On the connection between correlated equilibria and sunspot equilibria*

Julio Dávila

Departamento de Economía e Historia Económica, Universidad Autónoma de Barcelona, 08193 Bellaterra, Spain

Abstract. The conjecture stated in Maskin-Tirole (1987) about the existence of a close connection between correlated equilibria of market games modelling economies with asymmetric information about an extrinsic uncertainty and finite markovian stationary sunspot equilibria of related overlapping generations economies is reconsidered in this paper. It is shown that there is no robust example of an overlapping generations economy with a non-negligible set of finite markovian stationary sunspot equilibria which can be identified with correlated equilibria of the corresponding economy à la Maskin-Tirole, and vice versa. This result qualifies the conjectured connection between the two equilibrium concepts in these two frameworks.

JEL Classification Numbers: D52, E10

Keywords: Correlated equilibrium, sunspot equilibrium, overlapping generations economy

1 Introduction

The concepts of correlated equilibrium [Aumann (1974, 1987)] and sunspot equilibrium [Shell (1977), Cass-Shell (1983)] have been put in relation with each other very often in the economics literature since actually both cope with the role of extrinsic uncertainty on the outcomes of simultaneous optimizing behavior. As pointed out in Forges-Peck (1995), the difficulty for the assessment of the precise link between them comes from their original development in different frameworks: on the one hand a game-theoretic one for the correlated equilibrium concept and on the other hand a competitive

*I thank Eric Maskin, Piero Gottardi, James Bergin, Karl Shell and James Peck for their remarks and discussions on this subject, and Andreu Mas-Colell for useful comments. This research was started while visiting Harvard University as Post-doctoral Fellow and has since then benefited from remarks from attendants at several conferences and seminars. Funding from the Spanish Ministry of Education and Science from the research projects PB92-0120-C02-091, PB96-1160-C02-02 and a post-doctoral fellowship is gratefully acknowledged.

one for the sunspot equilibrium idea. Comparisons between them have been made by means of translating one of the two equilibrium concepts into the world of the other, either considering a game-theoretic avatar of the sunspot equilibrium in market games à la Shapley-Shubik (1977) [see Peck-Shell (1991), Forges-Peck (1995)] or considering a competitive version of the correlated equilibrium idea [see Maskin-Tirole (1987)].

In Maskin-Tirole (1987) it was conjectured that the concepts of a (finite-states markovian stationary) sunspot equilibrium in a simple overlapping generations economy and a competitive version of a correlated equilibrium in the economy with asymmetric information they study may well be closely connected. Specifically, under some conditions, a sunspot equilibrium of the former economy could be seen as a correlated equilibrium of the corresponding latter economy and vice versa. The goal of this paper is to make precise what the conditions are under which this is so and, accordingly, to assess the degree of equivalence between the two concepts in the particular frameworks of overlapping generations economies and the one considered in Maskin-Tirole (1987).

The investigation of this issue closest to the one developed in this paper is Forges-Peck (1995). In that paper the authors study the connection between the two equilibrium concepts by means of a market game à la Shapley-Shubik which mimics the overlapping generations economy. They get an equivalence result under the assumption of a continuum of players at each stage and a general sunspot process which may not be markovian. As for markovian sunspot equilibria, the scope of the equivalence shortens but is still shown to hold for sunspots fluctuating between two values. Interestingly enough, the case of sunspots taking just two values is the only one in which the lack of robustness of the equivalence of markovian sunspot equilibria to correlated equilibria shown in Section 3 does not hold for reasons which will be apparent in the proofs.

The main statements of this paper are in Propositions 1 and 2, in Sections 3 and 4, respectively. Their message can be summarized as follows: arbitrarily close to any overlapping generations economy, there is another economy for which the set of markovian stationary sunspot equilibria, equivalent to some correlated equilibrium of a Maskin-Tirole economy is negligible, and conversely. Auxiliary lemmas are collected in the Appendix.

2 The OG economy and the MT economy

The discussion that follows compares sunspot equilibria[1] of an overlapping generations economy à la Samuelson (1958) and correlated equilibria of the economy with asymmetric information studied in Maskin-Tirole (1987). Specifically, the overlapping generations economy considered is the sim-

[1] More precisely, finite states markovian stationary sunspot equilibria.

plest one: it consists of countably many generations of a representative agent, each living for two consecutive periods, one consumption commodity per period which can be produced with the leisure endowment of the contemporary "young" agent by means of a linear technology[2] with productivity 1 and which is consumed by the contemporary "old" generation in exchange of their "money" holdings. Besides, at each period a sunspot signal (*i.e.*, carrying no information on the fundamentals of the economy) takes one out of k possible values following a first order Markov chain and is publicly observed by the contemporary agents who, according to a shared belief on a perfect correlation between sunspots and prices, may make their decisions contingent to the future values of the sunspot (see, for instance, Azariadis-Guesnerie (1986) for details) and their expectations on future prices turn out to be nonetheless rational at a sunspot equilibrium.

The economy in Maskin-Tirole (1987) consists of two agents who have to decide how much to produce of a commodity (each of them producing a distinct commodity) with their leisure endowment in order to exchange it for the commodity produced by the other agent[3], with imperfect information about which is the state of the world. The state of the world is completely characterized by the values taken by two signals taking randomly a finite number k of values with a joint probability distribution. Each agent observes one signal distinct from that observed by the other agent. Even though the uncertainty is extrinsic (i.e., innocuous for the fundamentals of the economy), since the exchange value of the commodity produced by one agent depends on the quantity produced by the other agent of his own good, if the agents make their decisions contingent on the state of the world, the prices and allocation may turn out to be contingent on the extrinsic uncertainty and thus their expectations rational at a correlated equilibrium (see Maskin-Tirole (1987) for details).

3 From sunspot equilibria to correlated equilibria

Let U denote the class of C^2, strictly concave real-valued functions u on $\mathbb{R}^2_{++}$ monotone in the sense [4] that $D_1u(x) < 0$ and $D_2u(x) > 0$ at any $x \in \mathbb{R}^2_{++}$, endowed with the topology of C^2 uniform convergence on compacta. Given a utility function $u \in U$, if $l_1, l_2 > 0$ distinct and a Markov matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{1}$$

[2] Production is actually inessential. The economy is formally equivalent to an exchange economy.

[3] Their utility depends only on their consumption of the good produced by the other agent and their own leisure.

[4] In this sense, the first argument of a utility funcion of this class should be seen as labour supply and the second one as consumption.

satisfy the two equations [5]

$$
\begin{aligned}
m_{11}\big(D_1u(l_1,l_1) + \tfrac{l_1}{l_1}D_2u(l_1,l_1)\big) &+ \\
m_{12}\big(D_1u(l_1,l_2) + \tfrac{l_2}{l_1}D_2u(l_1,l_2)\big) &= 0,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
m_{21}\big(D_1u(l_2,l_1) + \tfrac{l_1}{l_2}D_2u(l_2,l_1)\big) &+ \\
m_{22}\big(D_1u(l_2,l_2) + \tfrac{l_2}{l_2}D_2u(l_2,l_2)\big) &= 0,
\end{aligned}
\tag{2.2}
$$

then $(l, M) \in \mathbb{R}^2_{++} \times (0,1)^2$ constitute a (2-states markovian stationary) *sunspot equilibrium* —a 2-SSE from now onwards— of a simple overlapping generations economy —an OG economy henceforth— (see Azariadis and Guesnerie (1986) for details and characterizations of its existence [6]) with a representative agent with utility function u and l_i being the amount of labor supplied by the young agent when the sunspot signal takes the value $i = 1, 2$. The k equations analogous to equations (2) but defining a k-SSE for any integer $k \geq 3$ can straightforwardly be written (in general, the support of a k-SSE has to be an off-diagonal point of $\mathbb{R}^k_{++}$, i.e., with at least two components distinct). Notice that for the sake of determining its k-SSE, an OG economy is completely characterized by the utility function u, thus we will speak of a u-OG economy in what follows.

As it was remarked in section 5 of Maskin-Tirole (1987), these conditions are very similar to those that a correlated equilibrium of the economy they consider has to satisfy. Specifically, $(l^1, l^2) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ and the joint probability distribution matrix for two signals s^1, s^2, taking one out of two values each,

$$
\Pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \tag{3}
$$

is a *correlated equilibrium* (l^1, l^2, Π) —a 2-CE henceforward— of the 2 periods, 2 commodities economy with asymmetric information in Maskin-Tirole (1987) —the MT economy, from now onwards— with two agents $i = 1, 2$, each with a utility function u^i of the class U and supplying an amount $l^i_{s^i}$ of labor whenever agent i (privately) observes the value $s^i = 1, 2$ of the signal he receives about the state of the world, if they satisfy the four equations (see Maskin-Tirole (1987) for details and characterizations

[5] Here, as well as in what follows, ratios trivially equal to 1 are made explicit for the symmetry of the equations to become more apparent.

[6] Equations 2.1-2.2 result from the utility maximization first order conditions and the feasibility of the equilibrium allocation condition.

of its existence [7])

$$\begin{aligned}&\pi_{11}\big(D_1u^1(l_1^1,l_1^2)+\tfrac{l_1^2}{l_1^1}D_2u^1(l_1^1,l_1^2)\big)+\\&\pi_{12}\big(D_1u^1(l_1^1,l_2^2)+\tfrac{l_2^2}{l_1^1}D_2u^1(l_1^1,l_2^2)\big)=0\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\pi_{21}\big(D_1u^1(l_2^1,l_1^2)+\tfrac{l_1^2}{l_2^1}D_2u^1(l_2^1,l_1^2)\big)+\\&\pi_{22}\big(D_1u^1(l_2^1,l_2^2)+\tfrac{l_2^2}{l_2^1}D_2u^1(l_2^1,l_2^2)\big)=0\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\pi_{11}\big(D_1u^2(l_1^2,l_1^1)+\tfrac{l_1^1}{l_1^2}D_2u^2(l_1^2,l_1^1)\big)+\\&\pi_{21}\big(D_1u^2(l_1^2,l_2^1)+\tfrac{l_2^1}{l_1^2}D_2u^2(l_1^2,l_2^1)\big)=0\end{aligned}\tag{5.1}$$

$$\begin{aligned}&\pi_{12}\big(D_1u^2(l_2^2,l_1^1)+\tfrac{l_1^1}{l_2^2}D_2u^2(l_2^2,l_1^1)\big)+\\&\pi_{22}\big(D_1u^2(l_2^2,l_2^1)+\tfrac{l_2^1}{l_2^2}D_2u^2(l_2^2,l_2^1)\big)=0.\end{aligned}\tag{5.2}$$

Again the $2k$ equations analogous to equations (4)–(5) but defining a k-CE for any integer $k \geq 3$ can straightforwardly be written too. Equations (4) (respectively, equations (5)) seem at first sight to be the equilibrium equations characterizing a sunspot equilibrium of an overlapping generations economy with identical agents with utility function u^1 (respectively, u^2) and a Markov matrix for the sunspot signal given by the conditional distribution of s^2 given s^1 (respectively, s^1 given s^2). Notice, nevertheless, that a careful inspection of the equations reveals that this is not exactly so, and that they are different from equations (2) characterizing a sunspot equilibrium by the fact that the supply of labor is now indexed not only by the value of the signal observed by each agent, but by the identity of the agent who observes it as well. Notice too that for the sake of determining its k-CE, an MT economy is completely characterized by the utility functions u^1 and u^2. Thus we will speak of a (u^1,u^2)-MT economy from now onwards.

A natural question arises from the comparison of the two sets of conditions (2) and (4)–(5): Is there a correlated equilibrium corresponding or "equivalent" to any sunspot equilibrium given by equations (2) as conjectured in Maskin-Tirole (1987)? To begin with, the question itself needs to be clarified. Notice that for our purposes a u-OG economy can be identified with a function u of the class U, while a (u^1,u^2)-MT economy can be equally identified with a couple of such functions u^1, u^2. Thus, what is

[7] Equations (4) result from the utility maximization first order conditions of the agent "observing the rows" of Π taking into account the feasibility of the equilibrium allocation condition. Similarly for equations (5) and the agent "observing the columns". Notice that the denominators have been dropped with no harm in the computation of conditional probabilities.

exactly meant by the "equivalence" between the equilibria of two such different economies? If we are given a sunspot equilibrium of a u-OG economy, which is the (u^1, u^2)-MT economy one of whose correlated equilibria is the "equivalent one"? If we interpret an overlapping generations economy as a sequence of agents which only trade with those two who are contiguous to them, the representative agent cares only about the rate at which he gives away, say, right-handed good in exchange for the left-handed good, and he does not bother at all whether the hand which receives his right-handed good belongs to a distinct agent than the hand which gives him left-handed good, as it is indeed the case in an OG economy, or to the same agent. The same is true for any of the two agents of the Maskin-Tirole economy, they just care about the terms of trade. This allows to identify, as it was implicitly done in Maskin-Tirole (1987), any u-OG economy with the corresponding (u, u)-MT economy, *i.e.* the one where both u^1 and u^2 are equal to u. Therefore, by "a correlated equilibrium equivalent to a given sunspot equilibrium" of a u-OG economy will be meant one of the (u, u)-MT economy such that its allocation of resources is related to the allocation of the sunspot equilibrium as follows: $l_1^i = l_1$ and $l_2^i = l_2$, for all $i = 1, 2$, *i.e.*, such that the correlated equilibrium treats symmetrically both agents. We will speak of two such equilibria as "giving the same allocation of resources", although we should keep in mind that, as a matter of fact, the two economies, and hence their allocations of resources, are quite distinct.

An answer to the question above is given by the next proposition.

Proposition 1 *There is no robust overlapping generations economy whose markovian stationary sunspot equilibria that both fluctuate between more than 2 states, and are equivalent to some correlated equilibrium of the corresponding Maskin-Tirole economy, constitute a non-negligible subset of that class of sunspot equilibria*[8].

Before making this claim precise, let us define a regular k-SSE $(l, M) \in \mathbb{R}_{++}^k \times (0,1)^{k^2-k}$ of a u-OG economy to be a k-SSE such that the perfect foresight steady state is not in its support. Notice that with the assumptions made on u this implies that for all $i = 1, \ldots, k$, $D_1u(l_i, l_i) + D_2u(l_i, l_i) \neq 0$, otherwise l_i would be the steady state labor supply. Proposition 1 above is but the interpretation of the following fact[9].

Proposition 1 (restated) *For any u in a dense subset of the set of functions in U for which the u-OG economy has 3-SSE, the set of 3-SSE equiva-*

[8] There is a good reason for not to try to simplify stating the seemingly equivalent statement that typically such set is negligible. The reason is that the usual sense given to "typically" would mean in this case "for any economy in a dense and open subset of the space of economies". Density is what is proved in Proposition 1, and not openness.

[9] Stated specifically for k-SSE with $k = 3$. As for $k > 3$ see the concluding remarks.

lent to some 3-CE of the corresponding (u,u)-MT economy is null measure in the set of all 3-SSE in a neighborhood of any regular 3-SSE.

The limitation of scope in the restatement of Proposition 1 to neighborhoods of regular 3-SSE is not very constraining, since most 3-SSE are regular in the sense that, while the set of regular 3-SSE is locally a 6-dimensional manifold, that of non-regular 3-SSE is locally contained in a finite union of 5-dimensional manifolds (see Lemma 7).

Proof. Let u be of the class U and (l,M) be a regular 3-SSE of u-OG. Then the set of 3-SSE is locally a manifold of dimension 6 in a neighborhood of (l,M) (see Lemma 3). For this 3-SSE to be equivalent to a 3-CE (l,l,Π) of (u,u)-MT, its correlation matrix Π should generate, either by rows or by columns, the Markov matrix M and satisfy the equilibrium conditions for a 3-CE not already implied by those of the 3-SSE. That is to say, there should exist $\pi_{ij} \in (0,1)$, for all $i,j=1,2,3$ such that

$$\sum_{i,j=1}^{3} \pi_{ij} = 1 \tag{6}$$

$$\frac{\pi_{ij}}{\sum_{h=1}^{3} \pi_{ih}} = m_{ij}, \forall i \neq j \tag{7}$$

$$\sum_{j=1}^{3} \pi_{ji} f_{ij}(l) = 0, \forall i = 1,2,3, \tag{8}$$

where f_{ij} is defined on R^3_{++} and such that

$$f_{ij}(x) = D_1 u(x_i,x_j) + \frac{x_j}{x_i} D_2 u(x_i,x_j).$$

Should there be such a Π, the previous system in π_{ij} would have to be linearly dependent, which can be proved to be equivalent to the singularity of the matrix $A = \big(m_{ji} f_{ij}(l)\big)$. But letting $B = \big(m_{ij} f_{ij}(l)\big)$ (notice the different order in the indices of the probabilities of transition in the entries of A and B), the equilibrium equations of the 3-SSE guarantee the singularity of this matrix [10], which implies that the singularity of A is equivalent to the fulfillment of

$$(m_{13}m_{21}m_{32} - m_{12}m_{23}m_{31})(f_{12}(l)f_{23}(l)f_{31}(l) - f_{13}(l)f_{21}(l)f_{32}(l)) = 0. \tag{9}$$

That is to say, should (l,M) be a 3-SSE of u-OG equivalent to a 3-CE of (u,u)-MT, then either

$$m_{13}m_{21}m_{32} - m_{12}m_{23}m_{31} = 0 \tag{10}$$

[10] Since they actually state that $(1,1,1)$ is in the null space of B.

or

$$f_{12}(l)f_{23}(l)f_{31}(l) - f_{13}(l)f_{21}(l)f_{32}(l) = 0. \tag{11}$$

But equations (10) and (11) are both transversal at (l, M) to the equations defining the 3-SSE for any u in a dense subset of the functions in U for which u-OG has 3-SSE (see Lemmas 4 and 5). Therefore, the subset of 3-SSE equivalent to a 3-CE is contained in the union of two manifolds of dimension 5 embedded in the manifold of dimension 6 constituted by the 3-SSE in a neighborhood of (l, M). The null measure of 3-SSE equivalent to a 3-CE follows immediately. ■

4 From correlated equilibria to sunspot equilibria

Going the other way round, in order to obtain a sunspot equilibrium (specifically, a 2-SSE of some u-OG economy) from a given correlated equilibrium of a (u^1, u^2)-MT economy, the latter must give the same allocation to both agents, as can easily be seen by comparing equations (2) and equations (4) (or (5)) in Section 3.

Thus, in order to interpret equations (4) as the conditions characterizing a 2-SSE of the u^1-OG economy necessarily the equations $l_1^1 = l_1^2$ and $l_2^1 = l_2^2$ must hold. The same is true if equations (5) are to be interpreted as the conditions of a sunspot equilibrium of the u^2-OG economy. Hence, there are two sunspot equilibria associated to a (u^1, u^2)-MT economy (one corresponding to equations (4) and the other to equations (5)) if and only if $l_1^1 = l_1^2$ and $l_2^1 = l_2^2$ hold for a given correlated equilibrium of it. Notice that, as a matter of fact, if u^1 and u^2 are distinct utility functions each of these two sunspot equilibria is an equilibrium of a different overlapping generations economy, either one with identical agents with utility function u^1 or another one with identical agents with utility function u^2. The same argument applies for any k-SSE with $k \geq 2$. As a conclusion, the only correlated equilibria of a (u^1, u^2)-MT economy that can be interpreted as finite markovian sunspot equilibria of some u-OG economy are those which treat symmetrically both agents. This allows us to state the next proposition.

Proposition 2 *There is no robust Maskin-Tirole economy whose finite correlated equilibria which are equivalent to finite markovian stationary sunspot equilibria of a corresponding overlapping generations economy constitute a non-negligible subset of that class of equilibria.*

The previous proposition is but the interpretation of the following fact.[11]

Proposition 2 (restated) *For* u^1, u^2 *in a dense subset of the set in* $U \times U$

[11] Stated specifically for k-CE with $k = 2$. As for $k > 2$, see the concluding remarks.

for which the (u^1, u^2)-MT economy has 2-CE, the set of symmetric 2-CE is null measure in the set of all 2-CE.

Proof. Let u^1, u^2 be of the class U and let (l, l, Π) be a symmetric 2-CE of (u^1, u^2)-MT. Then the set of 2-CE is locally a manifold of dimension 3 in a neighborhood of (l, l, Π) which is moreover, for any (u^1, u^2) in a dense subset of U^2, transversal at (l, l, Π) to the linearly independent equations $l_i^1 = l_i^2$, for $i = 1, 2$, satisfied by symmetric 2-CE (see Lemma 6). Thus the set of symmetric 2-CE of (u^1, u^2)-MT is a manifold of dimension 1 embedded in the manifold of dimension 3 of 2-CE in a neighborhood of (l, l, Π). The null measure follows immediately. ■

As a final remark, notice that if both utility functions of (u^1, u^2)-MT are the same, then the two k-SSE associated to any symmetric k-CE are equilibria of the same overlapping generations economy. Nevertheless, these two sunspot equilibria need not be the same: although they necessarily have the same allocation, their Markov matrices may be distinct. In general, these two Markov matrices coincide if and only if the joint distribution is symmetric and, only in this case, both sunspot equilibria are the same.

5 Conclusion

This paper states the conditions under which sunspot equilibria of a simple overlapping generations economy and correlated equilibria of a related economy with asymmetric information about some extrinsic uncertainty as in Maskin-Tirole (1987) can be considered to be equivalent in the sense that any sunspot equilibrium of the first economy can be naturally interpreted as a correlated equilibrium of the second one and vice versa. Specifically, it turns out that the translation of sunspot equilibria (3-SSE) to correlated equilibria (3-CE) is not robust whenever possible (and vice versa for 2-CE and 2-SSE) in these particular frameworks.

It is worth to be mentioned that the propositions have been stated for the minimum integer $k \geq 0$ which makes each of them hold. Actually, Proposition 1 does not hold for 2-SSE since in that case the singularity of matrix A is guaranteed by that of matrix B, which is not the case for $k = 3$ nor is it very likely for any $k > 3$, although this remains a conjecture (an obvious pattern for matrix B in the general case becomes easily apparent and makes this conjecture quite sensible, since for it not to be true some conditions on the values of the second order partial derivatives of u at the equilibrium allocation would have to be fulfilled, which needs not be the case). The same remark applies to the extension of Proposition 2 to any k-CE.

Propositions 1 and 2 may help to complete the understanding that the previous literature provided us about the links between both equilibrium concepts.

Appendix

Lemma 3 *Let u be any function in U such that* [12]

$$S_3 = \{(x, M) \in \mathbb{R}^3_{++} \setminus \Delta^3 \times (0,1)^6 \mid \forall i \in \{1,2,3\}, \sum_{j=1}^{3} m_{ij} f_{ij}(x) = 0\} \neq \varnothing \tag{12}$$

(f_{ij} being such that $f_{ij}(x) = D_1 u(x_i, x_j) + \frac{x_j}{x_i} D_2 u(x_i, x_j)$ and Δ^3 being the diagonal of R^3) and let $(a, N) \in S_3$ be such that $\forall i \in \{1,2,3\}, f_{ii}(a) \neq 0$. Then S_3 is locally a manifold of dimension 6 in a neighborhood of (a, N).

Proof. Given such u, let $F : \mathbb{R}^3_{++} \setminus \Delta^3 \times (0,1)^6 \to \mathbb{R}^3$ be such that

$$F(x, M) = \begin{pmatrix} \sum_{j=1}^3 m_{1j} f_{1j}(x) \\ \sum_{j=1}^3 m_{2j} f_{2j}(x) \\ \sum_{j=1}^3 m_{3j} f_{3j}(x) \end{pmatrix}$$

and let $(a, N) \in F^{-1}(0) = S_3$. The Jacobian of F at (a, N), $DF(a, N)$, is[13]

$$\left(\begin{array}{ccc} \sum_{j=1}^3 n_{1j} D_1 f_{1j}(a) & n_{12} D_2 f_{12}(a) & n_{13} D_3 f_{13}(a) \\ n_{21} D_1 f_{21}(a) & \sum_{j=1}^3 n_{2j} D_2 f_{2j}(a) & n_{23} D_3 f_{23}(a) \\ n_{31} D_1 f_{31}(a) & n_{32} D_2 f_{32}(a) & \sum_{j=1}^3 n_{3j} D_3 f_{3j}(a) \end{array} \right.$$

$$\begin{array}{ccccc} & f_{12}(a) - f_{11}(a) & f_{13}(a) - f_{11}(a) & & \\ \cdots & 0 & 0 & \cdots & \\ & 0 & 0 & & \end{array} \tag{13}$$

$$\begin{array}{cccc} & 0 & 0 & \\ \cdots & f_{21}(a) - f_{22}(a) & f_{23}(a) - f_{22}(a) & \cdots \\ & 0 & 0 & \end{array}$$

$$\left. \begin{array}{ccc} & 0 & 0 \\ \cdots & 0 & 0 \\ & f_{31}(a) - f_{33}(a) & f_{32}(a) - f_{33}(a) \end{array} \right)$$

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[12] To prove the existence of such functions and their characterization is actually the mathematical toil of the literature on k-SSE in OG economies (see Chiappori-Guesnerie(1989) and Grandmont(1986) among others).

[13] A 3×3 Markov matrix M is thought of as $(m_{12}, m_{13}, m_{21}, m_{23}, m_{31}, m_{32}) \in (0,1)^6$.

Should $DF(a,N)$ not be full rank, then necessarily $f_{i1}(a) = f_{i2}(a) = f_{i3}(a)$ for some $i \in \{1,2,3\}$, but then $(a,N) \in F^{-1}(0)$ would imply $f_{ii}(a) = 0$. Thus, if for all $i = 1,2,3$, $f_{ii}(a) \neq 0$, then $DF(a,N)$ is full rank and therefore S_3 constitutes a manifold of dimension 6 in a neighborhood of (a,N). ■

Lemma 4 *Let U' be the subset of those functions in U such that*

$$S_3' = \{(x,M) \in R^3_{++} \setminus \Delta^3 \times (0,1)^6 \mid \forall i \in \{1,2,3\}, \textstyle\sum_{j=1}^3 m_{ij} f_{ij}(x) = 0,$$
$$f_{12}(x)f_{23}(x)f_{31}(x) - f_{13}(x)f_{21}(x)f_{32}(x) = 0\} \neq \varnothing \tag{14}$$

(f_{ij} being as in Lemma 3). The set of functions $u \in U'$ such that S_3' is locally a manifold of dimension 5 in a neighborhood of any $(a,N) \in S_3'$ satisfying $f_{ii}(a) \neq 0$, for all $i \in \{1,2,3\}$, is dense in U'.

Proof. Let $u \in U'$, let $G : R^3_{++} \setminus \Delta^3 \times (0,1)^6 \to R^4$ be such that

$$G(x,M) = \begin{pmatrix} \sum_{j=1}^3 m_{1j} f_{1j}(x) \\ \sum_{j=1}^3 m_{2j} f_{2j}(x) \\ \sum_{j=1}^3 m_{3j} f_{3j}(x) \\ f_{12}(x)f_{23}(x)f_{31}(x) - f_{13}(x)f_{21}(x)f_{32}(x) \end{pmatrix} \tag{15}$$

and let $(a,N) \in G^{-1}(0) = S_3'$ be such that $f_{ii}(x) \neq 0$, for all $i \in \{1,2,3\}$. Then $DG(a,N)$ has as its three first rows $DF(a,N)$ computed in the proof of Lemma 3 and as fourth row a 9-tuple whose three first entries are expressions in terms of the second order partial derivatives of u at the points (a_i,a_j), $i,j = 1,2,3$, and the six last entries are zero. Now $f_{ii}(a) \neq 0$, for all $i \in \{1,2,3\}$, guarantees that the six last entries of the three first rows of $DG(a,N)$ are full rank and therefore should the forth row be a linear combination of the others, it would necessarily be the trivial one with zero scalars. Thus the three first entries of the forth row would have to be zero too, which requires the second order partial derivatives of u at the points (a_i,a_j) to satisfy three equations. But arbitrarily close to u there is another C^2 real-valued function $\tilde{u}$ in U' with the same gradients as u at the points (a_i,a_j) (guaranteeing thus that $(a,N) \in \tilde{G}^{-1}(0)$ for the corresponding $\tilde{f}$) but not satisfying the conditions on second order partial derivatives necessary to prevent full rank of $D\tilde{G}(a,N)$ and, thus, such that $\tilde{G}^{-1}(0) = \tilde{S}_3'$ is locally a manifold of dimension 5 in a neighborhood of (a,N). ■

Lemma 5 *Let U'' be the subset of those functions in U such that*

$$S_3'' = \{(x,M) \in \mathbb{R}^3_{++} \setminus \Delta^3 \times (0,1)^6 \mid \forall i \in \{1,2,3\}, \textstyle\sum_{j=1}^3 m_{ij} f_{ij}(x) = 0,$$
$$m_{13}m_{21}m_{32} - m_{12}m_{23}m_{31} = 0\} \neq \varnothing \tag{16}$$

(f_{ij} being as in Lemma 3). The set of functions $u \in U''$ such that S_3'' is locally a manifold of dimension 5 in a neighborhood of any $(a, N) \in S_3''$ satisfying $f_{ii}(x) \neq 0$, for all $i \in \{1, 2, 3\}$, is dense in U''.

Proof. Let $u \in U''$, let $H : \mathbb{R}^3_{++} \setminus \Delta^3 \times (0,1)^6 \to \mathbb{R}^4$ be such that

$$H(x, M) = \begin{pmatrix} \sum_{j=1}^3 m_{1j} f_{1j}(x) \\ \sum_{j=1}^3 m_{2j} f_{2j}(x) \\ \sum_{j=1}^3 m_{3j} f_{3j}(x) \\ m_{13}m_{21}m_{32} - m_{12}m_{23}m_{31} \end{pmatrix} \tag{17}$$

and let $(a, N) \in H^{-1}(0) = S_3''$ be such that $f_{ii}(x) \neq 0$, for all $i \in \{1, 2, 3\}$. Then $DH(a, N)$ has as its three first rows $DF(a, N)$ computed in the proof of Lemma 3 and as fourth row a 9-tuple whose three first entries are zeros and the six last entries are

$$(-n_{23}n_{31}, n_{21}n_{32}, n_{13}n_{32}, -n_{31}n_{12}, -n_{23}n_{12}, n_{13}n_{21}). \tag{18}$$

Should the three first columns of $DF(a, N)$ be linearly dependent, then arbitrarily close to u there is a function $\tilde{u}$ in U'' with the same gradients at the points (a_i, a_j) (guaranteeing that $(a, N) \in \tilde{H}^{-1}(0)$ for the corresponding $\tilde{f}$) making them linearly independent. Thus the forth row of $D\tilde{H}(a, N)$ is necessarily the trivial linear combination of the three first rows with scalars 0, which cannot be since $N \in (0,1)^6$. Therefore, $D\tilde{H}(a, N)$ is full rank and $\tilde{H}^{-1}(0) = \tilde{S}_3''$ is a manifold of dimension 5 in a neighborhood of (a, N). ■

Lemma 6 *Let W be the subset of those ordered pairs of functions in U such that* [14]

$$C_2 = \{(x^1, x^2, P) \in \mathbb{R}^2_{++} \setminus \Delta^2 \times \mathbb{R}^2_{++} \setminus \Delta^2 \times (0,1)^4 \mid \textstyle\sum_{i,j} p_{ij} = 1, \\ \forall i \in \{1, 2\}, \sum_{j=1}^2 p_{ij} f^1_{ij}(x) = 0, \sum_{j=1}^2 p_{ji} f^2_{ij}(x) = 0\} \neq \varnothing \tag{19}$$

(f^h_{ij} being such that $f^h_{ij}(x) = D_1 u^h(x^h_i, x^{h'}_j) + \frac{x^{h'}_j}{x^h_i} D_2 u^h(x^h_i, x^{h'}_j)$, where $h' = 2$ if $h = 1$ and vice versa). The set of $(u^1, u^2) \in W$ such that

(1) C_2 is locally a manifold of dimension 3 in a neighborhood of any $(a^1, a^2, \Pi) \in C_2$,

(2) the linearly independent equations $x^1_i - x^2_i = 0$, for all $i = 1, 2$, are transversal to such manifold at any intersection point,

[14] Maskin-Tirole(1987) provides a necessary characterization for such functions.

is dense in W.

Proof. Let $(u^1, u^2) \in W$, let $\Phi : \mathbb{R}^2_{++} \setminus \Delta^2 \times \mathbb{R}^2_{++} \setminus \Delta^2 \times (0,1)^4 \to \mathbb{R}^5$ be such that

$$\Phi(x^1, x^2, P) = \begin{pmatrix} \sum_{i,j} p_{ij} - 1 \\ \sum_{j=1}^2 p_{1j} f^1_{1j}(x) \\ \sum_{j=1}^2 p_{2j} f^1_{2j}(x) \\ \sum_{j=1}^2 p_{j1} f^1_{1j}(x) \\ \sum_{j=1}^2 p_{j2} f^2_{2j}(x) \end{pmatrix} \tag{20}$$

and let $(a^1, a^2, \Pi) \in \Phi^{-1}(0) = C_2$. Then $D\Phi(a^1, a^2, \Pi)$ is

$$\left(\begin{matrix} 0 & 0 \\ \pi_{11} D_1 f^1_{11}(a) + \pi_{12} D_1 f^1_{12}(a) & 0 \\ 0 & \pi_{21} D_1 f^1_{21}(a) + \pi_{22} D_1 f^1_{22}(a) \\ \pi_{11} D_2 f^2_{11}(a) & \pi_{21} D_2 f^2_{12}(a) \\ \pi_{12} D_2 f^2_{21}(a) & \pi_{22} D_2 f^2_{22}(a) \end{matrix}\right. \cdots$$

$$\cdots \begin{matrix} 0 & 0 \\ \pi_{11} D_2 f^1_{11}(a) & \pi_{12} D_2 f^1_{12}(a) \\ \pi_{21} D_2 f^1_{21}(a) & \pi_{22} D_2 f^1_{22}(a) \\ \pi_{11} D_1 f^2_{11}(a) + \pi_{21} D_1 f^2_{12}(a) & 0 \\ 0 & \pi_{12} D_1 f^2_{21}(a) + \pi_{22} D_1 f^2_{22}(a) \end{matrix} \cdots \tag{21}$$

$$\cdots \left.\begin{matrix} 1 & 1 & 1 & 1 \\ f^1_{11}(a) & f^1_{12}(a) & 0 & 0 \\ 0 & 0 & f^1_{21}(a) & f^1_{22}(a) \\ f^2_{11}(a) & 0 & f^2_{12}(a) & 0 \\ 0 & f^2_{21}(a) & 0 & f^2_{22}(a) \end{matrix}\right).$$

Should the lower left 4×4 submatrix in terms of the second order partial derivatives of u^1 and u^2 not be regular, there exist $(\tilde{u}^1, \tilde{u}^2)$ in W arbitrarily close to (u^1, u^2) in the product topology with the same gradients at points (a^m_i, a^n_j), $i, j, m, n = 1, 2$ (so that $(a^1, a^2, \Pi) \in \tilde{\Phi}^{-1}(0) = \tilde{C}_2$ for the corresponding $\tilde{f}$'s) which make this submatrix regular and such that the coordinates of $(1,0,-1,0)$ and $(0,1,0,-1)$ with respect to its rows are not in the (non trivial [15]) null space of the lower right 4×4 submatrix [16]. This has two consequences:

(1) First, $D\tilde{\Phi}(a^1, a^2, \Pi)$ is full rank (for it not to be so, looking at the first four columns we would conclude that the first row would have to be the trivial linear combination with zero scalars of the others, which clearly cannot be considering the last four columns). Hence, $\tilde{\Phi}^{-1}(0) = \tilde{C}_2$ is locally a manifold of dimension 3 in a neighborhood of (a^1, a^2, Π).

(2) Moreover, should (a^1, a^2, Π) satisfy the equations $x_i^1 - x_i^2 = 0$ too, for all $i = 1, 2$, these would be transversal to $\tilde{\Phi}^{-1}(0) = \tilde{C}_2$ at (a^1, a^2, Π) since their gradients are

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

Thus the subset of $\tilde{C}_2$ satisfying $x_i^1 - x_i^2 = 0$, for all $i = 1, 2$, is locally a manifold of dimension 1 in a neighborhood of (a^1, a^2, Π) embedded in the 3-dimensional manifold of $\tilde{\Phi}^{-1}(0)$. ■

Lemma 7 *Let U^* be the subset of those functions in U such that*

$$S_3 = \{(x, M) \in \mathbb{R}^3_{++} \setminus \Delta^3 \times (0,1)^6 \mid \forall i \in \{1,2,3\}, \sum_{j=1}^{3} m_{ij} f_{ij}(x) = 0\} \neq \varnothing \tag{23}$$

(f_{ij} being as in Lemma 3) and for any $u \in U^*$ and $i \in \{1,2,3\}$ let

$$D_i = \{(x, M) \in \mathbb{R}^3_{++} \setminus \Delta^3 \times (0,1)^6 \mid f_{ii}(x) = 0\}. \tag{24}$$

Then, for any $i \in \{1,2,3\}$, the set of functions $u \in U^*$ such that S_3 and D_i have a transversal intersection whenever nonempty is dense in U^*.

Proof: Let u be in U^* such that, for some $i = 1,2,3$, $S_3 \cap D_i \neq \varnothing$, and let

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[15] The non null vector $(\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22})$ is in it.

[16] Let A and B be respectively the lower left and right 4×4 submatrices of $D\Phi(a^1, a^2, \Pi)$, and perturb the second order partial derivatives of u^1 and u^2 at all (a_i^1, a_j^2) and (a_i^2, a_j^1) respectively in order to have A regular. The coordinates of $c = (1,0,-1,0)$ and $c' = (0,1,0,-1)$ in the basis constituted by the rows of A are $(A^t)^{-1}c$ and $(A^t)^{-1}c'$ respectively. The mappings associating $(A^t)^{-1}c$ and $(A^t)^{-1}c'$ to $(A^t)^{-1}$ are surjective and hence $(A^t)^{-1}$ can be perturbed in order to put both $(A^t)^{-1}c$ and $(A^t)^{-1}c'$ out of the null space of B in case this was not so yet. By continuity A^t will still be regular.

$(a, N) \in S_3 \cap D_i$. Let $\Psi_i : \mathbb{R}^3_{++} \setminus \Delta^3 \times (0,1)^6 \to \mathbb{R}^4$ be such that

$$\Psi_i(x, M) = \begin{pmatrix} \sum_{j=1}^3 m_{1j} f_{1j}(x) \\ \sum_{j=1}^3 m_{2j} f_{2j}(x) \\ \sum_{j=1}^3 m_{3j} f_{3j}(x) \\ f_{ii}(x) \end{pmatrix}. \tag{25}$$

Then $(a, N) \in \Psi_i^{-1}(0)$ and the fourth row of $D\Psi_i(a, N)$ is such that all its entries other than the i-th are zero. Should the i-th one, in terms of the second order partial derivatives of u at (a_i, a_i), be zero too, then arbitrarily close to u there is $\tilde{u}$ in U^* with the same gradient at the points (a_i, a_j) (and thus $(a, N) \in \tilde{\Psi}_i^{-1}(0)$ for the corresponding $\tilde{f}$), such that the forth row of $D\tilde{\Psi}_i(a, N)$ is non-null and hence this Jacobian is necessarily full rank (otherwise, if the fourth row of $D\tilde{\Psi}_i(a, N)$ were linearly dependent of the others, necessarily for all $h = 1, 2, 3$, $\tilde{f}_{h1}(a) = \tilde{f}_{h2}(a) = \tilde{f}_{h3}(a)$, which from $(a, N) \in \tilde{\Psi}_i^{-1}(0)$ implies $\tilde{f}_{hh}(a) = 0$, i.e. $-\frac{D_1\tilde{u}(a_h, a_h)}{D_2\tilde{u}(a_h, a_h)} = 1$ for all $h = 1, 2, 3$. But the strict quasi-concavity and monotonicity of $\tilde{u}$ guarantees that there is a unique $(\bar{a}, \bar{a})$ satisfying this condition. Thus, necessarily, $a_h = \bar{a}$ and hence a would not be off-diagonal.) The conclusion follows immediately.

References

Aumann, R.J., Subjectivity and correlation in randomized strategies, Journal of Mathematical Economics, 1974, Vol. 1, 67–96.

Aumann, R.J., Correlated equilibrium as an expression of Bayesian rationality, Econometrica, 1987, Vol. 55, 1–18.

Azariadis, C. and R. Guesnerie, Sunspots and cycles, Review of Economic Studies, 1986, Vol. 53, 725–736.

Cass, D. and K. Shell, Do sunspots matter?, Journal of Political Economy, 1983, Vol. 91, 193–227.

Chiappori, P.-A. and R. Guesnerie, On stationary sunspot equilibria of order k, in Economic Complexity: Chaos, Sunspots, Bubbles and Nonlinearity, W. Barnett, J. Geweke and K. Shell (Eds.), 1989.

Forges, F., Sunspot equilibrium as a game theoretical solution concept, in Proceedings of the Sixth International Symposium in Economic Theory and Econometrics, Barnett, W. (Ed.), 1991.

Forges, F. and J. Peck, Correlated equilibrium and sunspot equilibrium, Economic Theory, 1995, Vol. 5, 33–50.

Grandmont, J.-M., Stabilizing competitive business cycles, Journal of Economic Theory, 1986, Vol. 40, 57–76.

Maskin, E. and J. Tirole, Correlated equilibria and sunspots, Journal of Economic Theory, 1987, Vol. 43, 364–373.

Peck, J. and K. Shell, Market uncertainty: correlated and sunspot equilibria in imperfect competitive economies, Review of Economic Studies, 1991, Vol. 58, 1011–29.

Samuelson, P.A., An exact consumption-loan model of interest with or without the social contrivance of money, Journal of Political Economy, 1958, Vol. 66,467–482

Shapley, L. and M. Shubik, Trade using one commodity as medium of payment, Journal of Political Economy, 1977, Vol. 85, 937–968.

Shell, K., Monnaie et allocation intertemporelle, mimeo, 1977.