Co-smash products in semi-abelian algebra

Tim Van der Linden

Fonds de la Recherche Scientifique–FNRS
Université catholique de Louvain

Category Theory 2013
Sydney — 10th of July

Co-smash product

smash product

The smash product of pointed topological spaces

$$(X,x) + (Y,y) \longrightarrow (X,x) \times (Y,y) \longrightarrow\!\!\!\triangleright \, {\color{red}(X,x) \wedge (Y,y)} \longrightarrow 0$$

Example: smash product of one-spheres

$S^1 + S^1$

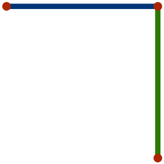
$S^1 \times S^1$

$S^1 \wedge S^1 \cong S^2$

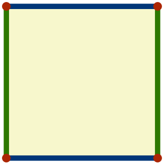
The smash product of pointed topological spaces

$$(X,x) + (Y,y) \longrightarrow (X,x) \times (Y,y) \longrightarrow (X,x) \wedge (Y,y) \longrightarrow 0$$

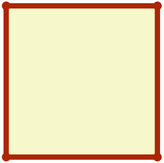
Example: smash product of one-spheres



$S^1 + S^1$



$S^1 \times S^1$



$S^1 \wedge S^1 \cong S^2$

The smash product of pointed objects in a lextensive category

[A Carboni & G Janelidze, 2003]

Let $\mathbb{X}$ be a category with a terminal object 1;
consider $(X, x\colon 1 \to X)$ and $(Y, y\colon 1 \to Y)$ in $(1 \downarrow \mathbb{X})$;
then

$$(X,x) +_1 (Y,y) \longrightarrow (X,x) \times (Y,y) \longrightarrow\!\!\!\triangleright\, (X,x) \wedge (Y,y) \longrightarrow 0$$

makes sense as soon as $\mathbb{X}$ has $+_1$ and $\times$ and cokernels.

Question

When is $\wedge$ part of a monoidal structure on $(1 \downarrow \mathbb{X})$?
In particular, when is $\wedge$ (canonically) associative?

Answer

1. When $\mathbb{X}$ is **lextensive** (= finitely complete with pullback-stable and disjoint sums) and products in it preserve certain pushouts
2. An important example is $\mathsf{CAlg}_R^{\mathrm{op}} = (1 \downarrow \mathsf{UCAlg}_R^{\mathrm{op}})$,
so $\mathbb{X} = \mathsf{UCAlg}_R^{\mathrm{op}} = (R \downarrow \mathsf{CRing})^{\mathrm{op}}$, in which case $\wedge = \otimes$

The smash product of pointed objects in a lextensive category

[A Carboni & G Janelidze, 2003]

Let $\mathbb{X}$ be a category with a terminal object 1;
consider $(X, x\colon 1 \to X)$ and $(Y, y\colon 1 \to Y)$ in $(1 \downarrow \mathbb{X})$;
then

$$(X,x) +_1 (Y,y) \longrightarrow (X,x) \times (Y,y) \longrightarrow (X,x) \wedge (Y,y) \longrightarrow 0$$

makes sense as soon as $\mathbb{X}$ has $+_1$ and $\times$ and cokernels.

Question

When is $\wedge$ part of a monoidal structure on $(1 \downarrow \mathbb{X})$?
In particular, when is $\wedge$ (canonically) associative?

Answer

1. When $\mathbb{X}$ is **lextensive** (= finitely complete with pullback-stable and disjoint sums) and products in it preserve certain pushouts
2. An important example is $\mathsf{CAlg}_R^{\mathrm{op}} = (1 \downarrow \mathsf{UCAlg}_R^{\mathrm{op}})$, so $\mathbb{X} = \mathsf{UCAlg}_R^{\mathrm{op}} = (R \downarrow \mathsf{CRing})^{\mathrm{op}}$, in which case $\wedge = \otimes$

The smash product of pointed objects in a lextensive category

[A Carboni & G Janelidze, 2003]

Let $\mathbb{X}$ be a category with a terminal object 1;
consider $(X, x\colon 1 \to X)$ and $(Y, y\colon 1 \to Y)$ in $(1 \downarrow \mathbb{X})$;
then

$$(X,x) +_1 (Y,y) \longrightarrow (X,x) \times (Y,y) \longrightarrow (X,x) \wedge (Y,y) \longrightarrow 0$$

makes sense as soon as $\mathbb{X}$ has $+_1$ and $\times$ and cokernels.

Question

When is $\wedge$ part of a monoidal structure on $(1 \downarrow \mathbb{X})$?
In particular, when is $\wedge$ (canonically) associative?

Answer

1. When $\mathbb{X}$ is **lextensive** (= finitely complete with pullback-stable and disjoint sums) and products in it preserve certain pushouts
2. An important example is $\mathsf{CAlg}_R^{\mathrm{op}} = (1 \downarrow \mathsf{UCAlg}_R^{\mathrm{op}})$, so $\mathbb{X} = \mathsf{UCAlg}_R^{\mathrm{op}} = (R \downarrow \mathsf{CRing})^{\mathrm{op}}$, in which case $\wedge = \otimes$

The case of commutative R-algebras

[A Carboni & G Janelidze, 2003]

Of course the smash product in $\mathrm{CAlg}_R^{\mathrm{op}}$
is nothing but a **co-**smash product in CAlg_R:

$$0 \longrightarrow X \wedge Y \rightarrowtriangle X + Y \longrightarrow X \times Y$$

This co-smash product is in fact the tensor product, because

- $X \times Y = X \oplus Y$
- $X + Y = X \oplus (X \otimes Y) \oplus Y$

What happens outside commutative algebras?
…in general semi-abelian categories?

The case of commutative R-algebras

[A Carboni & G Janelidze, 2003]

Of course the smash product in $\mathrm{CAlg}_R^{\mathrm{op}}$
is nothing but a **co-**smash product in CAlg_R:

$$0 \longrightarrow X \otimes Y \rightarrowtriangle X + Y \longrightarrow X \times Y$$

This co-smash product is in fact the tensor product, because

- $X \times Y = X \oplus Y$
- $X + Y = X \oplus (X \otimes Y) \oplus Y$

What happens outside commutative algebras?
...in general semi-abelian categories?

The case of commutative R-algebras

[A Carboni & G Janelidze, 2003]

Of course the smash product in $\mathrm{CAlg}_R^{\mathrm{op}}$
is nothing but a **co-**smash product in CAlg_R:

$$0 \longrightarrow X \otimes Y \rightarrowtail X + Y \longrightarrow X \times Y$$

This co-smash product is in fact the tensor product, because

- $X \times Y = X \oplus Y$
- $X + Y = X \oplus (X \otimes Y) \oplus Y$

What happens outside commutative algebras?
…in general semi-abelian categories?

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In semi-abelian categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtriangle X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In **semi-abelian** categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtriangle X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In semi-abelian categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtriangle X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In semi-abelian categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtriangle X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y \longrightarrow 0$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In semi-abelian categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtriangle X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y \longrightarrow 0$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In semi-abelian categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtail X + Y \xrightarrow{\left(\begin{smallmatrix} 1_X & 0 \\ 0 & 1_Y \end{smallmatrix}\right)} X \times Y \longrightarrow 0$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In semi-abelian categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtriangle X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y \longrightarrow 0$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In semi-abelian categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtail X + Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}} X \times Y \longrightarrow 0$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

Co-smash products in semi-abelian categories

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

In semi-abelian categories, co-smash products are **formal commutators**.

$$0 \longrightarrow X \otimes Y \rightarrowtail X + Y \xrightarrow{\left(\begin{smallmatrix} 1_X & 0 \\ 0 & 1_Y \end{smallmatrix}\right)} X \times Y \longrightarrow 0$$

- We keep the $\otimes$ notation. Sometimes $X \otimes Y$ is written $X \diamond Y$.
- In Gp, the object $X \otimes Y$
  - contains all $xyx^{-1}y^{-1}$
  - is actually generated as a subgroup by all such.
- These are used to extend the **Higgins commutator** [Higgins, 1956] from varieties of Ω-groups to semi-abelian categories.
- They can be used to characterise actions—see G Janelidze's talk.

The (binary) Higgins commutator

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider $K, L \leqslant X$ in a semi-abelian category.

$$
\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] & K+L \arrow[r, two heads, "\left(\begin{smallmatrix} 1_K & 0 \\ 0 & 1_L \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)"] & K \times L \arrow[r] & 0 \\
& & X & &
\end{tikzcd}
$$

- $[K, L]$ is called the **Higgins commutator** of K and L.
- $[K, L] = 0$ if and only if $\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ lifts over $K \times L$, so that k and l *cooperate* as in [F Borceux & D Bourn, 2004]. In fact, Borceux and Bourn do something slightly different…
- The Higgins commutator characterises normality of subobjects:
  - $K \leqslant X$ is normal in X if and only if $[K, X] \leqslant K$;
  - more precisely, the normal closure of $K \leqslant X$ is $K \vee [K, X]$.

The (binary) Higgins commutator

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider $K, L \leqslant X$ in a semi-abelian category.

$$\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, two heads, "\left(\begin{smallmatrix} 1_K & 0 \\ 0 & 1_L \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)"] & K \times L \arrow[r] & 0 \\
& {[K, L]} \arrow[r, tail] & X & &
\end{tikzcd}$$

- $[K, L]$ is called the **Higgins commutator** of K and L.
- $[K, L] = 0$ if and only if $\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ lifts over $K \times L$,
  so that k and l *cooperate* as in [F Borceux & D Bourn, 2004].
  In fact, Borceux and Bourn do something slightly different…
- The Higgins commutator characterises normality of subobjects:
  - $K \leqslant X$ is normal in X if and only if $[K, X] \leqslant K$;
  - more precisely, the normal closure of $K \leqslant X$ is $K \vee [K, X]$.

The (binary) Higgins commutator

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider K, $L \leqslant X$ in a semi-abelian category.

$$\begin{tikzcd} 0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, two heads, "\left(\begin{smallmatrix} 1_K & 0 \\ 0 & 1_L \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)"] & K \times L \arrow[r] & 0 \\ & {[K, L]} \arrow[r, tail] & X & & \end{tikzcd}$$

- $[K, L]$ is called the **Higgins commutator** of K and L.
- $[K, L] = 0$ if and only if $\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ lifts over $K \times L$, so that k and l *cooperate* as in [F Borceux & D Bourn, 2004]. In fact, Borceux and Bourn do something slightly different…
- The Higgins commutator characterises normality of subobjects:
  - $K \leqslant X$ is normal in X if and only if $[K, X] \leqslant K$;
  - more precisely, the normal closure of $K \leqslant X$ is $K \vee [K, X]$.

The (binary) Higgins commutator

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider $K, L \leqslant X$ in a semi-abelian category.

$$\begin{tikzcd} 0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, two heads, "\left(\begin{smallmatrix} 1_K & 0 \\ 0 & 1_L \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)"] & K \times L \arrow[r] & 0 \\ & {[K, L]} \arrow[r, tail] & X & & \end{tikzcd}$$

- $[K, L]$ is called the **Higgins commutator** of K and L.
- $[K, L] = 0$ if and only if $\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ lifts over $K \times L$, so that k and l *cooperate* as in [F Borceux & D Bourn, 2004]. In fact, Borceux and Bourn do something slightly different…
- The Higgins commutator characterises normality of subobjects:
  - $K \leqslant X$ is normal in X if and only if $[K, X] \leqslant K$;
  - more precisely, the normal closure of $K \leqslant X$ is $K \vee [K, X]$.

The (binary) Higgins commutator

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider $K, L \leqslant X$ in a semi-abelian category.

$$\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, two heads, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\begin{pmatrix} k \\ l \end{pmatrix}"] & K \times L \arrow[r] \arrow[dl, bend left, red] & 0 \\
 & {[K, L]} \arrow[r, tail] & X & &
\end{tikzcd}$$

- $[K, L]$ is called the **Higgins commutator** of K and L.
- $[K, L] = 0$ if and only if $\begin{pmatrix} k \\ l \end{pmatrix}$ lifts over $K \times L$, so that k and l *cooperate* as in [F Borceux & D Bourn, 2004].

  In fact, Borceux and Bourn do something slightly different…
- The Higgins commutator characterises normality of subobjects:
  - $K \leqslant X$ is normal in X if and only if $[K, X] \leqslant K$;
  - more precisely, the normal closure of $K \leqslant X$ is $K \vee [K, X]$.

The (binary) Higgins commutator

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider $K, L \leqslant X$ in a semi-abelian category.

$$\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K+L \arrow[r, two heads, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\begin{pmatrix} k \\ l \end{pmatrix}"] & K \times L \arrow[r] \arrow[dl, bend left, red] & 0 \\
& {[K,L]} \arrow[r, tail] & X & &
\end{tikzcd}$$

- $[K, L]$ is called the **Higgins commutator** of K and L.
- $[K, L] = 0$ if and only if $\begin{pmatrix} k \\ l \end{pmatrix}$ lifts over $K \times L$, so that k and l *cooperate* as in [F Borceux & D Bourn, 2004]. In fact, Borceux and Bourn do something slightly different…
- The Higgins commutator characterises normality of subobjects:
  - $K \leqslant X$ is normal in X if and only if $[K, X] \leqslant K$;
  - more precisely, the normal closure of $K \leqslant X$ is $K \vee [K, X]$.

The (binary) Higgins commutator

[S Mantovani & G Metere, 2010] [M Hartl & B Loiseau, 2013]

Consider $K, L \leqslant X$ in a semi-abelian category.

$$\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, "" , two heads] & K + L \arrow[r, two heads, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\binom{k}{l}"] & K \times L \arrow[r] & 0 \\
& {[K, L]} \arrow[r, tail] & X & &
\end{tikzcd}$$

- $[K, L]$ is called the **Higgins commutator** of K and L.
- $[K, L] = 0$ if and only if $\binom{k}{l}$ lifts over $K \times L$,
  so that k and l *cooperate* as in [F Borceux & D Bourn, 2004].
  In fact, Borceux and Bourn do something slightly different…
- The Higgins commutator characterises normality of subobjects:
  - $K \leqslant X$ is normal in X if and only if $[K, X] \leqslant K$;
  - more precisely, the normal closure of $K \leqslant X$ is $K \vee [K, X]$.

Higgins commutator vs. Huq commutator

Consider K, $L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

$$\begin{tikzcd} 0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, two heads, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\begin{pmatrix} k \\ l \end{pmatrix}"] & K \times L \arrow[r] & 0 \\ & {[K, L]} \arrow[r, tail] & X & & \end{tikzcd}$$

- $[K, L]_X$ is the **Huq commutator** of K and L [S A Huq, 1968].
- $[K, L]_X$ is the normal closure of $[K, L]$ in X, so that $[K, L]_X = [K, L] \vee [[K, L], X]$.
- In [A Cigoli, 2010] it is explained that when $K, L \lhd X$,
  - $[K, L] = [K, L]_X$ if $\mathbb{X}$ is a ***category of interest*** as in [G Orzech, 1972];
  - the equality need not hold in general: counterexample in NARng.

Higgins commutator vs. Huq commutator

Consider K, $L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

$$
\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K+L \arrow[r, two heads, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\begin{pmatrix} k \\ l \end{pmatrix}"] & K \times L \arrow[r] \arrow[d] & 0 \\
& {[K,L]} \arrow[r, tail] & X \arrow[r, two heads] & Q &
\end{tikzcd}
$$

- $[K,L]_X$ is the **Huq commutator** of K and L [S A Huq, 1968].
- $[K,L]_X$ is the normal closure of $[K,L]$ in X, so that $[K,L]_X = [K,L] \vee [[K,L],X]$.
- In [A Cigoli, 2010] it is explained that when $K, L \lhd X$,
  - $[K,L] = [K,L]_X$ if $\mathbb{X}$ is a ***category of interest*** as in [G Orzech, 1972];
  - the equality need not hold in general: counterexample in NARng.

Higgins commutator vs. Huq commutator

Consider $K, L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

$$\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, two heads, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\begin{pmatrix} k \\ l \end{pmatrix}"] & K \times L \arrow[r] \arrow[d] & 0 \\
 & {[K, L]} \arrow[r, tail] \arrow[d, tail] & X \arrow[r, two heads] & Q \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & 0 \\
0 \arrow[r] & {[K, L]_X} \arrow[ur, tail] & & &
\end{tikzcd}$$

- $[K, L]_X$ is the **Huq commutator** of K and L [S A Huq, 1968].
- $[K, L]_X$ is the normal closure of $[K, L]$ in X, so that $[K, L]_X = [K, L] \vee [[K, L], X]$.
- In [A Cigoli, 2010] it is explained that when $K, L \lhd X$,
  - $[K, L] = [K, L]_X$ if $\mathbb{X}$ is a ***category of interest*** as in [G Orzech, 1972];
  - the equality need not hold in general: counterexample in NARng.

Higgins commutator vs. Huq commutator

Consider K, $L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

$$\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, two heads, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\begin{pmatrix} k \\ l \end{pmatrix}"] & K \times L \arrow[r] \arrow[d] & 0 \\
 & {[K, L]} \arrow[r, tail] \arrow[d, tail] & X \arrow[r, two heads] & Q \arrow[r] & 0 \\
0 \arrow[r] & {[K, L]_X} \arrow[ur, tail] & & &
\end{tikzcd}$$

- $[K, L]_X$ is the **Huq commutator** of K and L [S A Huq, 1968].
- $[K, L]_X$ is the normal closure of $[K, L]$ in X, so that $[K, L]_X = [K, L] \vee [[K, L], X]$.
- In [A Cigoli, 2010] it is explained that when $K, L \lhd X$,
  - $[K, L] = [K, L]_X$ if $\mathbb{X}$ is a ***category of interest*** as in [G Orzech, 1972];
  - the equality need not hold in general: counterexample in NARng.

Higgins commutator vs. Huq commutator

Consider K, $L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

$$
\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}", two heads] \arrow[d, "\begin{pmatrix} k \\ l \end{pmatrix}"] & K \times L \arrow[r] \arrow[d] & 0 \\
& {[K, L]} \arrow[r, tail] \arrow[d, tail] & X \arrow[r, two heads] \arrow[ur, phantom, "\lrcorner", very near end] & Q \arrow[r] & 0 \\
0 \arrow[r] & {[K, L]_X} \arrow[ur, tail] & & &
\end{tikzcd}
$$

- $[K, L]_X$ is the **Huq commutator** of K and L [S A Huq, 1968].
- $[K, L]_X$ is the normal closure of $[K, L]$ in X, so that $[K, L]_X = [K, L] \vee [[K, L], X]$.
- In [A Cigoli, 2010] it is explained that when K, $L \lhd X$,
  - $[K, L] = [K, L]_X$ if $\mathbb{X}$ is a ***category of interest*** as in [G Orzech, 1972];
  - the equality need not hold in general: counterexample in NARng.

Higgins commutator vs. Huq commutator

Consider K, $L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

$$\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K + L \arrow[r, "\left(\begin{smallmatrix} 1_K & 0 \\ 0 & 1_L \end{smallmatrix}\right)", two heads] \arrow[d, "\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)"] & K \times L \arrow[r] \arrow[d] & 0 \\
& {[K, L]} \arrow[r, tail] \arrow[d, tail] & X \arrow[r, two heads] & Q \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & 0 \\
0 \arrow[r] & {[K, L]_X} \arrow[ur, tail] & & &
\end{tikzcd}$$

- $[K, L]_X$ is the **Huq commutator** of K and L [S A Huq, 1968].
- $[K, L]_X$ is the normal closure of $[K, L]$ in X, so that $[K, L]_X = [K, L] \vee [[K, L], X]$.
- In [A Cigoli, 2010] it is explained that when K, $L \lhd X$,
  - $[K, L] = [K, L]_X$ if $\mathbb{X}$ is a ***category of interest*** as in [G Orzech, 1972];
  - the equality need not hold in general: counterexample in NARng.

Higgins commutator vs. Huq commutator: condition **(NH)**

[A Cigoli, J R A Gray & TVdL, work-in-progress]

Consider K, $L \leqslant X$ in a semi-abelian category $\mathbb{X}$.

$$\begin{tikzcd}
0 \arrow[r] & K \otimes L \arrow[r, tail] \arrow[d, two heads] & K+L \arrow[r, two heads, "\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix}"] \arrow[d, "\binom{k}{l}"] & K \times L \arrow[r] \arrow[d] & 0 \\
 & {[K,L]} \arrow[r, tail] \arrow[d, tail] & X \arrow[r, two heads] \arrow[ur, phantom, "\lrcorner", very near end] & Q \arrow[r] & 0 \\
0 \arrow[r] & {[K,L]_X} \arrow[ur, tail] & & &
\end{tikzcd}$$

- $[K, L]_X$ is the **Huq commutator** of K and L [S A Huq, 1968].
- $[K, L]_X$ is the normal closure of $[K, L]$ in X, so that $[K, L]_X = [K, L] \vee [[K, L], X]$.
- In [A Cigoli, 2010] it is explained that when K, $L \lhd X$,
  - $[K, L] = [K, L]_X$ if $\mathbb{X}$ is a *category of interest* as in [G Orzech, 1972];
  - the equality need not hold in general: counterexample in NARng.

The Smith commutator

Smith

For equivalence relations R, S on X

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta_R" description] \arrow[r, "\Delta_S" description] & S, \arrow[l, shift right=2, "s_2"'] \arrow[l, shift left=2, "s_1"] \end{tikzcd}$$

the **Smith commutator** $[R,S]^{\mathrm{S}}$ is the kernel pair of t:

$$\begin{tikzcd} R+S \arrow[rr, "\begin{pmatrix} 1_R & \Delta_S\circ r_2 \\ \Delta_R\circ s_1 & 1_S \end{pmatrix}"] \arrow[d, "\binom{r_1}{s_2}"'] & & R\times_X S \arrow[d, dotted] \\ X \arrow[rr, dotted, "t"'] & & T \end{tikzcd}$$

Huq & **Higgins**

For $K, L \lhd X$, the **Huq commutator** $[K,L]_X$ is the kernel of q:

$$\begin{tikzcd} & K+L \arrow[r, -Triangle] \arrow[d, "\binom{k}{l}"'] & K\times L \arrow[d, dotted] \\ {[K,L]_X} \arrow[r, dotted, Triangle-] & X \arrow[r, dotted, "q"'] & Q \end{tikzcd}$$

The **Higgins commutator** $[K,L] \leqslant X$ is the image of $\binom{k}{l}\circ\iota_{K,L}$:

$$\begin{tikzcd} K\otimes L \arrow[r, "\iota_{K,L}", Triangle-] \arrow[d, dotted, -Triangle] & K+L \arrow[r, -Triangle] \arrow[d, "\binom{k}{l}"] & K\times L \\ {[K,L]} \arrow[r, dotted, tail] & X & \end{tikzcd}$$

The Smith commutator

Smith

For equivalence relations R, S on X

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta_R" description] \arrow[r, "\Delta_S" description] & S, \arrow[l, shift right=2, "s_2"'] \arrow[l, shift left=2, "s_1"] \end{tikzcd}$$

the **Smith commutator** $[R,S]^{\mathrm{S}}$ is the kernel pair of t:

$$\begin{tikzcd} R+S \arrow[r, "\begin{pmatrix} 1_R & \Delta_S \circ r_2 \\ \Delta_R \circ s_1 & 1_S \end{pmatrix}"] \arrow[d, "\binom{r_1}{s_2}"'] & R \times_X S \arrow[d, dotted] \\ X \arrow[r, dotted, "t"'] & T \end{tikzcd}$$

characterises internal groupoids amongst reflexive graphs

Huq & Higgins

For $K, L \lhd X$, the **Huq commutator** $[K,L]_X$ is the kernel of q:

$$\begin{tikzcd} & K+L \arrow[r] \arrow[d, "\binom{k}{l}"'] & K \times L \arrow[d, dotted] \\ {[K,L]_X} \arrow[r, dotted, tail] & X \arrow[r, dotted, "q"'] & Q \end{tikzcd}$$

The **Higgins commutator** $[K,L] \leqslant X$ is the image of $\binom{k}{l} \circ \iota_{K,L}$:

$$\begin{tikzcd} K \otimes L \arrow[r, "\iota_{K,L}"] \arrow[d, dotted] & K+L \arrow[r] \arrow[d, "\binom{k}{l}"] & K \times L \\ {[K,L]} \arrow[r, dotted, tail] & X & \end{tikzcd}$$

The Smith commutator

Smith

For equivalence relations R, S on X

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta_R" description] \arrow[r, "\Delta_S" description] & S, \arrow[l, shift right=2, "s_2"'] \arrow[l, shift left=2, "s_1"] \end{tikzcd}$$

the **Smith commutator** $[R, S]^{\mathrm{S}}$ is the kernel pair of t:

$$\begin{tikzcd} R+S \arrow[r, "\left(\begin{smallmatrix} 1_R & \Delta_S \circ r_2 \\ \Delta_R \circ s_1 & 1_S \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} r_1 \\ s_2 \end{smallmatrix}\right)"'] & R \times_X S \arrow[d, dotted] \\ X \arrow[r, dotted, "t"'] & T \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

characterises internal groupoids amongst reflexive graphs

Huq & Higgins

For $K, L \lhd X$, the **Huq commutator** $[K, L]_X$ is the kernel of q:

$$\begin{tikzcd} & K+L \arrow[r, -{Triangle[open]}] \arrow[d, "\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)"'] & K \times L \arrow[d, dotted, -{Triangle[open]}] \\ {[K,L]_X} \arrow[r, dotted, {Triangle[open, reversed]}->] & X \arrow[r, dotted, "q"'] & Q \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The **Higgins commutator** $[K, L] \leqslant X$ is the image of $\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right) \circ \iota_{K,L}$:

$$\begin{tikzcd} K \otimes L \arrow[r, {Triangle[open, reversed]}->, "\iota_{K,L}"] \arrow[d, dotted, -{Triangle[open]}] & K+L \arrow[r, -{Triangle[open]}] \arrow[d, "\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)"] & K \times L \\ {[K,L]} \arrow[r, dotted, >->] & X & \end{tikzcd}$$

The *Smith is Huq* condition (SH)

$$K \xrightarrow{r_1 \circ \ker(r_2)} X \xleftarrow{s_1 \circ \ker(s_2)} L \quad \text{normalisations of} \quad \begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X & S \arrow[l, shift right, "s_2"'] \arrow[l, shift left, "s_1"] \end{tikzcd}$$

$$[K, L]^{\mathrm{S}} \quad \text{normalisation of} \quad [R, S]^{\mathrm{S}}$$

- $[K, L]^{\mathrm{S}} = 0$ implies $[K, L]_X = 0$ [D Bourn & M Gran, 2002]
- The converse is the **Smith is Huq** condition $[K, L]^{\mathrm{S}} = [K, L]_X$,
- which does not hold for *digroups* or *loops*.

Under (SH), $\begin{tikzcd} X \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G \arrow[l, "e" description] \end{tikzcd}$ is a groupoid iff $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0$.

Theorem [N Martins–Ferreira & TVdL, 2010]
(SH) $\Leftrightarrow$ every internal $*$-multiplicative graph is an internal groupoid

$$\begin{tikzcd} & \cdot \arrow[dl, "\beta"'] & \\ \cdot & & 0 \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl, "\beta"'] & \\ \cdot & & \cdot \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd}$$

Hence under (SH), the "simplified description"
of internal crossed modules from [G Janelidze, 2003] is valid.

What happens when (SH) does *not* hold?

The *Smith is Huq* condition (SH)

$$K \xrightarrow{r_1 \circ \ker(r_2)} X \xleftarrow{s_1 \circ \ker(s_2)} L \quad \text{normalisations of} \quad \begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X & S \arrow[l, shift right, "s_2"'] \arrow[l, shift left, "s_1"] \end{tikzcd}$$

$$[K, L]^{\mathsf{S}} \quad \text{normalisation of} \quad [R, S]^{\mathsf{S}}$$

- $[K, L]^{\mathsf{S}} = 0$ implies $[K, L]_X = 0$ [D Bourn & M Gran, 2002]
- The converse is the **Smith is Huq** condition $[K, L]^{\mathsf{S}} = [K, L]_X$,
- which does not hold for *digroups* or *loops*.

Under (SH), $\begin{tikzcd} X \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G \arrow[l, "e" description] \end{tikzcd}$ is a groupoid iff $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0$.

Theorem [N Martins–Ferreira & TVdL, 2010]
(SH) $\Leftrightarrow$ every internal $*$-multiplicative graph is an internal groupoid

$$\begin{tikzcd} & \cdot \arrow[dl, "\beta"'] & \\ \cdot & & 0 \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl, "\beta"'] & \\ \cdot & & \cdot \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd}$$

Hence under (SH), the “simplified description”
of internal crossed modules from [G Janelidze, 2003] is valid.

What happens when (SH) does *not* hold?

The *Smith is Huq* condition (SH)

$$K \overset{r_1 \circ \ker(r_2)}{\rightarrowtail} X \overset{s_1 \circ \ker(s_2)}{\leftarrowtail} L \quad \text{normalisations of} \quad R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \underset{s_1}{\overset{s_2}{\leftleftarrows}} S$$

$$[K, L]^{\mathrm{S}} \quad \text{normalisation of} \quad [R, S]^{\mathrm{S}}$$

- $[K, L]^{\mathrm{S}} = 0$ implies $[K, L]_X = 0$ [D Bourn & M Gran, 2002]
- The converse is the **Smith is Huq** condition $[K, L]^{\mathrm{S}} = [K, L]_X$,
- which does not hold for *digroups* or *loops*.

Under (SH), $\begin{tikzcd} X \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G \arrow[l, "e" description] \end{tikzcd}$ is a groupoid iff $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0$.

Theorem [N Martins–Ferreira & TVdL, 2010]
(SH) $\Leftrightarrow$ every internal $*$-multiplicative graph is an internal groupoid

$$\begin{tikzcd} & \cdot \arrow[dl, "\beta"'] & \\ \cdot & & 0 \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl, "\beta"'] & \\ \cdot & & \cdot \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd}$$

Hence under (SH), the "simplified description"
of internal crossed modules from [G Janelidze, 2003] is valid.

What happens when (SH) does *not* hold?

The *Smith is Huq* condition (SH)

$$K \xrightarrow{r_1\circ\ker(r_2)} X \xleftarrow{s_1\circ\ker(s_2)} L \qquad \text{normalisations of} \qquad \begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X & S \arrow[l, shift right, "s_2"'] \arrow[l, shift left, "s_1"] \end{tikzcd}$$

$$[K, L]^{\mathrm{S}} \qquad \text{normalisation of} \qquad [R, S]^{\mathrm{S}}$$

- $[K, L]^{\mathrm{S}} = 0$ implies $[K, L]_X = 0$ [D Bourn & M Gran, 2002]
- The converse is the **Smith is Huq** condition $[K, L]^{\mathrm{S}} = [K, L]_X$,
- which does not hold for *digroups* or *loops*.

Under (SH), $\begin{tikzcd} X & G \arrow[l, shift right=2, "d"'] \arrow[l, shift left=2, "c"] \arrow[from=l, "e"] \end{tikzcd}$ is a groupoid iff $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0$.

Theorem [N Martins–Ferreira & TVdL, 2010]

(SH) $\Leftrightarrow$ every internal $*$-multiplicative graph is an internal groupoid

$$\begin{tikzcd} & \cdot \arrow[dl, "\beta"'] & \\ \cdot & & 0 \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl, "\beta"'] & \\ \cdot & & \cdot \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd}$$

Hence under (SH), the "simplified description" of internal crossed modules from [G Janelidze, 2003] is valid.

What happens when (SH) does *not* hold?

Associativity: towards higher Higgins commutators

[A Carboni & G Janelidze, 2003] [M Hartl & TVdL, 2013]

The higher-order co-smash product

$$\begin{tikzcd} \bigotimes_{k=1}^{n} X_k \arrow[r, tail] & \coprod_{k=1}^{n} X_k \arrow[r, "r"] & \prod_{k=1}^{n} \coprod_{j\neq k} X_j \end{tikzcd}$$

allows us to express associativity canonically (here for $n=3$):

$$\begin{tikzcd} 0 \arrow[r] & (X \otimes Y) \otimes Z \arrow[r, tail, "\iota_{X\otimes Y,Z}"] \arrow[d, dotted] & (X\otimes Y) + Z \arrow[r] \arrow[d, "\iota_{X,Y}+1_Z"] & (X\otimes Y) \times Z \arrow[d, "\iota_{X,Y}\times(\iota_Z,\iota_Z)"] \\ 0 \arrow[r] & X \otimes Y \otimes Z \arrow[r, tail, "\iota_{X,Y,Z}"'] & X+Y+Z \arrow[r] & (X+Y)\times(X+Z)\times(Y+Z) \end{tikzcd}$$

This gives us **higher-order Higgins commutators**: for $K, L, M \leqslant X$,

$$\begin{tikzcd} 0 \arrow[r] & K \otimes L \otimes M \arrow[r, tail] \arrow[d] & K+L+M \arrow[d, "\begin{pmatrix} k \\ l \\ m \end{pmatrix}"] \\ & {[K,L,M]} \arrow[r, tail] & X \end{tikzcd}$$

Associativity: towards higher Higgins commutators

[A Carboni & G Janelidze, 2003] [M Hartl & TVdL, 2013]

The higher-order co-smash product

$$\bigotimes_{k=1}^{n} X_k \rightarrowtail \coprod_{k=1}^{n} X_k \xrightarrow{r} \prod_{k=1}^{n} \coprod_{j\neq k} X_j$$

allows us to express associativity canonically (here for $n = 3$):

$$\begin{tikzcd}
0 \arrow[r] & (X\otimes Y)\otimes Z \arrow[r, tail, "\iota_{X\otimes Y,Z}"] \arrow[d, dotted] & (X\otimes Y)+Z \arrow[r] \arrow[d, "\iota_{X,Y}+1_Z"] & (X\otimes Y)\times Z \arrow[d, "\iota_{X,Y}\times(\iota_Z,\iota_Z)"] \\
0 \arrow[r] & X\otimes Y\otimes Z \arrow[r, tail, "\iota_{X,Y,Z}"'] & X+Y+Z \arrow[r] & (X+Y)\times(X+Z)\times(Y+Z)
\end{tikzcd}$$

This gives us **higher-order Higgins commutators**: for $K, L, M \leqslant X$,

$$\begin{tikzcd}
0 \arrow[r] & K\otimes L\otimes M \arrow[r, tail] \arrow[d, two heads] & K+L+M \arrow[d, "\begin{pmatrix} k \\ l \\ m \end{pmatrix}"] \\
& {[K, L, M]} \arrow[r, tail] & X
\end{tikzcd}$$

Groupoids and crossed modules via co-smash products

[M Hartl & TVdL, 2013]

Theorem

If $K, L \lhd X$ then $[K, L]^{\mathrm{S}} = [K, L] \vee [K, L, X]$. Hence a reflexive graph

$$\begin{tikzcd} X \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] & G \arrow[l, "e" description] \end{tikzcd}$$

is a groupoid if and only if $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0 = [\mathrm{Ker}(d), \mathrm{Ker}(c), X]$.

Theorem

An internal crossed module is determined by a quadruple (G, A, μ, ∂) where G and A are objects, $\mu \colon A \otimes G \to A$ is an action core, and $\partial \colon A \to G$ is a morphism, such that

$$\begin{tikzcd} A \otimes G \arrow[r, "\mu"] \arrow[d, "\partial \otimes 1_G"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "\partial"] \\ G \otimes G \arrow[r, "c^{G,G}"'] & G \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "c^{A,A}"] \arrow[d, "1_A \otimes \partial"'] \arrow[dr, phantom, "(2)"] & A \arrow[d, equal] \\ A \otimes G \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes G \arrow[r, "\mu_{2,1}"] \arrow[d, "1_A \otimes \partial \otimes 1_G"'] \arrow[dr, phantom, "(3)"] & A \arrow[d, equal] \\ A \otimes G \otimes G \arrow[r, "\mu_{1,2}"'] & A \end{tikzcd}$$

Groupoids and crossed modules via co-smash products

[M Hartl & TVdL, 2013]

Theorem

If $K, L \lhd X$ then $[K, L]^{\mathrm{S}} = [K, L] \vee [K, L, X]$. Hence a reflexive graph

$$\begin{tikzcd} X \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G \arrow[l, "e" description] \end{tikzcd}$$

is a groupoid if and only if $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0 = [\mathrm{Ker}(d), \mathrm{Ker}(c), X]$.

Theorem

An internal crossed module is determined by a quadruple (G, A, μ, ∂) where G and A are objects, $\mu\colon A \otimes G \to A$ is an action core, and $\partial\colon A \to G$ is a morphism, such that

$$\begin{tikzcd} A \otimes G \arrow[r, "\mu"] \arrow[d, "\partial\otimes 1_G"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "\partial"] \\ G \otimes G \arrow[r, "c^{G,G}"'] & G \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "c^{A,A}"] \arrow[d, "1_A\otimes\partial"'] \arrow[dr, phantom, "(2)"] & A \arrow[d, equal] \\ A \otimes G \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes G \arrow[r, "\mu_{2,1}"] \arrow[d, "1_A\otimes\partial\otimes 1_G"'] \arrow[dr, phantom, "(3)"] & A \arrow[d, equal] \\ A \otimes G \otimes G \arrow[r, "\mu_{1,2}"'] & A \end{tikzcd}$$

"All" commutators expressed as Higgins commutators

- How does this work?

  Key idea: general decomposition formula [M Hartl & TVdL, 2013]

  $$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K, L, M \leqslant X$$

- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$

  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?

- For $K, L \lhd X$, always

  $$\begin{aligned}[K, L] \leqslant [K, L]_X &= [K, L] \vee [[K, L], X] \\ &\leqslant [K, L]^{S} = [K, L] \vee [K, L, X].\end{aligned}$$

  Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{array}{ccccc} & & W & & \\ & & \downarrow{\scriptstyle w} & & \\ X & \xrightarrow[x]{} & D & \xleftarrow[y]{} & Y \end{array}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

“All” commutators expressed as Higgins commutators

- How does this work?
  Key idea: general decomposition formula [M Hartl & TVdL, 2013]

$$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K, L, M \leqslant X$$

- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$
  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?
- For $K, L \lhd X$, always

$$\begin{aligned}[K, L] &\leqslant [K, L]_X = [K, L] \vee [[K, L], X] \\ &\leqslant [K, L]^{\mathrm{S}} = [K, L] \vee [K, L, X].\end{aligned}$$

Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

“All” commutators expressed as Higgins commutators

- How does this work?
  Key idea: general decomposition formula [M Hartl & TVdL, 2013]
  $$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K,\ L,\ M \leqslant X$$
- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$
  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?
- For $K,\ L \lhd X$, always
  $$\begin{aligned}[K, L] &\leqslant [K, L]_X = [K, L] \vee [[K, L], X]\\ &\leqslant [K, L]^{\mathrm{S}} = [K, L] \vee [K, L, X].\end{aligned}$$
  Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

“All” commutators expressed as Higgins commutators

- How does this work?
  Key idea: general decomposition formula [M Hartl & TVdL, 2013]

  $$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K,\ L,\ M \leqslant X$$

- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$
  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?
- For K, $L \lhd X$, always

  $$\begin{aligned} [K, L] \leqslant [K, L]_X &= [K, L] \vee [[K, L], X] \\ &\leqslant [K, L]^S = [K, L] \vee [K, L, X]. \end{aligned}$$

  Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

"All" commutators expressed as Higgins commutators

- How does this work?
  Key idea: general decomposition formula [M Hartl & TVdL, 2013]

  $$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K, L, M \leqslant X$$

- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$
  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?

- For $K, L \lhd X$, always

  $$\begin{aligned}[K, L] &\leqslant [K, L]_X = [K, L] \vee [[K, L], X] \\ &\leqslant [K, L]^{S} = [K, L] \vee [K, L, X].\end{aligned}$$

  Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

"All" commutators expressed as Higgins commutators

- How does this work?
  Key idea: general decomposition formula [M Hartl & TVdL, 2013]
  $$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K,\ L,\ M \leqslant X$$
- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$
  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?
- For $K,\ L \lhd X$, always
  $$\begin{aligned}[K, L] &\leqslant [K, L]_X = [K, L] \vee [[K, L], X] \\ &\leqslant [K, L]^S = [K, L] \vee [K, L, X].\end{aligned}$$
  Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

“All” commutators expressed as Higgins commutators

- How does this work?
  Key idea: general decomposition formula [M Hartl & TVdL, 2013]

  $$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K,\ L,\ M \leqslant X$$

- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$
  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?
- For $K,\ L \lhd X$, always

$$\begin{aligned}[K, L] \leqslant [K, L]_X &= [K, L] \vee [[K, L], X]\\ \leqslant [K, L]^{\mathrm{S}} &= [K, L] \vee [K, L, X].\end{aligned}$$

Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

"All" commutators expressed as Higgins commutators

- How does this work?
  Key idea: general decomposition formula [M Hartl & TVdL, 2013]

$$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K,\ L,\ M \leqslant X$$

- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$
  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?
- For $K,\ L \lhd X$, always

$$\begin{aligned}[K, L] &\leqslant [K, L]_X = [K, L] \vee [[K, L], X] \\ &\leqslant [K, L]^{\mathrm{S}} = [K, L] \vee [K, L, X].\end{aligned}$$

Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

“All” commutators expressed as Higgins commutators

- How does this work?
  Key idea: general decomposition formula [M Hartl & TVdL, 2013]

$$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M] \qquad \text{for } K, L, M \leqslant X$$

- In Gp, $[K, L, M]$ is generated by elements $klk^{-1}l^{-1}mlkl^{-1}k^{-1}m^{-1}$
  Question: when is $[K, L, M] = [K, [L, M]] \vee [M, [K, L]] \vee [L, [M, K]]$?
- For $K, L \lhd X$, always

$$\begin{aligned}[K, L] &\leqslant [K, L]_X = [K, L] \vee [[K, L], X] \\ &\leqslant [K, L]^{\mathrm{S}} = [K, L] \vee [K, L, X].\end{aligned}$$

  Equalities in *categories of interest* [A Cigoli, 2009] [A Montoli, 2010].

Theorem [N Martins–Ferreira & TVdL, 2013]

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd}$$

Given any weighted cospan (D, X, x, Y, y, W, w), the *weighted subobject commutator* $[(X, x), (Y, y)]_{(W,w)}$ from [M Gran, G Janelidze & A Ursini, 2012] decomposes as $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$.

Application 1: peri-abelian categories

Looking for a context where *universal central extensions* are well-behaved, that is, a certain condition we called (UCE) holds

Proposition [J R A Gray & TVdL]

For a semi-abelian category $\mathbb{X}$, the following are equivalent:

1. $\mathbb{X}$ is **peri-abelian** [D Bourn, 2010];
2. for any object Y of $\mathbb{X}$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Pt}_Y(\mathbb{X}) \arrow[r, "\mathrm{Ker}"] \arrow[d, "\mathrm{Ab}"'] & \mathbb{X} \arrow[d, "\mathrm{Ab}"] \\ \mathrm{Ab}(\mathrm{Pt}_Y(\mathbb{X})) \arrow[r, "\mathrm{Ker}"'] & \mathrm{Ab}(\mathbb{X}) \end{tikzcd}$$

3. $[K, K] = [K, K]^{\mathrm{S}}$ for $K \lhd X$ protosplit normal in $\mathbb{X}$;
4. $[K, K] = [K, K]^{\mathrm{S}}$ for $K \lhd X$ in $\mathbb{X}$.

So (NH) + (SH) $\Rightarrow$ (PA) $\Rightarrow$ (UCE).

Examples: all *categories of interest*, so e.g. Gp, Lie_R, Rng, AAlg_R

Counterexamples: DiGp, Loop, NAAlg_R

Application 1: peri-abelian categories

Looking for a context where *universal central extensions* are well-behaved, that is, a certain condition we called (UCE) holds

Proposition [J R A Gray & TVdL]

For a semi-abelian category $\mathbb{X}$, the following are equivalent:

1. $\mathbb{X}$ is **peri-abelian** [D Bourn, 2010];
2. for any object Y of $\mathbb{X}$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Pt}_Y(\mathbb{X}) \arrow[r, "\mathrm{Ker}"] \arrow[d, "\mathrm{Ab}"'] & \mathbb{X} \arrow[d, "\mathrm{Ab}"] \\ \mathrm{Ab}(\mathrm{Pt}_Y(\mathbb{X})) \arrow[r, "\mathrm{Ker}"'] & \mathrm{Ab}(\mathbb{X}) \end{tikzcd}$$

3. $[K,K] = [K,K]^S$ for $K \lhd X$ protosplit normal in $\mathbb{X}$;
4. $[K,K] = [K,K]^S$ for $K \lhd X$ in $\mathbb{X}$.

So (NH) + (SH) $\Rightarrow$ (PA) $\Rightarrow$ (UCE).

Examples: all *categories of interest*, so e.g. Gp, Lie_R, Rng, AAlg_R

Counterexamples: DiGp, Loop, NAAlg_R

Application 1: peri-abelian categories

Looking for a context where *universal central extensions* are well-behaved, that is, a certain condition we called (UCE) holds

Proposition [J R A Gray & TVdL]

For a semi-abelian category $\mathbb{X}$, the following are equivalent:

1. $\mathbb{X}$ is **peri-abelian** [D Bourn, 2010];
2. for any object Y of $\mathbb{X}$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Pt}_Y(\mathbb{X}) \arrow[r, "\mathrm{Ker}"] \arrow[d, "\mathrm{Ab}"'] & \mathbb{X} \arrow[d, "\mathrm{Ab}"] \\ \mathrm{Ab}(\mathrm{Pt}_Y(\mathbb{X})) \arrow[r, "\mathrm{Ker}"'] & \mathrm{Ab}(\mathbb{X}) \end{tikzcd}$$

3. $[K, K] = [K, K]^{\mathrm{S}}$ for $K \lhd X$ protosplit normal in $\mathbb{X}$;
4. $[K, K] = [K, K]^{\mathrm{S}}$ for $K \lhd X$ in $\mathbb{X}$.

So (NH) + (SH) $\Rightarrow$ (PA) $\Rightarrow$ (UCE).

Examples: all *categories of interest*, so e.g. Gp, Lie_R, Rng, AAlg_R

Counterexamples: DiGp, Loop, NAAlg_R

Application 1: peri-abelian categories

Looking for a context where *universal central extensions* are well-behaved, that is, a certain condition we called (UCE) holds

Proposition [J R A Gray & TVdL]

For a semi-abelian category $\mathbb{X}$, the following are equivalent:

1. $\mathbb{X}$ is **peri-abelian** [D Bourn, 2010];
2. for any object Y of $\mathbb{X}$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Pt}_Y(\mathbb{X}) \arrow[r, "\mathrm{Ker}"] \arrow[d, "\mathrm{Ab}"'] & \mathbb{X} \arrow[d, "\mathrm{Ab}"] \\ \mathrm{Ab}(\mathrm{Pt}_Y(\mathbb{X})) \arrow[r, "\mathrm{Ker}"'] & \mathrm{Ab}(\mathbb{X}) \end{tikzcd}$$

3. $[K,K] = [K,K]^{\mathrm{S}}$ for $K \lhd X$ protosplit normal in $\mathbb{X}$;
4. $[K,K] = [K,K]^{\mathrm{S}}$ for $K \lhd X$ in $\mathbb{X}$.

So (NH) + (SH) $\Rightarrow$ (PA) $\Rightarrow$ (UCE).

Examples: all *categories of interest*, so e.g. Gp, Lie_R, Rng, AAlg_R

Counterexamples: DiGp, Loop, NAAlg_R

Application 1: peri-abelian categories

Looking for a context where *universal central extensions* are well-behaved, that is, a certain condition we called (UCE) holds

Proposition [J R A Gray & TVdL]

For a semi-abelian category $\mathbb{X}$, the following are equivalent:

1. $\mathbb{X}$ is **peri-abelian** [D Bourn, 2010];
2. for any object Y of $\mathbb{X}$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Pt}_Y(\mathbb{X}) \arrow[r, "\mathrm{Ker}"] \arrow[d, "\mathrm{Ab}"'] & \mathbb{X} \arrow[d, "\mathrm{Ab}"] \\ \mathrm{Ab}(\mathrm{Pt}_Y(\mathbb{X})) \arrow[r, "\mathrm{Ker}"'] & \mathrm{Ab}(\mathbb{X}) \end{tikzcd}$$

3. $[K, K] = [K, K]^{\mathrm{S}}$ for $K \lhd X$ protosplit normal in $\mathbb{X}$;
4. $[K, K] = [K, K]^{\mathrm{S}}$ for $K \lhd X$ in $\mathbb{X}$.

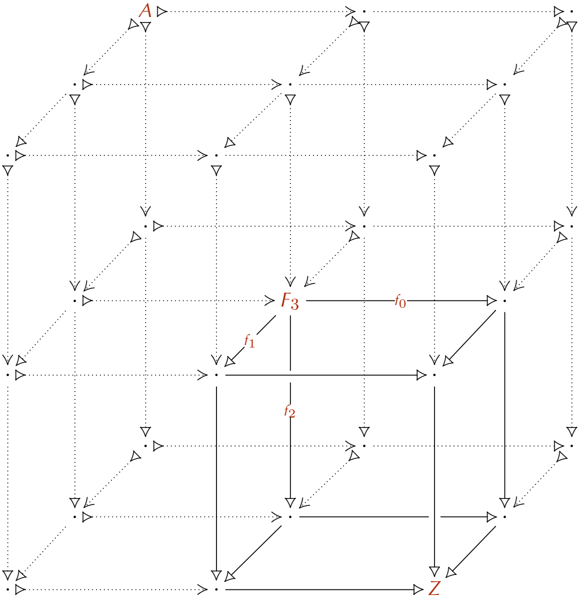
So $(\mathrm{NH}) + (\mathrm{SH}) \Rightarrow (\mathrm{PA}) \Rightarrow (\mathrm{UCE})$.

Examples: all *categories of interest*, so e.g. Gp, Lie_R, Rng, AAlg_R

Counterexamples: DiGp, Loop, NAAlg_R

Application 2: higher central extensions and cohomology

[M Gran, T Everaert & TVdL, 2008] [D Rodelo & TVdL, 2011–2012]



Application 2: higher central extensions and cohomology

[M Gran, T Everaert & TVdL, 2008] [D Rodelo & TVdL, 2011–2012]

Theorem

Let F be an n-fold extension. Then $1 \Rightarrow 2 \Rightarrow 3$:

1. F is central; $\leftarrow$ **Galois theory**
2. F is an n-torsor; $\leftarrow$ **Duskin–Glenn cohomology**
3. F satisfies $\Big[\bigwedge_{i\in I} \mathrm{Ker}(f_i), \bigwedge_{i\in n\setminus I} \mathrm{Ker}(f_i)\Big] = 0$ for all $I \subseteq n$.

If (SH) holds then also $3 \Rightarrow 1$.

Theorem

Let Z be an object, A an abelian object in a semi-abelian category which satisfies (SH). There is an isomorphism

$$\mathrm{H}^{n+1}(Z,A) \cong \mathrm{Centr}^n(Z,A)$$

for all $n \geqslant 1$.

Application 2: higher central extensions and cohomology

[M Gran, T Everaert & TVdL, 2008] [D Rodelo & TVdL, 2011–2012]

Theorem

Let F be an n-fold extension. Then $1 \Rightarrow 2 \Rightarrow 3$:

1. F is central; $\leftarrow$ **Galois theory**
2. F is an n-torsor; $\leftarrow$ **Duskin–Glenn cohomology**
3. F satisfies $\Big[\bigwedge_{i\in I} \mathrm{Ker}(f_i), \bigwedge_{i\in n\setminus I} \mathrm{Ker}(f_i)\Big] = 0$ for all $I \subseteq n$.

If (SH) holds then also $3 \Rightarrow 1$.

Theorem

Let Z be an object, A an abelian object in a semi-abelian category which satisfies (SH). There is an isomorphism

$$\mathrm{H}^{n+1}(Z, A) \cong \mathrm{Centr}^n(Z, A)$$

for all $n \geqslant 1$.

Application 2: higher central extensions and cohomology

[M Gran, T Everaert & TVdL, 2008] [D Rodelo & TVdL, 2011–2012]

Theorem

Let F be an n-fold extension. Then $1 \Rightarrow 2 \Rightarrow 3$:

1. F is central; $\leftarrow \bigvee_{I_1 \cup \cdots \cup I_k = n} \Big[\bigwedge_{i \in I_1} \mathrm{Ker}(f_i), \dots, \bigwedge_{i \in I_k} \mathrm{Ker}(f_i)\Big] = 0$
2. F is an n-torsor; $\leftarrow$ **Duskin–Glenn cohomology**
3. F satisfies $\Big[\bigwedge_{i \in I} \mathrm{Ker}(f_i), \bigwedge_{i \in n \setminus I} \mathrm{Ker}(f_i)\Big] = 0$ for all $I \subseteq n$.

If (SH) holds then also $3 \Rightarrow 1$.

Theorem

Let Z be an object, A an abelian object in a semi-abelian category which satisfies (SH). There is an isomorphism

$$\mathrm{H}^{n+1}(Z, A) \cong \mathrm{Centr}^n(Z, A)$$

for all $n \geqslant 1$.

Application 2: higher central extensions and cohomology

[M Gran, T Everaert & TVdL, 2008] [D Rodelo & TVdL, 2011–2012]

Theorem

Let F be an n-fold extension. Then $1 \Rightarrow 2 \Rightarrow 3$:

1. F is central; $\leftarrow$ **Galois theory**
2. F is an n-torsor; $\leftarrow$ **Duskin–Glenn cohomology**
3. F satisfies $\Big[\bigwedge_{i\in I} \mathrm{Ker}(f_i), \bigwedge_{i\in n\setminus I} \mathrm{Ker}(f_i)\Big] = 0$ for all $I \subseteq n$.

If (SH) holds then also $3 \Rightarrow 1$.

Theorem

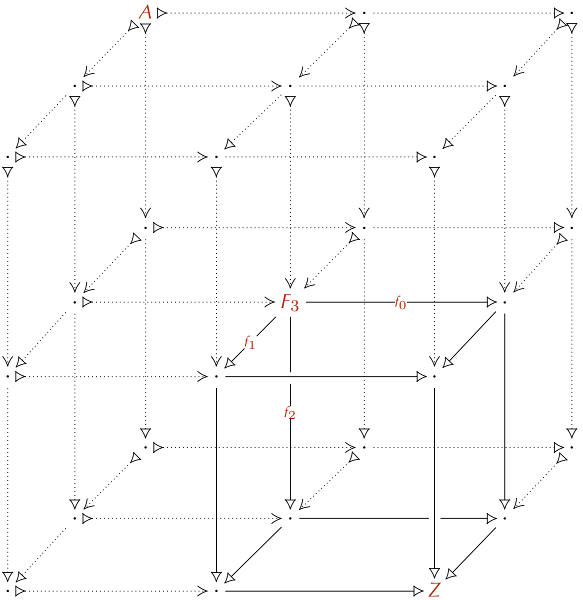
Let Z be an object, A an abelian object in a semi-abelian category which satisfies (SH). There is an isomorphism

$$\mathrm{H}^{n+1}(Z, A) \cong \mathrm{Centr}^n(Z, A)$$

for all $n \geqslant 1$.

Application 2: higher central extensions and cohomology

[M Gran, T Everaert & TVdL, 2008] [D Rodelo & TVdL, 2011–2012]



Some examples

	(NH)	(SH)	(PA)
Gp	Y	Y	Y
Lie_R	Y	Y	Y
all *categories of interest*	Y	Y	Y
DiGp		N	N
NARng	N	Y	N
all varieties of Ω-groups	N	N	N
HSLat	Y	Y	Y
Loop		N	N
$\mathbb{C}$	Y	N	

$\mathbb{C}$ is the variety of abelian groups with a symmetric distributive ternary operation t satisfying $t(t(x, y, z), u, v) = t(t(x, u, v), y, z)$.