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Thermodynamics Analysis of the Distributed Parameter Model of Counterflow Heat Exchanger

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Abstract: In this paper we are interested in the thermodynamic analysis of a counterflow heat exchanger, when the dynamics are described by two hyperbolic partial differential equations (PDEs). First from the first and second law of thermodynamics we derive the different models of the heat exchanger. Next we perform an analysis of the equilibrium profiles from a certain condition on the system parameters that guarantees the existence and uniqueness of the solution of the PDEs model. In this analysis we show the importance of the thermal pinch as an energy efficiency factor. Finally we study the passivity and the asymptotic stability of the considered heat exchanger essentially basing ourselves on the entropy as a storage function, entropy production as a dissipation function, and the thermodynamic availability function as a Lyapunov function.

Keywords: Non-equilibrium thermodynamics, heat exchangers, distributed parameter systems, stability, passivity.

1. INTRODUCTION

In industrial applications where energy is vital, heat exchangers have acquired major economic importance. Almost all the produced or collected thermal energy passes at least once through a heat exchanger. They are among the most important equipment in industry where irreversible thermodynamic processes and transformations occur. The analysis of irreversibilities that lead to the degradation of energy has been performed by several authors. For example, Bejan (1977) analyzes a counterflow heat exchanger for gas-to-gas application. He introduces a dimensionless number, called "number of entropy production", to quantify the irreversibilities. In (Bejan, 1996, Chapter 5), Mohamed (2005), Guo et al. (2010), authors studied chronologically in various situations the entropy production in balanced and imbalanced counterflow heat exchangers, and whether viscous dissipation phenomena are considered or not. They show that in the absence of viscous dissipation phenomena, balanced exchangers, i.e. whose fluids have the same thermal heat capacity flow rates, correspond to the equipartition of the transferred flow, but not of the thermodynamic driving force or of the entropy production. Recall that according to Ilya Prigogine, a system or process that achieves equipartition over time is a so-called steady-state process, and therefore appears to minimize the overall entropy production (Prigogine (1947)). Systems where entropy production is thus equidistributed are not feasible in practice. Their interest is that they serve as reference cases, that we seek to approach. Furthermore it should be specified that all the studies cited above are made using the number of entropy production, as a criterion for evaluating irreversibility, which only takes into account the thermomechanical properties of each fluid at the inlet and at the outlet of the heat exchanger.

In this paper we perform a thermodynamic analysis of a counterflow heat exchanger described by a system of two coupled hyperbolic partial differential equations. First (see Section 3), from a general result on the existence and uniqueness of solution of hyperbolic systems (see Prieur and Winkin (2018)), we derive an inequality from which we perform a steady-state thermodynamic analysis. Specifically we show that the thermodynamic analysis of the heat exchanger from the well-posedness condition is equivalent to that made from the thermal capacity flows rates of the fluids by Bejan (1977); Balaji et al. (2021); Ogunedo (2020). Moroever unlike the works that use the number of entropy production as a criterion for measuring irreversibility, we use the entropy production (Sandoval (2021)). Next in the Section 4, we investigate and establish the passivity of the heat exchanger by using entropy as the storage function and entropy production as the dissipation function. Finally, we study the stability of the heat exchanger in a thermodynamic sense. Recall that the existing results in the literature on this subject (see Besson et al. (2006), Kazaku et al. (2022a,b)) use Lyapunov functions constructed from the square of the intensive quantities, typically temperatures. Such functions are not energy functions in the physical or thermodynamic sense. In this paper we use as Lyapunov function the thermodynamic availability, which is a function that depends on the entropy as well as the irreversible entropy production of the system (Alonso and Ydstie (2001)).

Before that, let us start by recalling the different thermodynamic representations of the counterflow heat exchanger.

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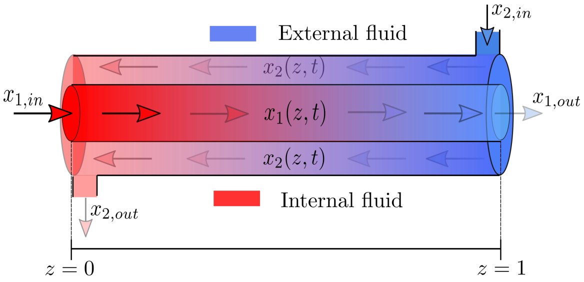

Fig. 1. A counterflow heat exchanger

2. SYTEM MODEL

In this section we will establish the mathematical models of the counterflow heat exchanger shown in Figure 1. To do this, it is necessary first to admit a certain number of simplifying assumptions and some constitutive laws:

- The outer tube sufficiently insulates the exchanger, so that heat losses are not taken into account.
- Axial heat diffusion is negligible in fluids and in walls.
- The transport velocities v_j (with $j = 1, 2$) of component j are assumed to be constant and uniform along the exchanger.
- Fluids are incompressible so that viscous dissipation phenomena can be neglected.
- The pressure is assumed to be constant and uniform.
- The heat exchange with the double jacket is characterized by the following equation: $q_j(z,t) = A_s U_g (x_k(z,t) - x_j(z,t))$, where $x_k(z,t)$ (with $k = 2, 1$) is the distributed temperature of the jacket, A_s and U_g the heat transfer surface area and the overall heat transfer coefficient, respectively.

The total energy balance is established on the basis of the first law of thermodynamics. Given the modelling assumptions, the total energy corresponds to the internal energy. As the pressure is assumed to be constant, this balance is equal to the enthalpy balance, and it is written:

$$\frac{\partial h}{\partial t} = \Lambda \frac{\partial h}{\partial z} + g q(z,t), \qquad \text{with } g = (-1 \quad 1)^\top. \tag{1}$$

where $q(z,t) = A_s U_g (x_1(z,t) - x_2(z,t))$, $\Lambda = \mathrm{diag}(-v_1, v_2)$, $h(z,t) = (h_1(z,t); h_2(z,t))^\top$ is the enthalpy vector. Throughout this paper we consider that $h(z,t) \in X$ and $x(z,t) \in X$, with $X = L^2(0,1) \times L^2(0,1)$ a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$, and where $L^2(0,1) = L^2((0,1);\mathbb{R})$ denotes the Hilbert space of measurable square-integrable function with values in $\mathbb{R}$.

The system of equations (1) written as a function of the enthalpies can be expressed as a function of the temperatures using the expressions of the partial mass enthalpies for an ideal mixture: $h_j = \rho_j S_j c_{p_j}(x_j - x_{ref}) + h_{j_{ref}}$, where $\rho_j, c_{p_j}, S_j, x_{ref}$ and h_{ref} represent the density, the heat capacity, the section of the tube, the reference temperature, and the enthalpy of fluid j, respectively. Thus (1) becomes (Maidi et al. (2009)):

$$\frac{\partial x}{\partial t} = \underbrace{\begin{pmatrix} -v_1 & 0 \\ 0 & v_2 \end{pmatrix}}_{\Lambda} \frac{\partial x}{\partial z} + \underbrace{\begin{pmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{pmatrix}}_{M} x(z,t), \tag{2}$$

with $x(z,t) = (x_1(z,t)\ x_2(z,t))^\top$ and $\alpha_j[s^{-1}] = \frac{A_s U_g}{\rho_j S_j c_{p_j}}$ the heat transfer coefficient of fluid j. The system (2) is completed by the following boundary conditions:

$$x_1(0,t) = x_{1,in}(t), \qquad x_2(1,t) = x_{2,in}(t), \tag{3}$$

where $x_{1,in} \in \mathbb{R}$ and $x_{2,in} \in \mathbb{R}$ are the inlet temperatures of the heat exchanger.

Under the classical non equilibrium thermodynamics assumption of local equilibrium (de Groot et al., 1984, Chapter 3.2) [1], the Gibbs relation is $ds = w^\top dh$, with $w^\top = \begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_2} \end{pmatrix}$ vector of intensive variables, and s the entropy function. Therefore, the entropy balances of each fluid are:

$$\begin{cases} \dfrac{\partial s_1}{\partial t} = -v_1 \dfrac{\partial s_1}{\partial z} - \dfrac{q(z,t)}{x_1(z,t)} \\ \dfrac{\partial s_2}{\partial t} = v_2 \dfrac{\partial s_2}{\partial z} + \dfrac{q(z,t)}{x_2(z,t)}. \end{cases} \tag{4}$$

The total entropy per unit length define as $s(z,t) = s_1(z,t) + s_2(z,t)$ has the following dynamical behaviour:

$$\frac{Ds}{Dt} = q\left(\frac{1}{x_2} - \frac{1}{x_1}\right) = A_s U_g \frac{(x_1 - x_2)^2}{\langle x_m \rangle^2} := \sigma(z,t), \tag{5}$$

with $\langle x_m \rangle = \left(\int_0^1 x_1 x_2 dz \right)^{1/2}$ the average temperature between x_1 and x_2, $\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + v_1 \frac{\partial s_1}{\partial z} - v_2 \frac{\partial s_2}{\partial z}$ is the material derivative of entropy. $\sigma(z,t)$ is the density of entropy production per unit of length inside the heat exchanger due to heat transfer between the two fluids (Kjelstrup et al., 2010, Chapter 9, page 165-166). It is obtained by identification with the dynamic entropy equations (4). Therefore, the total entropy production in the heat exchanger is (Sandoval (2021)):

$$\Sigma_s(t) = \int_0^1 \sigma(z,t) dz. \tag{6}$$

3. STEADY-STATE ANALYSIS

3.1 Equilibrium profiles analysis

The stationary problem of (2)-(3) is defined by

$$\begin{cases} \Lambda \dfrac{d\overline{x}}{dz} + M\overline{x}(z) = 0 \\ x_1(0) = x_{1,in}, \ x_2(L) = x_{2,in} \end{cases} \tag{7}$$

whose solutions $\overline{x}(z)$ represent the profile of temperatures at equilibrium, which are given for all $\overline{x}(z)$ and $z \in [0, L]$ by $\overline{x}(z) = \mathrm{e}^{\Lambda^{-1}\tilde{M}z} x(0)$, with $\tilde{M} = -M$. Taking into account the configuration of the system, the boundary condition is expressed by

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \multicolumn{2}{c}{(0\ 1) \cdot \exp(\Lambda^{-1}\tilde{M})} \end{pmatrix}^{-1} \begin{pmatrix} x_1(0) \\ x_2(L) \end{pmatrix}. \tag{8}$$

Thus a straightforward computation yield the following expressions for the components $\overline{x}_1(z)$ and $\overline{x}_2(z)$:

$$\begin{aligned} \overline{x}_1(z) = & \frac{1}{\beta_1 - \beta_2 \, \mathrm{e}^{(\beta_2 - \beta_1)L}} \left(\beta_1 \, \mathrm{e}^{(\beta_2-\beta_1)z} - \beta_2 \, \mathrm{e}^{(\beta_2-\beta_1)L} \right) \\ & x_1(0) + \frac{\beta_1}{\beta_1 - \beta_2 \, \mathrm{e}^{(\beta_2-\beta_1)L}} \left(1 - \mathrm{e}^{(\beta_2-\beta_1)z} \right) x_2(L), \\ \overline{x}_2(z) = & \frac{\beta_2}{\beta_1 - \beta_2 \, \mathrm{e}^{(\beta_2-\beta_1)L}} \left(\mathrm{e}^{(\beta_2-\beta_1)z} - \mathrm{e}^{(\beta_2-\beta_1)L} \right) x_1(0) \\ & + \frac{1}{\beta_1 - \beta_2 \, \mathrm{e}^{(\beta_2-\beta_1)L}} \left(\beta_1 - \beta_2 \, \mathrm{e}^{(\beta_2-\beta_1)z} \right) x_2(L), \end{aligned} \tag{9}$$

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[1] The local and instantaneous relations of a irreversible system are the same as those of a system at thermodynamic equilibrium.

with $\beta_1 = \frac{\alpha_1}{v_1}$, and $\beta_2 = \frac{\alpha_2}{v_2}$.
According to (Prieur and Winkin, 2018, Lemma 1) the distributed parameter model (2)-(3) is well posed, i.e. for all initial conditions $x_0(z) = (x_{10}(z), x_{20}(z))^\top \in X$ there exists a unique solution of model (2)-(3) if and only if the matrix given at the equation (8) is invertible; in other words if:

$$\beta_1 \neq \beta_2. \tag{10}$$

We are going to show that this condition on the physical parameters of the system plays an important role in understanding the thermodynamics of heat exchangers, and is equivalent to the inequality defined on the thermal capacity flow rates of each fluid, found in the litterature ((Bejan and Kraus, 2003, Chapter 11), Forsberg (2021)). Three cases can be studied. If $\beta_1 > \beta_2$, the amount of heat that can be transmitted is limited by fluid 1. It is this fluid that undergoes the greatest temperature variation along the heat exchanger. Its outlet temperature $x_{1,out}$ is close to the inlet temperature $x_{2,in}$, as shown in Figure 2a. If $\beta_2 > \beta_1$ we have the reverse scenario, see Figure 2b. From Figures 2a and 2b we note also that the outlet of the cold fluid is greater than that of the hot fluid; this is one of the main advantages of the counterflow configuration. In contrast, the thermal pinch $\Delta\mathrm{T}_{\min}$ [2] is located at the outlet of the hot or cold fluid, depending on whether $\beta_1 > \beta_2$ or $\beta_2 > \beta_1$, respectively. Note that in general for $\beta_1 \neq \beta_2$, the temperature difference along the heat exchanger is obtained from the expressions of the trajectories (9), and is equal to:

$$\Delta\mathrm{T}(z) = \frac{(\beta_1 - \beta_2)\,\mathrm{e}^{(\beta_2-\beta_1)z}}{\beta_1 - \beta_2\,\mathrm{e}^{(\beta_2-\beta_1)L}}\,(x_1(0) - x_2(L)). \tag{11}$$

Then we observe that when $L \mapsto +\infty$, $\Delta\mathrm{T}(z) \mapsto 0$ (see Figure 1 for the case $\beta_1 > \beta_2$ and for the length $L = 3$). This shows that *the lower the thermal pinch, the higher the thermal heat transferred.* It can be deduced that the thermal pinch is an efficiency factor for an imbalanced exchanger.

Now let us see what happens to the condition (10) when it is not verified. For $\beta_1 = \beta_2 = \beta$ the problem defined by (7) becomes singular. In that case, the temperature profiles are calculated using a series expansion (of first order) of $\mathrm{e}^{\Lambda^{-1}\tilde{M}z} = I + \Lambda^{-1}\tilde{M}z$. Then we obtain:

$$\begin{cases} \overline{x}_1(z) = \left(1 - \beta z + \dfrac{\beta^2 L}{1+\beta L} z\right) x_1(0) + \dfrac{\beta z}{1+\beta L} x_2(L), \\ \overline{x}_2(z) = \beta\left(\dfrac{L(1+\beta z)}{1+\beta L} - z\right) x_1(0) + \dfrac{1+\beta z}{1+\beta L} x_2(L). \end{cases} \tag{12}$$

Figure 2d shows that the temperature difference remains uniform along the heat exchanger by the fact that we have symmetrical flows. In other words, the counterflow heat exchanger works particularly well when $\beta_1 = \beta_2$ because warming varies in the same way as cooling (absence of pinch thermal; indeed $\Delta\mathrm{T}(z) = \frac{1}{1+\beta L}(x_1(0) - x_2(L)$ is constant). However, even if the length is infinite the heat exchanger is still irreversible; in that case the trajectories of $x_1(z)$ and $x_2(z)$ are collinear ($x_1(L) = x_2(L), x_2(0) = x_1(0)$).

Remark 3.1. Some authors talk about balanced counterflow heat exchanger to refer to case $\beta_1 = \beta_2$, and imbalanced to refer to case $\beta_1 \neq \beta_2$ (Bejan, 1996, Chapter 5), Mohamed (2005), Guo et al. (2010). Although they do not explicitly express it, this can explained by the fact that the temperature difference along a counterflow heat exchanger is isothermal in the first case, while in the second case it is nonisothermal.

3.2 *Entropy analysis*

We have just seen that depending on the conditions on the system parameters, three different behaviors can be identified. We shall now see what this implies about the entropy production. At steady state, equation (5) is written

$$\sigma(z) = A_s U_g \frac{(x_1(z) - x_2(z))^2}{\langle x_m(z)\rangle^2}. \tag{13}$$

Thus the local entropy production is easily obtained by successively substituting in (13) the expressions of the trajectories (9) and (12). Figure (3) represents the local entropy production profiles $\sigma(z)$ in the cases $\beta_1 \neq \beta_2$ and $\beta_1 = \beta_2$. In general, we observe that this function is positive and convex. It is mainly for this reason that the entropy production can be used in some cases as a Lyapunov function (see e.g. Favache and Dochain (2009), García-Sandoval et al. (2016)). Furthermore, from the Figure 3 an important conclusion is that, although the heat flux transferred $q(z)$ is uniform in the case of a balanced heat exchanger ($\beta_1 = \beta_2$), the entropy production along the heat exchanger axis is not. This is due in particular to the fact that the thermodynamic driving force $\left(\frac{1}{x_2(z)} - \frac{1}{x_1(z)}\right)$ is non-uniform. We note also that the balanced counterflow heat exchanger corresponds to a minimum entropy production regime.
In Figure 4 we also see that the irreversibility decreases when $L \mapsto +\infty$. Thus, related to the analysis of temperature profiles, we conclude that, to harness the full potential of heat recovery or heat transfer it is essential to carry out an analysis upstream in order to find the right balance between "length (or heat transfer surface area)" and "thermal pinch", and ultimately to reduce investment costs.

4. DYNAMICAL ANALYSIS

4.1 *Passivity analysis*

Let us start by recalling the definition of passivity (der Schaft (1996)).

Definition 4.1. Let us consider a dynamical system $\Gamma(t, \xi(t))$ with input variable $u(t) \in \mathbb{R}^\ell$, output variable $y(t) \in \mathbb{R}^p$ and the state variable $\xi \in L^2(0,1)$. We consider as a supply rate function $w(t) \in \mathbb{R}^\ell \times \mathbb{R}^p \mapsto \mathbb{R}$. We say that $\Gamma(t, \xi(t))$ is passive with respect to the supply rate function $w(t)$ if there exists a non-negative function $W(\xi(t))$ called storage function, such that for all $u(t) \in \mathbb{R}^\ell$, $y(t) \in \mathbb{R}^p$ and $\xi_0 \in L^2(0,1)$:

$$W(\xi(T)) - W(\xi_0) \leq \int_0^T u^\top(t) y(t) dt. \tag{14}$$

Proposition 4.1. The thermodynamic model (4) of counterflow heat exchanger is passive with the entropy functional as the storage function, and the entropy production functional as the dissipation function.

Proof. Let us consider the following auxiliary variables:

$$W(t) := \overline{S} - S(t),$$

and

$$\omega(z,t) := \overline{s} - s(z,t),$$

where $\overline{S}$ and $\overline{s}$ are arbitrarily large constant values of entropy such that for $t \in [0, +\infty)$, $W(t)$ and $\omega(z,t)$ are nonnegative. Then

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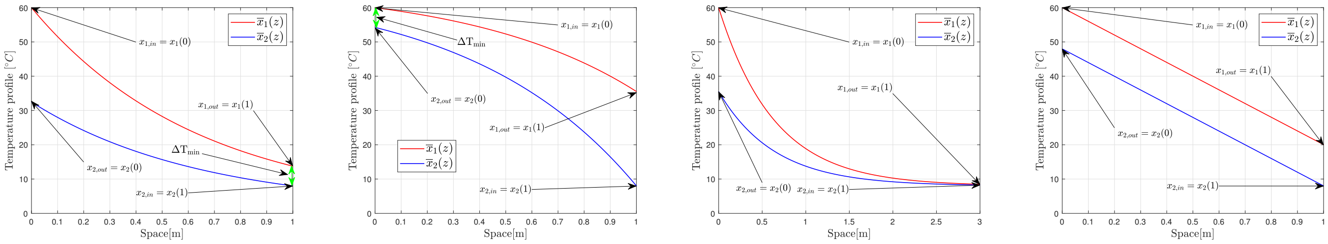
[2] Minimum temperature differences between the hot and cold fluids in the heat exchanger.



(a) Equilibrium profiles in the case $\beta_1 > \beta_2$. (b) Equilibrium profiles in the case $\beta_2 > \beta_1$. (c) Equilibrium profiles: $\beta_1 > \beta_2$, for $L = 3m$. (d) Equilibrium profiles in the case $\beta_1 = \beta_2$.

Fig. 2. Equilibrium profiles for inlet temperatures $x_{1,in} = 60°C$, $x_{2,in} = 8°C$, and parameter values: $v_1 = 0.9749m/s$, $v_2 = 1.6513m/s$, $\alpha_1 = 3.2389s^{-1}$ and $\alpha_2 = 2.903s^{-1}$.

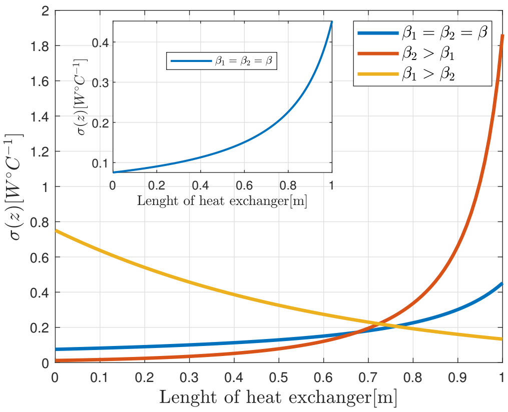


Fig. 3. Local entropy production $\sigma(z)$ of balanced and imbalanced counterflow heat exchangers.

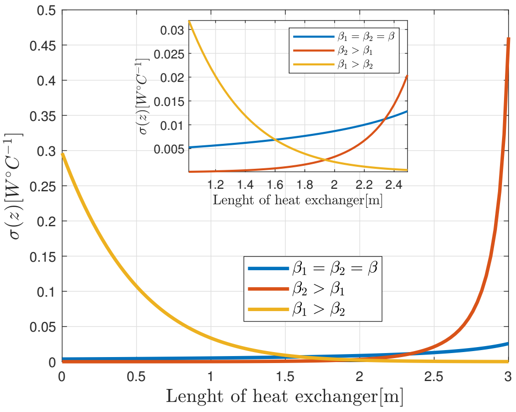


Fig. 4. Local entropy production $\sigma(z)$ of balanced and imbalanced counterflow heat exchangers for $L = 3m$.

$$\begin{aligned}\frac{dW(t)}{dt} &= -\int_0^1 \left(\frac{\partial s_1}{\partial t} + \frac{\partial s_2}{\partial t}\right) dz \\ &= -\int_0^1 \left(-v_1 \frac{\partial s_1}{\partial z} + v_2 \frac{\partial s_2}{\partial z} + \left(\frac{1}{x_2} - \frac{1}{x_1}\right) q(z,t)\right) dz \\ &= -\int_0^1 \left(-v_1 \frac{\partial s_1}{\partial z} + v_2 \frac{\partial s_2}{\partial z} + \sigma(z,t)\right) dz \\ &= [v_1 s_1(z,t) - v_2 s_2(z,t)]_0^1 - \int_0^1 \sigma(z,t) dz.\end{aligned} \tag{15}$$

Given that the dissipation function $\sigma(z,t) \geq 0$, then according to definition 4.1 the following inequality of the passivity holds:

$$\begin{aligned}\frac{dW(t)}{dt} &\leq [v_1 s_1(z,t) - v_2 s_2(z,t)]_0^1 \\ &= \omega\left(s_{in}(t), s_{out}(t)\right),\end{aligned}$$

and for $t \in [0,T]$ its integral form is:

$$W(T) - W(0) \leq \int_0^T \omega\left(s_{in}(t), s_{out}(t)\right) dt.$$

Note that $\omega\left(s_{in}(t), s_{out}(t)\right)$ can be seen as a bounded supply rate function. ■

In this proof, the interest of the variable $W(t)$ is to formalize the passivity in the sense of definition 4.1. Note also that for an isolated system (with $v_1 = v_2$), $s_{in}(t) = s_{out}(t)$, and that implies that the supply rate function $\omega(t)$ is zero, and $\frac{dW(t)}{dt} = -\int_0^1 \sigma(z,t)dz = -\Sigma_s(t)$.

4.2 Stability analysis

In this paragraph we are interested in the stability in the thermodynamic sense of the counterflow heat exchanger, by using thermodynamic availability function for which we recall the definition and the properties.

Consider $\mathbf{h}(z,t) \in X$ as the state vector (extensive properties vector) of the entropy function (as state function of $\mathbf{h}(z,t)$) $s(z,t) \in X$.

Definition 4.2. (Local availability). Let $\tilde{w}(z,t) = w(\mathbf{h}(z,t)) - w(\mathbf{h}_d(z))$ be the difference between the profile of the intensive variables of the system $w(\mathbf{h}(z,t))$ and that of the desired time invariant intensive variables $w(\mathbf{h}_d(z))$, where the desired steady state $\mathbf{h}_d(z)$ is defined in a Hilbert space X.

We call local thermodynamic availability the function $a(\mathbf{h}(z,t))$ defined with respect to $\mathbf{h}_d(z)$ as follows:

$$a(\mathbf{h}(z,t)) = -\tilde{w}(z,t)^\top \mathbf{h}(z,t). \tag{16}$$

Therefore the global availability is given by

$$\mathcal{A}(t) = \int_0^1 a(\mathbf{h}(z,t))dz, \qquad t \in [0,+\infty). \tag{17}$$

The time derivative of the global availability function $\mathcal{A}(t)$ has the following expression (Hoang and Dochain (2013), Zhou et al. (2021)):

$$\frac{d\mathcal{A}(t)}{dt} = \int_0^1 \frac{\partial a}{\partial t} dz. \tag{18}$$

We give some properties of local availability that will be used later.

Properties

- **P1:** The differential form of $a(\mathbf{h}(z,t)) = -\tilde{w}(z,t)^\top \mathbf{h}(z,t)$ is homogeneous function of degree 1 with respect to $\mathbf{h}(z,t)$.
- **P2:** If the entropy is strictly concave, then
  - (1) $a(\mathbf{h}(z))$ is strictly convex,
  - (2) $a(\mathbf{h}(z)) > 0$, $\mathbf{h}(z) \neq \mathbf{h}_d(z)$,
  - (3) $a(\mathbf{h}(z)) = 0$, $\mathbf{h}(z) = \mathbf{h}_d(z)$,
- **P3:** Let the function $f(\mathbf{h}, \mathbf{h}_d) = -\tilde{w}\tilde{\mathbf{h}}$ with $\tilde{w} = w(\mathbf{h}) - w(\mathbf{h}_d)$ and $\tilde{\mathbf{h}} = \mathbf{h} - \mathbf{h}_d$. If s is strictly concave, then:
  - (1) $0 \leq a(\mathbf{h}) \leq f(\mathbf{h}, \mathbf{h}_d)$,
  - (2) $\tilde{w} = -\mathcal{P}\tilde{\mathbf{h}}$, where $\mathcal{P} \in X$ is positive define operator.

The above properties are established in Alonso and Ydstie (2001).

Proposition 4.2. The dynamical system (1) of imbalanced heat exchanger is globally asymptotically stable at h_d.

Proof. Let us consider $\tilde{w}^\top(z,t) = \begin{pmatrix} \frac{1}{\tilde{x}_1} & \frac{1}{\tilde{x}_2} \end{pmatrix}$, with $\tilde{x}_1(z,t) = x_1(z,t) - x_{1d}(z)$, $\tilde{x}_2(z,t) = x_2(z,t) - x_{2d}(z)$, and $\tilde{h} = (\tilde{h}_1\ \tilde{h}_2)^\top$ with $\tilde{h}_1 = h_1 - h_{1d}$ and $\tilde{h}_2 = h_2 - h_{2d}$. Substituting the dynamical system (1) into the expression of the global availability derivative (18) we get the following:

$$\begin{aligned} \frac{d\mathcal{A}}{dt} &= \int_0^1 \frac{\partial a}{\partial t} dz = -\int_0^1 \tilde{w}^\top(z,t) \frac{\partial \tilde{h}}{\partial t} dz \\ &= -\int_0^1 \tilde{w}^\top(z,t) \left(\Lambda \frac{\partial \tilde{h}}{\partial z} + gq(z,t) \right) dz. \end{aligned}$$

By integrating by part in the domain $[0,1]$ the first term on the right which represents the convection flux, we obtain:

$$\begin{aligned} \frac{d\mathcal{A}}{dt} = &-\left[\tilde{w}^\top(z,t)\Lambda\tilde{h}(z,t) \right]_0^1 + \int_0^1 \Big(\tilde{h}^\top(z,t)\Lambda \frac{\partial \tilde{w}}{\partial z} \\ &+ \tilde{w}^\top(z,t) gq(z,t) \Big) dz. \end{aligned}$$

Note that $\int_0^1 \tilde{w}^\top gq dz = \int_0^1 \left(\frac{1}{\tilde{x}_2} - \frac{1}{\tilde{x}_1} \right) q dz = \tilde{\Sigma}_s(t) \geq 0$ corresponds to the irreversible entropy production at the reference point (h_{1d}, h_{2d}). Thus,

$$\frac{d\mathcal{A}}{dt} \leq -\left[\tilde{w}^\top(z,t)\Lambda\tilde{h}(z,t) \right]_0^1 + \int_0^1 \left(\tilde{h}^\top(z,t)\Lambda \frac{\partial \tilde{w}}{\partial z} \right) dz.$$

Property 3 tells us that $\tilde{w}(z,t) = -\mathcal{P}\tilde{h}(z,t)$, with $\mathcal{P} \in \mathbb{R}^{2\times 2}$ a symmetric positive definite matrix. So we can write:

$$\begin{aligned} \frac{d\mathcal{A}}{dt} &= \left[\tilde{h}^\top(z,t)\Lambda\mathcal{P}\tilde{h}(z,t) \right]_0^1 - \int_0^1 \tilde{h}^\top(z,t)\Lambda\mathcal{P} \frac{\partial \tilde{h}}{\partial z} dz \\ &\leq -\int_0^1 \tilde{h}^\top(z,t)\Lambda\mathcal{P} \frac{\partial \tilde{h}}{\partial z} dz \leq 0. \end{aligned}$$

Finally, if $h \mapsto h_d$, $\tilde{h}(\cdot,t) \mapsto 0$, then $\lim_{t \mapsto +\infty} \mathcal{A}(t) = 0$, which implies the global asymptotically stability of heat exchanger system. ■

4.3 Numerical simulations

In order to assess the stability of the heat exchanger, simulation tests have been carried out. The simulation parameters are listed in Table 1, and the initial condition inspired by Besson et al. (2006) was considered:

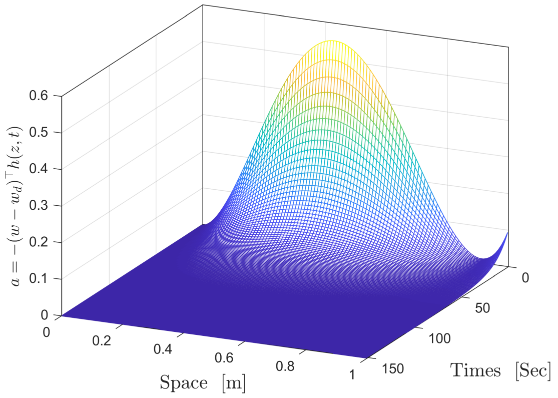


Fig. 5. Local availability.

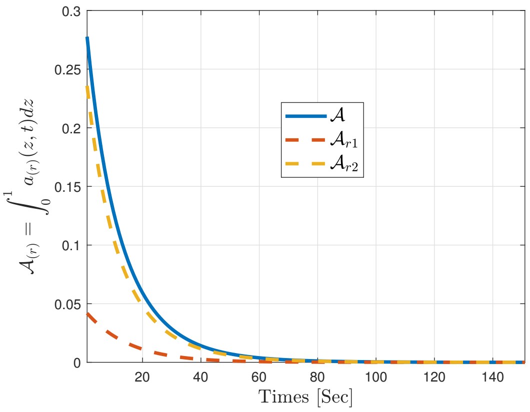


Fig. 6. Global availability $\mathcal{A}$ and global reduced availability $\mathcal{A}_{r1}$ and $\mathcal{A}_{r2}$.

$$\begin{cases} x_1(z) = x_{1,in} \left[\left(\mathrm{e}^1 \dfrac{x_{2,in}}{x_{1,in}} - 1 \right) z^2 + 1 \right] \mathrm{e}^{-z^2}, \\ x_2(z) = x_{2,in} \exp \left[(1-z)^2 \ln \left(\dfrac{x_{1,in}}{x_{2,in}} \right) \right]. \end{cases} \tag{19}$$

Figure 5 shows the time and spatial evolutions of the local availability function $a(z,t)$. It converges globally and asymptotically to zero along the counterflow heat exchanger. Figure 6 shows that the global availability function $\mathcal{A}(t)$ converges asymptotically to zero. Note that this function can be split in two contributions:

Table 1. Heat exchanger parameters & simulation conditions

Description	Symbol	Value
Boundary condition for internal fluid	$x_1(0,t)$	60°C
Boundary condition for exttternal fluid	$x_2(1,t)$	8°C
Heat transfer coefficient of internal fluid	α_1	$0.9011 s^{-1}$
Heat transfer coefficient of external fluid	α_2	$0.9091 s^{-1}$
Velocity of internal fluid	v_1	$0.0938 m/s$
Velocity of internal fluid	v_2	$0.0946 m/s$

$$\begin{aligned} \mathcal{A}(t) &= -\int_0^1 \begin{pmatrix} \frac{1}{\tilde{x}_1} & \frac{1}{\tilde{x}_2} \end{pmatrix} \begin{pmatrix} \tilde{h}_1 \\ \tilde{h}_2 \end{pmatrix} dz \\ &= -\int_0^1 \frac{1}{\tilde{x}_1} \tilde{h}_1(z,t) dz - \int_0^1 \frac{1}{\tilde{x}_2} \tilde{h}_2(z,t) dz, \end{aligned} \tag{20}$$

the first of which, noted $\mathcal{A}_{r1}(t)$, corresponds to the thermal contribution of the internal fluid, and the second, noted $\mathcal{A}_{r2}(t)$, to the thermal contribution of the external fluid. These global reduced availability functions are depicted in Figure 6. We note that they converge towards zero with a smaller amplitude than $\mathcal{A}(t)$. In particular, the amplitude of $\mathcal{A}_{r2}$ is greater than that of $\mathcal{A}_{r1}$. This is due to the fact that according to the simulation conditions listed in Table 1, the following condition on the thermodynamic driving forces of the heat exchanger is verified: $(1/x_2) > (1/x_1)$. In other words, all of the entropy produced $\Sigma_s(t)$ is transferred to the external fluid.

5. CONCLUSION

This article was dedicated to the thermodynamic analysis of a counterflow heat exchanger, when the dynamics are described by two hyperbolic partial differential equations. These devices are essential for energy applications. First, from the two first thermodynamics laws we have derived the models of the heat exchanger in entropic and energetic representation. Next we have performed an analysis of the equilibrium profiles from a certain condition on the system parameters, that guarantees the existence and uniqueness of the solution of the PDEs model. This analysis highlights the importance of the thermal pinch as a factor in energy efficiency. Finally we successively studied passivity and asymptotic stability, by using the entropy as a storage function entropy production as a dissipation function for to proof the passivity, and the thermodynamic availability function for the stability.

REFERENCES

Alonso, A. and Ydstie, B. (2001). Stabilisation of distributed systems using irreversible thermodynamics. *Automatica*, 37(11), pp:1739 – 1755.

Balaji, C., Srinivasan, B., and Gedupudi, S. (2021). Chapter 7 - heat exchangers. In C. Balaji, B. Srinivasan, and S. Gedupudi (eds.), *Heat Transfer Engineering*, 199–231. Academic Press. doi:https://doi.org/10.1016/B978-0-12-818503-2.00007-1.

Bejan, A. (1977). The Concept of Irreversibility in Heat Exchanger Design: Counterflow Heat Exchangers for Gas-to-Gas Applications. *Journal of Heat Transfer*, 99(3), 374–380. doi:10.1115/1.3450705. URL `https://doi.org/10.1115/1.3450705`.

Bejan, A. (1996). *Entropy generation minimization : the method of thermodynamic optimization of finite-size systems and finite-time processes*. Boca Raton (Fla.) : CRC press.

Bejan, A. and Kraus, A.D. (2003). *Heat Transfer Handbook*. John Wiley & Sons, Inc., Hoboken, New Jersey.

Besson, T., Tchousso, A., and Xu, C.Z. (2006). Exponential stability of a class of hyperbolic pde models from chemical engineering. In *Proceedings of the 45th IEEE Conference on Decision and Control*, 3974–3978. doi:10.1109/CDC.2006.377214.

de Groot, S.R., Mazur, P., and Choi, S. (1984). *Non-equilibrium thermodynamics*. Dover, New York.

der Schaft, A.V. (1996). *L_2-Gain and passivity techniques in nonlinear control*. London.

Favache, A. and Dochain, D. (2009). Thermodynamics and chemical systems stability: The cstr case study revisited. *Journal of Process Control*, 19(3), 371–379. doi:https://doi.org/10.1016/j.jprocont.2008.07.007.

Forsberg, C.H. (2021). Chapter 8 - heat exchangers. In C.H. Forsberg (ed.), *Heat Transfer Principles and Applications*, 305–341. Academic Press. doi:https://doi.org/10.1016/B978-0-12-802296-2.00008-1.

García-Sandoval, J., Hudon, N., Dochain, D., and González-Álvarez, V. (2016). Stability analysis and passivity properties of a class of thermodynamic processes: An internal entropy production approach. *Chemical Engineering Science*, 139, 261–272. doi:https://doi.org/10.1016/j.ces.2015.07.039.

Guo, Z., Liu, X., Tao, W., and Shah, R. (2010). Effectiveness–thermal resistance method for heat exchanger design and analysis. *International Journal of Heat and Mass Transfer*, 53(13), 2877–2884. doi:https://doi.org/10.1016/j.ijheatmasstransfer.2010.02.008.

Hoang, N.H. and Dochain, D. (2013). A thermodynamic approach to the passive boundary control of tubular reactors. *IFAC Proceedings Volumes*, 46(23), 383–388. doi:https://doi.org/10.3182/20130904-3-FR-2041.00018. 9th IFAC Symposium on Nonlinear Control Systems.

Kazaku, J.K., Dochain, D., Kahilu, M.M., and Kalenga Kaunde Kasongo, J. (2022a). Optimal observer-based lq-feedback regulation for hyperbolic model of a countercurrent heat exchanger. In *2022 American Control Conference (ACC)*, 1679–1684. doi:10.23919/ACC53348.2022.9867502.

Kazaku, J.K., Dochain, D., Kahilu, M.M., and Kasongo, J.K.K. (2022b). P-i boundary control of countercurrent heat exchanger. *IFAC-PapersOnLine*, 55(40), 49–54. doi:https://doi.org/10.1016/j.ifacol.2023.01.047. 1st IFAC Workshop on Control of Complex Systems COSY 2022.

Kjelstrup, S., Bedeaux, D., Johannessen, E., and Gross, J. (2010). *Non-Equilibrium Thermodynamics for Engineers*. WORLD SCIENTIFIC. doi:10.1142/7869.

Maidi, A., Diaf, M., and Corriou, J. (2009). Boundary geometric control of a counter-current heat exchanger. *Journal of process control*, 19, 297 – 313.

Mohamed, H.A. (2005). Entropy Generation in Counter Flow Gas to Gas Heat Exchangers. *Journal of Heat Transfer*, 128(1), 87–92. doi:10.1115/1.2130407. URL `https://doi.org/10.1115/1.2130407`.

Ogunedo, B.M.O. (2020). Chapter 4 - characterization of heat exchanger dynamic behavior. In L. Pekař (ed.), *Advanced Analytic and Control Techniques for Thermal Systems with Heat Exchangers*, 133–147. Academic Press. doi:https://doi.org/10.1016/B978-0-12-819422-5.00004-9.

Prieur, C. and Winkin, J.J. (2018). Boundary feedback control of linear hyperbolic systems: Application to the saint-venant–exner equations. *Automatica*, 89, 44 – 51.

Prigogine, I. (1947). *Étude thermodynamique des phénomènes irréversibles*. (Collection des thèses d'agrégation de l'enseignement supérieur de l'Université Libre de Bruxelles). Dunod.

Sandoval, J.P.G. (2021). Towards the control of heat exchangers based on thermodynamic principles. In *Memorias del Congreso Nacional de Control Automático*, pp:80 – 85. Guanajuato, Mexico.

Zhou, W., Hamroun, B., Le Gorrec, Y., and Couenne, F. (2021). A thermodynamic approach to the stabilization of tubular reactors. *Journal of Process Control*, 108, 98–111. doi:https://doi.org/10.1016/j.jprocont.2021.11.006.