot \triangleright = m \cdot (m \otimes 1_B) \cdot (1_B \otimes \mu \otimes S) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X). \quad (6.4)$$

This re-expression of the condition (6.1) is proven thanks to the commutativity of the following diagram

$$\begin{array}{ccccccc}
 BX & \xrightarrow{\Delta 1_X} & B^2 X & \xrightarrow{1_B \sigma} & BXB & \xrightarrow{1_{BX} S} & BXB & \xrightarrow{1_B \mu 1_B} & B^3 \\
 \parallel & & & & \downarrow \Delta 1_{XB} & & \downarrow \Delta 1_{XB} & & \downarrow m 1_B \\
 & & & & B^2 XB & & B^2 XB & & \\
 & & & & \downarrow 1_B \sigma 1_B & & \downarrow 1_B \sigma 1_B & & \\
 & & & & BXB^2 & \xrightarrow{1_{BX} S} & BXB^2 & \xrightarrow{\triangleright 1_{B^2}} & XB^2 & \xrightarrow{\mu 1_{B^2}} & B^3 & \xrightarrow{m 1_B} & B^2 \\
 & & & & \uparrow 1_{BX} \Delta & & \uparrow 1_{BX} \Delta & & \downarrow \mu m & & \downarrow \mu m & & \downarrow m \\
 & & & & B^2 X & \xrightarrow{1_B \sigma} & BXB & \xrightarrow{\triangleright \epsilon_B} & X & \xrightarrow{\mu u_B} & B^2 & \xrightarrow{m} & B \\
 & & & & \uparrow \Delta 1_X & & \uparrow \Delta 1_X & & \parallel & & \parallel & & \parallel \\
 BX & \xrightarrow{\Delta 1_X} & B^2 X & \xrightarrow{1_B \sigma} & BXB & \xrightarrow{\triangleright \epsilon_B} & X & \xrightarrow{\mu} & B^2 & \xrightarrow{m} & B
 \end{array}$$

(1.6)
(6.1)
(1.11)
(6.2)
(1.5)
(1.3)

Note that the equality (6.4) can also be re-expressed as

$$\mu \cdot \triangleright = m \cdot (1_B \otimes m) \cdot (1_B \otimes \mu \otimes S) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X)$$

thanks to (6.3).

**Definition 6.0.2.** A morphism of precrossed modules of Hopf algebras between  $(\mu: X \rightarrow B, \triangleright: B \otimes X \rightarrow X)$  and  $(\mu': X' \rightarrow B', \triangleright': B' \otimes X' \rightarrow X')$  is given by a morphism of actions between  $\triangleright$  and  $\triangleright'$ ,  $(f: X \rightarrow X', g: B \rightarrow B')$  such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\mu} & B \\ \downarrow f & & \downarrow g \\ X' & \xrightarrow{\mu'} & B'. \end{array} \quad (6.5)$$

These two definitions form the category  $\text{PXMod}(\text{Hopf})$  of precrossed modules of Hopf algebras.

We define the category  $\text{SRGraph}(\text{Hopf})$  of  $S$ -reflexive graphs of Hopf algebras. The objects and the morphisms of this category are defined as follows.

**Definition 6.0.3.** An  $S$ -reflexive graph of Hopf algebras is a graph of the form

$$X \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\kappa} \end{array} A \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} B, \quad (6.6)$$

where  $X, A, B$  are Hopf algebras,  $\kappa, \alpha, \beta$  and  $e$  are morphisms of Hopf algebras satisfying the following identity

$$\beta \cdot e = 1_B \quad (6.7)$$

and  $X \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\kappa} \end{array} A \begin{array}{c} \xleftarrow{e} \\ \xrightarrow{\alpha} \end{array} B$  is a split extension of Hopf algebras (as in Definition 5.1.1).

The terminology “ $S$ -reflexive graphs” was introduced in [20]. The authors defined  $S$ -reflexive graphs for a specific class  $S$  of split extensions. This class  $S$  needs to satisfy several properties, for example, the class  $S$  is assumed to be stable under pullback. An  $S$ -reflexive graph in [20] is

defined as a reflexive graph  $A \begin{array}{c} \xleftarrow{\beta} \\ \xrightarrow{\alpha} \end{array} B$  where the point  $(\alpha, \beta)$  is in

the class  $S$ . Even if the theory developed in [20] does not apply in our context, since we do not know if the split extensions of Hopf algebras are stable under pullbacks, we use this terminology to emphasize the difference with the reflexive graphs defined in 1.6.2.

**Definition 6.0.4.** A morphism between two  $S$ -reflexive graphs of Hopf algebras  $X \xrightleftharpoons[\kappa]{\lambda} A \xrightleftharpoons[\beta]{\alpha} B$  and  $X' \xrightleftharpoons[\kappa']{\lambda'} A' \xrightleftharpoons[\beta']{\alpha'} B'$  is given by 3 morphisms of Hopf algebras  $g: B \rightarrow B'$ ,  $v: X \rightarrow X'$ , and  $p: A \rightarrow A'$  such that the squares of the corresponding arrows of the following diagram commute

$$\begin{array}{ccccc}
 X & \xrightleftharpoons[\kappa]{\lambda} & A & \xrightleftharpoons[\beta]{\alpha} & B \\
 v \downarrow & & p \downarrow & & g \downarrow \\
 X' & \xrightleftharpoons[\kappa']{\lambda'} & A' & \xrightleftharpoons[\beta']{\alpha'} & B'
 \end{array}$$

We build step by step the equivalence of categories between the category  $\text{SRGraph}(\text{Hopf})$  and the category  $\text{PXMod}(\text{Hopf})$ .

**Proposition 6.0.5.** *There is a functor*

$$\hat{F}: \text{SRGraph}(\text{Hopf}) \rightarrow \text{PXMod}(\text{Hopf}).$$

*Proof.* Let us consider an  $S$ -reflexive graph  $X \xrightleftharpoons[\kappa]{\lambda} A \xrightleftharpoons[\beta]{\alpha} B$ , thanks to Theorem 5.1.9, we can construct, from this  $S$ -reflexive graph of Hopf algebras, an action of Hopf algebras  $\triangleright := \lambda \cdot m \cdot (e \otimes \kappa): B \otimes X \rightarrow X$ . Moreover, this action endowed with the morphism  $\mu := \beta \cdot \kappa: X \rightarrow B$  is a precrossed module of Hopf algebras. Indeed, by applying  $\beta$  to (4.12), we obtain the following commutative diagram showing that  $\mu$  satisfies (6.1)

$$\begin{array}{ccccccc}
BX & \xrightarrow{\Delta 1_X} & B^2X & \xrightarrow{1_B \sigma} & BXB & \xrightarrow{\triangleright 1_B} & XB \xlongequal{\quad} XB \\
\parallel & & & & & & \downarrow \kappa e \quad (6.7) \\
& & & & & & A^2 \\
& & & & & & \downarrow m \\
& & & & & & A \\
& & & & & & \downarrow \beta \\
& & & & & & B^2 \\
& & & & & & \downarrow m \\
& & & & & & B
\end{array}$$

(4.12) (6.7)

Moreover, to prove (6.2) and (6.3) we use the “partial associativity” of the underlying split extension  $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$  of the  $S$ -reflexive graph, as we can see in the following diagram

$$\begin{array}{ccccccc}
XB^2 & \xrightarrow{\mu 1_B 1_B} & B^3 & \xrightarrow{1_B m} & B^2 \\
\parallel & \searrow \kappa e e \quad (6.7) & \nearrow \beta \beta \beta & & \downarrow m \\
XB^2 & \xrightarrow{\kappa e e} & A^3 & \xrightarrow{1_A m} & A^2 \xrightarrow{m} A \xrightarrow{\beta} B \\
\parallel & & & (6') & \parallel & \parallel \\
XB^2 & \xrightarrow{\kappa e e} & A^3 & \xrightarrow{m 1_A} & A^2 \xrightarrow{m} A \xrightarrow{\beta} B \\
\parallel & \nearrow \kappa e e \quad (6.7) & \searrow \beta \beta \beta & & \uparrow m \\
XB^2 & \xrightarrow{\mu 1_B 1_B} & B^3 & \xrightarrow{m 1_B} & B^2
\end{array}$$

A similar computation proves (6.3) by using the condition (8') of the definition of split extensions of Hopf algebras (Definition 5.1.1) and we conclude that  $(\mu := \beta \cdot \kappa, \triangleright := \lambda \cdot m \cdot (e \otimes \kappa))$  is a precrossed module of Hopf algebras.

Let  $(v, p, g)$  be a morphism of  $S$ -reflexive graphs, then  $(v, g)$  is a morphism of actions (see Theorem 5.1.9), and it is clear that (6.5) is satisfied for  $\mu = \beta \cdot \kappa$ , turning  $(v, g)$  into a morphism of precrossed modules.  $\square$

In the previous proposition, we built a precrossed module of Hopf algebras from an  $S$ -reflexive graph of Hopf algebras. Conversely, the following proposition allows us to construct an  $S$ -reflexive graph from a precrossed module.

**Proposition 6.0.6.** *There is a functor*

$$\hat{G}: \text{PMod}(\text{Hopf}) \rightarrow \text{SReflex}(\text{Hopf}).$$

*Proof.* From a precrossed module of Hopf algebras  $(\mu: X \rightarrow B, \triangleright: B \otimes X \rightarrow X)$ , thanks to Theorem 5.1.9, we construct the following split extension of Hopf algebras  $X \xrightleftharpoons[i_1]{\pi_1} X \rtimes B \xrightleftharpoons[i_2]{\pi_2} B$ . Via the morphism  $\mu$ , this construction can be transformed into an  $S$ -reflexive graph of Hopf algebras,

$$X \xrightleftharpoons[i_1]{\pi_1} X \rtimes B \xrightleftharpoons[\tilde{\beta}]{i_2 \xrightarrow{\pi_2}} B,$$

where  $\tilde{\beta} = m \cdot (\mu \otimes 1_B): X \rtimes B \rightarrow B$ . It is clear that  $\tilde{\beta} \cdot i_2 = 1_B$  and that  $\tilde{\beta}$  is a morphism of coalgebras. We check that  $\tilde{\beta}$  is a morphism of Hopf algebras thanks to the commutativity of the two following diagrams.

$$\begin{array}{c}
 \xrightarrow{\quad m_{X \rtimes B} \quad} \\
 \begin{array}{ccccccc}
 (XB)^2 & \xrightarrow{1_X \Delta 1_{XB}} & XB^2XB & \xrightarrow{1_{XB} \sigma 1_B} & (XB)^2B & \xrightarrow{1_X \triangleright 1_{B^2}} & X^2B^2 \xrightarrow{m 1_{B^2}} XB^2 \xrightarrow{1_X m} XB \\
 \parallel & & & & \downarrow \mu^2 1_{B^2} & & \downarrow \mu^2 1_{B^2} \\
 (XB)^2 & \xrightarrow{(\mu 1_B)^2} & B^4 & \xrightarrow{1_B m 1_B} & B^3 & \xrightarrow{m 1_{B^2}} & B^3 \xrightarrow{m 1_B} B^2 \xrightarrow{m} B \\
 \parallel & & & & \downarrow m 1_B & & \downarrow m 1_B \\
 (XB)^2 & \xrightarrow{(\mu 1_B)^2} & B^4 & \xrightarrow{m 1_{B^2}} & B^3 & \xrightarrow{m 1_B} & B^2 \xrightarrow{m} B \\
 \downarrow (\mu 1_B)^2 & & & & & & \downarrow m \\
 B^4 & \xrightarrow{m^2} & B^2 & \xrightarrow{m} & B
 \end{array}
 \end{array}$$

(6.1) (6.2) (6.3)

$$\begin{array}{ccccccc}
& & & S_{X \rtimes B} & & & \\
& \swarrow & & \searrow & & & \\
XB & \xrightarrow{\sigma} & BX & \xrightarrow{SS} & BX & \xrightarrow{\Delta 1_X} & B^2 X \xrightarrow{1_B \sigma} BXB \xrightarrow{\triangleright 1_B} XB \\
\downarrow \mu 1_B & & \downarrow 1_B \mu & & \downarrow 1_B \mu & & \downarrow \mu 1_B \\
B^2 & \xrightarrow{\sigma} & B^2 & \xrightarrow{SS} & B^2 & & B^2 \\
\downarrow m & & & & \downarrow m & & \downarrow m \\
B & \xrightarrow{S} & B & & B & \xlongequal{\quad} & B
\end{array} \quad (6.1)$$

Furthermore, let  $(f, g)$  be a morphism of precrossed modules. Then  $(f, f \otimes g, g)$  is a morphism of  $S$ -reflexive graphs

$$\begin{array}{ccccc}
X & \xleftarrow[\pi_1]{i_1} & X \rtimes B & \xleftarrow[\tilde{\beta}]{\pi_2} & B \\
\downarrow f & & \downarrow f \otimes g & & \downarrow g \\
X' & \xleftarrow[\pi_1]{i_1} & X' \rtimes B' & \xleftarrow[\tilde{\beta}']{\pi_2} & B'
\end{array}$$

Thanks to Theorem 5.1.9, we only need to check that the square with  $\tilde{\beta}$  commutes, which is clear via the commutativity of the following diagram

$$\begin{array}{ccccc}
X \rtimes B & \xrightarrow{\mu \otimes 1_B} & B^2 & \xrightarrow{m} & B \\
\downarrow f \otimes g & & \downarrow g \otimes g & & \downarrow g \\
X' \rtimes B' & \xrightarrow{\mu' \otimes 1_{B'}} & B'^2 & \xrightarrow{m} & B'
\end{array}$$

□

In the previous propositions, we built the functors  $\hat{G}: \text{PXMod}(\text{Hopf}) \rightarrow \text{SRGraph}(\text{Hopf})$  and  $\hat{F}: \text{SRGraph}(\text{Hopf}) \rightarrow \text{PXMod}(\text{Hopf})$ . To show that the functors give rise to an equivalence of categories, we will now use Theorem 5.1.9 and check that the constructions and the morphisms involved in the proof of Theorem 5.1.9 give the desired equivalence.

**Theorem 6.0.7.** *There is an equivalence of categories between the category  $\text{PXMod}(\text{Hopf})$  and the category  $\text{SRGraph}(\text{Hopf})$ .*

*Proof.* We need to prove that there is an isomorphism between the

$S$ -reflexive graph  $X \xrightleftharpoons[\kappa]{\lambda} A \xrightleftharpoons[\beta]{\alpha} B$  and the  $S$ -reflexive graph

$\hat{G} \cdot \hat{F} ( X \xrightleftharpoons[\kappa]{\lambda} A \xrightleftharpoons[\beta]{\alpha} B )$ . By using Theorem 5.1.9, we want

that  $\varphi$  and  $\psi$  of (4.11) induce a morphism of  $S$ -reflexive graphs

$$\begin{array}{ccccc}
 X & \xrightleftharpoons[\kappa]{\lambda} & A & \xrightleftharpoons[\beta]{\alpha} & B \\
 \parallel & & \uparrow \varphi & \downarrow \psi & \parallel \\
 X & \xrightleftharpoons[\tilde{\beta}]{\pi_1} & X \rtimes B & \xrightleftharpoons[\tilde{\beta}]{\pi_2} & B,
 \end{array} \tag{6.8}$$

where  $\varphi = m \cdot (\kappa \otimes e)$ ,  $\psi = (\lambda \otimes \alpha) \cdot \Delta$ , and  $\tilde{\beta} = m \cdot ((\beta \cdot \kappa) \otimes 1_B)$ .

In the previous chapter, we noticed that  $(1_X, \varphi, 1_B)$  and  $(1_X, \psi, 1_B)$  are isomorphisms of split extensions of Hopf algebras (see Theorem 5.1.9). They are also morphisms of  $S$ -reflexive graphs of Hopf algebras since we have the following identities,

$$\beta \cdot m \cdot (\kappa \otimes e) = m \cdot (\beta \otimes \beta) \cdot (\kappa \otimes e) = \tilde{\beta}.$$

Moreover, since  $\hat{F} \cdot \hat{G}(\mu, \triangleright) := (m \cdot (\mu \otimes 1_B) \cdot i_1, \pi_1 \cdot m_{X \rtimes B} \cdot (i_2 \otimes i_1)) = (\mu, \triangleright)$ , we have the equivalence between the two categories.  $\square$

The previous theorem can be applied in particular to associative Hopf algebras leading to the following corollary.

**Corollary 6.0.8.** *There is an equivalence of categories between the following two categories:*

- the category  $\text{PXMod}(\text{AssHopf})$  of precrossed modules of associative Hopf algebras, whose objects are the actions of associative Hopf algebras with a morphism  $\mu$  satisfying the condition (6.1);

- the category  $\text{SRGraph}(\text{AssHopf})$  of  $S$ -reflexive graphs of associative Hopf algebras as defined in Definition 6.0.3, where the underlying split extension  $X \xrightleftharpoons[\kappa]{\lambda} A \xrightleftharpoons[\alpha]{e} B$  is a split extension of associative Hopf algebras.

Hence, the associative case follows from this section by erasing all the associative conditions.

**Remark 6.0.9.** We notice that when restricting to associative Hopf algebras, the definition of  $S$ -reflexive graphs (Definition 6.0.3) does not coincide with the one given by the authors in [43] of the  $\text{pre-cat}^1$ -Hopf algebras. Indeed, in [43], the authors defined a  $\text{pre-cat}^1$ -Hopf algebra

as a reflexive graph (as defined in Definition 1.6.2)  $A \xrightleftharpoons[\alpha]{\beta} B$

where  $A$  and  $B$  are associative Hopf algebras, and the following equalities hold  $(\beta \otimes 1_A) \cdot \sigma \cdot \Delta = (\beta \otimes 1_A) \cdot \Delta$  and  $(\alpha \otimes 1_A) \cdot \sigma \cdot \Delta = (\alpha \otimes 1_A) \cdot \Delta$ . This definition is not equivalent to Definition 6.0.3 for associative Hopf algebras, since

$$I \xrightleftharpoons[u_A]{\epsilon_A} A \xrightleftharpoons[1_A]{1_A} A$$

is an  $S$ -reflexive graph of Hopf algebras (if  $A$  is a Hopf algebra), but it is a  $\text{pre-cat}^1$ -Hopf algebra (in the sense of [43]) if and only if  $A$  is a cocommutative Hopf algebra.

In [43], the notion of  $\text{pre-cat}^1$ -Hopf algebras was used to define what they called “ $\text{cat}^1$ -Hopf algebras”, which they proved to be equivalent to their definition of “crossed modules”.

We recommend also examining Proposition 2.4 of [7]. The author used a definition that is close to the one given in [43]. This result is based on Proposition 1.5 in [7], which was discussed in Section 5.3.2. The results in [43] and [7] coincide with Corollary 6.0.8 in the particular case of cocommutative Hopf algebras in a symmetric monoidal category.



# Conclusion and outlook

## Conclusion

In this thesis, we have proved some new results relating categorical algebra and the theory of Hopf algebras. This dissertation allowed us to shed some new light on some categorical notions in the category of Hopf algebras such as crossed modules, crossed squares, commutator, actions and split extensions.

The notations, basic definitions, and basic properties were given in Chapter 1. The next two chapters were devoted to the category of cocommutative Hopf algebras over any field  $K$ .

In Chapter 2, we studied the properties of the category of cocommutative Hopf  $K$ -algebras. Thanks to a result of Newman [79], we proved that the category of cocommutative Hopf  $K$ -algebras,  $\mathbf{Hopf}_{K,\text{coc}}$  is semi-abelian and action representable. These results, Theorem 2.2.8 and Proposition 2.2.11, generalized the results obtained in the case of a field of characteristic zero [48] to any characteristic. We also described the *Huq commutator*, which coincides with the commutator defined in [96], and observed that  $\mathbf{Hopf}_{K,\text{coc}}$  satisfies the *Smith is Huq* condition. We proved that the Zassenhaus Lemma holds for cocommutative Hopf algebras of arbitrary dimension, which provides new classes of Hopf algebras for which the Lemma holds beyond those covered by the results of [78].

In Chapter 3, we recalled a well-known equivalence of categories between actions and split extensions of cocommutative Hopf algebras. We studied the internal crossed modules (in the sense of [57]) in the category of

cocommutative Hopf  $K$ -algebras. Since the category  $\mathbf{Hopf}_{K,\text{coc}}$  is semi-abelian, the internal crossed modules in  $\mathbf{Hopf}_{K,\text{coc}}$  exist and are equivalent to the internal groupoids in  $\mathbf{Hopf}_{K,\text{coc}}$ . We showed that the internal crossed modules in  $\mathbf{Hopf}_{K,\text{coc}}$  coincide with the definition of crossed modules of cocommutative Hopf algebras defined independently in [70] and [43]. Although these definitions are different, they coincide when we restrict them to the cocommutative Hopf algebras.

We used again the fact that  $\mathbf{Hopf}_{K,\text{coc}}$  is semi-abelian to observe that the category of crossed modules of cocommutative Hopf  $K$ -algebras is semi-abelian [46, 15]. Hence, there is a notion of crossed module in this category. We defined the notion of Hopf crossed square, which extends the notions of crossed squares of groups [28] and Lie algebras [37]. We proved that the Hopf crossed squares are the internal crossed modules in the category of crossed modules of cocommutative Hopf  $K$ -algebras, by showing that the Hopf crossed squares are equivalent to internal double groupoids in  $\mathbf{Hopf}_{K,\text{coc}}$ .

These two chapters offered a new understanding of the category of cocommutative Hopf  $K$ -algebras.

The questions were at that point: “What does it happen for general Hopf algebras?” and “What is a good notion of split extensions of general Hopf algebras?”. We wanted to understand this notion in a general framework.

These questions motivate Chapter 4, in which we studied the link between actions and split epimorphisms of (non-associative) bialgebras in any symmetric monoidal category. Inspired by [47], we defined split extensions of non-associative bialgebras, and we proved that they form exact sequences and they are equivalent to what we called actions of non-associative bialgebras. We showed that this equivalence restricts to the associative case. This chapter ended with the proof of the Split Short Five Lemma for the split extensions of (non-associative) bialgebras.

In Chapter 5, similarly to the previous chapter, we defined the split extensions of (non-)associative Hopf algebras and proved an equivalence between split extensions and actions of (non-)associative Hopf algebras.

We concluded this chapter with some insightful examples and a comparison with other definitions of “split extensions of Hopf algebras” given by Andruskiewitsch [4] and Böhm [7].

In the last chapter, we used the definitions of split extensions and actions of non-associative Hopf algebras in a symmetric monoidal category (described in Chapter 5) to define precrossed modules and  $S$ -reflexive graphs of (non-associative) Hopf algebras. We showed that these two notions are equivalent.

This thesis has strengthened the links between Hopf algebras and category theory, providing a thorough study of the categorical notions of actions, precrossed modules and crossed modules of Hopf algebras in various contexts.

## Outlook

This thesis gives rise to several new questions and perspectives. We present some of them in this section.

### Crossed modules of Hopf algebras

The results obtained in Chapter 6 motivate us to define crossed modules of (non-associative) Hopf algebras and describe their equivalent internal structures.

A natural way to define crossed modules of (non-associative) Hopf algebras is the following:

**Definition 6.0.10.** *A crossed module of (non-associative) Hopf algebras is given by a pair  $(\mu: X \rightarrow B, \triangleright: B \otimes X \rightarrow X)$  where  $\mu$  is a morphism of (non-associative) Hopf algebras and  $\triangleright$  is an action of (non-associative) Hopf algebras such that*

$$\begin{aligned}
 m \cdot (\mu \otimes 1_B) \cdot (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X) &= m \cdot (1_B \otimes \mu), \\
 m \cdot (\triangleright \otimes 1_X) \cdot (\mu \otimes 1_X \otimes 1_X) \cdot (1_X \otimes \sigma) \cdot (\Delta \otimes 1_X) &= m, \\
 m \cdot (1_B \otimes m) \cdot (\mu \otimes 1_B \otimes 1_B) &= m \cdot (m \otimes 1_B) \cdot (\mu \otimes 1_B \otimes 1_B), \\
 m \cdot (1_B \otimes m) \cdot (1_B \otimes \mu \otimes 1_B) &= m \cdot (m \otimes 1_B) \cdot (1_B \otimes \mu \otimes 1_B).
 \end{aligned} \tag{6.9}$$

In other words, a crossed module of (non-associative) Hopf algebras is a precrossed module of (non-associative) Hopf algebras such that (6.9) is satisfied, which corresponds to the so-called *Peiffer condition*.

We can observe that from a crossed module of (non-associative) Hopf algebras, we can construct an  $S$ -reflexive graph of (non-associative) Hopf algebras satisfying an additional property. For the following results and definition, we consider Hopf algebras in a symmetric monoidal category which has equalizers.

**Proposition 6.0.11.** *From a crossed module of (non-associative) Hopf algebras, the  $S$ -reflexive graph of (non-associative) Hopf algebras obtained by the functor of Proposition 6.0.6, i.e.*

$$X \begin{array}{c} \xleftarrow{\pi_1} \\ \xrightarrow{i_1} \end{array} X \rtimes B \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{i_2} \\ \xrightarrow{\tilde{\beta}} \end{array} B ,$$

where  $\tilde{\beta} := m \cdot (\mu \otimes 1_B): X \rtimes B \rightarrow B$ , satisfies the following property

$$m_{X \rtimes B} \cdot (i_1 \otimes \kappa_{\tilde{\beta}}) = m_{X \rtimes B} \cdot \sigma_{XB, XB} \cdot (i_1 \otimes \kappa_{\tilde{\beta}}),$$

where  $\kappa_{\tilde{\beta}}$  stands for the following equalizer in  $\mathcal{C}$

$$K_{\tilde{\beta}} \xrightarrow{\kappa_{\tilde{\beta}}} X \otimes B \begin{array}{c} \xrightarrow{u_B \otimes 1_{X \otimes B}} \\ \xrightarrow{(\tilde{\beta} \otimes 1_{X \otimes B}) \cdot \Delta_{X \otimes B}} \end{array} B \otimes X \otimes B .$$

A first interesting perspective would be to understand this particular kind of structure, i.e. an  $S$ -reflexive graph  $X \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\kappa} \end{array} A \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{e} \\ \xrightarrow{\beta} \end{array} B$  of (non-associative) Hopf algebras satisfying the condition

$$m \cdot (\kappa \otimes \kappa_{\beta}) = m \cdot \sigma \cdot (\kappa \otimes \kappa_{\beta}), \quad (6.10)$$

where  $\kappa_{\beta}$  stands for the following equalizer in  $\mathcal{C}$

$$K_{\beta} \xrightarrow{\kappa_{\beta}} A \begin{array}{c} \xrightarrow{u_B \otimes 1_A} \\ \xrightarrow{(\beta \otimes 1_A) \cdot \Delta} \end{array} B \otimes A .$$

**Remark 6.0.12.** *There is a converse result of Proposition 6.0.11. Indeed, we can show that from an  $S$ -reflexive graph of (non-associative) Hopf algebras satisfying the condition (6.10), we can construct a crossed module of (non-associative) Hopf algebras. It can also be shown that there is an equivalence of categories between the crossed modules and these internal structures.*

As explained in Chapter 3, G. Janelidze [57] defined internal crossed modules and proved that they are equivalent to the internal groupoids in any semi-abelian category. In a Mal'tsev category, it is well-known [27] that, for a reflexive graph, it is equivalent to being a reflexive multiplicative graph, an internal category, an internal groupoid, or a reflexive graph having a double centralizing equivalence relation on the kernel pairs of its two “legs” (domain and codomain arrows). Moreover, when the category has the so-called “Smith is Huq” condition, the condition of centralization for equivalence relations exactly corresponds to the centralization of their associated normal subobjects, i.e. the Huq commutator is trivial. (see Proposition 1.6.9).

In particular, in the context of cocommutative Hopf algebras, the above structures are described in Proposition 3.2.3 and they are all equivalent.

Hence, it would be interesting to understand if the  $S$ -reflexive graphs satisfying (6.10) can be linked to the centrality of the kernels in the sense of Huq and to the internal groupoids of general Hopf algebras.

## Crossed modules of bialgebras

Another natural perspective is the study of the precrossed modules and crossed modules of (non-associative) bialgebras. We are confident that we can easily obtain the same equivalence between precrossed modules of (non-associative) bialgebras and  $S$ -reflexive graphs of (non-associative) bialgebras as in Chapter 6. The definitions of precrossed modules of bialgebras and  $S$ -reflexive graphs of bialgebras would be as expected, in the sense that we would have the following definitions.

**Definition 6.0.13.** *A precrossed module of (non-associative) bialgebras is given by a pair  $(\mu: X \rightarrow B, \triangleright: B \otimes X \rightarrow X)$  where  $\mu$  is a morphism*

of (non-associative) bialgebras and  $\triangleright$  is an action of (non-associative) bialgebras such that

$$\begin{aligned} m \cdot (\mu \otimes 1_B) \cdot (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X) &= m \cdot (1_B \otimes \mu), \\ m \cdot (1_B \otimes m) \cdot (\mu \otimes 1_B \otimes 1_B) &= m \cdot (m \otimes 1_B) \cdot (\mu \otimes 1_B \otimes 1_B), \\ m \cdot (1_B \otimes m) \cdot (1_B \otimes \mu \otimes 1_B) &= m \cdot (m \otimes 1_B) \cdot (1_B \otimes \mu \otimes 1_B). \end{aligned}$$

**Definition 6.0.14.** An  $S$ -reflexive graph of (non-associative) bialgebras is the following graph

$$X \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\kappa} \end{array} A \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} B, \quad (6.11)$$

where  $X, A, B$  are (non-associative) bialgebras,  $\kappa, \alpha, \beta$  and  $e$  are morphisms of bialgebras satisfying the following identity

$$\beta \cdot e = 1_B$$

and  $X \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\kappa} \end{array} A \begin{array}{c} \xleftarrow{e} \\ \xrightarrow{\alpha} \end{array} B$  is a split extension of (non-associative) bialgebras (as in Definition 4.1.3).

The definition of crossed modules of associative bialgebras would be very similar to the definition of crossed modules of Hopf algebras. Indeed, if we define a crossed module of associative bialgebras exactly as a crossed module of associative Hopf algebras, with “bialgebras” replacing “Hopf algebras”, then we can prove Proposition 6.0.11. For the non-associative bialgebras case, if we do not add any condition in the definition of crossed modules of non-associative bialgebras, Proposition 6.0.11 does not hold.

A solution to obtain this result for non-associative bialgebras is asking that a crossed module satisfies the following equation:

$$\triangleright \cdot (m \otimes 1_X) \cdot (\mu \otimes 1_B \otimes 1_X) = \triangleright \cdot (1_B \otimes \triangleright) \cdot (\mu \otimes 1_B \otimes 1_X).$$

An appropriate definition of a crossed module of (non-associative) bialgebras could be the following one:

**Definition 6.0.15.** A crossed module of bialgebras is given by  $(\mu: X \rightarrow B, \triangleright: B \otimes X \rightarrow X)$  where  $\mu$  is a morphism of bialgebras and  $\triangleright$  is an action of bialgebras such that

$$m \cdot (\mu \otimes 1_B) \cdot (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma) \cdot (\Delta \otimes 1_X) = m \cdot (1_B \otimes \mu),$$

$$\begin{aligned}
m \cdot (\triangleright \otimes 1_X) \cdot (\mu \otimes 1_X \otimes 1_X) \cdot (1_X \otimes \sigma) \cdot (\Delta \otimes 1_X) &= m, \\
m \cdot (1_B \otimes m) \cdot (\mu \otimes 1_B \otimes 1_B) &= m \cdot (m \otimes 1_B) \cdot (\mu \otimes 1_B \otimes 1_B), \\
m \cdot (1_B \otimes m) \cdot (1_B \otimes \mu \otimes 1_B) &= m \cdot (m \otimes 1_B) \cdot (1_B \otimes \mu \otimes 1_B), \\
\triangleright \cdot (m \otimes 1_X) \cdot (\mu \otimes 1_B \otimes 1_X) &= \triangleright \cdot (1_B \otimes \triangleright) \cdot (\mu \otimes 1_B \otimes 1_X).
\end{aligned}$$

We will need to define an appropriate “internal structure” in the category of (non-associative) bialgebras to obtain an equivalence of categories.

Another interesting perspective would be to explore the paper [90] (see also [7]), where the author studies crossed modules of monoids whose domain is a group. By considering crossed modules, where the domain is a Hopf algebra, we may obtain an easier equivalence as the one we expect in the category of (non-associative) bialgebras.

## Crossed squares and 2-crossed modules

In [40], the author studied the notion of *2-crossed module of cocommutative Hopf algebras*, which is close to the concept of crossed square. He defined a concept which unifies the 2-crossed modules of groups and Lie algebras. Moreover, similarly to the case of groups and Lie algebras, he showed an equivalence between the category of simplicial cocommutative Hopf algebras with Moore complex of length 2 and the category of 2-crossed modules of cocommutative Hopf  $K$ -algebras. We know that this notion and the notion of Hopf crossed square are not equivalent. Indeed, in the particular case of groups, Conduché [32] explained that from a crossed square we can form two crossed 2-modules whose associated simplicial groups (see Proposition 3.4 in [32]) are homotopic, but they are not isomorphic. However, it would be interesting to study the link between these two generalizations of crossed modules in two dimensions.

Another possibility would be to go one step further and show an equivalence between  $\text{cat}^n$ -Hopf  $K$ -algebras and crossed  $\text{cube}^n$  Hopf  $K$ -algebras. Indeed, similarly to what was observed in Lemma 3.3.1, the construction can be repeated in any dimension.

## Non-abelian tensor product of cocommutative Hopf algebras

Another interesting perspective would be to describe the non-abelian tensor products of cocommutative Hopf  $K$ -algebras by following the approach using the crossed squares given in [35]. The authors used the concept of crossed squares in semi-abelian categories to obtain an intrinsic approach to the non-abelian tensor products. The notions of non-abelian tensor products of groups [22] and Lie algebras [38] would be special cases of this notion of non-abelian tensor products of cocommutative Hopf  $K$ -algebras. As a first step, it would be interesting to study the internal notion of compatible actions defined in any semi-abelian category (which satisfies the *Smith is Huq* property) [34], in the category of cocommutative Hopf  $K$ -algebras. This notion is strongly related to the crossed modules.

## $S$ -protomodularity

In the category of groups, split extensions have a lot of interesting properties. One of them is the fact that the category of split extensions is equivalent to the category of group actions. Unfortunately, this does not hold in any category, for example the category of monoids does not have this property. The category of groups is a protomodular category, while the category of monoids is not. However, the class of the *Schreier split epimorphisms* turned out to have interesting properties, similar to the points in groups. For example, they are equivalent to the actions of monoids, the Split Short Five Lemma holds, etc. [20]. The authors in [20], introduced the notion of  $S$ -protomodularity, that is the protomodularity with respect to a class  $S$  of split epimorphisms. The main example is the category of monoids with the class of *Schreier split epimorphisms*.

In [45], the authors showed that the category of  $K$ -bialgebras, the category of Hopf  $K$ -algebras, and even the category of cocommutative  $K$ -bialgebras are not protomodular categories. Nevertheless, in the category of (non-associative) Hopf algebras or (non-associative) bialgebras, we have a class of split epimorphisms that has interesting properties. In particular, the Split Short Five Lemma holds when it is restricted to the split extensions of (non-associative) bialgebras or (non-associative) Hopf algebras. Hence, it seems natural to expect that the categories

of (non-associative) Hopf algebras and (non-associative) bialgebras are  $S$ -protomodular categories, for the class of split extensions defined in Chapter 4 and Chapter 5, respectively.

To be an  $S$ -protomodular category, the class  $S$  of split epimorphisms needs to satisfy some properties. Among them, the fact that any split epimorphism in  $S$  has to be a *strong point* and that the class  $S$  has to be stable under pullbacks (and under all the finite limits). These conditions are not trivial. In particular, the product of two general bialgebras (or Hopf algebras) is not given by the monoidal product, but rather by a much more difficult construction using the cofree coalgebra construction (see [1] in the category  $\mathbf{Vect}_K$ ). One perspective would be to understand this construction and then prove the  $S$ -protomodularity.

With this result, we could apply the theory of  $S$ -protomodular categories to the categories of (non-associative) Hopf algebras and bialgebras. For example, these categories would be  $S$ -Mal'tsev [14] and we could obtain a  $3 \times 3$  lemma for the split extensions of (non-associative) Hopf algebras and bialgebras [21].

It would be also interesting to study other properties, similar to the ones observed in [20] and [47], for the split extensions of (non-associative) bialgebras and Hopf algebras. In particular, we could examine the composition of such split extensions.



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