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Michel Denuit, Christian Y. Robert

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ISBA
Voie du Roman Pays 20 - L1.04.01
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Michel Denuit
Institute of Statistics, Biostatistics and Actuarial Science (ISBA)
Louvain Institute of Data Analysis and Modeling (LIDAM)
UCLouvain
Louvain-la-Neuve, Belgium
michel.denuit@uclouvain.be

Christian Y. Robert
Laboratoire de Sciences Actuarielle et Financière (SAF)
Institut de science financière et d'assurances (ISFA)
Université Claude Bernard Lyon 1
Lyon, France
christian.robert@univ-lyon1.fr

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Abstract

This paper proposes a new fully-funded risk-sharing mechanism among economic agents. Individual contributions are deterministic and selected ex ante, while ex-post payoffs depend on realized individual losses and aggregate surplus or deficit at the level of the pool. Full compensation is awarded to claiming participants and surplus is distributed as long as the aggregate losses remain smaller than the accumulated contributions, whereas compensations are reduced proportionally in case of deficit. The paper establishes the existence of actuarially-fair contributions, ensuring that each agent's contribution equals the corresponding expected payoff. The risk-reduction properties of the proposed mechanism are analyzed with respect to the convex order. Particular attention is devoted to the case where total contributions are equal to the expected aggregate loss. In this case, limiting values of individual contributions are derived as well as the asymptotic distribution of the number of participants who are only partially compensated. Refined results are obtained in the homogeneous case, allowing for a better understanding of the proposed risk-sharing mechanism. Numerical illustrations based on Monte-Carlo methods demonstrate how the theoretical results can be applied in practice.

Keywords: Risk sharing; Actuarially-fair contributions; Fixed-point methods; Convex order; Large-pool asymptotics.

1 Introduction

In the classical insurance paradigm, economic agents transfer their risks to a "central" insurer. In this setting, each policyholder pays in advance a premium to the insurer and, in return, receives compensation for the realized losses. The aggregate risk of the portfolio is entirely borne by the insurer. Regulatory authorities typically impose that insurers must be able to pay for the claims with high probability. This is possible by relying on conservative premium calculation, diversification across large and sufficiently homogeneous portfolios, and appropriate amount of solvency capital. As a centralized form of risk transfer, insurance thus comes at the cost of risk loadings, capital requirements, and dividends paid to the insurer's shareholders.

An alternative to centralized insurance consists of decentralized risk-sharing schemes where the aggregate risk remains with the participants and no external guarantor is involved. Decentralized risk sharing refers to the situation where economic agents join to form a pool to mutually protect each other against the monetary consequences of a given peril. To this end, individual losses are re-allocated among the agents in a mutually beneficial way. Recent research has renewed interest in these mechanisms, motivated by mutual insurance, collaborative (or peer-to-peer) insurance, and mutual aid. The reader is referred, e.g., to the book by Feng (2023) for a general presentation as well as to Feng et al. (2024), Anthropelos et al. (2026), and the references therein for recent developments.

Decentralized risk sharing is often restricted to niche players but there are nevertheless successful initiatives on the market. Let us mention Medishare offering an alternative to traditional health insurance. It is the US largest health care sharing community, with more than 400 000 members. As another example, broodfondsen (literally translated as bread funds) in the Netherlands allow self-employed workers to mutually protect against the economic consequences of disability by starting a "gift circle".

Emerging risks can also be covered through dedicated schemes. This is the case with Mutual Insurance and Reinsurance for Information Systems (MIRIS) based in Brussels, owned by its members and operating cyber risk pooling exclusively for the members. Existing insurance offer can also be supplemented with a risk-sharing scheme. For instance, parametric insurance where benefits are calculated from a suitable reference index (or parameter) can be supplemented with a risk-sharing scheme to pool departures of individual losses from parametric insurance benefits. See, e.g., Niakh et al. (2026). These decentralized schemes grow in importance, even if they remain less common compared to traditional, centralized insurance.

In this paper, we propose a new risk-sharing scheme where participants pay ex-ante contributions to the pool while the collected money is redistributed ex post, contingent on the losses occurring from pre-defined adverse events. Full compensation is awarded when total contributions exceed realized losses, and a cash-back mechanism operates, whereas only partial compensation is possible when realized losses exceed total contributions. The central question addressed in this paper is how losses can be pooled and shared in a way that is well defined ex ante, actuarially fair to participants, and effective in reducing individual risk. This is achieved by a proportional decrease in benefits in deficit situations, based on the individual excesses of actual losses over contributions paid to the pool. This mechanism ensures that the scheme is fully funded in all cases.

The proposed risk-sharing scheme is inspired from so-called endowment contingency funds introduced by Denuit and Robert (2025) to pool binary losses. As the name indicates, participants to an endowment contingency fund contribute money in advance to a mutual fund established over a given time period, with the understanding that the total amount collected is going to be distributed in arrear, among those participants who experienced a predefined event during that period. This approach is called "risk sharing via compensations" according to the terminology used by Dhaene and Milevsky (2024) or "compensation-based risk-sharing" by Dhaene et al. (2025). This is in contrast with "contribution-based risk-sharing", in the terminology proposed by Dhaene et al. (2025) where random contributions are paid at the end of the period, by sharing total losses once they have been observed. This is the dominant practice in risk sharing where each participant is fully compensated for the realized loss, but subsequently contributes to the pool according to a rule agreed ex ante.

Dhaene et al. (2025) propose and study compensation-based risk-sharing mechanisms for general losses, extending previous results by Denuit and Robert (2025) and Dhaene and Milevsky (2024) where only binary losses were considered. In their framework, the contributions paid by participants at inception are subsequently redistributed at maturity. Solvency is ensured by a full-allocation condition requiring that total compensations equal the fund value. Benefits awarded to participants may fall short of, match, or exceed their realized losses, depending on aggregate outcomes. The present paper builds on this line of research, but adopts a complementary perspective aiming to strengthen the link between compensations and losses compared to Dhaene et al. (2025). Specifically, the paper considers agents exposed to a non-negative random loss and contributing a deterministic amount to a common pool. Ex post, realized losses and aggregate imbalances are redistributed through a fully-funded risk-sharing rule. This paper studies actuarially-fair contributions as solutions of a fixed-point problem, ensuring that each participant's contribution coincides with the expected payoff under the sharing mechanism. This notion of fairness is endogenous, distribution-sensitive, and compatible with heterogeneity across participants. From a conceptual viewpoint, the approach proposed in this paper bridges contribution-based and compensation-based perspectives. Contributions are deterministic and set ex ante, as in classical insurance, but no external guarantor is present and all payments are financed internally, as in decentralized schemes. The risk-sharing scheme proposed in this paper aims to provide a cheaper protection compared to insurance contracts, and to solve the conflict of interest inherent to centralized insurance since all money collected from participants is returned to claimants at the end of the period. It also avoids any counter-party risk inherent to traditional risk-sharing solutions but compensations paid to claimants remain unknown until the end of the coverage period and can be reduced proportionally in case of deficit. The fixed-point characterization of actuarially-fair contributions plays a role analogous to actuarial fairness conditions in classical risk-sharing schemes, while being tailored to settings in which contributions, rather than compensations, are the primary design variables.

The risk-sharing scheme proposed in this paper is fully funded. But this comes at the cost of some randomness in the compensation awarded to claiming participants. The risk associated to the net position is nevertheless reduced with respect to the convex order, for every participant. In the expected utility setting, this means that all risk-averse agents prefer joining the pool over standing alone and facing individual risk. It seems reasonable to expect that the randomness in the benefits awarded ex post to claimants disappears when the number of participants is large enough. This question is rigorously addressed by studying diversification effects within large pools, under mutual independence. It is shown that insurance at fair price is recovered at the limit, when the number of participants tends to infinity.

After having considered heterogeneous losses and large-pool asymptotics, it is equally important to analyze the properties of the mechanism in the homogeneous case. When agents are exposed to identically distributed losses, the structure of the risk-sharing rule considerably simplifies and allows for sharper results, both at the individual level and at the aggregate level. Homogeneity provides a natural benchmark that makes it possible to disentangle the intrinsic effects of the mechanism from those induced by heterogeneity in risk profiles or contribution levels. It also facilitates explicit computations and closed-form characterizations, which are often unavailable in the general case.

Beyond its analytical convenience, the homogeneous setting is economically meaningful. It corresponds to situations where agents face comparable risks and agree to participate in a mutualized arrangement under symmetric conditions. In this context, the risk-sharing mechanism can be studied in a transparent way, and the impact of the pool size can be assessed without confounding effects. Understanding the risk-sharing mechanism in the case of an homogeneous pool therefore offers valuable insight into its core properties and serves as a benchmark for interpreting the more complex results obtained in heterogeneous environments.

The remainder of the paper is organized as follows. Section 2 introduces the risk-sharing mechanism and defines actuarially-fair contributions. Such contributions are the solutions to a fixed-point problem and their existence is carefully discussed. Section 3 studies the heterogeneous case, including risk-reduction and asymptotic results in the critical regime. Section 4 focuses on the homogeneous setting and highlights the impact of symmetry. Section 5 provides numerical illustrations based on Monte-Carlo approximations of the fixed-point operator. Exponentially-distributed losses are considered throughout the paper to illustrate the theoretical results and to support simulation studies. Section 6 concludes by summarizing the main contributions of the paper and discussing additional research topics. Proofs and further technical developments are collected in Appendix.

2 The risk-sharing mechanism

2.1 Description of the mechanism

We consider a group of n participants exposed over a fixed period of time, (0,1) say, to non-negative random losses $X_1, X_2, \ldots, X_n$, assumed to have finite expectations. These losses may represent damage claims, health expenditures, or more generally any adverse financial consequences materializing at the end of the period. The objective is to design a decentralized risk-sharing mechanism that is fully funded, budget-balanced in all states of the world, and capable of reducing individual risk through mutualization.

At time 0, the n agents join to form a common pool. Each agent i contributes a deterministic amount $c_i \geq 0$ to the pool. The total amount of available resources is denoted by $w = \sum_{i=1}^n c_i$. It represents the maximum aggregate compensation that can be delivered at time 1. The value of w is fixed ex ante and reflects the collective level of risk that participants are willing to mutualize. In particular, w may be chosen as a quantile of the aggregate loss distribution so that full compensation can be awarded with a controlled probability. For instance, participants may agree about the quantile at probability level 75%, so that moderate aggregate fluctuations are fully absorbed by the pool. As another example, participants may decide to contribute the total expected loss to the pool.

Notice that contributions c_i may be supplemented with fees covering running expenses (that are not considered in our analysis). The amount c_i may be invested and accumulate to $c_i' = c_i(1+r)$ at time 1. To simplify the exposition, we work here under $r = 0$. Thus, the total contributions w paid in advance by the n participants is the amount to be allocated at time 1 among those participants who experienced losses over (0,1). Formulas can easily be extended to the case $r > 0$.

At time 1, losses are realized and aggregated into $S_n = \sum_{i=1}^n X_i$. The redistribution of the pool depends on the comparison between the realized aggregate loss S_n and the available resources w. The time-1 payoff for participant i is denoted by

$$H_i = h_i(\boldsymbol{c}; \boldsymbol{X}),\ \ i = 1, 2, \ldots, n,$$

where the functions $h_1, h_2, \ldots, h_n$ map the vector of individual contributions $\boldsymbol{c} = (c_1, \ldots, c_n)$ and the random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ of individual losses to the respective compensation $H_1, H_2, \ldots, H_n$ awarded to the n participants. The dependence of H_i in $\boldsymbol{c}$ and $\boldsymbol{X}$ complicates the analysis of the system but the relevant information confines to the first few moments in large pool of independent losses. The risk-sharing mechanism is designed so that, for any realization of losses, the full-allocation condition

$$\sum_{i=1}^n H_i = \sum_{i=1}^n h_i(\boldsymbol{c}; \boldsymbol{X}) = w \tag{2.1}$$

is satisfied, ensuring solvency by construction. Throughout the paper, equality and inequality between random variables or random vectors are assumed to hold with probability 1, that is, to hold almost surely, unless stated otherwise.

The redistribution rule materializes in the functions $h_1, h_2, \ldots, h_n$. As explained earlier, agents first set the level w of total resources so that the contribution vector belongs to the simplex Δ_w of admissible contributions defined as

$$\Delta_w = \Big\{ \boldsymbol{c} \in \mathbb{R}_+^n :\ \sum_{i=1}^n c_i = w \Big\}. \tag{2.2}$$

For a given $\boldsymbol{c} \in \Delta_w$, redistribution is based on the individual shortfall random variables $(c_i - X_i)_+$, $i = 1, 2, \ldots, n$, and on the individual excess random variables $(X_i - c_i)_+$, $i = 1, 2, \ldots, n$, where $(\cdot)_+$ denotes the positive part of the real number appearing within the brackets. These quantities measure, respectively, how much participant i has contributed in excess of his or her loss and how much his or her loss exceeds the corresponding contribution.

Two situations may occur with the proposed risk-sharing scheme:

Surplus situation If the realized aggregate loss satisfies $S_n < w$ then total contributions exceed total losses and the pool generates a surplus $w - S_n > 0$. In this case, all participants are fully indemnified for their losses and the surplus is redistributed among participants in proportion to their individual shortfalls. The intuition is that participants whose realized losses are small relative to their contributions should be rewarded and are therefore entitled to a larger share of the surplus remaining after losses have been compensated.

Deficit situation Conversely, if $S_n > w$ then the aggregate losses exceed available resources and the pool experiences a deficit. Full indemnification is no longer possible and losses must be partially absorbed by the participants. The proposed risk-sharing scheme allocates the aggregate shortfall proportionally to the individual excesses. Participants whose losses exceed their initial contributions to a larger extent bear a larger fraction of the deficit, reflecting their higher exposure to the pooled risk.

Formally, for given w and $\boldsymbol{c} \in \Delta_w$, the time-1 payoff for participant i is defined as

$$h_i(\boldsymbol{c}; \boldsymbol{X}) = \begin{cases} X_i + \dfrac{(c_i - X_i)_+}{\sum_{j=1}^n (c_j - X_j)_+}(w - S_n), & \text{if } S_n < w, \\ X_i, & \text{if } S_n = w, \\ X_i - \dfrac{(X_i - c_i)_+}{\sum_{j=1}^n (X_j - c_j)_+}(S_n - w), & \text{if } S_n > w, \end{cases} \tag{2.3}$$

with the convention that the ratios are evaluated only when the corresponding denominators are strictly positive. The piece-wise definition (2.3) emphasizes the symmetry of the proposed mechanism: the same proportional rule applies to surpluses and deficits, with the sign of $w - S_n$ determining whether extra-resources are redistributed or benefits are rationed compared to actual losses.

Several important properties immediately follow from (2.3). First, individual payoffs are always non-negative and bounded above by w. Second, the mechanism is fully funded and involves no counterparty risk, since payments never exceed the available resources w. Third, the redistribution rule is local and transparent as adjustments only depend on individual deviations $X_i - c_i$ and on aggregate imbalances captured by $\sum_{j=1}^n (c_j - X_j)_+$ and $\sum_{j=1}^n (X_j - c_j)_+$. As in any risk-sharing scheme and traditional insurance, submitted claims must be checked for correctness to avoid any over-estimation. This can be done by appointing claim-handlers or by contracting with an independent office specialized in claim settlement.

From an economic perspective, the mechanism can be interpreted as a mutual aid arrangement with an endogenous deductible at the aggregate level. Total losses are fully compensated until the resources w are exhausted, while adverse scenarios trigger proportional sharing of deficits. The individual shortfall random variables $(c_i - X_i)_+$ and excess random variables $(X_i - c_i)_+$ play a central role throughout the paper, as they govern both the redistribution of resources ex post and the sensitivity of the mechanism to changes in the contribution vector $\boldsymbol{c}$. In particular, they will be instrumental in the analysis of actuarial fairness and large-pool asymptotics developed in the following sections.

When the number of participants becomes large, the behavior of the fully-funded risk-sharing mechanism cannot be understood solely through the analysis of a finite pool gathering a given number n of participants. Large pools are precisely the setting in which diversification is expected to be most effective, and where comparisons with classical insurance arrangements become meaningful. They correspond to limits taken for $n \to \infty$. From a probabilistic perspective, increasing the number of agents induces concentration phenomena for aggregate quantities such as total losses and aggregate buffers, which in turn shape the frequency and severity of surplus and deficit situations. Analyzing the mechanism in large pools therefore provides insight into its stability, its risk-reduction capacity, and the way individual positions react to aggregate fluctuations when diversification is strong.

A key distinction arises according to the level of aggregate contributions relative to expected aggregate losses. Three asymptotic regimes naturally emerge:

Subcritical regime In the subcritical regime, total contributions exceed expected aggregate losses, so that surplus situations dominate asymptotically and full compensation becomes systematic.

Supercritical regime In the supercritical regime, total contributions fall short of expected aggregate losses, leading to persistent aggregate deficits and systematic partial indemnification.

Critical regime Between these two extremes lies the critical regime, in which total contributions are exactly equal to the expected aggregate loss. This regime plays a central role, as it corresponds to the classical actuarial benchmark where agents collectively contribute the expectation of their losses, as under fair insurance pricing. In this case, neither surplus nor deficit dominates asymptotically but both occur with non-vanishing probability, and aggregate fluctuations remain relevant even in large pools.

Understanding the behavior of the mechanism in the critical regime is therefore essential, as it captures the knife-edge situation where actuarial fairness, risk sharing, and aggregate uncertainty subtly interact. It is precisely in this regime that non-trivial limits are obtained and where the mechanism departs most sharply from both full insurance and systematic under-insurance.

2.2 Actuarially-fair contributions

The fully-funded sharing rule defined by the functions (2.3) depends on the vector of deterministic contributions $\boldsymbol{c}$ paid at time 0, with $\boldsymbol{c} \in \Delta_w$. A natural endogenous notion of fairness for such a decentralized mechanism, referred to as actuarial fairness in the insurance literature, is to require that each agent's ex-ante contribution coincides with the expected value of the corresponding time-1 payoff. This leads to the following definition.

Definition 2.1. *Let $h_1(\boldsymbol{c};\boldsymbol{X}), h_2(\boldsymbol{c};\boldsymbol{X}), \ldots, h_n(\boldsymbol{c};\boldsymbol{X})$ denote the respective payoffs induced by the sharing rule* (2.3) *for some loss vector $\boldsymbol{X}$ and contribution vector $\boldsymbol{c} \in \Delta_w$. The system is actuarially fair if*

$$c_i = \mathrm{E}[h_i(\boldsymbol{c};\boldsymbol{X})] \text{ for } i = 1, 2, \ldots, n. \tag{2.4}$$

The next result shows that whatever the total resources w accumulated by the participants in the pool, the fully-funded risk-sharing scheme (2.3) admits at least one vector of actuarially-fair contributions.

Theorem 2.2 (Existence of actuarially-fair contributions)**.** *Assume that* $X_i \geq 0$ *and* $\mathrm{E}[X_i] < \infty$ *for* $i = 1, 2, \ldots, n$*. Then, for every* $w > 0$*, there exists* $\boldsymbol{c}^\star \in \Delta_w$ *such that* (2.4) *holds true.*

The proof of Theorem 2.2 is given in Appendix A. It consists in establishing the existence of a fixed point for the mapping $\mathcal{T}$ defined on Δ_w by

$$\mathcal{T}(\boldsymbol{c}) = \mathrm{E}[\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X})] \tag{2.5}$$

where $\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X}) = (h_1(\boldsymbol{c}; \boldsymbol{X}), h_2(\boldsymbol{c}; \boldsymbol{X}), \ldots, h_n(\boldsymbol{c}; \boldsymbol{X}))$ denotes the payoff vector resulting from the sharing rule (2.3) when the loss vector is $\boldsymbol{X}$ and the contribution vector is $\boldsymbol{c} \in \Delta_w$. Any contribution vector $\boldsymbol{c}^\star \in \Delta_w$ satisfying $\mathcal{T}(\boldsymbol{c}^\star) = \boldsymbol{c}^\star$, that is, every fixed-point $\boldsymbol{c}^\star$ of $\mathcal{T}$ is thus a vector of actuarially-fair contributions according to Definition 2.1.

Theorem 2.2 ensures existence but does not by itself imply uniqueness of the actuarially-fair contribution vector, nor contractivity in the neighborhood of a fixed point. These issues are non-trivial because the sharing rule involves normalized positive-part vectors (ratios with individual excesses or shortfalls in the numerator and corresponding sums in the denominator), which may prevent global contractivity on Δ_w. Under additional regularity and lower-bound conditions on the aggregate buffers $\sum_{i=1}^n (c_i - X_i)_+$ and $\sum_{i=1}^n (X_i - c_i)_+$ in a neighborhood of an actuarially-fair contribution vector, local uniqueness is established as well as convergence of the Picard iteration $\boldsymbol{c}^{(k+1)} = \mathcal{T}(\boldsymbol{c}^{(k)})$ when initialized sufficiently close to the fixed point. We refer to the Appendix B for precise statements and proofs.

3 Heterogeneous pools

3.1 Risk reduction properties

A key question is to understand to what extent the fully-funded risk-sharing mechanism introduced in the previous section reduces individual risk, beyond mere budget balance and actuarial fairness. A convenient object to study is the *net position* $c_i + X_i - H_i$ of participant i, where $H_i = h_i(\boldsymbol{c}; \boldsymbol{X})$. This quantity represents the effective cost borne by agent i once both the initial contribution and the ex-post compensation delivered by the pool are taken into account. It may be compared to the case of a mechanism that would provide full insurance and where the net position would be equal to c_i, or to the case without risk sharing where agents stand alone so that the net position would be equal to X_i. Hence, the net position captures the extent to which the mechanism transforms the original risk X_i into a more diversified position.

Let $\mathbb{1}[\cdot]$ denote the indicator function, equal to 1 if the condition appearing within the brackets is satisfied and to 0 otherwise. Using the definition of the payoff rule given in (2.3) and noting that

$$w - S_n = \sum_{j=1}^n (c_j - X_j)_+ - \sum_{j=1}^n (X_j - c_j)_+,$$

the net position can be written as the weighted average

$$c_i + X_i - H_i = \big(1 - \rho_i(\boldsymbol{c}; \boldsymbol{X})\big)\, X_i + \rho_i(\boldsymbol{c}; \boldsymbol{X})\, c_i, \tag{3.1}$$

where the random weight $\rho_i(\boldsymbol{c};\boldsymbol{X}) \in [0,1]$ is given by

$$\rho_i(\boldsymbol{c};\boldsymbol{X}) = \min\left\{\frac{\sum_{j=1}^n (X_j - c_j)_+}{\sum_{j=1}^n (c_j - X_j)_+}, 1\right\} \mathbb{1}[X_i \le c_i] + \min\left\{\frac{\sum_{j=1}^n (c_j - X_j)_+}{\sum_{j=1}^n (X_j - c_j)_+}, 1\right\} \mathbb{1}[X_i > c_i],$$

This representation highlights the endogenous and state-dependent nature of the proposed risk-sharing scheme and makes the risk-reduction effect of the mechanism transparent. The weight $\rho_i(\boldsymbol{c};\boldsymbol{X})$ measures the intensity with which the realized loss X_i is pulled towards the deterministic benchmark c_i, depending both on the individual deviation $X_i - c_i$ and on the aggregate imbalance of the pool.

Several properties directly follow from the representation (3.1). First, the inequalities

$$\min\{X_i, c_i\} \le c_i + X_i - H_i \le \max\{X_i, c_i\}$$

hold true, so that the net position always lies between the realized loss and the contribution. Second, the transformation from X_i to the net position $c_i + X_i - H_i$ is a state-by-state contraction towards the constant c_i. Hence, the farther X_i lies from c_i, the stronger the redistribution effect induced by the proposed risk-sharing mechanism.

Representation (3.1) shows that the fully-funded sharing rule (2.3) induces a systematic shrinkage of individual losses towards deterministic benchmarks that are endogenously determined through the fairness condition. Let us now establish that any risk-sharing mechanism resulting in a net position of the form (3.1) reduces risk in terms of convex order. Recall that given two random variables Y and Z with equal means, supposed to be finite, Y is said to be smaller than Z in the convex order, henceforth denoted by $Y \preceq_{\mathrm{CX}} Z$, if the inequality $\mathrm{E}[(Y-t)_+] \le \mathrm{E}[(Z-t)_+]$ holds for all t. Thus, $\preceq_{\mathrm{CX}}$ only applies to random variables with the same expected value. The term "convex" is used since

$$Y \preceq_{\mathrm{CX}} Z \Leftrightarrow \mathrm{E}[\varphi(Y)] \le \mathrm{E}[\varphi(Z)] \text{ for all convex } \varphi \text{ (whenever expectations exist).} \tag{3.2}$$

The stochastic inequality $Y \preceq_{\mathrm{CX}} Z$ intuitively means that Y and Z have on average the same "size" (as $\mathrm{E}[Y] = \mathrm{E}[Z]$ holds) but that Z is "more variable" than Y. For instance, the variance of Z is larger than the variance of Y. In the expected utility setting, the convex order expresses the common preferences by all risk-averse economic agents about losses with the same expected value. For a thorough description of the convex order, we refer the reader to Shaked and Shanthikumar (2007).

The following general result of independent interest is needed to establish the risk-reducing effect of the proposed fully-funded risk-sharing mechanism.

Lemma 3.1. *Consider a pair (Z,R) of non-negative random variables with $\mathrm{E}[Z] < \infty$, $0 \le R \le 1$ and $\mathrm{E}[R] > 0$. Define the linear combination*

$$Y = (1-R)Z + Rb$$

for some constant $b > 0$ such that

$$\mathrm{E}[Y] = \mathrm{E}[Z] \Leftrightarrow b\,\mathrm{E}[R] = \mathrm{E}[RZ] \tag{3.3}$$

holds true. Then, $Y \preceq_{\mathrm{CX}} Z$. Moreover, if R is non-degenerate and Z is non-degenerate then the inequality (3.2) is strict for some convex functions φ.

The proof of Lemma 3.1 is given in Appendix C. The main result of this section then directly follows from representation (3.1) which allows us to use Lemma 3.1 with $R = \rho_i(\boldsymbol{c}; \boldsymbol{X})$, $Z = X_i$, and $b = c_i$ for $i = 1, 2, \ldots, n$. Condition (3.3) is fulfilled since $\boldsymbol{c}$ is a vector of actuarially-fair contributions. This shows that the proposed fully-funded risk-sharing rule improves the net position for every participant with respect to the convex order, as precisely stated next.

Proposition 3.2 (Risk reduction)**.** *Assume that $X_i \geq 0$, that $\mathrm{E}[X_i] < \infty$ for $i = 1, 2, \ldots, n$ and that $\boldsymbol{c}$ is a vector of actuarially-fair contributions. Then,*

$$c_i + X_i - H_i \preceq_{\mathrm{CX}} X_i \text{ for } i = 1, 2, \ldots, n.$$

In the expected utility setting, Proposition 3.2 ensures that all risk-averse agents are willing to join a pool enforcing the proposed risk-sharing rule. This is because they consider the net position $c_i + X_i - H_i$ when joining the pool less risky (in the $\preceq_{\mathrm{CX}}$-sense) compared to the stand-alone position where they face individual loss X_i. Notice that this result holds true whatever the distribution of the losses comprised in the pool and their dependence structure as long as their expectations are finite. This is possible because the risk-sharing rule (2.3) with an actuarially-fair contribution vector makes pooling attractive in all circumstances, in terms of convex order.

3.2 Growing number of agents in the critical regime with independent losses

Throughout this section, we work in the critical regime introduced earlier, where $w_n = \mathrm{E}[S_n] = \sum_{j=1}^{n} \mathrm{E}[X_j]$, and we let n tend to infinity. Notice that we have added a subscript n to total resources in order to emphasize the dependence on varying pool size n. For each n, the contribution vector $\boldsymbol{c}_n$ is assumed to be actuarially fair, that is, (2.4) holds true. The asymptotic analysis relies on the idea that aggregate surplus and deficit buffers typically grow linearly with the number of agents in large pools, while aggregate loss fluctuations remain of smaller order. This makes the redistribution effect induced by the mechanism asymptotically negligible at the level of a fixed individual.

3.2.1 Asymptotic behavior of individual contributions

We first investigate the asymptotic behavior of actuarially-fair contributions. This question is of central importance, as it allows us to understand how the endogenous contributions selected by the mechanism relate to classical actuarial benchmarks in large pools. In particular, one would expect that when diversification becomes strong, individual contributions should tend to the corresponding expected losses, as under fair insurance pricing. Establishing such a convergence provides a strong consistency property for the proposed mechanism. To formalize this intuition, we need the following assumptions.

Assumption 3.3 (Large-pool regime)**.** *Assume that individual losses $X_1, X_2, \ldots$ are mutually independent and satisfy the following conditions:*

(A1) There exist a constant $\gamma > 0$ and a sequence $\delta_n \downarrow 0$ such that

$$\mathrm{P}\left[\min\left\{\sum_{j=1}^{n}(c_{j,n} - X_j)_+, \sum_{j=1}^{n}(X_j - c_{j,n})_+\right\} \geq \gamma n\right] \geq 1 - \delta_n.$$

(A2) Aggregate loss variability grows at most linearly, that is,

$$\mathrm{Var}[S_n] = \sum_{j=1}^{n}\mathrm{Var}[X_j] \leq n\nu \text{ for some } \nu < \infty.$$

In Assumption 3.3, (A1) formalizes the idea that the pool is large enough for aggregate buffers to be of order n with high probability. This is typically satisfied under mild heterogeneity conditions by a large-deviation theorem. Condition (A2) prevents excessive aggregate volatility and is standard in large-portfolio asymptotics.

Proposition 3.4 (Convergence of actuarially-fair contributions)**.** *Assume that $X_i \geq 0$, that $\mathrm{E}[X_i] < \infty$ for $i = 1, 2, \ldots, n$, that $w_n = \mathrm{E}[S_n]$, and that Assumption 3.3 holds true. Let $\boldsymbol{c}_n$ be a vector of actuarially-fair contributions, that is, $\boldsymbol{c}_n$ solves* (2.4) *for every n. Then, for every fixed index i,*

$$|c_{i,n} - \mathrm{E}[X_i]| \to 0 \text{ as } n \to \infty.$$

More precisely,

$$|c_{i,n} - \mathrm{E}[X_i]| \leq \frac{4}{\gamma}\sqrt{\frac{\nu}{n}}\sqrt{\mathrm{Var}[X_i]} + 4\sqrt{\mathrm{E}[X_i^2]}\sqrt{\delta_n} \text{ for } n \geq \frac{\nu}{\gamma^2},$$

so that $|c_{i,n} - \mathrm{E}[X_i]| = O(n^{-1/2}) + O(\sqrt{\delta_n})$.

The proof of Proposition 3.4 is given in Appendix D. Appendix E strengthens Proposition 3.4 by establishing a global ℓ_2-rate for the solution of the fixed-point equation. More precisely, it is shown there that, at criticality, actuarially-fair contributions deviate from the mean vector $\mathrm{E}[\boldsymbol{X}] = (\mathrm{E}[X_1], \ldots, \mathrm{E}[X_n])$ at order $n^{-1/2}$ in ℓ_2-norm.

3.2.2 Number of partially-compensated agents

In a fully-funded risk-sharing mechanism, aggregate solvency is guaranteed by construction, but individual indemnification may be only partial when aggregate losses exceed the resources available at pool level. In such deficit situations, some participants receive a payoff that is strictly smaller than their realized loss. Understanding how many agents are only partially compensated is therefore a key issue, both from a quantitative and from a welfare perspective. This question is particularly relevant in large pools, where diversification is expected to mitigate individual exposures, but where aggregate fluctuations may still generate non-negligible shortfalls.

From an operational standpoint, identifying the number of agents who are not fully compensated provides an intuitive measure of the severity of deficit situations. From a theoretical viewpoint, this quantity is intrinsically linked to the structure of the sharing rule in the sense that agents are not fully compensated precisely when their loss exceeds their contribution and the aggregate loss exceeds total contributions at the level of the pool. As such, the distribution of the number of participants who are only partially compensated reflects the joint tail properties of individual risks and their aggregation.

Define the indicator variables

$$I_i = \mathbb{1}[X_i > c_i], \;\; i = 1, 2, \ldots, n,$$

which signal whether agents experience losses exceeding contributions. The total number of agents who are not fully compensated ex post is then

$$M_n = \sum_{i=1}^{n} I_i \text{ conditional on the event } \{S_n > w_n\}.$$

The objective is to characterize the asymptotic behavior of M_n given that the system is in deficit.

Since both S_n and M_n are sums of independent (but heterogeneous) random variables, a joint asymptotic analysis naturally relies on central-limit arguments. To this end, define

$$p_i = \mathrm{E}[I_i] = \mathrm{P}[X_i > c_i], \qquad \mu_{M,n} = \mathrm{E}[M_n] = \sum_{i=1}^{n} p_i, \qquad \sigma^2_{M,n} = \sum_{i=1}^{n} p_i(1 - p_i),$$

as well as $\sigma^2_{S,n} = \mathrm{Var}[S_n]$ and $\sigma_{SM,n} = \mathrm{Cov}[S_n, M_n]$.

Assumption 3.5 (Lindeberg-type conditions)**.** *Assume that losses $X_1, X_2, \ldots$ are mutually independent, that $\sigma^2_{S,n} \to \infty$ and $\sigma^2_{M,n} \to \infty$ as $n \to \infty$, and that the following conditions are fulfilled:*

(i) for every $\varepsilon > 0$,

$$\frac{1}{\sigma^2_{S,n}} \sum_{i=1}^{n} \mathrm{E}\big[(X_i - \mathrm{E}[X_i])^2 \mathbb{1}[|X_i - \mathrm{E}[X_i]| > \varepsilon \sigma_{S,n}]\big] \to 0.$$

(ii)

$$\max_{1 \le i \le n} \frac{p_i(1 - p_i)}{\sigma^2_{M,n}} \to 0.$$

In Assumption 3.5, item (i) is the classical Lindeberg condition ensuring asymptotic Normality of the aggregate loss S_n. Item (ii) plays an analogous role for M_n and prevents degeneracy driven by a vanishing number of dominant agents. Together, these conditions guarantee a joint Gaussian approximation for (S_n, M_n) after suitable normalization. This is precisely stated next.

Proposition 3.6 (Conditional Gaussian approximation)**.** *Assume that the limit*

$$\tau = \lim_{n \to \infty} \frac{\sigma_{SM,n}}{\sigma_{S,n} \sigma_{M,n}}$$

exists. Under Assumption 3.5,

$$\left(\frac{S_n - w_n}{\sigma_{S,n}}, \frac{M_n - \mu_{M,n}}{\sigma_{M,n}}\right) \xrightarrow{\mathrm{d}} (Z_1, Z_2) \sim \text{Normal}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \tau \\ \tau & 1 \end{pmatrix}\right).$$

Moreover,

$$\begin{aligned} \mathrm{E}[M_n \mid S_n > w_n] &= \mu_{M,n} + \frac{\sigma_{SM,n}}{\sigma_{S,n}}\sqrt{\frac{2}{\pi}} + o(\sigma_{M,n}), \\ \mathrm{Var}[M_n \mid S_n > w_n] &= \sigma^2_{M,n}\left(1 - \frac{\tau^2}{2\pi}\right) + o(\sigma^2_{M,n}), \end{aligned}$$

and, for any fixed $x \in \mathbb{R}$,

$$\mathrm{P}\left[M_n > \mu_{M,n} + \frac{\sigma_{SM,n}}{\sigma_{S,n}}\sqrt{\frac{2}{\pi}} + x\sigma_{M,n}\sqrt{1 - \frac{\tau^2}{2\pi}} \;\middle|\; S_n > w_n\right] \to 1 - \Phi(x),$$

where $\Phi(\cdot)$ denotes the distribution function of the Normal(0,1) law.

The proof of Proposition 3.6 is given in Appendix F. The next example considers a simple setting where analytical developments are available. It thus helps to understand how the system operates.

Example 3.7 (Heterogeneous Exponentially-distributed losses)**.** *Assume that individual losses are independent and Exponentially distributed, that is, $\mathrm{P}[X_i > t] = \exp(-\lambda_i t)$ for some $\lambda_i > 0$, $i = 1, \ldots, n$, which is henceforth denoted as $X_i \sim \mathrm{Exp}(\lambda_i)$. This setting thus allows for heterogeneity across agents through distinct rate parameters λ_i. Recall that $\mathrm{E}[X_i] = 1/\lambda_i$ and $\mathrm{Var}(X_i) = 1/\lambda_i^2$. We work in the critical regime where $w_n = \mathrm{E}[S_n] = \sum_{i=1}^n 1/\lambda_i$. By Theorem E.1, we have for each fixed i that*

$$c_{i,n} = \mathrm{E}[X_i] + O\left(\frac{1}{n}\right) = \frac{1}{\lambda_i} + O\left(\frac{1}{n}\right).$$

In all asymptotic expressions below, contributions may therefore be replaced by $\mathrm{E}[X_i]$ without affecting first-order results.

Using the Exponential distribution and the above approximation,

$$p_i = \mathrm{E}[I_i] = \mathrm{P}[X_i > c_{i,n}] = \exp(-\lambda_i c_{i,n}) = e^{-1} + O\left(\frac{1}{n}\right),$$

uniformly in i. Hence,

$$\mathrm{E}[M_n] = \sum_{i=1}^n p_i = ne^{-1} + O(1), \qquad \mathrm{Var}[M_n] = ne^{-1}(1 - e^{-1}) + O(1).$$

Moreover, using the memoryless property of the Exponential distribution,

$$\mathrm{E}[X_i I_i] = \mathrm{E}[X_i \mid X_i > c_{i,n}]\,\mathrm{P}[X_i > c_{i,n}] = \left(c_{i,n} + \frac{1}{\lambda_i}\right)e^{-\lambda_i c_{i,n}},$$

which yields

$$\mathrm{Cov}[X_i, I_i] = c_{i,n} e^{-\lambda_i c_{i,n}} = \frac{e^{-1}}{\lambda_i} + O\left(\frac{1}{n}\right).$$

Therefore,

$$\mathrm{Cov}[S_n, M_n] = \sum_{i=1}^{n} \mathrm{Cov}[X_i, I_i] = e^{-1} \sum_{i=1}^{n} \frac{1}{\lambda_i} + O(1) = e^{-1} w_n + O(1).$$

All assumptions of Proposition 3.6 and Theorem E.1 are satisfied provided the sequence $(\lambda_i)_{i\geq 1}$ *remains bounded away from* 0 *and* ∞*. As a consequence,*

$$\mathrm{E}[M_n \mid S_n > w_n] = ne^{-1} + \frac{e^{-1} w_n}{\sqrt{\sum_{i=1}^{n} \lambda_i^{-2}}} \sqrt{\frac{2}{\pi}} + o(\sqrt{n}),$$

and the conditional fluctuations of M_n *are asymptotically Gaussian. This example illustrates that, in large heterogeneous pools with Exponentially-distributed losses, the frequency and severity of partial compensation events are primarily driven by first moments, while heterogeneity only enters through second-order dispersion effects.*

4 Homogeneous pools

4.1 Finite pools

We first specialize the analysis to the homogeneous case with a pool gathering a given number n of participants and we omit the subscript n. In this setting, all agents face identically distributed losses and contribute symmetrically to the pool. Actuarial fairness then implies a common contribution level $c > 0$ for all agents, so that $w = nc$. The objective of this section is to characterize the distribution of the payoff $H_1 = h_1(c; \boldsymbol{X})$ received by a representative agent. This distribution is a central object as it fully describes the individual risk borne under the proposed mechanism and allows for direct comparisons with classical insurance contracts in terms of variability and welfare.

Let $F(\cdot)$ be the common distribution function of $X_1, X_2, \ldots, X_n$. Let us denote by X a general loss with distribution F and let $\mu = \mathrm{E}[X]$ be the common mean, assumed to be finite. The payoff H_1 depends on the realization of losses through the comparison of S_n with w and through the relative position of X_1 with respect to c as well as the aggregate surplus and deficit random variables. Henceforth, we denote

$$U_n = U_n(c) = \sum_{i=1}^{n} (c - X_i)_+ \text{ and } D_n = D_n(c) = \sum_{i=1}^{n} (X_i - c)_+,$$

so that the identity $w - S_n = U_n - D_n$ holds and

$$H_1 = X_1 + \frac{(c - X_1)_+}{U_n}(w - S_n)\mathbb{1}[S_n \leq w] + \frac{(X_1 - c)_+}{D_n}(w - S_n)\mathbb{1}[S_n > w], \tag{4.1}$$

with the convention that the ratios are set to zero whenever the denominators vanish. This representation emphasizes that deviations of H_1 from X_1 arise only through aggregate imbalance and are proportional to the individual deviation from the contribution level c. Let us introduce the aggregated quantities excluding agent 1:

$$U_n^{\setminus 1} = \sum_{j=2}^{n} (c - X_j)_+, \text{ and } D_n^{\setminus 1} = \sum_{j=2}^{n} (X_j - c)_+.$$

For each $j \geq 2$, we have

$$(c - X_j)_+ (X_j - c)_+ = 0,$$

so that each individual loss contributes either to $U_n^{\setminus 1}$ or to $D_n^{\setminus 1}$, but never to both. Moreover, the aggregate loss of the subpool $\{2, \ldots, n\}$ can be written as

$$\sum_{j=2}^{n} X_j = (n-1)c - U_n^{\setminus 1} + D_n^{\setminus 1}.$$

Hence, the pair $(U_n^{\setminus 1}, D_n^{\setminus 1})$ fully characterizes the financial position of the pool excluding agent 1.

Proposition 4.1 (Joint law of $(U_n^{\setminus 1}, D_n^{\setminus 1})$)**.** *Let $X_2, \ldots, X_n$ be independent with common distribution function $F(\cdot)$. For $s, t \geq 0$, the joint Laplace transform of $(U_n^{\setminus 1}, D_n^{\setminus 1})$ is given by*

$$\mathrm{E}\left[e^{-sU_n^{\setminus 1} - tD_n^{\setminus 1}} \right] = \left(\int_0^c e^{-s(c-x)} \, \mathrm{d}F(x) + \int_c^\infty e^{-t(x-c)} \, \mathrm{d}F(x) \right)^{n-1}. \tag{4.2}$$

Equivalently, letting

$$K^{\setminus 1} := \sum_{j=2}^{n} \mathbb{1}[X_j > c],$$

we have $K^{\setminus 1} \sim \mathrm{Binomial}(n-1, q)$ with $q = \mathrm{P}[X > c]$. Conditionally on $K^{\setminus 1} = m$, the random variables $U_n^{\setminus 1}$ and $D_n^{\setminus 1}$ are independent, with

- *$U_n^{\setminus 1}$ equal in distribution to the sum of $n-1-m$ independent copies of $(c-X) \mid (X \leq c)$,*
- *$D_n^{\setminus 1}$ equal in distribution to the sum of m independent copies of $(X-c) \mid (X > c)$.*

The proof of Proposition 4.1 is given in Appendix G. Conditionally on $(X_2, X_3, \ldots, X_n)$, or equivalently on $(U_n^{\setminus 1}, D_n^{\setminus 1})$, the payoff H_1 becomes a deterministic function of $X_1 = x$, and will now be denoted by $h_1(x)$. Let us distinguish between the following situations:

- In the surplus regime, $x - U_n^{\setminus 1} + D_n^{\setminus 1} < c$, the payoff h_1 is given by

$$h_1(x) = \begin{cases} x + \dfrac{c-x}{(c-x) + U_n^{\setminus 1}} \left(c - x + U_n^{\setminus 1} - D_n^{\setminus 1} \right), & \text{if } x < c, \\ x, & \text{if } x \geq c. \end{cases}$$

- In the deficit regime, $x - U_n^{\setminus 1} + D_n^{\setminus 1} > c$, the payoff h_1 is given by

$$h_1(x) = \begin{cases} x, & \text{if } x \leq c, \\ x + \dfrac{x-c}{(x-c) + D_n^{\setminus 1}} \big(c - x + U_n^{\setminus 1} - D_n^{\setminus 1}\big), & \text{if } x > c. \end{cases}$$

- On the boundary $x - U_n^{\setminus 1} + D_n^{\setminus 1} = c$, we have $h_1(x) = x$.

As a consequence, the conditional distribution of H_1 given $(X_2, X_3, \ldots, X_n)$ is the image of $F(\cdot)$ by the random transformation

$$x \longmapsto h_1\big(x; U_n^{\setminus 1}, D_n^{\setminus 1}\big)$$

defined piece-wise above. Hence, the unconditional distribution of H_1 admits the mixture representation

$$\mathrm{P}[H_1 \leq y] = \mathrm{E}\Big[\mathbb{1}\big[h_1\big(X_1; U_n^{\setminus 1}, D_n^{\setminus 1}\big) \leq y\big]\Big]. \tag{4.3}$$

This representation highlights that $H_1 = X_1$ on a non-negligible region of the state space. Adjustments only occur when agent 1 is eligible to participate in the surplus ($x < c$ and $x - U_n^{\setminus 1} + D_n^{\setminus 1} < c$) or in the deficit ($x > c$ and $x - U_n^{\setminus 1} + D_n^{\setminus 1} > c$).

Example 4.2 (Homogeneous Exponentially-distributed losses)**.** *Assume that* $X \sim \mathrm{Exp}(\lambda)$*. Then,* $q = \mathrm{P}[X > c] = \mathrm{e}^{-\lambda c}$ *and* $K^{\setminus 1} \sim \mathrm{Binomial}(n-1, \mathrm{e}^{-\lambda c})$*. By the memoryless property of the Exponential distribution,* $(X - c) \mid (X > c) \sim \mathrm{Exp}(\lambda)$*, so that conditionally on* $K^{\setminus 1} = m$*,*

$$D_n^{\setminus 1} \sim \mathrm{Gamma}(m, \lambda).$$

Moreover, the conditional Laplace transform of $(c - X) \mid (X \leq c)$ *is explicit, which yields a closed-form expression for the joint Laplace transform of* $(U_n^{\setminus 1}, D_n^{\setminus 1})$ *via* (4.2)*. As a consequence, all quantities entering the mixture representation* (4.3) *can be evaluated explicitly in this case.*

In the case of an homogeneous pool of given size n, this section has thus shown that the fully-funded risk-sharing mechanism induces a nontrivial transformation of the individual loss distribution. The payoff H_1 can be viewed as a mixture of the original loss and a state-dependent truncation driven by aggregate imbalance. This characterization provides a probabilistic description of individual risk under the mechanism and serves as a foundation for the asymptotic and numerical analyses developed in the following sections.

4.2 Large-pool asymptotics

This section considers a sequence of pools of increasing size n with available resources $w_n = nc$, where $c > 0$ is a fixed per-capita contribution level. A fundamental distinction arises depending on the sign of $c - \mu$. This yields three asymptotic regimes: *subcritical* ($c > \mu$), *supercritical* ($c < \mu$), and *critical* ($c = \mu$). Outside the critical regime, the law of large numbers forces the pool to be asymptotically almost surely in surplus or almost surely in deficit, and the payoff $H_{1,n}$ converges to a non-degenerate limit. At criticality, by contrast, surplus ($S_n < w_n \Leftrightarrow U_n > D_n$) and deficit ($S_n > w_n \Leftrightarrow D_n > U_n$) both occur with non-vanishing probability, and the relevant fluctuations are driven by central-limit effects.

Henceforth, let X denote a generic random variable with distribution function $F(\cdot)$. To make the regime classification explicit, introduce the deterministic quantities

$$\bar{u} = \bar{u}(c) = \mathrm{E}[(c - X)_+] \text{ and } \bar{d} = \bar{d}(c) = \mathrm{E}[(X - c)_+] \text{ with } \bar{u} - \bar{d} = c - \mathrm{E}[X] = c - \mu.$$

These quantities correspond to expected per-capita shortfall and per-capita excess that would be generated by truncating the loss distribution at level c. By the law of large numbers, we have

$$\frac{U_n}{n} \xrightarrow{\mathrm{P}} \bar{u} \text{ and } \frac{D_n}{n} \xrightarrow{\mathrm{P}} \bar{d},$$

so that

$$\mathrm{P}[S_n < w_n] = \mathrm{P}[U_n > D_n] \to \begin{cases} 1, & \text{if } c > \mu \quad \text{(subcritical / asymptotic surplus)}, \\ \\ 0, & \text{if } c < \mu \quad \text{(supercritical / asymptotic deficit)}, \end{cases}$$

and $\mathrm{P}[S_n = w_n] \to 0$ under a mild regularity condition (for instance, that S_n possesses a density for large n). Moreover, away from the critical case $c = \mu$, the ratios that appear in the payoff rule stabilize: on the asymptotic surplus event ($c > \mu$),

$$\frac{D_n}{U_n} \xrightarrow{\mathrm{P}} \frac{\bar{d}}{\bar{u}} \text{ as } n \to \infty \text{ provided } \bar{u} > 0,$$

and on the asymptotic deficit event ($c < \mu$),

$$\frac{U_n}{D_n} \xrightarrow{\mathrm{P}} \frac{\bar{u}}{\bar{d}} \text{ as } n \to \infty \text{ provided } \bar{d} > 0.$$

This means that, for large pools, the randomness in the sharing rule is essentially driven by the individual loss X_1 (according to whether $X_1 \le c$ or $X_1 > c$), while the aggregate ratios converge to deterministic constants.

Proposition 4.3 (Limit law outside the critical case)**.** *Assume $c \neq \mu$ and set $w_n = nc$. Then, $H_{1,n}$ converges in distribution to a random variable H_∞ defined as follows:*

- *If $c > \mu$ (asymptotic surplus), then*

$$H_\infty = \begin{cases} X, & \text{if } X > c, \\ c - (c - X)\dfrac{\bar{d}}{\bar{u}}, & \text{if } X \le c. \end{cases} \tag{4.4}$$

- *If $c < \mu$ (asymptotic deficit), then*

$$H_\infty = \begin{cases} X, & \text{if } X \le c, \\ c + (X - c)\dfrac{\bar{u}}{\bar{d}}, & \text{if } X > c. \end{cases} \tag{4.5}$$

The proof of Proposition 4.3 is given in Appendix H.

We now turn to the critical case $c = \mu$, so that $\bar{u} = \bar{d}$ and $w_n = n\mu$ matches the classical actuarial benchmark of pricing at the fair premium. This regime is the most delicate and informative asymptotically: unlike the subcritical and supercritical cases, the event $\{S_n < w_n\}$ does not concentrate, because the aggregate loss fluctuates around its mean at the $\sqrt{n}$-scale. Assuming $\mathrm{Var}[X_1] = \sigma^2 \in (0, \infty)$ and a mild regularity condition ensuring $\mathrm{P}[S_n = w_n] \to 0$ (for instance, S_n possesses a density), the central-limit theorem yields

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{\mathrm{d}} Z \sim \mathrm{Normal}(0, 1), \text{ so that } \mathrm{P}[S_n < w_n] \to \mathrm{P}[Z < 0] = \tfrac{1}{2}.$$

However, even though surplus and deficit both occur asymptotically, the impact of the mechanism on a given participant becomes negligible at first order.

The next result is related to Proposition 3.4 where the convergence of the contribution to the expected loss was established in the general, heterogeneous case.

Proposition 4.4 (First-order limit at criticality)**.** *Assume* $\mathrm{E}[X_1^2] < \infty$ *and set* $c = \mu$ *so that* $w_n = n\mu$*. Then,*

$$H_{1,n} - X_1 \xrightarrow{\mathrm{P}} 0 \text{ as } n \to \infty.$$

The proof of Proposition 4.4 can be found in Appendix I. This first-order result shows that, at the actuarially-fair per-capita level $c = \mu$, the mechanism is asymptotically indistinguishable from the "insurance mechanism" for any given agent. To capture nontrivial corrections one must therefore analyze the $\sqrt{n}$-scale. Notice that

$$\bar{u}(\mu) = \mathrm{E}[(\mu - X)_+] = \mathrm{E}[(X - \mu)_+] = \bar{d}(\mu),$$

which is strictly positive as soon as X is non-degenerate. Denote

$$\beta = \bar{u}(\mu) = \bar{d}(\mu).$$

By the law of large numbers, $U_n/n \to \beta$ and $D_n/n \to \beta$ almost surely. Since $U_n - D_n = w_n - S_n = n\mu - S_n$, one can write

$$\frac{D_n}{U_n} = 1 + \frac{S_n - n\mu}{U_n}, \qquad \frac{U_n}{D_n} = 1 - \frac{S_n - n\mu}{D_n},$$

and linearize these ratios around 1 using $U_n = n\beta + o_P(n)$ and $D_n = n\beta + o_P(n)$, yielding expansions of the form

$$\frac{D_n}{U_n} = 1 + \frac{S_n - n\mu}{n\beta} + o_P(n^{-1/2}) \text{ and } \frac{U_n}{D_n} = 1 - \frac{S_n - n\mu}{n\beta} + o_P(n^{-1/2}).$$

The next result summarizes the induced $\sqrt{n}$-order fluctuations.

Proposition 4.5 ($\sqrt{n}$-fluctuation heuristic at criticality)**.** *Assume* $\mathrm{E}[X_1^2] < \infty$*,* $\sigma^2 = \mathrm{Var}[X_1] \in (0, \infty)$*, and* $\mathrm{P}[S_n = w_n] \to 0$*. Set* $c = \mu$ *and* $w_n = n\mu$*. Then,*

$$\sqrt{n}\,(H_{1,n} - X_1) = -\frac{S_n - n\mu}{\sqrt{n}\,\beta}\Big((\mu - X_1)_+ \mathbb{1}[S_n < w_n] + (X_1 - \mu)_+ \mathbb{1}[S_n > w_n]\Big) + o_P(1).$$

In particular, $\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{\mathrm{d}} Z \sim \mathrm{Normal}(0,1)$ *by the central-limit theorem and asymptotic independence between* X_1 *and* $(S_n - n\mu)/\sqrt{n}$*, gives the limit in distribution*

$$\sqrt{n}\,(H_{1,n} - X_1) \;\xrightarrow{\mathrm{d}}\; -\frac{\sigma}{\beta}\,Z\Big((\mu - X)_+\mathbb{1}[Z<0] + (X-\mu)_+\mathbb{1}[Z>0]\Big),$$

where Z *and* X *are mutually independent.*

Proposition 4.5 is stated as a heuristic expansion because it requires justifying uniform integrability and controlling the coupling between the ratio events $\{U_n > D_n\}$ and $\{D_n > U_n\}$. Nevertheless, it captures the correct scaling: at criticality, the mechanism modifies an individual payoff only through aggregate fluctuations of size $\sqrt{n}$, while the denominators U_n and D_n are of size n. This explains why the distributional limit in Proposition 4.4 is trivial at first order, and why second-order effects are the appropriate object for refined asymptotic comparisons. This can be achieved with, for instance, higher-order moments or moderate deviations.

Example 4.6 (Homogeneous Exponentially-distributed losses)**.** *For* $X \sim \mathrm{Exp}(\lambda)$*, elementary integration yields*

$$\bar{u} = c - \frac{1}{\lambda} + \frac{e^{-\lambda c}}{\lambda}, \;\; \bar{d} = \frac{e^{-\lambda c}}{\lambda}, \;\; \textit{and}\; \bar{u} - \bar{d} = c - \frac{1}{\lambda}.$$

For $c \neq 1/\lambda$*,* $H_{1,n}$ *converges in distribution to a random variable* H_∞ *defined as follows. If* $c > 1/\lambda$ *then* $S < w_n$ *holds with probability tending to* 1 *and* H_∞ *is given by* (4.4)*. If* $c < 1/\lambda$ *then* $S > w_n$ *holds with probability tending to* 1 *and* H_∞ *is given by* (4.5)*.*

5 Numerical illustration

This section proposes a numerical approach to compute the vector of actuarially-fair contributions as function of w. The aim is to determine, for each level of resources w, a contribution vector $\boldsymbol{c}^\star(w)$ belonging to the the simplex Δ_w and fulfilling the fixed-point condition $\boldsymbol{c}^\star(w) = \mathcal{T}\big(\boldsymbol{c}^\star(w)\big)$ corresponding to actuarial fairness.

For this numerical illustration, we assume that individual losses $X_1, X_2, \ldots, X_n$ are independent with $X_i \sim \mathrm{Exp}(\lambda_i)$ for distinct rate parameters $\lambda_i > 0$. This specification provides a flexible yet tractable benchmark, with finite moments of all orders and closed-form sampling, while preserving substantial heterogeneity in tail behavior. Throughout the numerical experiments, the number of agents is fixed to $n = 10$ and the rate parameters λ_i are evenly distributed, starting at 0.6 and ending at 1.8.

Given a contribution vector $\boldsymbol{c}$, the payoff rule is defined by (2.3). This rule is budget-balanced by construction as (2.1) holds true for any $\boldsymbol{c}$. The operator $\mathcal{T}$ does not admit a closed-form expression and is therefore approximated numerically using Monte-Carlo simulation. For a given contribution vector $\boldsymbol{c}$, independent realizations of $\boldsymbol{X}$ are generated according to the Exponential distribution, and the empirical mean of the corresponding payoffs provides an unbiased estimator $\widehat{\mathcal{T}}(\boldsymbol{c})$. To control Monte-Carlo variability, mini-batches of a few thousands simulations are used at each evaluation of $\mathcal{T}$, and antithetic sampling based on a uniform representation of Exponentially-distributed random variables is employed to reduce variance. Componentwise sample variances are computed simultaneously, yielding standard error estimates that are later exploited in the optimization procedure.

Actuarially-fair contributions are obtained by solving the fixed-point equation $\boldsymbol{c} = \mathcal{T}(\boldsymbol{c})$ on the simplex Δ_w. Because the operator $\mathcal{T}$ is evaluated with simulation noise and may fail to be globally contractive, a stochastic approximation approach is adopted. Specifically, a projected Robbins–Monro algorithm is implemented, in which successive iterates are updated according to

$$\boldsymbol{c}^{(k+1)} = \Pi_{\Delta_w}\left(\boldsymbol{c}^{(k)} + e_k \left(\widehat{\mathcal{T}}(\boldsymbol{c}^{(k)}) - \boldsymbol{c}^{(k)}\right)\right),$$

where Π_{Δ_w} denotes the Euclidean projection onto the simplex Δ_w and $(e_k)_{k\geq 0}$ is a decreasing step-size sequence. To improve numerical stability in the presence of heterogeneous risks, the update direction is preconditioned componentwise using the Monte-Carlo standard errors of $\widehat{\mathcal{T}}(\boldsymbol{c}^{(k)})$, effectively normalizing each coordinate by an estimate of its sampling variability. In addition, Polyak–Ruppert averaging is applied after a short burn-in period, substantially reducing asymptotic variance and yielding smoother convergence.

The algorithm is run for a fixed number of iterations (2,000), with step-size parameters chosen according to standard stochastic approximation guidelines. Convergence is monitored by evaluating the ℓ^1-residual $\|\boldsymbol{c} - \widehat{\mathcal{T}}(\boldsymbol{c}^{(k)})\|_1$ on independent Monte-Carlo batches. For each value of w, the final averaged iterate is taken as an approximation of the fair contribution vector $\boldsymbol{c}^\star(w)$.

The procedure is repeated for a grid $w \in \{6, 7, \ldots, 15\}$ to obtain contributions in function of total resources w. A continuation strategy is used. Specifically, the solution obtained for a given w is rescaled to satisfy the simplex constraint at the next level of resources and used as the initial condition of the algorithm. This warm-start mechanism ensures numerical stability and produces smooth trajectories of $c_i^\star(w)$ as functions of w. The resulting curves are visible in Figure 1. They provide a detailed illustration of how fair contributions adjust with total resources w under heterogeneous Exponentially-distributed losses.

Considering the results displayed in Figure 1, agent 1 brings to the pool the largest loss (with respect to stochastic dominance, with mean value 5/3). As expected, agent 1 thus pays the highest contribution to the pool. Contributions then decrease with the mean value of the loss brought to the pool. When w increases, we see that all agents must contribute to a larger extent to the pool, with steeper increase for agents bringing larger losses to the pool.

6 Conclusion

This paper has thoroughly analyzed a new fully-funded risk-sharing mechanism that accommodates heterogeneous losses while preserving budget balance and a natural notion of fairness based on fixed-point contributions. The probabilistic structure of the mechanism allows for a detailed study of both finite-pool properties and large-pool asymptotics.

Our results show that the mechanism induces a genuine reduction of individual risk. In particular, we have established convex-order dominance of net positions over initial losses, providing a strong stochastic notion of risk reduction. In the critical regime where total

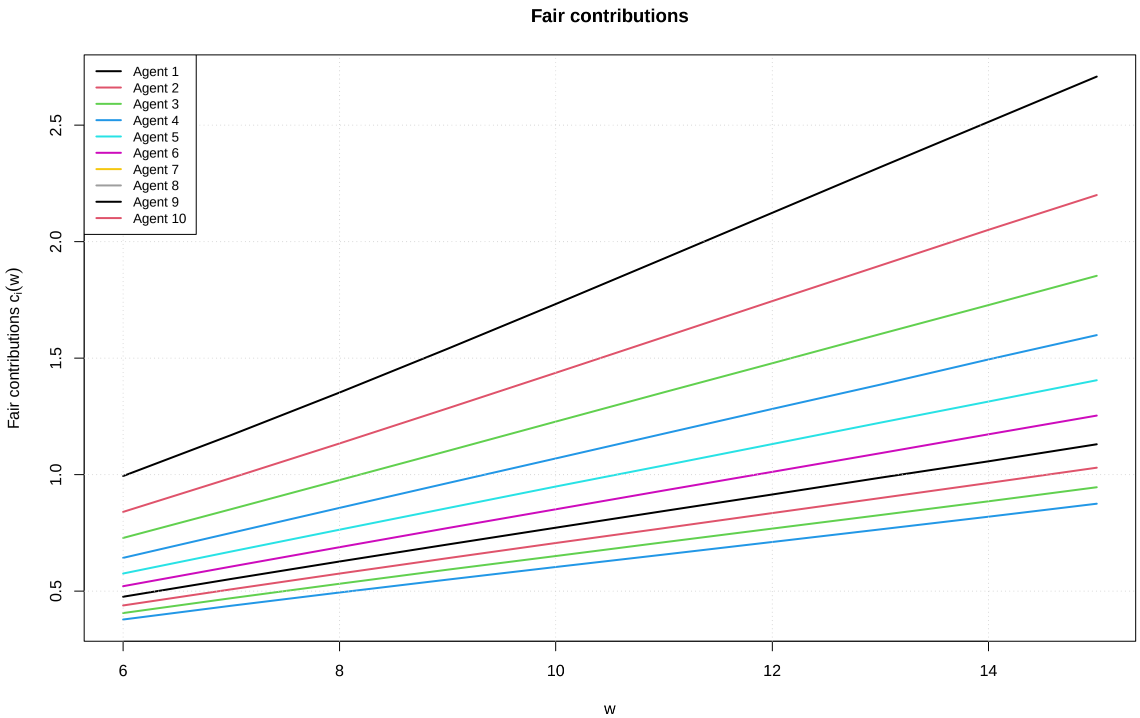

Figure 1: Actuarially-fair contributions as functions of total resources w.

contributions match the expected aggregate loss, we have derived asymptotic results for the number of agents who are not fully compensated and shown that meaningful non-degenerate limits can be obtained. Moreover, under natural large-pool conditions, actuarially-fair contributions converge to the corresponding expected losses, thereby linking the endogenous equilibrium contributions to classical actuarial quantities. The homogeneous case highlights the role of symmetry and leads to sharper and more explicit characterizations of the fixed point and its risk-reduction properties. Together with the numerical illustrations, the results derived in this paper confirm both the theoretical robustness and the practical relevance of the proposed fully-funded risk-sharing scheme.

The paper has made several contributions. First, we have established existence results for actuarially-fair contributions in a general heterogeneous setting and provided sufficient conditions under which the associated operator is locally contractive. This yields uniqueness and stability of the solution and connects the problem to classical fixed-point arguments. Second, we have analyzed the risk reduction achieved by the mechanism with respect to the convex order. Third, we have investigated asymptotic regimes in which the number of participants grows, with particular emphasis on the critical case where total contributions equal the expected aggregate loss. In this regime, we have derived limiting values for individual contributions in large pools and characterized the number of participants who are not fully compensated. The homogeneous case has been treated separately, allowing for sharper results and a clear comparison with existing models.

Several extensions deserve further investigation, including dynamic or multi-period versions of the mechanism and the incorporation of alternative fairness criteria reflecting risk aversion or regulatory constraints. From a broader perspective, the framework developed in this paper provides a sound basis for the analysis of collective risk-sharing arrangements in insurance mathematics. While this paper favored a proportional reduction in compensation in deficit situations, other approaches are possible. Following Schmeiser and Orozco-Garcia (2021), we could reduce compensations to $(X_i - t)_+$ in case of deficit where $t = t(w)$ solves $w = \sum_{i=1}^{n}(X_i - t)_+$. If ex-ante contributions are actuarially fair then the same deductible t can be applied ex post to all participants. We leave these topics for future research.

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Appendix

A Proof of Theorem 2.2

Formula (2.4) means that an actuarially-fair vector of contributions $\boldsymbol{c}$ is a fixed point of the mapping $\mathcal{T} : \Delta_w \to \mathbb{R}^n$ defined by (2.5). The proof of Theorem 2.2 thus consists in proving that $\mathcal{T}$ has a fixed point on Δ_w. First, let us show that $\mathcal{T}$ maps Δ_w into itself. Indeed, for any $\boldsymbol{c} \in \Delta_w$, (2.1) ensures that the sharing rule is budget-balanced state by state. Taking expectations yields $\mathcal{T}_i(\boldsymbol{c}) = \mathrm{E}[h_i(\boldsymbol{c}; \boldsymbol{X})] \geq 0$ for every i and

$$\sum_{i=1}^{n} \mathcal{T}_i(\boldsymbol{c}) = \mathrm{E}\left[\sum_{i=1}^{n} h_i(\boldsymbol{c}; \boldsymbol{X})\right] = w.$$

Hence, $\mathcal{T}(\boldsymbol{c}) \in \Delta_w$.

Second, let us show that $\mathcal{T}$ is continuous on Δ_w. Consider a sequence $\boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)}, \ldots$ in Δ_w such that $\boldsymbol{c}^{(m)} \to \boldsymbol{c} \in \Delta_w$. For any fixed $\boldsymbol{x} \in \mathbb{R}_+^n$ with $S_n(\boldsymbol{x}) := \sum_{i=1}^n x_i \neq w$, we are in a strict regime ($S_n(\boldsymbol{x}) < w$ or $S_n(\boldsymbol{x}) > w$); since $\boldsymbol{c} \mapsto (c_i - x_i)_+$ and $\boldsymbol{c} \mapsto (x_i - c_i)_+$ are continuous, it follows that $h_i(\boldsymbol{c}^{(m)}; \boldsymbol{x}) \to h_i(\boldsymbol{c}; \boldsymbol{x})$ for each i. On the boundary set $\{S_n(\boldsymbol{x}) = w\}$, the definition $h_i(\boldsymbol{c}; \boldsymbol{x}) = x_i$ ensures that the payoff is well-defined for all $\boldsymbol{c}$. Moreover, the inequalities $0 \leq h_i(\boldsymbol{c}^{(m)}; \boldsymbol{X}) \leq w$ are valid for all m and $w < \infty$. The dominated convergence theorem therefore yields $\mathrm{E}[h_i(\boldsymbol{c}^{(m)}; \boldsymbol{X})] \to \mathrm{E}[h_i(\boldsymbol{c}; \boldsymbol{X})]$ for each i, which is precisely $\mathcal{T}(\boldsymbol{c}^{(m)}) \to \mathcal{T}(\boldsymbol{c})$.

Since Δ_w is a nonempty compact convex subset of $\mathbb{R}^n$ and $\mathcal{T} : \Delta_w \to \Delta_w$ is continuous, Brouwer's fixed point theorem guarantees the existence of $\boldsymbol{c}^\star \in \Delta_w$ such that $\mathcal{T}(\boldsymbol{c}^\star) = \boldsymbol{c}^\star$, that is, (2.4) holds true. This ends the proof.

B Sufficient conditions for the contractivity of the fixed-point operator $\mathcal{T}$

B.1 Notation and the fairness map

Fix $n \geq 2$ and $w > 0$. For $\boldsymbol{c} \in \Delta_w$, define $u_i(\boldsymbol{c}) = (c_i - X_i)_+$ and $d_i(\boldsymbol{c}) = (X_i - c_i)_+$, with $U_n(\boldsymbol{c}) = \sum_{j=1}^n u_j(\boldsymbol{c})$ and $D_n(\boldsymbol{c}) = \sum_{j=1}^n d_j(\boldsymbol{c})$. The (time-1) payoff vector is

$$\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{\alpha}(\boldsymbol{c}; \boldsymbol{X})\,(w - S_n), \tag{B.1}$$

where the weight vector $\boldsymbol{\alpha}(\boldsymbol{c}; \boldsymbol{X}) \in \mathbb{R}_+^n$ is defined by

$$\alpha_i(\boldsymbol{c}; \boldsymbol{X}) = \begin{cases} \dfrac{u_i(\boldsymbol{c})}{U_n(\boldsymbol{c})}, & \text{if } S_n < w, \\ 0, & \text{if } S_n = w, \\ \dfrac{d_i(\boldsymbol{c})}{D_n(\boldsymbol{c})}, & \text{if } S_n > w. \end{cases} \tag{B.2}$$

With this convention, the case $S_n = w$ produces $\boldsymbol{h}(\boldsymbol{c};\boldsymbol{X}) = \boldsymbol{X}$ and avoids any 0/0 indeterminacy. The fairness map $\mathcal{T} : \Delta_w \to \Delta_w$ is defined by (2.5). An actuarially-fair contribution vector is a fixed point $\boldsymbol{c}^\star$ of $\mathcal{T}$, that is, $\boldsymbol{c}^\star = \mathcal{T}(\boldsymbol{c}^\star)$.

B.2 A deterministic ℓ^1 Lipschitz bound for normalized vectors

The only source of instability is the sensitivity of ratios $\boldsymbol{u}/U_n$ or $\boldsymbol{d}/D_n$, where $\boldsymbol{u} = (u_1, \dots, u_n)$ and $\boldsymbol{d} = (d_1, \dots, d_n)$. The following deterministic lemma is the key tool.

Lemma B.1 (Normalization bound in ℓ^1)**.** *Let $\boldsymbol{a}$ and $\boldsymbol{a}' \in \mathbb{R}^n_+$ with $A = \sum_{i=1}^n a_i > 0$ and $A' = \sum_{i=1}^n a'_i > 0$. Then,*

$$\left\| \frac{\boldsymbol{a}}{A} - \frac{\boldsymbol{a}'}{A'} \right\|_1 \le \min\left(\frac{1}{A}, \frac{1}{A'}\right) \|\boldsymbol{a} - \boldsymbol{a}'\|_1.$$

Proof. Write

$$\frac{\boldsymbol{a}}{A} - \frac{\boldsymbol{a}'}{A'} = \frac{\boldsymbol{a} - \boldsymbol{a}'}{A} + \boldsymbol{a}'\left(\frac{1}{A} - \frac{1}{A'}\right).$$

Taking ℓ^1 norms and using $\|\boldsymbol{a}'\|_1 = A'$ yields

$$\left\| \frac{\boldsymbol{a}}{A} - \frac{\boldsymbol{a}'}{A'} \right\|_1 \le \frac{\|\boldsymbol{a} - \boldsymbol{a}'\|_1}{A} + A'\left|\frac{1}{A} - \frac{1}{A'}\right| = \frac{\|\boldsymbol{a} - \boldsymbol{a}'\|_1}{A} + \frac{|A - A'|}{A}.$$

Since $|A - A'| \le \|\boldsymbol{a} - \boldsymbol{a}'\|_1$, we obtain $\left\| \frac{\boldsymbol{a}}{A} - \frac{\boldsymbol{a}'}{A'} \right\|_1 \le \frac{2}{A}\|\boldsymbol{a} - \boldsymbol{a}'\|_1$. By symmetry (exchanging $(\boldsymbol{a}, A)$ and $(\boldsymbol{a}', A')$) and taking the better bound, we get the stated inequality. □

Lemma B.2 (Positive-part vectors are 1-Lipschitz in ℓ^1)**.** *For every realization of $\boldsymbol{X}$ and all $\boldsymbol{c}$ and $\boldsymbol{c}' \in \Delta_w$, we have*

$$\|\boldsymbol{u}(\boldsymbol{c}) - \boldsymbol{u}(\boldsymbol{c}')\|_1 \le \|\boldsymbol{c} - \boldsymbol{c}'\|_1 \text{ and } \|\boldsymbol{d}(\boldsymbol{c}) - \boldsymbol{d}(\boldsymbol{c}')\|_1 \le \|\boldsymbol{c} - \boldsymbol{c}'\|_1.$$

Proof. For each coordinate, the map $t \mapsto (t - x)_+$ is 1-Lipschitz. Hence,

$$|(c_i - X_i)_+ - (c'_i - X_i)_+| \le |c_i - c'_i|.$$

Summing over i gives the first inequality. The second follows similarly from $t \mapsto (x - t)_+$ being 1-Lipschitz. □

B.3 A practical (high-probability) contraction criterion

Let $\mathbb{1}[\cdot]$ denote the indicator function, equal to 1 if the condition appearing within the brackets is satisfied and to 0 otherwise.

Proposition B.3 (Local contraction via a high-probability lower bound)**.** *Fix $\boldsymbol{c}^\star \in \Delta_w$ and a neighborhood $\mathcal{N} \subset \Delta_w$. Assume that there exist constants ε_U and $\varepsilon_D > 0$ and an event E with complement E^c such that, if $\mathbb{1}[E] = 1$,*

$$U_n(\boldsymbol{c}) \ge \varepsilon_U \text{ and } D_n(\boldsymbol{c}) \ge \varepsilon_D \text{ for all } \boldsymbol{c} \in \mathcal{N}. \tag{B.3}$$

Further, assume there exists $\eta \geq 0$ and a constant $M < \infty$ such that

$$\sup_{\boldsymbol{c}\in\mathcal{N}} \mathrm{E}\big[(w - S_n)_+ \, \mathbb{1}[E^c]\big] \leq \eta, \quad \sup_{\boldsymbol{c}\in\mathcal{N}} \mathrm{E}\big[(S_n - w)_+ \, \mathbb{1}[E^c]\big] \leq \eta, \tag{B.4}$$

and

$$\sup_{\boldsymbol{c}\in\mathcal{N}} \mathrm{E}\left[\frac{(w - S_n)_+}{U_n(\boldsymbol{c})} \, \mathbb{1}[E^c]\right] \leq M\eta, \quad \sup_{\boldsymbol{c}\in\mathcal{N}} \mathrm{E}\left[\frac{(S_n - w)_+}{D_n(\boldsymbol{c})} \, \mathbb{1}[E^c]\right] \leq M\eta. \tag{B.5}$$

Then, $\mathcal{T}$ is ℓ^1-Lipschitz on $\mathcal{N}$. Precisely,

$$\|\mathcal{T}(\boldsymbol{c}) - \mathcal{T}(\boldsymbol{c}')\|_1 \leq L_{\mathcal{N}} \, \|\boldsymbol{c} - \boldsymbol{c}'\|_1 \text{ where } L_{\mathcal{N}} \leq 2\left(\frac{\mathrm{E}[(w - S_n)_+]}{\varepsilon_U} + \frac{\mathrm{E}[(S_n - w)_+]}{\varepsilon_D}\right) + 4M\eta. \tag{B.6}$$

In particular, if the right-hand side of (B.6) *is less than 1, then $\mathcal{T}$ is a contraction on $\mathcal{N}$. Hence, the fixed point in $\mathcal{N}$ is unique and the Picard iteration $\boldsymbol{c}^{(k+1)} = \mathcal{T}(\boldsymbol{c}^{(k)})$ converges to it for any starting point $\boldsymbol{c}^{(0)} \in \mathcal{N}$.*

Proof. Consider $\boldsymbol{c}$ and $\boldsymbol{c}' \in \mathcal{N}$. On $\{S_n < w\}$, we have

$$\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X}) - \boldsymbol{h}(\boldsymbol{c}'; \boldsymbol{X}) = (w - S_n)\left(\frac{\boldsymbol{u}(\boldsymbol{c})}{U_n(\boldsymbol{c})} - \frac{\boldsymbol{u}(\boldsymbol{c}')}{U_n(\boldsymbol{c}')}\right).$$

On $\{S_n > w\}$, we have

$$\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X}) - \boldsymbol{h}(\boldsymbol{c}'; \boldsymbol{X}) = (w - S_n)\left(\frac{\boldsymbol{d}(\boldsymbol{c})}{D_n(\boldsymbol{c})} - \frac{\boldsymbol{d}(\boldsymbol{c}')}{D_n(\boldsymbol{c}')}\right).$$

Thus,

$$\|\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X}) - \boldsymbol{h}(\boldsymbol{c}'; \boldsymbol{X})\|_1 \leq (w - S_n)_+ \left\|\frac{\boldsymbol{u}(\boldsymbol{c})}{U_n(\boldsymbol{c})} - \frac{\boldsymbol{u}(\boldsymbol{c}')}{U_n(\boldsymbol{c}')}\right\|_1 + (S_n - w)_+ \left\|\frac{\boldsymbol{d}(\boldsymbol{c})}{D_n(\boldsymbol{c})} - \frac{\boldsymbol{d}(\boldsymbol{c}')}{D_n(\boldsymbol{c}')}\right\|_1.$$

Applying Lemma B.1 and Lemma B.2 gives

$$\begin{aligned}\|\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X}) - \boldsymbol{h}(\boldsymbol{c}'; \boldsymbol{X})\|_1 \quad \leq \quad & \Bigg((w - S_n)_+\Big(\frac{1}{U_n(\boldsymbol{c})} + \frac{1}{U_n(\boldsymbol{c}')}\Big) \\ & + (S_n - w)_+\Big(\frac{1}{D_n(\boldsymbol{c})} + \frac{1}{D_n(\boldsymbol{c}')}\Big)\Bigg) \|\boldsymbol{c} - \boldsymbol{c}'\|_1\end{aligned}$$

on the event where denominators are positive. Let us now split the right-hand side over E and E^c. On E, we use (B.3) and symmetry to bound the quantity appearing within brackets by $2\left(\frac{(w-S_n)_+}{\varepsilon_U} + \frac{(S_n-w)^+}{\varepsilon_D}\right)$. On E^c, we use (B.5). Taking expectations and noting that

$$\|\mathcal{T}(\boldsymbol{c}) - \mathcal{T}(\boldsymbol{c}')\|_1 \leq \mathrm{E}[\|\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X}) - \boldsymbol{h}(\boldsymbol{c}'; \boldsymbol{X})\|_1]$$

yields (B.6). The final statement follows from Banach's fixed-point theorem. $\square$

Let us briefly explain how to build E and $\mathcal{N}$. A common choice is

$$E = \{\min\{U_n(\boldsymbol{c}^\star), D_n(\boldsymbol{c}^\star)\} \geq 2\varepsilon\} \text{ for some } \varepsilon > 0.$$

Since $U_n(\cdot)$ and $D_n(\cdot)$ are 1-Lipschitz in ℓ^1, choosing $\mathcal{N} = \{\boldsymbol{c} : \|\boldsymbol{c} - \boldsymbol{c}^\star\|_1 \leq \varepsilon\}$ ensures that the inequality $\min\{U_n(\boldsymbol{c}), D_n(\boldsymbol{c})\} \geq \varepsilon$ holds true on E. The tail terms in (B.4)–(B.5) are then controlled by $\mathrm{P}[E^c]$ and moments of $(w - S_n)_+$ and $(S_n - w)_+$ using Hölder or Cauchy–Schwarz inequalities.

Numerical illustration. The sufficient condition of this section can be checked numerically in the exponential setting of Section 5 when the pool is large enough. Consider independent losses $X_i \sim \mathrm{Exp}(\lambda_i)$, with $n = 100$ and parameters λ_i equally spaced over $[0.6, 1.8]$, and take the critical budget $w = \sum_{i=1}^n \lambda_i^{-1}$. Using the same construction as above, we define

$$E_t = \{U_n(c^\star) \geq t,\ D_n(c^\star) \geq t\} \quad \text{and} \quad N_r = \{c : \|c - c^\star\|_1 \leq r\},$$

so that $\varepsilon_U = \varepsilon_D = t - r$ whenever $r < t$. In a Monte Carlo experiment, the choice $t = 22.2805$ and $r = 0.4456$, yields $\varepsilon_U = \varepsilon_D = 21.8349$, $\eta = 17.4183$, $M = 0.0028514$, with $\mathrm{P}[E_t^c] \approx 0.05$. The corresponding bound in (B.6) is then $L_{\mathcal{N}} = 0.9021 < 1$. Hence, in this example, the local contraction condition required in Proposition B.3 is indeed satisfied. This numerical illustration suggests that, although the criterion may be conservative for small pools, it becomes effective in the same exponential framework when the number of participants is sufficiently large.

B.4 A Jacobian-based local criterion (restricted to the simplex tangent space)

Define the tangent space of the simplex as

$$\Lambda = \{\boldsymbol{v} \in \mathbb{R}^n :\ \mathbf{1}^\top \boldsymbol{v} = 0\},$$

endowed with the $\|\cdot\|_1$-norm. A differential criterion for local contraction is obtained by bounding the operator norm of the derivative $\mathcal{D}\mathcal{T}(\boldsymbol{c}^\star)$ on Λ.

Regularity assumptions Assume:

(R1) $\mathrm{P}[X_i = c_i] = 0$ for all i and all $\boldsymbol{c}$ in a neighborhood of $\boldsymbol{c}^\star$ (this is the case, for instance, if $\boldsymbol{X}$ possesses a density).

(R2) There exists a neighborhood $\mathcal{N}(\boldsymbol{c}^\star)$ such that $\sup_{\boldsymbol{c} \in \mathcal{N}(\boldsymbol{c}^\star)} \mathrm{E}\left[\frac{(w-S_n)_+}{U_n(\boldsymbol{c})^2} + \frac{(S_n-w)_+}{D_n(\boldsymbol{c})^2}\right] < \infty$.

These assumptions allow for differentiation under the expectation and ensure that $\mathcal{T}$ is Fréchet differentiable at $\boldsymbol{c}^\star$. We write $\|\mathcal{D}T(c^\star)\|_{\Lambda\to\Lambda,1}$ for the operator norm of the Fréchet derivative $\mathcal{D}T(c^\star) : \Lambda \to \Lambda$, namely

$$\|\mathcal{D}T(c^\star)\|_{\Lambda\to\Lambda,1} = \sup_{\substack{v \in \Lambda \\ \|v\|_1 = 1}} \|\mathcal{D}T(c^\star)v\|_1.$$

Proposition B.4 (Jacobian bound implies local contraction in ℓ^1)**.** *Under (R1)–(R2), $\mathcal{T}$ is Fréchet differentiable at $\boldsymbol{c}^\star$ and*

$$\|\mathcal{D}\mathcal{T}(\boldsymbol{c}^\star)\|_{\Lambda\to\Lambda,1} \ \leq\ 2\,\mathrm{E}\left[\frac{(w - S_n)_+}{U_n(\boldsymbol{c}^\star)} + \frac{(S_n - w)_+}{D_n(\boldsymbol{c}^\star)}\right]. \tag{B.7}$$

In particular, if the right-hand side of (B.7) *is strictly smaller than* 1*, then there exists a neighborhood $\mathcal{N} \subset \Delta_w$ of $\boldsymbol{c}^\star$ on which $\mathcal{T}$ is a contraction (with respect to $\|\cdot\|_1$ restricted to Δ_w). Hence, the fixed point is locally unique.*

Proof sketch. On the event $\{S_n < w\}$, the weight vector in (B.2) is $\boldsymbol{\alpha}(\boldsymbol{c}) = \boldsymbol{u}(\boldsymbol{c})/U_n(\boldsymbol{c})$. By assumption (R1) (excluding the kinks of $(\cdot)_+$), the Jacobian of $\boldsymbol{u}(\boldsymbol{c})$ is the diagonal matrix $\boldsymbol{M}_1(\boldsymbol{c}) = \operatorname{diag}(\mathbb{1}[c_i > X_i])$ and $\nabla U_n(\boldsymbol{c}) = \mathbf{1}^\top \boldsymbol{M}_1(\boldsymbol{c})$. A direct calculation gives

$$\nabla\boldsymbol{\alpha}(\boldsymbol{c})\,\boldsymbol{v} = \frac{1}{U_n(\boldsymbol{c})}\Big(\boldsymbol{M}_1(\boldsymbol{c})\boldsymbol{v} - \boldsymbol{\alpha}(\boldsymbol{c})\,\mathbf{1}^\top \boldsymbol{M}_1(\boldsymbol{c})\boldsymbol{v}\Big) \text{ for } \boldsymbol{v} \in \mathbb{R}^n.$$

Similarly, on $\{S_n > w\}$ the weights are $\boldsymbol{\alpha}(\boldsymbol{c}) = \boldsymbol{d}(\boldsymbol{c})/D_n(\boldsymbol{c})$ and

$$\nabla\boldsymbol{\alpha}(\boldsymbol{c})\,\boldsymbol{v} = \frac{1}{D_n(\boldsymbol{c})}\Big(-\boldsymbol{M}_2(\boldsymbol{c})\boldsymbol{v} + \boldsymbol{\alpha}(\boldsymbol{c})\,\mathbf{1}^\top \boldsymbol{M}_2(\boldsymbol{c})\boldsymbol{v}\Big) \text{ where } \boldsymbol{M}_2(\boldsymbol{c}) = \operatorname{diag}(\mathbb{1}[X_i > c_i]).$$

Since $\mathcal{T}(\boldsymbol{c}) = \mathrm{E}[\boldsymbol{X} + \boldsymbol{\alpha}(\boldsymbol{c})(w - S_n)]$, differentiation under the expectation yields $\mathcal{D}\mathcal{T}(\boldsymbol{c}^\star) = \mathrm{E}[(w - S_n)\nabla\boldsymbol{\alpha}(\boldsymbol{c}^\star)]$.

To bound the ℓ^1 operator norm on Λ, note that $\boldsymbol{M}_1(\boldsymbol{c})$ and $\boldsymbol{M}_2(\boldsymbol{c})$ are diagonal projections with $\|\boldsymbol{M}_1(\boldsymbol{c})\boldsymbol{v}\|_1 \le \|\boldsymbol{v}\|_1$ and $\|\boldsymbol{M}_2(\boldsymbol{c})\boldsymbol{v}\|_1 \le \|\boldsymbol{v}\|_1$, while $\|\boldsymbol{\alpha}(\boldsymbol{c})\|_1 = 1$ on both events $\{S_n < w\}$ and $\{S_n > w\}$ and $|\mathbf{1}^\top \boldsymbol{M}_1(\boldsymbol{c})\boldsymbol{v}| \le \|\boldsymbol{M}_1(\boldsymbol{c})\boldsymbol{v}\|_1 \le \|\boldsymbol{v}\|_1$. For all $\boldsymbol{v} \in \Lambda$, we thus have

$$\|\nabla\boldsymbol{\alpha}(\boldsymbol{c})\,\boldsymbol{v}\|_1 \le \frac{2}{U_n(\boldsymbol{c})}\|\boldsymbol{v}\|_1 \text{ on } \{S_n < w\}$$

and

$$\|\nabla\boldsymbol{\alpha}(\boldsymbol{c})\,\boldsymbol{v}\|_1 \le \frac{2}{D_n(\boldsymbol{c})}\|\boldsymbol{v}\|_1 \text{ on } \{S_n > w\}.$$

Multiplying by $(w - S_n)$ on the corresponding regimes and taking expectations gives (B.7). Finally, if $\|\mathcal{D}\mathcal{T}(\boldsymbol{c}^\star)\|_{\Lambda\to\Lambda,1} < 1$, standard inverse function and mean-value arguments imply that $\mathcal{T}$ is a contraction on a sufficiently small neighborhood of $\boldsymbol{c}^\star$ in Δ_w. □

Numerical illustration. The sufficient condition based on (B.7) can be checked numerically in the exponential framework of Section 5. Consider independent losses $X_i \sim \mathrm{Exp}(\lambda_i)$, where the parameters λ_i are equally spaced over $[0.6, 1.8]$, and take the critical budget $w = \sum_{i=1}^n \lambda_i^{-1}$. For each value of n, we compute an approximation of the actuarially fair contribution vector $c^\star$ and then estimate by Monte Carlo the right-hand side of (B.7), namely

$$2\,\mathrm{E}\left[\frac{(w - S_n)^+}{U_n(c^\star)} + \frac{(S_n - w)^+}{D_n(c^\star)}\right].$$

For $n = 10$, 12 and 15, the corresponding estimates are 0.9721, 0.9095, 0.8376, while the associated upper 95% Monte Carlo bounds are 0.9740, 0.9113, 0.8393 respectively. In all three cases, the upper bound remains strictly smaller than 1. Hence, in this exponential example, the majorant appearing in (B.7) is strictly less than 1, which provides numerical evidence that the local contraction condition holds already for moderate pool sizes.

Remark B.5. *Proposition B.4 yields a local condition evaluated at the candidate fixed point* $\boldsymbol{c}^\star$*, whereas Proposition B.3 provides a more "robust" (but potentially looser) criterion on a neighborhood. Both are driven by the same phenomenon: instability occurs when* $U_n(\boldsymbol{c})$ *or* $D_n(\boldsymbol{c})$ *becomes too small on scenarios carrying non-negligible mass for* $(w - S_n)_+$ *and* $(S_n - w)_+$.

C Proof of Lemma 3.1

Consider a convex function φ such that the expectations below exist. Since Y is a convex combination of Z and b with weight R, convexity of φ yields the pointwise inequality

$$\varphi(Y) = \varphi\big((1-R)Z + Rb\big) \leq (1-R)\varphi(Z) + R\,\varphi(b).$$

Taking expectations gives

$$\mathrm{E}[\varphi(Y)] \leq \mathrm{E}[(1-R)\varphi(Z)] + \mathrm{E}[R]\,\varphi(b). \tag{C.1}$$

Under (3.3), we have

$$b = \frac{\mathrm{E}[RZ]}{\mathrm{E}[R]} = \frac{1}{\mathrm{E}[R]}\int_0^\infty \int_0^1 zr\mathrm{d}F_{(Z,R)}(z,r) := \mathrm{E}[Z']$$

where the random variable Z' is distributed according to

$$\mathrm{P}[Z' \leq z] = \frac{1}{\mathrm{E}[R]}\int_0^z \int_0^1 r\mathrm{d}F_{(Z,R)}(z,r) = \frac{\mathrm{E}\big[R\mathbb{1}[Z \leq z]\big]}{\mathrm{E}[R]}.$$

Jensen's inequality allows us to write

$$\varphi(b) = \varphi(\mathrm{E}[Z']) \leq \mathrm{E}[\varphi(Z')] = \frac{\mathrm{E}[R\,\varphi(Z)]}{\mathrm{E}[R]}.$$

Multiplying both sides by $\mathrm{E}[R]$ yields

$$\mathrm{E}[R]\,\varphi(b) \leq \mathrm{E}[R\,\varphi(Z)]. \tag{C.2}$$

Plugging (C.2) into (C.1) yields

$$\mathrm{E}[\varphi(Y)] \leq \mathrm{E}[(1-R)\varphi(Z)] + \mathrm{E}[R\,\varphi(Z)] = \mathrm{E}[\varphi(Z)]$$

so that the stochastic inequality $Y \preceq_{\mathrm{CX}} Z$ holds true, as announced.

If R is not almost surely constant and Z is not almost surely constant then Y is not almost surely equal to Z, and the above inequalities are strict for some strictly convex φ. This ends the proof.

D Proof of Proposition 3.4

Recall from (B.1) that the payoff satisfies the identity

$$H_{i,n} - X_i = (w_n - S_n)\left(\frac{(c_{i,n} - X_i)_+}{U_n(\boldsymbol{c}_n)}\mathbb{1}[S_n < w_n] - \frac{(X_i - c_{i,n})_+}{D_n(\boldsymbol{c}_n)}\mathbb{1}[S_n > w_n]\right), \tag{D.1}$$

with the usual convention that the ratios are zero when the denominators vanish. Since $\boldsymbol{c}_n$ is a fixed-point of the operator $\mathcal{T}$, we have $c_{i,n} = \mathrm{E}[H_{i,n}]$ and therefore

$$|c_{i,n} - \mathrm{E}[X_i]| = \big|\mathrm{E}[H_{i,n} - X_i]\big| \leq \mathrm{E}\left[|H_{i,n} - X_i|\right].$$

Using representation (D.1) yields

$$|H_{i,n} - X_i| \le |w_n - S_n| \left(\frac{(c_{i,n} - X_i)_+}{U_n(\boldsymbol{c}_n)} + \frac{(X_i - c_{i,n})_+}{D_n(\boldsymbol{c}_n)} \right).$$

Now, we have

$$\frac{(c_{i,n} - X_i)_+}{U_n(\boldsymbol{c}_n)} + \frac{(X_i - c_{i,n})_+}{D_n(\boldsymbol{c}_n)} \le \frac{2|X_i - c_{i,n}|}{\gamma n} \text{ on } \{\min\{U_n(\boldsymbol{c}_n), D_n(\boldsymbol{c}_n)\} \ge \gamma n\}$$

and therefore

$$|H_{i,n} - X_i| \le \frac{2}{\gamma n} |w_n - S_n|\, |X_i - c_{i,n}| \text{ on } \{\min\{U_n(\boldsymbol{c}_n), D_n(\boldsymbol{c}_n)\} \ge \gamma n\}.$$

Taking expectations and applying the Cauchy–Schwarz inequality gives

$$\mathrm{E}\big[|H_{i,n} - X_i| \mathbb{1}[\min\{U_n(\boldsymbol{c}_n), D_n(\boldsymbol{c}_n)\} \ge \gamma n]\big] \le \frac{2}{\gamma n} \sqrt{\mathrm{E}[(w_n - S_n)^2]} \sqrt{\mathrm{E}[(X_i - c_{i,n})^2]}.$$

Since $w_n = \mathrm{E}[S_n]$ and $\mathrm{Var}[S_n] \le n\nu$ by (A2), the second factor is bounded by $\sqrt{n\nu}$. Moreover,

$$\sqrt{\mathrm{E}[(X_i - c_{i,n})^2]} \le \sqrt{\mathrm{Var}(X_i)} + |c_{i,n} - \mathrm{E}[X_i]|$$

which yields

$$\mathrm{E}\big[|H_{i,n} - X_i| \mathbb{1}[\min\{U_n(\boldsymbol{c}_n), D_n(\boldsymbol{c}_n)\} \ge \gamma n]\big] \le \frac{2}{\gamma} \sqrt{\frac{\nu}{n}} \sqrt{\mathrm{Var}(X_i)} + \frac{2}{\gamma} \sqrt{\frac{\nu}{n}} |c_{i,n} - \mathrm{E}[X_i]|.$$

On $\{\min\{U_n(\boldsymbol{c}_n), D_n(\boldsymbol{c}_n)\} < \gamma n\}$, we use $|H_{i,n} - X_i| \le X_i + c_{i,n} \le 2X_i$, so that

$$\begin{aligned} \mathrm{E}\big[|H_{i,n} - X_i| \mathbb{1}[\min\{U_n(\boldsymbol{c}_n), D_n(\boldsymbol{c}_n)\} < \gamma n]\big] &\le 2\sqrt{\mathrm{E}[X_i^2]} \sqrt{\mathrm{P}[\min\{U_n(\boldsymbol{c}_n), D_n(\boldsymbol{c}_n)\} < \gamma n]} \\ &\le 2\sqrt{\mathrm{E}[X_i^2]} \sqrt{\delta_n}. \end{aligned}$$

Combining the two bounds completes the proof.

E Global ℓ_2-rate for actuarially-fair contributions in the critical regime

E.1 Main result

In Proposition 3.4, we established that, in the critical regime, i.e. when $w_n = \mathrm{E}[S_n]$, each actuarially-fair contribution $c_{i,n}$ converges to $\mathrm{E}[X_i]$ as $n \to \infty$. That result is pointwise in i and provides a first-order consistency statement.

The purpose of the present appendix is to strengthen this result by proving a global ℓ_2-rate for the full contribution vector $\boldsymbol{c}_n^\star = (c_{1,n}^\star, \dots, c_{n,n}^\star)$, solution of the fixed-point equation $\boldsymbol{c}_n^\star = \mathcal{T}_n(\boldsymbol{c}_n^\star)$. More precisely, we show that, at criticality, deviations from the mean vector $\mathrm{E}[\boldsymbol{X}] = (\mathrm{E}[X_1], \dots, \mathrm{E}[X_n])$ are of order $n^{-1/2}$ in ℓ_2-norm. This provides a quantitative control of heterogeneity effects in large pools.

This is formally stated as follows.

Theorem E.1 (Critical ℓ_2-rate)**.** *Assume that* $w_n = \mathrm{E}[S_n]$ *and that Assumptions E.2–E.5 below hold. Then there exists a constant* $C < \infty$ *such that, for all sufficiently large* n*,*

$$\left\|\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}]\right\|_2 \;\le\; C\, n^{-1/2}.$$

E.2 Assumptions

Throughout this appendix section, we define

$$u_i := (\mathrm{E}[X_i] - X_i)_+, \qquad d_i := (X_i - \mathrm{E}[X_i])_+,$$

and

$$U := \sum_{i=1}^n u_i, \qquad D := \sum_{i=1}^n d_i.$$

We also write $\bar{u}_i := \mathrm{E}[u_i]$, $\bar{d}_i := \mathrm{E}[d_i]$, $\bar{U} := \mathrm{E}[U]$, $\bar{D} := \mathrm{E}[D]$. Note that $\bar{u}_i = \bar{d}_i$ and $\bar{U} = \bar{D}$. Finally, define for each i,

$$S_{-i} := \sum_{j \neq i} X_j, \qquad w_{-i} := \mathrm{E}[S_{-i}] = w_n - \mathrm{E}[X_i].$$

Assumption E.2 (Moments and variance growth)**.** *There exist constants* $0 < c_\sigma \le C_\sigma < \infty$ *and* $M_6 < \infty$ *such that, for all large* n*,*

$$c_\sigma n \;\le\; \mathrm{Var}[S_n] = \sum_{j=1}^n \mathrm{Var}[X_j] \;\le\; C_\sigma n, \qquad \max_{1 \le j \le n} \mathrm{E}|X_j - \mathrm{E}[X_j]|^6 \le M_6.$$

This assumption ensures a classical $\sqrt{n}$-scale for aggregate fluctuations and uniform control of higher moments, which is required for ratio and Taylor expansions.

Assumption E.3 (Linear buffers with high probability)**.** *There exist constants* $u_0, d_0 > 0$, $c_U, c_D > 0$ *and* $p > 8$ *such that, for all large* n*,*

$$\mathrm{E}[U] \ge u_0 n, \qquad \mathrm{E}[D] \ge d_0 n,$$

and

$$\mathrm{P}[U \le c_U n] \le n^{-p}, \qquad \mathrm{P}[D \le c_D n] \le n^{-p}.$$

This assumption guarantees that aggregate surplus and deficit buffers are typically of linear order in n, preventing instability of the normalized weights.

Assumption E.4 (Small-ball condition for leave-one-out sums)**.** *There exists* $C_{\mathrm{sb}} < \infty$ *such that, for all large* n*, all* i*, and all* $t \ge 0$*,*

$$\sup_{x \in \boldsymbol{R}} \mathrm{P}\big[S_{-i} \in [x, x+t]\big] \;\le\; C_{\mathrm{sb}} \frac{t}{\sqrt{n}},$$

where $S_{-i} = \sum_{j \neq i} X_j$*.*

This anti-concentration assumption controls the probability that the aggregate loss lies close to the critical threshold once one component is removed. It replaces any Gaussian approximation and is the key input of the leave-one-out argument that will be used next. Assumption E.4 is the minimal quantitative input needed in the proof. It serves to bound the probability that S_{-i} falls in a short interval. It holds, for instance, if the X_j have uniformly bounded densities, or under standard non-lattice local limit conditions.

Assumption E.5 (Local stability of the fixed-point operator)**.** *There exist $r > 0$ and $\rho \in (0,1)$ such that, for all large n, the mapping $\mathcal{T}_n$ is Fréchet differentiable on*

$$\mathcal{N}_n = \big\{\boldsymbol{c} \in \Delta_{w_n} : \|\boldsymbol{c} - \mathrm{E}[\boldsymbol{X}]\|_2 \leq r\big\},$$

and

$$\|P_n \mathcal{D}\mathcal{T}_n(\boldsymbol{c}) P_n\|_{\mathrm{op}} \leq \rho \quad \textit{for all } \boldsymbol{c} \in \mathcal{N}_n,$$

with $P_n = I - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$. Moreover, $\boldsymbol{c}_n^\star \in \mathcal{N}_n$.

Here $\|P_n \mathcal{D}\mathcal{T}_n(\boldsymbol{c}) P_n\|_{\mathrm{op}}$ stands for the Euclidean operator norm of the restriction of $\mathcal{D}\mathcal{T}_n(\boldsymbol{c})$ to $\Lambda_n = \{\boldsymbol{v} \in \mathbb{R}^n : \ \mathbf{1}^\top \boldsymbol{v} = 0\}$, namely

$$\|P_n \mathcal{D}\mathcal{T}_n(\boldsymbol{c}) P_n\|_{\mathrm{op}} = \sup_{\substack{v \in \Lambda_n \\ \|v\|_2 = 1}} \|P_n \mathcal{D}\mathcal{T}_n(\boldsymbol{c}) v\|_2.$$

This assumption ensures local contractivity of the fixed-point equation on the simplex, allowing deviations at the mean to propagate linearly.

E.3 Elementary lemmas and proof of Theorem E.1

Throughout this subsection, we let

$$\mathrm{E}[\boldsymbol{X}] = (\mu_1, \ldots, \mu_n) \text{ with } \mu_i := \mathrm{E}[X_i].$$

E.3.1 Uniform moment bounds

Lemma E.6 (Uniform moment bounds for u_i, d_i)**.** *Under Assumption E.2, we have*

$$\sup_{i \geq 1} \Big(\mathrm{E}[u_i^k] + \mathrm{E}[d_i^k]\Big) < \infty \qquad \textit{for every integer } 1 \leq k \leq 6.$$

Proof. For every real x, $0 \leq (x)_+ \leq |x|$. Hence,

$$0 \leq u_i = (\mu_i - X_i)_+ \leq |\mu_i - X_i|, \qquad 0 \leq d_i = (X_i - \mu_i)_+ \leq |X_i - \mu_i|.$$

Raising to the power $k \leq 6$ and taking expectations yields

$$\mathrm{E}[u_i^k] \leq \mathrm{E}|X_i - \mu_i|^k \leq \big(\mathrm{E}|X_i - \mu_i|^6\big)^{k/6} \leq M_6^{k/6},$$

and similarly for d_i. Taking the supremum over i gives the announced result. □

E.3.2 Ratio L^2 bounds

Define

$$W_i^- := \frac{u_i}{U} - \frac{\bar{u}_i}{\bar{U}}, \qquad W_i^+ := \frac{d_i}{D} - \frac{\bar{d}_i}{\bar{D}}.$$

Lemma E.7 (Ratio L^2 bounds)**.** *Under Assumptions E.2 and E.3, there exists $C_{\mathrm{rat}} < \infty$ such that, for all sufficiently large n and all i,*

$$\mathrm{E}\left[\left(\frac{u_i}{U} - \frac{\bar{u}_i}{\bar{U}}\right)^2\right] \le \frac{C_{\mathrm{rat}}}{n^2}, \qquad \mathrm{E}\left[\left(\frac{d_i}{D} - \frac{\bar{d}_i}{\bar{D}}\right)^2\right] \le \frac{C_{\mathrm{rat}}}{n^2}.$$

Proof. We prove the bound for u_i/U; the proof for d_i/D is similar.

Write the decomposition

$$\frac{u_i}{U} - \frac{\bar{u}_i}{\bar{U}} = \underbrace{\frac{u_i - \bar{u}_i}{\bar{U}}}_{A} + \underbrace{u_i\left(\frac{1}{U} - \frac{1}{\bar{U}}\right)}_{B}.$$

Then $(A+B)^2 \le 2(A^2 + B^2)$.

Term A. By Assumption E.3, $\bar{U} \ge u_0 n$ for all large n, hence

$$\mathrm{E}[A^2] = \frac{\mathrm{Var}[u_i]}{\bar{U}^2} \le \frac{\mathrm{E}[u_i^2]}{u_0^2 n^2} \le \frac{C}{n^2},$$

using Lemma E.6.

Term B Let $E := \{U \ge c_U n\}$. On E,

$$\left|\frac{1}{U} - \frac{1}{\bar{U}}\right| = \frac{|U - \bar{U}|}{U\bar{U}} \le \frac{|U - \bar{U}|}{(c_U n)(u_0 n)} = \frac{|U - \bar{U}|}{c_U u_0\, n^2}.$$

Therefore,

$$\mathrm{E}\big[B^2 \mathbb{1}[E]\big] \le \frac{1}{c_U^2 u_0^2 n^4}\, \mathrm{E}\big[u_i^2 (U - \bar{U})^2\big] \le \frac{1}{c_U^2 u_0^2 n^4} \Big(\mathrm{E}[u_i^4]\mathrm{E}[(U - \bar{U})^4]\Big)^{1/2}.$$

By Lemma E.6, $\sup_i \mathrm{E}[u_i^4] < \infty$. Moreover $U = \sum_{j=1}^n u_j$ is a sum of independent terms with uniformly bounded fourth moments, so $\mathrm{E}[(U - \bar{U})^4] = O(n^2)$. Hence $\mathrm{E}\big[B^2 \mathbb{1}[E]\big] = O(n^{-3})$.

On E^c, we use the crude bound $\big|\frac{u_i}{U} - \frac{\bar{u}_i}{\bar{U}}\big| \le 2$, so $\mathrm{E}\big[B^2 \mathbb{1}[E^c]\big] \le 4\mathrm{P}[E^c] \le 4n^{-p}$, and $p > 8$ implies $n^{-p} = o(n^{-2})$. Thus $\mathrm{E}[B^2] = O(n^{-2})$. $\square$

E.3.3 Indicator perturbations: strip bounds and small-ball input

Lemma E.8 (Strip bound for indicator differences)**.** *For any real numbers y and δ,*

$$\big|\, \mathbb{1}[y < -\delta] - \mathbb{1}[y < 0] \,\big| \le \mathbb{1}[|y| \le |\delta|].$$

Proof. The two indicators can differ only if y lies between $-\delta$ and 0, which is equivalent to $|y| \le |\delta|$. $\square$

Lemma E.9 (Indicator perturbation from the small-ball condition)**.** *Assume that Assumption E.4 holds true. Then, for all sufficiently large n and all i,*

$$\mathrm{E}\Big[\big|\mathbb{1}[S_n < w_n] - \mathbb{1}[S_{-i} < w_{-i}]\big|\Big] \le \frac{C_{\mathrm{sb}}}{\sqrt{n}}\mathrm{E}\big[|X_i - \mu_i|\big].$$

The same bound holds with "$<$" replaced by "$>$" in the indicator functions.

Proof. Note that

$$S_n < w_n \iff S_{-i} < w_{-i} + (\mu_i - X_i).$$

Thus, the indicators can differ only if S_{-i} lies in an interval of length $|\mu_i - X_i|$:

$$\big|\mathbb{1}[S_n < w_n] - \mathbb{1}[S_{-i} < w_{-i}]\big| \le \mathbb{1}\Big[S_{-i} \in [w_{-i} - |\mu_i - X_i|,\ w_{-i} + |\mu_i - X_i|]\Big].$$

Conditioning on X_i (or equivalently, on $\Delta_i := X_i - \mu_i$) and applying Assumption E.4 with $t = 2|\Delta_i|$ yields

$$\mathrm{P}\big[S_{-i} \in [w_{-i} - |\Delta_i|, w_{-i} + |\Delta_i|] \mid X_i\big] \le 2C_{\mathrm{sb}}\frac{|\Delta_i|}{\sqrt{n}}.$$

Taking expectations gives the stated inequality (absorbing constants into C_{sb}). □

E.3.4 Core leave-one-out control of the mean drift at $\mathrm{E}[\boldsymbol{X}]$

Let $\mathcal{T}_n(\boldsymbol{c}) = \mathrm{E}[\boldsymbol{h}(\boldsymbol{c}; \boldsymbol{X})]$ be the fairness operator. We consider $\mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) - \mathrm{E}[\boldsymbol{X}]$ componentwise.

Proposition E.10 (Key bound: componentwise $1/n$)**.** *Assume that Assumptions E.2, E.3, and E.4 hold true. Then, there exists $C_\varepsilon < \infty$ such that, for all sufficiently large n and all i,*

$$\big|\mathcal{T}_{n,i}(\mathrm{E}[\boldsymbol{X}]) - \mu_i\big| \le \frac{C_\varepsilon}{n}.$$

Proof. Fix i. The sharing rule evaluated at $\boldsymbol{c} = \mathrm{E}[\boldsymbol{X}]$ yields the decomposition

$$\mathcal{T}_{n,i}(\mathrm{E}[\boldsymbol{X}]) - \mu_i = \varepsilon_{i,n} := \mathrm{E}\Big[(w_n - S_n)\Big(W_i^{-}\mathbb{1}[S_n < w_n] + W_i^{+}\mathbb{1}[S_n > w_n]\Big)\Big],$$

where

$$W_i^{-} = \frac{u_i}{U} - \frac{\bar{u}_i}{\bar{U}}, \qquad W_i^{+} = \frac{d_i}{D} - \frac{\bar{d}_i}{\bar{D}}.$$

We treat the "$<$" part, considering

$$\varepsilon_{i,n}^{-} := \mathrm{E}\big[(w_n - S_n)W_i^{-}\mathbb{1}[S_n < w_n]\big].$$

Define $\Delta_i := X_i - \mu_i$ and $Y_{-i} := S_{-i} - w_{-i}$. Then

$$w_n - S_n = -(\Delta_i + Y_{-i}), \qquad \mathbb{1}[S_n < w_n] = \mathbb{1}[\Delta_i + Y_{-i} < 0].$$

Let $A_i := \mathbb{1}[Y_{-i} < 0] = \mathbb{1}[S_{-i} < w_{-i}]$. We decompose (leave-one-out)

$$\varepsilon_{i,n}^{-} = \mathrm{E}\big[(w_n - S_n)W_i^{-}A_i\big] + \mathrm{E}\big[(w_n - S_n)W_i^{-}\big(\mathbb{1}[\Delta_i + Y_{-i} < 0] - A_i\big)\big] =: T_{1,i} + T_{2,i}.$$

Step 1: Boundary strip control for $T_{2,i}$. By Lemma E.8,

$$\left|\mathbb{1}[\Delta_i + Y_{-i} < 0] - A_i\right| = \left|\mathbb{1}[Y_{-i} < -\Delta_i] - \mathbb{1}[Y_{-i} < 0]\right| \le \mathbb{1}[|Y_{-i}| \le |\Delta_i|].$$

On the event $\{|Y_{-i}| \le |\Delta_i|\}$,

$$|w_n - S_n| = |\Delta_i + Y_{-i}| \le 2|\Delta_i|.$$

Hence

$$|T_{2,i}| \le 2\mathrm{E}\left[|\Delta_i|\,|W_i^-|\,\mathbb{1}[|Y_{-i}| \le |\Delta_i|]\right].$$

The strip event is

$$\{|Y_{-i}| \le |\Delta_i|\} = \{S_{-i} \in [w_{-i} - |\Delta_i|,\ w_{-i} + |\Delta_i|]\},$$

an interval of length $2|\Delta_i|$. By Assumption E.4,

$$\mathrm{P}\left[|Y_{-i}| \le |\Delta_i| \mid X_i\right] \le 2C_{\mathrm{sb}}\frac{|\Delta_i|}{\sqrt{n}}.$$

Therefore,

$$|T_{2,i}| \le \frac{C}{\sqrt{n}}\,\mathrm{E}\left[|\Delta_i|^2|W_i^-|\right].$$

Split on $E := \{U \ge c_U n\}$. On E,

$$|W_i^-| \le \frac{u_i}{U} + \frac{\bar{u}_i}{\bar{U}} \le \frac{C}{n}.$$

Thus,

$$\mathrm{E}\left[|\Delta_i|^2|W_i^-|\mathbb{1}[E]\right] \le \frac{C}{n}\mathrm{E}\left[|\Delta_i|^2\right] \le \frac{C}{n}.$$

On E^c, use $|W_i^-| \le 1$ and $\mathrm{P}[E^c] \le n^{-p}$ with $p > 8$, hence by Hölder,

$$\mathrm{E}\left[|\Delta_i|^2|W_i^-|\mathbb{1}[E^c]\right] \le \left(\mathrm{E}|\Delta_i|^6\right)^{1/3}(\mathrm{P}[E^c])^{2/3} \le Cn^{-2p/3} = o(n^{-1}).$$

So $\mathrm{E}[|\Delta_i|^2|W_i^-|] \le C/n$, implying $|T_{2,i}| \le C/n$.

Step 2: Main term $T_{1,i}$. Using $w_n - S_n = -(\Delta_i + Y_{-i})$,

$$T_{1,i} = -\mathrm{E}\left[\Delta_i W_i^- A_i\right] - \mathrm{E}\left[Y_{-i}W_i^- A_i\right] =: -(T_{1,i}^{(a)} + T_{1,i}^{(b)}).$$

<u>Step 2a: bound $|T_{1,i}^{(a)}|$.</u> On $E = \{U \ge c_U n\}$, we have that $|W_i^-| \le C/n$. Hence,

$$\mathrm{E}\left[|\Delta_i||W_i^-|A_i\mathbb{1}[E]\right] \le \frac{C}{n}\mathrm{E}[|\Delta_i|] \le \frac{C}{n}.$$

On E^c, Cauchy–Schwarz yields

$$\mathrm{E}\left[|\Delta_i||W_i^-|A_i\mathbb{1}[E^c]\right] \le (\mathrm{E}[\Delta_i^2])^{1/2}(\mathrm{P}[E^c])^{1/2} \le Cn^{-p/2} = o(n^{-1}).$$

Thus $|T_{1,i}^{(a)}| \le C/n$.

<u>Step 2b: bound $|T_{1,i}^{(b)}|$ via a leave-one-out gain.</u> Let $\mathcal{F}_{-i} := \sigma\{X_j : j \neq i\}$. Then Y_{-i} and A_i are $\mathcal{F}_{-i}$-measurable, so

$$T_{1,i}^{(b)} = \mathrm{E}\big[Y_{-i} A_i \, \mathrm{E}[W_i^- \mid \mathcal{F}_{-i}]\big] .$$

By Cauchy–Schwarz,

$$|T_{1,i}^{(b)}| \le \mathrm{E}[|Y_{-i}|] \, \mathrm{E}\left[\big|\mathrm{E}[W_i^- \mid \mathcal{F}_{-i}]\big|\right) .$$

Under Assumption E.2,

$$\mathrm{E}[|Y_{-i}|] \le (\mathrm{E}[Y_{-i}^2])^{1/2} \le (\mathrm{Var}[Y_{-i}])^{1/2} \le C\sqrt{n}.$$

It remains to show

$$\mathrm{E}\left[\big|\mathrm{E}[W_i^- \mid \mathcal{F}_{-i}]\big|\right] \le \frac{C}{n^{3/2}}.$$

Write $U = U_{-i} + u_i$ with $U_{-i} := \sum_{j\neq i} u_j$ $\mathcal{F}_{-i}$-measurable. Then

$$\mathrm{E}[W_i^- \mid \mathcal{F}_{-i}] = \mathrm{E}\left[\frac{u_i}{U_{-i} + u_i} \,\middle|\, \mathcal{F}_{-i}\right] - \frac{\bar{u}_i}{\bar{U}}.$$

For $a > 0$, define $g_a(u) := \frac{u}{a+u}$. For $u \ge 0$,

$$g_a(u) = \frac{u}{a} - \frac{u^2}{a^2} + r(u,a),$$

where

$$|r(u,a)| \le \frac{u^3}{a^3}.$$

With $a = U_{-i}$, conditional expectations give

$$\mathrm{E}\left[\frac{u_i}{U_{-i} + u_i} \,\middle|\, \mathcal{F}_{-i}\right] = \frac{\bar{u}_i}{U_{-i}} - \frac{\mathrm{E}[u_i^2]}{U_{-i}^2} + R_{i,n}, \qquad |R_{i,n}| \le \frac{\mathrm{E}[u_i^3]}{U_{-i}^3}.$$

Let $E_{-i} := \{U_{-i} \ge (c_U/2)n\}$. Then, $\mathrm{P}[E_{-i}^c] = O(n^{-p})$ and on E_{-i}, the last two terms are $O(n^{-2})$ in absolute value. Hence, on E_{-i},

$$\mathrm{E}[W_i^- \mid \mathcal{F}_{-i}] = \bar{u}_i \left(\frac{1}{U_{-i}} - \frac{1}{\bar{U}}\right) + O(n^{-2}).$$

Using $\bar{U} = \bar{U}_{-i} + \bar{u}_i$ with $\bar{U}_{-i} := \mathrm{E}[U_{-i}]$, we expand on E_{-i}:

$$\frac{1}{U_{-i}} - \frac{1}{\bar{U}_{-i}} = -\frac{U_{-i} - \bar{U}_{-i}}{\bar{U}_{-i}^2} + O\!\left(\frac{(U_{-i} - \bar{U}_{-i})^2}{n^3}\right).$$

Taking expectations and using $\mathrm{E}[|U_{-i} - \bar{U}_{-i}|] \le \sqrt{\mathrm{Var}[U_{-i}]} = O(\sqrt{n})$ and $\mathrm{E}[(U_{-i} - \bar{U}_{-i})^2] = O(n)$, we obtain

$$\mathrm{E}\left[\left|\frac{1}{U_{-i}} - \frac{1}{\bar{U}_{-i}}\right| \mathbb{1}[E_{-i}]\right] \le \frac{C}{n^{3/2}}.$$

Also $|1/\bar{U}_{-i} - 1/\bar{U}| \le C/n^2$. Therefore,

$$\mathrm{E}\big[\big|\mathrm{E}[W_i^- \mid \mathcal{F}_{-i}]\big|\mathbb{1}[E_{-i}]\big] \le \frac{C}{n^{3/2}}.$$

On E_{-i}^c, $|\mathrm{E}[W_i^- \mid \mathcal{F}_{-i}]| \le 1$ gives a contribution $O(n^{-p})$. Hence, overall,

$$\mathrm{E}\big|\mathrm{E}[W_i^- \mid \mathcal{F}_{-i}]\big| \le \frac{C}{n^{3/2}}.$$

Consequently $|T_{1,i}^{(b)}| \le (C\sqrt{n})(Cn^{-3/2}) = C/n$.

Conclusion for the "<" part. We have shown $|T_{2,i}| \le C/n$ and $|T_{1,i}^{(a)}| \le C/n$, $|T_{1,i}^{(b)}| \le C/n$, hence $|\varepsilon_{i,n}^-| \le C/n$.

The ">" part. The same argument applies to

$$\varepsilon_{i,n}^+ := \mathrm{E}\big[(w_n - S_n)W_i^+\mathbb{1}[S_n > w_n]\big]$$

by replacing $(u_i, U, \bar{u}_i, \bar{U})$ with $(d_i, D, \bar{d}_i, \bar{D})$. Thus $|\varepsilon_{i,n}^+| \le C/n$, and finally $|\varepsilon_{i,n}| \le C_\varepsilon/n$. □

E.3.5 From componentwise $1/n$ to ℓ_2 and to the fixed point

Proposition E.11 (Mean drift at the mean)**.** *Under the assumptions of Proposition E.10, there exists $K_0 < \infty$ such that, for all sufficiently large n,*

$$\big\|\mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) - \mathrm{E}[\boldsymbol{X}]\big\|_2 \le K_0\, n^{-1/2}.$$

Proof. By Proposition E.10, for each coordinate,

$$|\mathcal{T}_{n,i}(\mathrm{E}[\boldsymbol{X}]) - \mu_i| \le \frac{C_\varepsilon}{n}.$$

Therefore,

$$\big\|\mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) - \mathrm{E}[\boldsymbol{X}]\big\|_2^2 = \sum_{i=1}^n \big(\mathcal{T}_{n,i}(\mathrm{E}[\boldsymbol{X}]) - \mu_i\big)^2 \le n\left(\frac{C_\varepsilon}{n}\right)^2 = \frac{C_\varepsilon^2}{n}.$$

Taking square roots yields the result with $K_0 = C_\varepsilon$. □

Lemma E.12 (Fixed-point displacement)**.** *Assume that Assumption E.5 holds true. Then, for all sufficiently large n,*

$$\big\|\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}]\big\|_2 \le \frac{1}{1-\rho}\,\big\|\mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) - \mathrm{E}[\boldsymbol{X}]\big\|_2.$$

Proof. Since $\boldsymbol{c}_n^\star, \mathrm{E}[\boldsymbol{X}] \in \Delta_{w_n}$, we have $\mathbf{1}^\top(\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}]) = 0$, hence $\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}] = P_n(\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}])$. Using the fixed-point relation $\boldsymbol{c}_n^\star = \mathcal{T}_n(\boldsymbol{c}_n^\star)$,

$$\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}] = \mathcal{T}_n(\boldsymbol{c}_n^\star) - \mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) + \big(\mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) - \mathrm{E}[\boldsymbol{X}]\big).$$

Let $\boldsymbol{c}_t := \mathrm{E}[\boldsymbol{X}] + t(\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}])$, $t \in [0,1]$. By Fréchet differentiability on $\mathcal{N}_n$,

$$\mathcal{T}_n(\boldsymbol{c}_n^\star) - \mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) = \int_0^1 \mathcal{D}\mathcal{T}_n(\boldsymbol{c}_t)\,(\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}])\,\mathrm{d}t.$$

Projecting with P_n and using $\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}] = P_n(\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}])$,

$$\|\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}]\|_2 \le \int_0^1 \|P_n\mathcal{D}\mathcal{T}_n(\boldsymbol{c}_t)P_n\|_{\mathrm{op}}\,\mathrm{d}t\;\|\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}]\|_2 + \|\mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) - \mathrm{E}[\boldsymbol{X}]\|_2.$$

Assumption E.5 gives $\|P_n\mathcal{D}\mathcal{T}_n(\boldsymbol{c}_t)P_n\|_{\mathrm{op}} \le \rho$, so that

$$\|\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}]\|_2 \le \rho\,\|\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}]\|_2 + \|\mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) - \mathrm{E}[\boldsymbol{X}]\|_2.$$

Rearranging yields the announced result. □

Proof of Theorem E.1. Combine Proposition E.11 and Lemma E.12:

$$\|\boldsymbol{c}_n^\star - \mathrm{E}[\boldsymbol{X}]\|_2 \le \frac{1}{1-\rho}\,\|\mathcal{T}_n(\mathrm{E}[\boldsymbol{X}]) - \mathrm{E}[\boldsymbol{X}]\|_2 \le \frac{K_0}{1-\rho}\,n^{-1/2}.$$

This proves the critical ℓ_2-rate. □

F Proof of Proposition 3.6

The result follows from the multivariate Lindeberg–Feller theorem applied to the independent centered vectors $(X_i - \mathrm{E}[X_i],\, I_i - p_i)$. Joint convergence yields a bivariate Normal limit. Conditioning on $\{Z_1 > 0\}$ and using standard properties of truncated Normal distributions leads to the stated conditional moments and tail approximation after rescaling.

G Proof of Proposition 4.1

For a generic loss X, define

$$Y := (c-X)_+, \qquad Z := (X-c)_+.$$

Then (Y,Z) takes values in $(\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)$ and

$$(U_n^{\setminus 1}, D_n^{\setminus 1}) = \sum_{j=2}^n (Y_j, Z_j),$$

where (Y_j, Z_j) are independent copies of (Y,Z). By independence,

$$\mathrm{E}\Big[e^{-sU_n^{\setminus 1} - tD_n^{\setminus 1}}\Big] = \big(\mathrm{E}[e^{-sY-tZ}]\big)^{n-1}.$$

Since $Y = (c-X)$ on $\{X \le c\}$ and $Z = (X-c)$ on $\{X > c\}$,

$$\mathrm{E}[e^{-sY-tZ}] = \int_0^c e^{-s(c-x)}\,\mathrm{d}F(x) + \int_c^\infty e^{-t(x-c)}\,\mathrm{d}F(x),$$

which yields (4.2). The mixture representation follows by conditioning on $K^{\setminus 1}$.

H Proof of Proposition 4.3

In the non-critical regimes, $\mathbb{1}[U_n > D_n] \to 1$ (resp. 0) in probability and the ratios D_n/U_n and U_n/D_n converge almost surely to $\bar{d}/\bar{u}$ and $\bar{u}/\bar{d}$, respectively. Slutsky's theorem then yields the stated distributional limits.

I Proof of Proposition 4.4

Writing the payoff using (4.1), the adjustment term is always of the form $\pm(c - X_1)_+ \frac{D_n}{U_n}$ on $\{S_n < w_n\}$ or $\pm(X_1 - c)_+ \frac{U_n}{D_n}$ on $\{S_n > w_n\}$. At criticality, $w_n - S_n = O_P(\sqrt{n})$ by the central-limit theorem, while U_n and D_n are sums of independent, identically distributed, non-negative random variables with nonzero mean and therefore are $O_P(n)$. Hence the multiplicative factors $(w_n - S_n)/U_n$ and $(w_n - S_n)/D_n$ are $O_P(n^{-1/2})$, yielding $H_{1,n} - X_1 = o_P(1)$.