Coherent and ideal actions in ideally exact categories with an application to varieties of universal algebras

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Group action

An **action** of a group G on a group X is a morphism $\phi\colon G \to \mathrm{Aut}(X)$ in Grp.

- The category of the actions of a group G is equivalent to the category of the split epis over G.
- The equivalence is estabilished via semidirect products.

In 2005, Borceux, Janelidze and Kelly extended the correspondance between **actions** and **split extensions** from the setting of groups to more general *semi-abelian categories*.

Definition

A finitely complete category C is a *category with semidirect products* if, for all $p : E \to B$, the functor

$$p^* \colon \mathrm{Pt}(B) \to \mathrm{Pt}(E)$$

has a left adjoint and is monadic.

Equivalence between **split epis** over B and **algebras on split epis** over E.

$$\begin{tikzcd}
& (X,\xi) \;\in \arrow[dl, mapsto, dashed] & \mathrm{Pt}(E)^{T_p} \arrow[d] \\
(X,\xi)\rtimes(B,p) \;\in & \mathrm{Pt}(B) \arrow[ur, "K"] \arrow[r, "p^*"', "\perp"] & \mathrm{Pt}(E) \arrow[l, bend right]
\end{tikzcd}$$

The $B\flat$-algebras are called **internal actions**.

If C pointed: choose $p : 0 \to B$, $\mathrm{Pt}(0) \cong \mathsf{C}$ and equivalence between **split epis** over B and **algebras on their kernels** in C.

$$\begin{tikzcd}
& (X,\xi) \;\in \arrow[dl, mapsto, dashed] & \mathsf{C}^{B\flat} \arrow[d] \\
(X,\xi)\rtimes B \;\in & \mathrm{Pt}(B) \arrow[ur, "K"] \arrow[r, "\mathrm{ker}"', "\perp"] & \mathsf{C} \arrow[l, bend right]
\end{tikzcd}$$

where $B\flat X = \mathrm{ker}(\, B + X \xrightarrow{[1,0]} B \,)$

Case: non-pointed categories

Problem: Non-pointed categories *don't* have kernels, but we can see them in a bigger category.

A **basic setting** for relative U-ideals is an adjunction $\begin{tikzcd} \mathsf{U} \arrow[r, "U"'] & \mathsf{V} \arrow[l, bend right, "F"', "\perp"] \end{tikzcd}$, where V is homological and U is conservative and faithful.

$k \colon X \to U(A)$ in V is a **relative U-ideal** of an object A in U if

$$\exists \begin{array}{c} A \\ \downarrow f \\ B \end{array} \text{ s.t. } \begin{tikzcd} X \arrow[r, "k"] \arrow[d] & U(A) \arrow[d, "U(f)"] \\ 0 \arrow[r] & U(B) \end{tikzcd} \text{ is a pullback.}$$

- **The varietal case:** $\mathsf{V} = (V, \Sigma_V, Z_V)$, $\mathsf{U} = (U, \Sigma_U, Z_U)$ algebraic varieties, V homological (and hence semi-abelian), $\Sigma_V \subseteq \Sigma_U$, $Z_V \subseteq Z_U$ and $U : \mathsf{U} \to \mathsf{V}$ forgetful functor.

Definition

A category $\mathbb{U}$ is **ideally exact** if it is Barr exact, protomodular, has finite coproducts and $\iota\colon 0 \to 1$ is a regular epi.

Theorem (G. Janelidze)

$\mathbb{U}$ is ideally exact iff it is Barr exact, has finite coproducts and $\exists U : \mathbb{U} \to \mathbb{V}$ monadic, with $\mathbb{V}$ semi-abelian.

- If $\mathbb{U}$ ideally exact, the pullback functor $\mathbb{U} \to (\mathbb{U} \downarrow 0)$ along $0 \to 1$ is monadic
- A monadic adjunction $\begin{tikzcd} \mathbb{U} \arrow[r, "U"', shift right] \arrow[r, phantom, "\perp"] & \mathbb{V} \arrow[l, bend right, "F"'] \end{tikzcd}$, where $\mathbb{U}$ ideally exact and $\mathbb{V}$ semi-abelian, is associated to $0 \to 1$ iff the unit of the adjunction is cartesian.
- A non-trivial algebraic variety is ideally exact iff it is protomodular.

▷ **Ideally exact context**: a monadic adjunction with cartesian unit $\begin{tikzcd} \mathbb{U} \arrow[r, "U"', shift right] \arrow[r, phantom, "\perp"] & \mathbb{V} \arrow[l, bend right, "F"'] \end{tikzcd}$, with $\mathbb{U}$ ideally exact and $\mathbb{V}$ semi-abelian.

A **hoop** is an algebra $(A, \cdot, \rightarrow, 1)$ s.t.

- $x \rightarrow x = 1$
- $x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$
- $x \cdot y \rightarrow z = x \rightarrow (y \rightarrow z)$

A **Wajsberg hoop** is a hoop s.t.

- $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

A **bounded hoop** is a hoop with a constant 0 s.t.

- $0 \rightarrow x = 1$

We denote by Hoops, WHoops and BWHoops the corresponding categories.

- Every hoop is endowed with a partial order $\leqslant$ s.t. $x \leqslant y$ iff $x \rightarrow y = 1$.
- Bounded Wajsberg hoops are term equivalent to *MV-algebras*.
- In BWHoops, initial object: $L_2 = \{0, 1\}$, terminal object: trivial algebra $\{1\}$.

Hoops and WHoops are semi-abelian, while BWHoops is protomodular (and hence ideally exact).

Consider the adjunction
$$\begin{tikzcd} \mathbf{BWHoops} \arrow[r, "U"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathbf{WHoops} \arrow[l, bend right, "M"'] \end{tikzcd}$$
, where

- U is the functor that forgets the constant 0
- M is the *MV-closure*, i.e. it maps every Wajsberg hoop W to the bounded Wajsberg hoop

$$M(W) = (W \times \{0, 1\}, \cdot, \to, 0, 1),$$

where $0 := (1, 0)$, $1 := (1, 1)$,

$$(a,i)\cdot(b,j) = \begin{cases} (a\cdot b, 1), & \text{if } i=j=1,\\ (a\to b, 0), & \text{if } i=1,\ j=0,\\ (b\to a, 0), & \text{if } i=0,\ j=1,\\ ((a\to(a\cdot b))\to b, 0), & \text{if } i=j=0. \end{cases} \qquad (a,i)\to(b,j) = \begin{cases} (a\to b, 1), & \text{if } i=j=1,\\ (a\cdot b, 0), & \text{if } i=1,\ j=0,\\ ((a\to(a\cdot b))\to b, 1), & \text{if } i=0,\ j=1,\\ (b\to a, 1), & \text{if } i=j=0. \end{cases}$$

▻ The unit $\eta : 1_{\mathrm{WHoops}} \Rightarrow UM$ of the adjunction is cartesian, and hence the adjunction is associated with $L_2 \to \{1\}$

▻ It gives an ideally exact context

Definition

Let B be an object of U.

- A split epi $A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$ in V is **ideal** if there exist a split epi $A' \underset{s'}{\overset{p'}{\rightleftarrows}} B$ in U, and an isomorphism $\sigma\colon U(A') \to A$ that induces an isomorphism of points.
- A morphism $h\colon A_1 \to A_2$ of split epis over $U(B)$ is ideal if there exists h' s.t. $h \circ \sigma_1 = \sigma_2 \circ U(h')$.

Definition

Let $\xi\colon U(B)\flat X \to X$ be the action associated with the split epi $A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$, and

$\xi_0\colon UF(0)\flat X \to X$ the action associated with the canonical $UF(X) \underset{UF(\iota)}{\overset{UF(\tau)}{\rightleftarrows}} UF(0)$. We say that

- ξ is a **coherent action** if $\xi \circ (U(\iota)\flat \operatorname{id}_X) = \xi_0$
- ξ is an **ideal action** if the corresponding split epi is ideal.

Lemma

ξ coherent iff $\exists f : UF(X) \to A$ in $\mathbb{V}$ such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, equal] & UF(X) \arrow[r, shift left, "UF(\tau)"] \arrow[d, "f"] & UF(0) \arrow[l, shift left, "UF(\iota)"] \arrow[d, "U(\iota)"] \\
X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & U(B). \arrow[l, shift left, "s"]
\end{tikzcd}$$

is a morphism of split extensions in $\mathbb{V}$. Moreover, the square on the right is a split pullback.

Theorem (M. Mancini, G. Metere, F. P.)

In an ideally exact context, all ideal actions are coherent.

Definition

An ideally exact context admits a **good theory of actions** (it is **BAT**) if all coherent actions are ideal, and all morphisms of such actions are ideal.

Ideally exact context: a varietal example

- U, V: non-trivial varieties, signatures $\Sigma_{\mathsf{U}}, \Sigma_{\mathsf{V}}$, identities $Z_{\mathsf{U}}, Z_{\mathsf{V}}$
- V semi-abelian, unique constant 0; $\quad \Sigma_{\mathsf{V}} \subseteq \Sigma_{\mathsf{U}}, \quad Z_{\mathsf{V}} \subseteq Z_{\mathsf{U}}$
- $\Sigma_{\mathsf{U}} \backslash \Sigma_{\mathsf{V}}$: nullary operations, interpreted as constants
- $\Gamma := I_{\mathsf{U}} \backslash 0_{\mathsf{V}} = I_{\mathsf{U}} \backslash T_{\mathsf{U}}$: constants of U not in V
- $U \colon \mathsf{U} \to \mathsf{V}$ forgets Γ and the identities $Z_{\mathsf{U}} \backslash Z_{\mathsf{V}}$

Remark

U is faithful, conservative, right adjoint to $F \colon \mathsf{V} \to \mathsf{U}$ (freely adds the constants of Γ):

$$\begin{tikzcd} \mathsf{U} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathsf{V} \arrow[l, bend right=40, "F"'] \end{tikzcd}$$

This is the adjunction associated with $I_{\mathsf{U}} \to T_{\mathsf{U}} \quad \Longleftrightarrow \quad \eta \colon 1_{\mathsf{V}} \Rightarrow UF$ is **cartesian**.

▻ It follows that U is a protomodular variety, and therefore an ideally exact category.

Theorem (M. Mancini, F. P.)

The ideally exact context

$$\begin{tikzcd} \mathsf{U} \arrow[r, "U"', ""{name=A}] & \mathsf{V} \arrow[l, bend right, "F"', ""{name=B}] \arrow[from=A, to=B, phantom, "\perp"] \end{tikzcd}$$

is BAT if and only if

$$\omega_A = \omega(\omega_A, \dots, \omega_A, s(c_1^B), \dots, s(c_q^B)),$$

for any coherent split epimorphism $(A, U(B), p, s)$, *for any identity* $\omega = 0$ *in* $Z_{\mathsf{U}} \backslash Z_{\mathsf{V}}$ *and for every* $a_1, \dots, a_p \in A$, *where* ω_A *denotes the element* $\omega(a_1, \dots, a_p, s(c_1^B), \dots, s(c_q^B)) \in A$.

Theorem (M. Mancini, G. Metere, F. P.)

The ideally exact context $\begin{tikzcd} \mathbf{BWHoops} \arrow[r, "U"', ""{name=A}] & \mathbf{WHoops} \arrow[l, bend right, "M"', ""{name=B}] \arrow[from=A, to=B, phantom, "\perp"] \end{tikzcd}$ *is BAT.*

A counterexample

Let BoolGrp be the variety of Boolean groups, i.e. abelian groups satisfying

$$x + x = 0.$$

Let

$$\mathbf{BoolGrp}[u]$$

be the variety obtained by adding one unary operation symbol u, with no identities imposed on u.
Thus an algebra is a Boolean group A together with an arbitrary function

$$u^A : A \to A.$$

The operation u^A is not required to be a group homomorphism.

- The variety = $\mathbf{BoolGrp}[u]$ is ideally exact.

Theorem (M. Abbadini, Z. Janelidze, G. Metere, F. P.)

The canonical ideally exact context associated with $\mathbf{BoolGrp}[u]$ *is not BAT.*

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Thank you for your attention!