

Fertility, Heterogeneity and the Golden Rule*

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Abstract

According to Phelps's (1961) Golden Rule, a rise in fertility decreases the optimal capital intensity, because a higher fertility increases the investment required to sustain a given capital intensity (the capital dilution effect). Using a matrix population model embedded in a two-period OLG setting, we examine the robustness of that relationship to the partitioning of the population into two subpopulations having distinct fertility behaviors and entering the production process as distinct inputs. We show that, unlike what prevails under a homogeneous population, a rise in fertility (caused by a change in type-specific fertility) does not necessarily reduce the Golden Rule capital intensity. The intuition is that changes in type-specific fertility modify the composition of the labor force, which affects the marginal productivity of capital and the capital dilution effect. When the composition effect induced by the fertility change outweighs the standard capital dilution effect prevailing under a fixed partition of the population, a rise in fertility increases the optimal capital intensity. These results are robust to a finer description of heterogeneity, that is, a partitioning of the population into a larger number of subpopulations.

Keywords: Golden Rule, capital accumulation, fertility, OLG models, matrix population models, heterogeneity.

JEL classification codes: E13, E21, E22, J13.

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1 Introduction

According to Phelps's (1961) Golden Rule of capital accumulation, consumption per capita at the stationary equilibrium is maximized when the capital to labour ratio is such that the marginal productivity of capital in intensive terms is equal to the sum of the population growth rate and the depreciation rate of capital. Phelps's Golden Rule states an unambiguous relationship between the optimal accumulation rule and fertility. Everything else being left constant, a rise in the strength of fertility reduces the optimal capital to labour ratio, because a higher fertility increases the investment required to sustain a given capital intensity (i.e., the capital dilution effect or Solow effect, following Solow 1956). On the contrary, a decline in fertility justifies a higher capital intensity.

This result is most relevant when considering optimal accumulation in Western economies, which have exhibited a fertility transition towards low fertility levels since the 1870s. In the light of Phelps's Golden Rule, this declining trend in fertility, by lowering the capital dilution effect, justifies, from a long-run perspective, a higher capital intensity.

Whereas the relationship between the optimal accumulation and fertility is unambiguous in economies with a homogeneous population, one may wonder whether this relationship still holds in economies composed of heterogeneous subpopulations having distinct fertility behaviors. In that more general context, does a decline in total fertility imply a higher optimal capital intensity?

Examining the robustness of the Golden Rule to introducing heterogeneity in fertility is motivated by the increasing attention paid by growth theorists and economic historians to fertility differentials across subpopulations, and to the impact of the resulting dynamics of heterogeneity on long-run economic outcomes. Galor and Moav (2002) and Galor (2011) studied, from a theoretical perspective, the key role played by fertility differentials across subpopulations in long-run economic dynamics. Moreover, economic historians showed evidence of significant fertility differentials across subpopulations, as well as of their effects on various economic outcomes (Clark 2007, Clark and Cummins, 2015, de la Croix et al 2019, Cummins 2020). This motivates us to study the robustness of Phelps's Golden Rule to introducing heterogeneity in fertility.

Although Phelps's (1961) Golden Rule has been reexamined in the context of heterogeneous agents, the existing literature has mainly focused on two dimensions of heterogeneity: (i) heterogeneity in labour endowments; (ii) heterogeneity in preferences. As underlined by Kuhle (2012), Phelps's Golden Rule continues to maximize the society's consumption possibilities (i.e., consumption per capita) under these sources of heterogeneity. The intuition underlying that robustness result is that Phelps's Golden Rule is obtained from maximizing consumption per capita defined from the economy's aggregate resource constraint, which aggregates type-specific labour quantities in total output and type-specific consumptions in total consumption, and, as such, is robust to introducing these sources of heterogeneity. But the existing literature has not so far examined the robustness of the Golden Rule to introducing heterogeneity in fertility.

The goal of this paper is to reexamine the relationship between optimal

capital accumulation and fertility, by studying its robustness to the partitioning of the population into subpopulations having distinct fertility behaviors. For that purpose, we study a matrix population model embedded in a two-period overlapping generations (OLG) setting. In that framework, the population is composed of two subpopulations having their own fertility and mobility patterns, and who act as distinct inputs in the production process. Hence, in that model, fertility differentials across types affect the long-run production possibilities of the economy, through their impact on the composition of the labour force.¹

We derive the Golden Rule of capital accumulation - the level of capital intensity that maximizes consumption per capita at the stationary equilibrium - within that framework, and we show that this extended Golden Rule collapses to Phelps's Golden Rule in the special case where the two subpopulations exhibit the same fertility and mobility patterns. We also show that, unlike what prevails under a homogeneous population, a rise in fertility does not necessarily decrease the optimal capital intensity, but can increase it.² The underlying intuition is that a change in a type-specific fertility rate leads not only to the standard capital dilution effect prevailing under a fixed partition of the population, but, also, to a composition effect, that is, a change in the long-run partition of the population into subpopulations. This composition effect takes two forms: first, a change in the marginal productivity of capital due to a variation in the composition of the labour force; second, a change in the level of the investment required to maintain capital intensity constant, that is, a change in the capital dilution effect. If (i) the composition effect takes a sign opposite to the one of the standard capital dilution effect prevailing under a fixed partition of the population, and (ii) its size is larger, in absolute value, than the standard capital dilution effect, the relationship between optimal capital and fertility can be inverted in comparison to Phelps's original setting.

At this early stage, it should be stressed that the results of this paper are general, and do not have to assume a particular cause of heterogeneity of fertility within the population. Actually, heterogeneity in fertility behaviors among the population could have various causes: rural *versus* urban, skilled *versus* unskilled, traditional *versus* modern, religious *versus* secular, etc. Obviously, each particular kind of heterogeneity would lead to a particular parametrization of the model, with some specific values assigned to structural parameters of the economy. But the main conclusion of the paper - the general relationship between Golden Rule capital and fertility under heterogeneity in fertility - does

¹As such, our framework allows for (potential) differences in productivity and/or in labour supply across types of workers. The relation between the Golden Rule and heterogeneous labour endowments being studied in details in Kuhle (2012), we will leave it aside here and focus instead on heterogeneity in fertility differentials.

²These results are obtained while defining 'capital intensity' as the ratio of the capital stock to the number of workers of type 2 in the baseline model (Sections 2 to 4) or, alternatively, as the ratio of the capital stock to the number of workers of type 3 in the extended model (Section 5). It should be stressed here that our main result - the ambiguous relation between optimal capital intensity and fertility - does not depend on a particular definition of 'capital intensity': adopting alternative normalizations would just affect the form of formulae (i.e., the precise formulation of the composition effect), but without modifying our results.

not need to assume a particular cause of heterogeneity in fertility behaviors.

This paper is related to the literature extending Phelps's Golden Rule of capital accumulation in various directions, concerning, among others, the production technology, the demography or other fundamental characteristics of the economy (Phelps 1965, Davis 1969, Cass 1972, Zilcha 1990, Galor and Ryder 1991, de la Croix and Ponthiere 2010, Kuhle 2012, Mertens and Rubinchik 2015, Pestieau and Ponthiere 2014, Strulik 2021). With respect to that literature, the contribution of this paper is to introduce heterogeneity in fertility patterns, and to characterize an extended Golden Rule in that setting. By focusing on the robustness of Phelps's Golden Rule to introducing heterogeneity in fertility, the present paper is particularly complementary to Kuhle (2012), who focused on heterogeneity in other dimensions (labour endowments and preferences).

The paper is organized as follows. The model is presented in Section 2. Section 3 derives the optimal accumulation policy. The relationship between optimal capital and fertility is reexamined in Section 4. Section 5 studies the robustness of our results to a finer description of heterogeneity, by extending the framework to the partition of the population into 3 subpopulations. Section 6 concludes.

2 The model

Let us consider a two-period OLG economy with physical capital accumulation, where time is discrete, and goes from 0 to $+\infty$. A human life is composed of two periods of adulthood.³ The first period is young adulthood, during which individuals consume and save for old days, while the second period is the old age, which is here assumed to be reached with certainty.⁴ There is here no labour during the old age.

While the present framework shares many features with OLG models with physical capital (see de la Croix and Michel 2002), a key difference with respect to the standard OLG model is that the population is here partitioned in two subpopulations 1 and 2, who contribute to the production process as two distinct labour inputs.

Population heterogeneity Let us denote by N_t^i the number of young adults of type i at time t . The young adult population at period t can be represented by the vector:

$$\mathbf{N}_t = \begin{pmatrix} N_t^1 \\ N_t^2 \end{pmatrix}$$

The partition of the young adult population into subpopulations varies over time, depending on fertility behaviors and social mobility across subpopulations. Let us denote by n^i the average number of children born from a young adult

³As usual in OLG models, there is an implicit childhood period. See de la Croix and Michel (2002).

⁴See de la Croix and Ponthiere (2010) on the Golden Rule under risky lifetime.

of type i . These children of type i may or may not keep their type i once they become adults, depending on the strength of social mobility across types. Let us denote by $m^i \in]0, 1[$ the probability for a child born from an individual of type i to escape from subpopulation i at the adult age, and to join the other subpopulation $j \neq i$. We denote by $\bar{m}^i$ the probability that a child born from an individual of type i remains in the subpopulation i at the adult age.⁵

Taken together, the parameters $\{n^1, n^2, m^1, m^2\}$ determine the dynamics of the structure of the population. To see this, let us define the matrix $\mathbf{M}$ as:

$$\mathbf{M} = \begin{pmatrix} n^1 \bar{m}^1 & n^2 m^2 \\ n^1 m^1 & n^2 \bar{m}^2 \end{pmatrix}$$

The matrix $\mathbf{M}$ can be used to obtain the structure of the young adult population at period $t + 1$ from the structure of the adult population at period t :

$$\mathbf{M}\mathbf{N}_t = \mathbf{N}_{t+1} \quad (1)$$

This expression can be rewritten in detailed form as:

$$\begin{pmatrix} n^1 \bar{m}^1 & n^2 m^2 \\ n^1 m^1 & n^2 \bar{m}^2 \end{pmatrix} \begin{pmatrix} N_t^1 \\ N_t^2 \end{pmatrix} = \begin{pmatrix} N_{t+1}^1 \\ N_{t+1}^2 \end{pmatrix} \quad (2)$$

Let us write the number of young adults of type 1 at time $t + 1$ as follows:

$$N_{t+1}^1 = N_t^1 n^1 \bar{m}^1 + N_t^2 n^2 m^2 \quad (3)$$

The number of young adults of type 1 at time $t + 1$, N_{t+1}^1 , has two components, which are the two terms of the right-hand side (RHS): on the one hand, the number of children who were born from an individual of type 1 at t , who remained in subpopulation 1 once adult (first term of the RHS), and, on the other hand, the number of children who were born from an individual of type 2 at t , and joined subpopulation 1 once adult (second term of the RHS). Figure 1 presents the life cycle graph of our model.⁶

⁵We have $\bar{m}^i = 1 - m^i$. Whereas the reliance on the notation $\bar{m}^i$ is not necessary from a mathematical perspective, the use of the notation $\bar{m}^i$ is standard in matrix population models, because this allows to shorten and simplify formulae (see Caswell 2001).

⁶The life cycle graph deliberately omits the old age, since it is assumed that individuals, once adults, keep their type during the remaining of their life. Our analysis of heterogeneity can thus focus only on young adults.

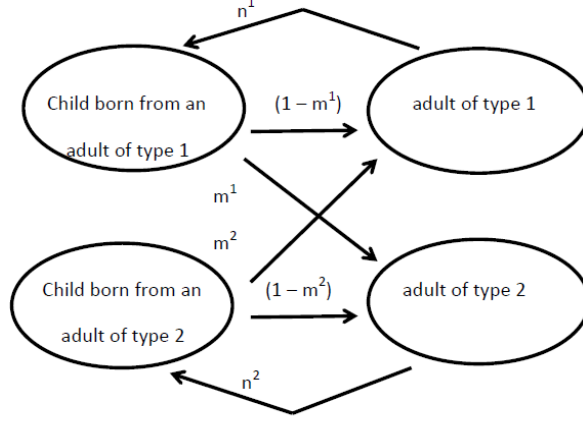


Figure 1: The life cycle graph.

Production The production of an output Y_t involves capital K_t and two types of labor L_t^1 and L_t^2 , according to the function:

$$Y_t = F(K_t, L_t^1, L_t^2) \quad (4)$$

We assume that all young adults supply one unit of labour of their type, so that $L_t^1 = N_t^1$ and $L_t^2 = N_t^2$. The production function $F(K_t, L_t^1, L_t^2)$ is supposed to be increasing in its three arguments and homogeneous of degree 1.

Dividing the output by N_t^2 , the production process can be rewritten in intensive terms as:⁷

$$y_t = F(k_t, x_t, 1) \equiv f(k_t, x_t) \quad (5)$$

where $y_t \equiv \frac{Y_t}{N_t^2}$ is output per worker of type 2, $k_t \equiv \frac{K_t}{N_t^2}$ is capital stock per worker of type 2, and $x_t \equiv \frac{N_t^1}{N_t^2}$ is the ratio of the number of workers of the two types. The variable x_t synthesizes the composition of the population of young adults at time t .

The capital stock depreciates at a constant rate δ , with $0 < \delta < 1$.

The resource constraint of the economy is:

$$F(K_t, L_t^1, L_t^2) = c_t^1 L_t^1 + d_t^1 L_{t-1}^1 + c_t^2 L_t^2 + d_t^2 L_{t-1}^2 + K_{t+1} - (1 - \delta)K_t \quad (6)$$

where c_t^1 (resp. d_t^1) denotes the consumption of young (resp. old) individuals of type 1, and c_t^2 (resp. d_t^2) denotes the consumption of young (resp. old) individuals of type 2.

⁷Note that alternative ways of rewriting variables in intensive terms would only lead to introducing other notations, without changing our results. To show this, Section 5 examines an extended framework with three subpopulations, where the normalization used is a division of variables by the number of young adults of type 3.

Dividing the LHS and RHS of the resource constraint by N_t^2 , and noting that $x_t \equiv \frac{N_t^1}{N_t^2}$, the resource constraint can be rewritten as:

$$f(k_t, x_t) = c_t^1 x_t + d_t^1 \frac{N_{t-1}^1}{N_t^2} + c_t^2 + d_t^2 \frac{N_{t-1}^2}{N_t^2} + \frac{K_{t+1}}{N_t^2} - (1 - \delta) \frac{K_t}{N_t^2} \quad (7)$$

Noting that $N_t^2 = N_{t-1}^2 n^2 \bar{m}^2 + N_{t-1}^1 n^1 m^1$, we have:

$$\begin{aligned} \frac{N_{t-1}^1}{N_t^2} &= \frac{N_{t-1}^1}{N_{t-1}^2 n^2 \bar{m}^2 + N_{t-1}^1 n^1 m^1} = \frac{1}{\frac{n^2 \bar{m}^2}{x_{t-1}} + n^1 m^1} \\ \frac{N_{t-1}^2}{N_t^2} &= \frac{N_{t-1}^2}{N_{t-1}^2 n^2 \bar{m}^2 + N_{t-1}^1 n^1 m^1} = \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1} \end{aligned}$$

Hence, since $\frac{N_{t+1}^2}{N_t^2} = \frac{N_t^2 n^2 \bar{m}^2 + N_t^1 n^1 m^1}{N_t^2} = n^2 \bar{m}^2 + x_t n^1 m^1$, the resource constraint can be written as:

$$y_t = c_t^1 x_t + \frac{d_t^1}{\frac{n^2 \bar{m}^2}{x_{t-1}} + n^1 m^1} + c_t^2 + \frac{d_t^2}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1} + k_{t+1} (n^2 \bar{m}^2 + x_t n^1 m^1) - (1 - \delta) k_t \quad (8)$$

Stationary state Let us now characterize a stationary equilibrium in our economy. Throughout this paper, we focus on stationary states with a constant composition of the population, that is, with a constant ratio $x_t \equiv \frac{N_t^1}{N_t^2} = x_{t+1}$. The long-run partition of the population into subpopulations can be studied by examining the properties of the matrix $\mathbf{M}$. Let us first notice the following property of matrix $\mathbf{M}$.

Proposition 1 *The matrix $\mathbf{M}$ is irreducible and primitive.*

Proof. See the Appendix. ■

Proposition 1 allows us to use both the Perron-Frobenius Theorem and the Strong Ergodic Theorem for the analysis of the long-run composition of the young adult population. The Perron-Frobenius Theorem states that, under conditions of irreducibility and primitivity of a non-negative matrix, there exists in general one eigenvalue, called the "dominant eigenvalue", that is larger than or equal to any of the other eigenvalue of that matrix (Caswell 2001). According to the Strong Ergodic Theorem, that dominant eigenvalue of the population matrix determines the ergodic properties of population growth. The Strong Ergodic Theorem states that if the population matrix is primitive, then, regardless of initial conditions, the population will, in the long-run, grow at a rate given by the dominant eigenvalue, with a stable population structure proportional to the eigen vector associated to that eigenvalue (the influence of other eigenvalues being negligible).⁸

⁸See Caswell (2001), p. 84-85.

Building on Proposition 1, Proposition 2 gives us the long-run interior partition of the young adult population, as well as the associated ratio x .⁹

Proposition 2 *The structure of the young adult population converges asymptotically towards a stable structure. The long-run structure of the young adult population is defined, up to a constant $c > 0$, by:*

$$\begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = \begin{pmatrix} c \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2n^1 - (n^1 \bar{m}^1 + n^2 \bar{m}^2) + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}} \\ c \frac{2n^1 (1 - \bar{m}^1)}{2n^1 - (n^1 \bar{m}^1 + n^2 \bar{m}^2) + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}} \end{pmatrix}$$

while the associated long-run ratio x is

$$x = \frac{N^1}{N^2} = \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2n^1 (1 - \bar{m}^1)}$$

Proof. See the Appendix. ■

Proposition 2 provides a closed-form solution for the long-run interior partition of the young adult population into subpopulations. The long-run partition of the young adult population does not depend on the level of initial conditions. Whatever the economy involves initially a small or a large fraction of the population belonging to, let us say, subpopulation 1, this has no effect on the long-run partition of the population. The long-run ratio x depends only on the parameters $\{n^1, n^2, m^1, m^2\}$ describing type-specific fertility and social mobility. The fact that the long-run level of x does not depend on initial conditions follows from the application of the Strong Ergodic Theorem to the particular context under study. The Strong Ergodic Theorem provides conditions on a population process under which the structure of the population stabilizes asymptotically independently of the initial structure of the population.

As a consequence of the stationarity of x_t , the resource constraint of the economy can, at the stationary equilibrium, be written as:

$$f(k, x) = c^1 x + \frac{d^1 x}{n^2 \bar{m}^2 + x n^1 m^1} + c^2 + \frac{d^2}{n^2 \bar{m}^2 + x n^1 m^1} + k (n^2 \bar{m}^2 + x n^1 m^1) - (1 - \delta)k \quad (9)$$

where $n^2 \bar{m}^2 + x n^1 m^1$ is the growth factor of the number of young adults of type 2 across two successive periods.¹⁰ That growth factor depends on the ratio of the two types of individuals, on type-specific fertility rates and on type-specific probabilities of having (or not) children of the same type as the type of parents.

⁹It should be stressed here that, if the model included no social mobility across types (i.e., if $m^1 = m^2 = 0$), the matrix $\mathbf{M}$ would not be irreducible any more, and Proposition 1 would not prevail. In that case (which is excluded by our assumptions), the long-run partition of the population into subpopulations would be, in general, a corner solution, with the population being composed entirely of a unique subpopulation. This corner case being rather special, our framework assumes social mobility across types. This assumption allows us to use the Frobenius Theorem and the Strong Ergodic Theorem to characterize a close-form solution for the long-run interior partition of the population into subpopulations.

¹⁰To see this, note that $n^2 \bar{m}^2 + x n^1 m^1$ can be rewritten as: $\frac{N_t^2 n^2 \bar{m}^2 + N_t^1 n^1 m^1}{N_t^2}$.

3 Optimal accumulation policy

From the economy's resource constraint, total consumption is:

$$C_t = F(K_t, L_t^1, L_t^2) - K_{t+1} + (1 - \delta)K_t \quad (10)$$

Consumption per capita at time t is defined as total consumption divided by the total population size at t :¹¹

$$c_t \equiv \frac{C_t}{N_t^1 + N_t^2 + N_{t-1}^1 + N_{t-1}^2}$$

This can be rewritten as:

$$c_t = \frac{\frac{C_t}{N_t^2}}{x_t + 1 + \frac{N_{t-1}^1}{N_t^2} + \frac{N_{t-1}^2}{N_t^2}} = \frac{\frac{C_t}{N_t^2}}{x_t + 1 + x_{t-1} \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1}}$$

Note that consumption per young individual of type 2 is:

$$\frac{C_t}{N_t^2} = f(k_t, x_t) - (n^2 \bar{m}^2 + x_t n^1 m^1) k_{t+1} + (1 - \delta) k_t$$

Thus, consumption per capita can be written as follows:

$$c_t = \frac{f(k_t, x_t) - (n^2 \bar{m}^2 + x_t n^1 m^1) k_{t+1} + (1 - \delta) k_t}{x_t + 1 + x_{t-1} \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1}} \quad (11)$$

Consumption per capita is a fraction $\frac{1}{x_t + 1 + x_{t-1} \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1}}$ of consumption per young individual of type 2. That fraction depends on the composition of the population among the young (i.e. x_t) and among the old (i.e. x_{t-1}).

At the steady-state, the composition of the population is a constant x , so that consumption per capita c can be defined as the function $\phi(k)$:

$$c = \phi(k) \equiv \frac{f(k, x) - (n^2 \bar{m}^2 + x n^1 m^1) k + (1 - \delta) k}{x + 1 + x \frac{1}{n^2 \bar{m}^2 + x n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x n^1 m^1}} \quad (12)$$

where

$$x = \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4 n^1 n^2 (1 - m^2 - m^1)}}{2 n^1 (1 - \bar{m}^1)}$$

is the equilibrium composition of the young adult population.

¹¹Here again, we deliberately ignore consumption during childhood, so that consumption per capita is equal to consumption per adult.

In order to discuss the conditions necessary and sufficient for the existence of a Golden Rule capital level, let us first differentiate consumption per capita $\phi(k)$ with respect to capital:

$$\phi'(k) = \frac{f_k(k, x) - (n^2\bar{m}^2 + xn^1m^1) + (1 - \delta)}{x + 1 + x \frac{1}{n^2\bar{m}^2 + xn^1m^1} + \frac{1}{n^2\bar{m}^2 + x_{t-1}n^1m^1}} \quad (13)$$

The expression $\phi'(k) = 0$ defines an interior Golden Rule capital level only if $\phi(k)$ is neither always decreasing in k (implying that the capital level maximizing $\phi(k)$ is 0), nor always increasing in k (implying that the level of k maximizing $\phi(k)$ is infinite). The interiority of the solution requires the following condition, which guarantees that $\phi'(k)$ is positive when k tends to 0, but negative when it tends to $+\infty$.

Proposition 3 *Assume that:*

$$\begin{aligned} & \lim_{k \rightarrow 0+} f_k \left(k, \frac{n^1\bar{m}^1 - n^2\bar{m}^2 + \sqrt{(n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)}}{2n^1m^1} \right) \\ & > \frac{n^1\bar{m}^1 + n^2\bar{m}^2 + \sqrt{(n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)}}{2} - 1 + \delta \end{aligned}$$

and

$$\begin{aligned} & \lim_{k \rightarrow +\infty} f_k \left(k, \frac{n^1\bar{m}^1 - n^2\bar{m}^2 + \sqrt{(n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)}}{2n^1m^1} \right) \\ & < \frac{n^1\bar{m}^1 + n^2\bar{m}^2 + \sqrt{(n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)}}{2} - 1 + \delta \end{aligned}$$

Then there exists a level of k^{GR} that maximizes steady-state consumption per capita. That level k^{GR} satisfies:

$$\begin{aligned} & f_k \left(k^{GR}, \frac{n^1\bar{m}^1 - n^2\bar{m}^2 + \sqrt{(n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)}}{2n^1m^1} \right) \\ & = \frac{n^1\bar{m}^1 + n^2\bar{m}^2 + \sqrt{(n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)}}{2} - 1 + \delta \end{aligned}$$

Proof. See the Appendix. ■

Proposition 3 characterizes an extended version of Phelps's Golden Rule of capital accumulation for OLG economies having a population partitioned in two subpopulations with distinct fertility and mobility patterns. Like the standard Golden Rule, it states that the capital stock in intensive terms maximizing consumption possibilities at the stationary equilibrium equalizes the marginal

productivity of capital in intensive terms (here, per young individual of type 2) (LHS of the condition) with a sum of two terms (RHS): on the one hand, the growth rate of the subpopulation of young individuals of type 2, which is:

$$\frac{n^1 \bar{m}^1 + n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2} - 1$$

and, on the other hand, the depreciation rate of capital δ .

In order to see how the accumulation rule stated in Proposition 3 generalizes Phelps's Golden Rule, it is useful to examine a special case of our economy where the two subpopulations 1 and 2 exhibit the same structural parameters.

Proposition 4 *Consider an economy where the two subpopulations 1 and 2 exhibit the same structural parameters (i.e., $n^1 = n^2 = n$ and $m^1 = m^2 = m$). The rule for optimal capital accumulation stated in Proposition 3 becomes:*

$$f'_k(k^{GR}, 1) = n - 1 + \delta$$

that is, Phelps's Golden Rule.

Proof. Under $n^1 = n^2 = n$ and $m^1 = m^2 = m$, the stationary composition of the young adult population x is:

$$x = \frac{n\bar{m} - n\bar{m} + \sqrt[2]{(n\bar{m} + n\bar{m})^2 - 4nn(1 - m - m)}}{2nm} = 1$$

Moreover, under $n^1 = n^2 = n$ and $m^1 = m^2 = m$, we have also:

$$\frac{n\bar{m} + n\bar{m} + \sqrt[2]{(n\bar{m} + n\bar{m})^2 - 4nn(1 - m - m)}}{2} = n$$

Substituting for these terms in the accumulation rule of Proposition 3, one obtains:

$$f'_k(k, 1) = n - 1 + \delta$$

that is, Phelps's Golden Rule. ■

In the special case where the two subpopulations 1 and 2 have the same fertility and mobility patterns, the extended Golden Rule collapses to the equalization of the marginal productivity of capital to the population growth rate *plus* the depreciation rate of capital. But this special case - homogeneous subpopulations - constitutes a kind of exception from the perspective of long-run analysis. The case of heterogeneous subpopulations is the general case. In that general context, a key determinant of long-run dynamics concerns how the partition of the population into subpopulations evolves over time.

Proposition 3 tells us how the dynamics of heterogeneity affects the optimal accumulation policy. The extended Golden Rule states that the optimal capital k^{GR} depends on type-specific fertility rates (n^1 and n^2) and on type-specific mobility rates (m^1 and m^2), which determine the long-run partition of the young

adult population into subpopulations. The reason why the long-run partition x matters for optimal capital accumulation lies in the fact that the two subpopulations 1 and 2 enter the production process as two distinct production factors, and affect also the capital dilution effect in distinct ways. As a consequence, the long-run consumption possibilities of the economy depend, for a given capital intensity, on the exact proportions of the two subpopulations 1 and 2 in the long run. The level of k maximizing long-run consumption possibilities depends also on the long-run partition of the population into subpopulations 1 and 2. The precise influence of the long-run composition of the population on k^{GR} varies with the specific form taken by the production process, that is, the form of $f(k_t, x_t)$. The next section will examine that influence by studying the effect of a change in type-specific fertility on the optimal capital accumulation.

4 Fertility and optimal accumulation

Let us now examine how fertility affects the optimal accumulation policy.

Within Phelps's Golden Rule, the relationship between the optimal capital intensity and fertility is straightforward. Given the classical Golden Rule:

$$f'_k(k^{GR}, 1) = n - 1 + \delta$$

it follows that, the higher the fertility rate n is, the lower the optimal capital k^{GR} is. The reason why a rise in fertility reduces the optimal capital k^{GR} lies in the fact that a higher fertility raises the hypothetical investment required to maintain a given capital intensity constant (i.e. the capital dilution effect or Solow effect), which makes accumulation less desirable at the margin.

To see why a higher fertility decreases the optimal capital, remind that consumption per capita at the stationary equilibrium is, in the absence of heterogeneity (i.e., $n^1 = n^2 = n$, $m^1 = m^2 = m$ and $x = 1$):

$$c = \frac{f(k, 1) - (n\bar{m} + nm)k + (1 - \delta)k}{2 + \frac{1}{n\bar{m} + nm} + \frac{1}{n\bar{m} + nm}} = \frac{n}{2(1 + n)} [f(k, 1) - nk + (1 - \delta)k]$$

Differentiating c with respect to k and equalizing this to 0 yields:

$$\frac{\partial c}{\partial k} = \frac{n}{2(1 + n)} [f_k(k^{GR}, 1) - n + 1 - \delta] = 0$$

from which one can see that, in order to have the condition for optimal accumulation satisfied, a rise in fertility n requires a reduction of the capital intensity. Thus, *ceteris paribus*, a higher fertility reduces the optimal capital k^{GR} .

But that result is no longer necessarily true once one considers a more general setting where the population is composed of heterogeneous subpopulations. In the context of a heterogeneous population, an increase in a type-specific fertility rate n^1 or n^2 does not necessarily reduce the optimal capital k^{GR} .

To see why a rise in a type-specific fertility rate may increase the optimal capital in that general setting, remind that consumption per capita at the stationary equilibrium is:

$$\begin{aligned} c &= \frac{f(k, x) - (n^2 \bar{m}^2 + x n^1 m^1) k + (1 - \delta)k}{x + 1 + x \frac{1}{n^2 \bar{m}^2 + x n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x n^1 m^1}} \\ &= \frac{n^2 \bar{m}^2 + x n^1 m^1}{(1 + x)(1 + n^2 \bar{m}^2 + x n^1 m^1)} [f(k, x) - (n^2 \bar{m}^2 + x n^1 m^1) k + (1 - \delta)k] \end{aligned}$$

where the stationary composition of the young adult population x is:

$$x = \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2n^1 m^1}$$

Differentiating consumption per capita with respect to k and equalizing to 0 yields, after simplifications:

$$f_k(k^{GR}, x) = n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta$$

When considering the impact of a change in the fertility rate n^1 or n^2 on k^{GR} , two effects must be taken into account. On the one hand, the effect of a change in fertility on the investment required to maintain capital intensity constant *for a given composition x of the labour force* (as in the special case studied above), and, on the other hand, the effect of a change in fertility on the steady-state composition of the young adult population x .

That second effect - that can be called the "composition effect" - was absent in the case of a homogeneous population, but plays a key role in an economy with a heterogeneous population. The composition effect takes a double form: first, a change in the composition x affects the marginal productivity of capital intensity (LHS of the Golden Rule equation), and, also, it modifies the investment required to maintain capital intensity constant (second term of the RHS of the Golden Rule equation). These two new effects may outweigh the standard capital dilution effect prevailing under a fixed partition x that was already present in Phelps's Golden Rule, and, hence, modify the relation between optimal capital and fertility.

In order to examine the effect of changing a type-specific fertility rate n^i on the optimal capital k^{GR} , a first, preliminary step, consists of studying the sign of the effect of a change in a type-specific fertility rate n^i on the stationary composition of the population x . This is the task of Proposition 5.

Proposition 5 *Consider a marginal change in a type-specific fertility rate n^i on the stationary composition of the young adult population x . We have:*

$$\begin{aligned} \frac{\partial x}{\partial n^1} &\geq 0 \iff \frac{x}{1+x} \geq m^2 \\ \frac{\partial x}{\partial n^2} &\geq 0 \iff \frac{x}{1+x} \leq m^2 \end{aligned}$$

Proof. See the Appendix. ■

Proposition 5 states a necessary and sufficient condition under which a rise in a type-specific fertility rate n^i increases the long-run proportion of type 1 in the young adult population. Intuitively, one would expect that, *ceteris paribus*, an increase in the fertility rate of, let us say, type 1, increases the prevalence of the subpopulation of type 1 in the long run. However, this result does not necessarily hold. Actually, this result holds if and only if the proportion of type-1 individuals in the young adult population is sufficiently large in comparison to the mobility of type-2 individuals towards type 1. When the mobility of type 2 to type 1 is high, this result may not prevail. In a similar vein, a rise of n^2 does not necessarily reduce the proportion of type-1 individuals in the long run.

The conditions stated in Proposition 5 can be interpreted by means of marginalist reasoning. For that purpose, let us focus on the second condition, and let us rewrite it as: $\frac{\partial x}{\partial n^2} \geq 0 \iff \frac{1}{1+x} \geq \bar{m}^2$.¹² The term $\frac{1}{1+x}$ is the probability, at the equilibrium, of being a young individual of type 2, that is, $\frac{N^2}{N^1+N^2}$. A marginal rise in n^2 can be understood as an infinitely small increase in the number of individuals born from type-2 individuals. Moreover, the parameter $\bar{m}^2$ is the probability, for a child born from an individual of type 2, to remain of type 2 at adulthood. The infinitely small rise in the number of children born from individuals of type 2 leaves the equilibrium proportion of type 2 individuals in the young adult population unchanged if and only if the probability that these added children remain of type 2 at adulthood, i.e., $\bar{m}^2$, is exactly equal to the proportion of type-2 individuals in the young adult population $\frac{1}{1+x}$, that is, if and only if $\bar{m}^2 = \frac{1}{1+x}$. However, if the probability that these added children remain of type 2 at adulthood, i.e., $\bar{m}^2$, is higher than the proportion of type 2 individuals in the young adult population, that is, if $\bar{m}^2 > \frac{1}{1+x}$, the addition of a number of children born from type-2 parents increases the proportion of type 2, implying a decline in x , that is, $\frac{\partial x}{\partial n^2} < 0$.¹³

In the light of Proposition 5, it appears that a variation in a type-specific fertility rate has ambiguous effects on the long-run partition of the population into subpopulations, depending on the strength of social mobility across types. A rise in a type-specific fertility rate does not necessarily increase the long-run proportion of individuals of that type within the population. The sign of the composition effect induced by a change in a type-specific fertility rate being ambiguous, the total effect of a variation of a type-specific fertility rate on the Golden Rule capital level is even more complicated to study.

In order to study analytically the effect of a variation in a type-specific fertility rate n^i on the Golden Rule capital level k^{GR} , let us first assume a Cobb-Douglas production function:

$$F(K_t, L_t^1, L_t^2) = AK_t^\alpha (L_t^1)^\beta (L_t^2)^{1-\alpha-\beta} \implies f(k_t, x_t) = Ak_t^\alpha x_t^\beta \quad (14)$$

¹²The first condition of Proposition 5 can also be interpreted in a similar, marginalist, fashion.

¹³On the contrary, when $\bar{m}^2 < \frac{1}{1+x}$, the addition of children born from type-2 parents decreases the equilibrium proportion of type 2 in the young adult population, leading to a rise in x , that is, $\frac{\partial x}{\partial n^2} > 0$.

where $A > 0$, $0 < \alpha < 1$ is the elasticity of output with respect to capital, and $0 < \beta < 1$ is the elasticity of output with respect to labour of workers of type 1. Without loss of generality, we assume $\alpha + \beta < 1$ (so as to allow for a positive productivity of type-2 workers).

Under that production technology, the Golden Rule becomes:

$$A\alpha k^{\alpha-1}x^\beta = (n^2\bar{m}^2 + xn^1m^1) - 1 + \delta \quad (15)$$

implying that the Golden Rule capital level k^{GR} is:

$$k^{GR} = \left[\frac{A\alpha x^\beta}{(n^2\bar{m}^2 + xn^1m^1) - 1 + \delta} \right]^{\frac{1}{1-\alpha}} \quad (16)$$

If the long-run composition of the young adult population x were independent from type-specific fertility rates, a rise in a type-specific fertility rate would necessarily reduce the Golden Rule capital. However, things are less clear once one allows for adjustments in the composition of the population induced by a variation in type-specific fertility.

Proposition 6 states that the impact of a rise in a type-specific fertility rate on the Golden Rule capital is ambiguous. Thus, unlike in the basic setting with a homogeneous population, a rise in fertility does not necessarily reduce k^{GR} .

Proposition 6 *Consider an economy with a Cobb-Douglas technology: $f(k_t, x_t) = Ak_t^\alpha x_t^\beta$. We have:*

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^1} &\geq 0 \iff \frac{\partial x}{\partial n^1} [\beta (n^2\bar{m}^2 + \delta - 1) - (1 - \beta)n^1m^1x] \geq (x)^2 m^1 \\ \frac{\partial k^{GR}}{\partial n^2} &\geq 0 \iff \frac{\partial x}{\partial n^2} [\beta (n^2\bar{m}^2 + \delta - 1) - (1 - \beta)n^1m^1x] \geq x\bar{m}^2 \end{aligned}$$

where $\frac{\partial x}{\partial n^1} \geq 0 \iff \frac{x}{1+x} \geq m^2$ and where $\frac{\partial x}{\partial n^2} \geq 0 \iff \frac{x}{1+x} \leq m^2$.

Proof. See the Appendix. ■

Proposition 6 points to a major departure in comparison to what prevails in an economy with a homogeneous population. In the latter, a rise in fertility reduces the Golden Rule capital k^{GR} , because of the capital dilution effect. However, in a model with a heterogeneous population, a rise in a type-specific fertility rate (leading to a rise in total fertility) may either decrease or increase k^{GR} . This result is due to the fact that, once heterogeneity in labour is introduced, a variation of a type-specific fertility rate modifies the long-run composition of the young adult population, which affects also long-run consumption possibilities through two distinct channels: first, modifying the marginal productivity of capital, and second, modifying also the capital dilution effect. This composition effect can, in some cases, have a sign opposite to the one of the standard capital dilution effect prevailing under a given composition of the labour force, so that the relation between the optimal capital and fertility is inverted.

In Proposition 6, the LHS of the conditions captures the composition effect caused by a change in fertility, whereas the RHS corresponds to the standard

capital dilution effect caused by a variation in fertility (when the composition of the labour force is left unchanged). If the composition effect were absent (that is, if $\frac{\partial x}{\partial n^1} = \frac{\partial x}{\partial n^2} = 0$, which requires that the equality $\frac{x}{1+x} = m^2$ is satisfied), the conditions of Proposition 6 would imply that only the standard capital dilution effect would be at work, so that a rise in a type specific fertility rate would necessarily reduce the Golden Rule capital, that is, $\frac{\partial k^{GR}}{\partial n^1} < 0$ and $\frac{\partial k^{GR}}{\partial n^2} < 0$. But in the general case where there is a non-zero composition effect, a variation of a type-specific fertility rate may either increase or decrease the Golden Rule capital.

Two main forces drive the sign of $\frac{\partial k^{GR}}{\partial n^1}$ (resp. $\frac{\partial k^{GR}}{\partial n^2}$). First, the sign and extent of the derivative $\frac{\partial x}{\partial n^1}$ (resp. $\frac{\partial x}{\partial n^2}$), which was studied in Proposition 5. Assuming that $m^2 < \frac{x}{1+x}$, we have $\frac{\partial x}{\partial n^1} > 0$ (resp. $\frac{\partial x}{\partial n^2} < 0$). But it is only if the composition effect is sufficiently large that a rise in a type-specific fertility rate can increase the Golden Rule capital level, not otherwise. Second, the sign of $\frac{\partial k^{GR}}{\partial n^1}$ (resp. $\frac{\partial k^{GR}}{\partial n^2}$) depends also on the sign of the factor $[\beta(n^2\bar{m}^2 + \delta - 1) - (1 - \beta)n^1m^1x]$, which is ambiguous. The sign of that factor depends on the level of β , the elasticity of output with respect to the labour of type 1. The higher the elasticity β is, the larger that factor is, which, under $\frac{\partial x}{\partial n^1} > 0$ (resp. $\frac{\partial x}{\partial n^2} < 0$) supports $\frac{\partial k^{GR}}{\partial n^1} > 0$ (resp. $\frac{\partial k^{GR}}{\partial n^2} < 0$). The factor $[\beta(n^2\bar{m}^2 + \delta - 1) - (1 - \beta)n^1m^1x]$ is also increasing in the depreciation rate of capital δ , which pushes towards a positive effect of fertility on k^{GR} .

Whereas the previous results are obtained under a Cobb-Douglas technology, it is possible to consider a more general production function: a three-inputs nested CES production function:

$$Y_t = A \left[\alpha K_t^{-\rho} + (1 - \alpha) \left(\beta (L_t^1)^{-\sigma} + (1 - \beta) (L_t^2)^{-\sigma} \right)^{\frac{\rho}{\sigma}} \right]^{-\frac{1}{\rho}} \quad (17)$$

where the elasticity of substitution between the two nested inputs (labour of types 1 and 2) - known as the McFadden (1963) elasticity - is denoted by $\frac{1}{1+\sigma}$, while the elasticity of substitution between the nested inputs and the third input (capital) - known as the Uzawa (1962) elasticity of substitution - is $\frac{1}{1+\rho}$.

Dividing output by L_t^2 , the production can be rewritten in intensive form as:

$$\begin{aligned} y_t &= \frac{Y_t}{L_t^2} = A (L_t^2)^{-\frac{\rho}{\sigma}} \left[\alpha K_t^{-\rho} + (1 - \alpha) \left(\beta (L_t^1)^{-\sigma} + (1 - \beta) (L_t^2)^{-\sigma} \right)^{\frac{\rho}{\sigma}} \right]^{-\frac{1}{\rho}} \\ &= A \left[\alpha k_t^{-\rho} + (1 - \alpha) \left(\beta (x_t)^{-\sigma} + (1 - \beta) \right)^{\frac{\rho}{\sigma}} \right]^{-\frac{1}{\rho}} \end{aligned}$$

Using the equality $f_k(k^{GR}, x) = n^2\bar{m}^2 + xn^1m^1 - 1 + \delta$, the Golden Rule

capital is here given by:

$$k^{GR} = \left[\frac{\left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1+\rho}} - \alpha}{(1-\alpha) \left(\beta(x)^{-\sigma} + (1-\beta) \right)^{\frac{\rho}{\sigma}}} \right]^{\frac{1}{\rho}}$$

Proposition 7 extends the results of Proposition 6 for the case of a CES production technology.

Proposition 7 *Consider an economy with a CES technology:*

$$f(k_t, x_t) = A \left[\alpha k_t^{-\rho} + (1-\alpha) \left(\beta(x_t)^{-\sigma} + (1-\beta) \right)^{\frac{\rho}{\sigma}} \right]^{-\frac{1}{\rho}}$$

We have:

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^1} &\geq 0 \iff \frac{\partial x}{\partial n^1} \left[\frac{\rho \beta \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1+\rho}} - \alpha}{\frac{[\beta(x)^{-\sigma} + 1 - \beta] x^{\sigma+1}}{\rho n^1 m^1 \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-2\rho-1}{1+\rho}}}} - \frac{1}{(1+\rho)A\alpha} \right] \geq \frac{\rho x m^1 \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-1-2\rho}{1+\rho}}}{(1+\rho)A\alpha} \\ \frac{\partial k^{GR}}{\partial n^2} &\geq 0 \iff \frac{\partial x}{\partial n^2} \left[\frac{\rho \beta \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1+\rho}} - \alpha}{\frac{[\beta(x)^{-\sigma} + 1 - \beta] x^{\sigma+1}}{\rho n^1 m^1 \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-2\rho-1}{1+\rho}}}} - \frac{1}{(1+\rho)A\alpha} \right] \geq \frac{\rho \bar{m}^2 \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-1-2\rho}{1+\rho}}}{(1+\rho)A\alpha} \end{aligned}$$

where $\frac{\partial x}{\partial n^1} \geq 0 \iff \frac{x}{1+x} \geq m^2$ and where $\frac{\partial x}{\partial n^2} \geq 0 \iff \frac{x}{1+x} \leq m^2$.

Proof. See the Appendix. ■

As in Proposition 6, one can see that, when composition effects are small, the standard capital dilution effect under a fixed partition of the population (the RHS of the conditions) dominates the composition effect (the LHS of the conditions), so that a rise in a type-specific fertility rate decreases the Golden Rule capital level.

In order to illustrate the relation between fertility and optimal capital accumulation, let us conclude this section by some numerical simulations. For that purpose, we assume that the production function is $f(k_t, x_t) = 10k_t^{1/3}x_t^{1/3}$, as well as full depreciation of capital from one period to the next ($\delta = 1$). We assume that $n^2 = 1.5$, $m^1 = 0.1$ and $m^2 = 0.1$ (low mobility across types). We consider the impact of a marginal variation in the fertility rate n^1 on the Golden Rule capital k^{GR} , everything else being left unchanged. Figure 2 shows the effect of a change of n^1 from $n^1 = 1$ to $n^1 = 1.1$. In that case, the rise of n^1 increases the Golden Rule capital from $k^{GR} = 2.093$ to $k^{GR} = 2.268$. Figure 3 shows the effect of a change of n^1 from $n^1 = 2$ to $n^1 = 2.1$. In that case, the rise of n^1 decreases the Golden Rule capital from $k^{GR} = 3.830$ to $k^{GR} = 3.775$.

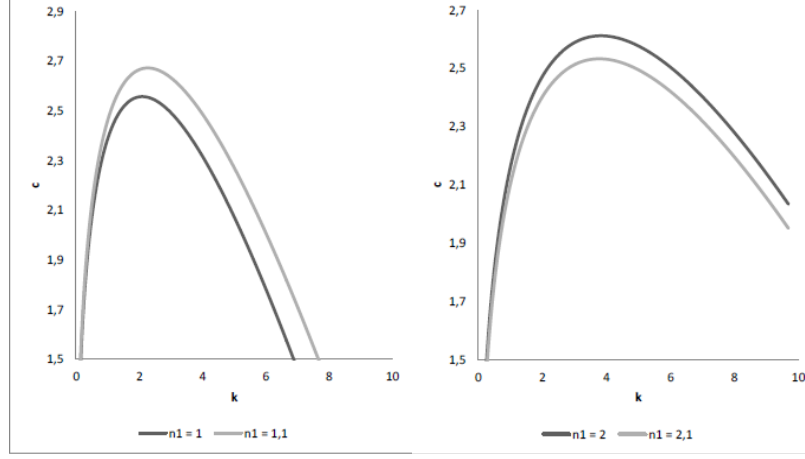


Figure 2. A rise of fertility causes a rise of k^{GR} . Figure 3. A rise of fertility causes a decline of k^{GR} .

These numerical examples illustrate that an increase in a type-specific fertility rate n^i (leading to a rise of total fertility) can have opposite effects on the Golden Rule capital k^{GR} . In some cases, illustrated by Figure 2, the rise of fertility increases the Golden Rule capital, whereas, in other cases, illustrated by Figure 3, the opposite takes place, and a rise of fertility reduces k^{GR} .

5 Extension: the 3-type case

Let us now consider how heterogeneity in fertility affects optimal capital accumulation under a finer description of heterogeneity, that is, a finer decomposition of the population. For that purpose, this section reexamines our results in the case where the population is partitioned into three subpopulations.

The extended model The structure of the young adult population is given by the vector:

$$\mathbf{N}_t = \begin{pmatrix} N_t^1 \\ N_t^2 \\ N_t^3 \end{pmatrix}$$

The population matrix $\mathbf{M}$ becomes:

$$\mathbf{M} = \begin{pmatrix} n^1 m^{1,1} & n^2 m^{2,1} & n^3 m^{3,1} \\ n^1 m^{1,2} & n^2 m^{2,2} & n^3 m^{3,2} \\ n^1 m^{1,3} & n^2 m^{2,3} & n^3 m^{3,3} \end{pmatrix}$$

where n^i is the type-specific fertility rate, while $m^{i,j} \in]0, 1[$ is the probability, for a child born from an individual of type i , to take type j once adult.

We have:

$$\mathbf{M}\mathbf{N}_t = \mathbf{N}_{t+1} \quad (18)$$

The production of an output Y_t involves capital K_t and three types of labor L_t^1 , L_t^2 and L_t^3 , according to the function:

$$Y_t = G(K_t, L_t^1, L_t^2, L_t^3) \quad (19)$$

We assume that all young adults supply one unit of labour of their type, so that $L_t^1 = N_t^1$, $L_t^2 = N_t^2$ and $L_t^3 = N_t^3$. The production function $G(K_t, L_t^1, L_t^2, L_t^3)$ is supposed to be increasing in its four arguments and homogeneous of degree 1.

Dividing the output by N_t^3 , the production process can be rewritten in intensive terms as:

$$y_t = G(k_t, x_t, z_t, 1) \equiv g(k_t, x_t, z_t) \quad (20)$$

where $y_t \equiv \frac{Y_t}{N_t^3}$ is output per worker of type 3, $k_t \equiv \frac{K_t}{N_t^3}$ is capital stock per worker of type 3, $x_t \equiv \frac{N_t^1}{N_t^3}$ and $z_t \equiv \frac{N_t^2}{N_t^3}$.

As above, the capital stock depreciates at a constant rate δ , with $0 < \delta < 1$.

Long-run population composition Since the matrix $\mathbf{M}$ is irreducible and primitive, one can, as in the baseline model, use the Perron-Frobenius Theorem and the Strong Ergodic Theorem to study the long-run partition of the young adult population. Proposition 8 summarizes our results.

Proposition 8 *Denote*

$$\begin{aligned} B &\equiv n^1 n^2 [m^{2,1} m^{1,2} - m^{1,1} m^{2,2}] + n^1 n^3 [m^{3,1} m^{1,3} - m^{1,1} m^{3,3}] \\ &\quad + n^2 n^3 [m^{3,2} m^{2,3} - m^{2,2} m^{3,3}] \\ D &\equiv n^1 n^2 n^3 \left(\begin{array}{c} m^{3,3} m^{1,1} m^{2,2} - m^{3,1} m^{1,3} m^{2,2} - m^{2,1} m^{1,2} m^{3,3} \\ + m^{2,1} m^{3,2} m^{1,3} + m^{3,1} m^{1,2} m^{2,3} - m^{1,1} m^{2,3} m^{3,2} \end{array} \right) \\ p &\equiv \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^2}{3} - B \\ q &\equiv -\frac{2[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^3}{27} - \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}] B}{3} - D \end{aligned}$$

Assume that $B < 0$. The structure of the young adult population converges asymptotically toward a stable structure. The long-run structure of the young adult population is defined, up to a constant $c > 0$, by:

$$\begin{pmatrix} N^1 \\ N^2 \\ N^3 \end{pmatrix} = \begin{pmatrix} \frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{c [n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \\ \frac{n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [n^1 m^{1,1} - \lambda]}{c [n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \\ \frac{(n^2 m^{2,2} - \lambda) (n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]}{c [n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \end{pmatrix}$$

where

$$\lambda = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4}{27} p^3}}{2}} + \sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4}{27} p^3}}{2}} + \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]}{3}$$

The associated long-run ratios x and z are:

$$\begin{aligned} x &= \frac{N^1}{N^3} = \frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{(n^2 m^{2,2} - \lambda) (n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]} \\ z &= \frac{N^2}{N^3} = \frac{n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [n^1 m^{1,1} - \lambda]}{(n^2 m^{2,2} - \lambda) (n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]} \end{aligned}$$

Proof. See the Appendix. ■

Proposition 8 states that the partition of the young adult population into subpopulations converges asymptotically towards a partition that does not depend on initial conditions, but depends only on type-specific fertility and mobility parameters. The unique difference with respect to the 2-type case (Proposition 2) concerns the structure of heterogeneity, in particular, the larger number of demographic parameters determining the long-run partition of the population.

Optimal accumulation policy The resource constraint of the economy is:

$$G(K_t, L_t^1, L_t^2, L_t^3) = c_t^1 L_t^1 + d_t^1 L_{t-1}^1 + c_t^2 L_t^2 + d_t^2 L_{t-1}^2 + c_t^3 L_t^3 + d_t^3 L_{t-1}^3 + K_{t+1} - (1-\delta)K_t \quad (21)$$

Dividing the LHS and the RHS of that constraint by N_t^3 , one obtains:

$$y_t = x_t c_t^1 + \frac{N_{t-1}^1}{N_t^3} d_t^1 + z_t c_t^2 + \frac{N_{t-1}^2}{N_t^3} d_t^2 + c_t^3 + \frac{N_{t-1}^3}{N_t^3} d_t^3 + \frac{K_{t+1}}{N_t^3} - (1-\delta)k_t$$

Notice that:

$$\begin{aligned} \frac{N_{t-1}^3}{N_t^3} &= \frac{N_{t-1}^3}{n^1 N_{t-1}^1 m^{1,3} + n^2 N_{t-1}^2 m^{2,3} + n^3 N_{t-1}^3 m^{3,3}} = \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} \\ \frac{N_{t-1}^1}{N_t^3} &= \frac{N_{t-1}^1}{N_t^3} \frac{N_{t-1}^3}{N_{t-1}^3} = x_{t-1} \frac{N_{t-1}^3}{N_t^3} = x_{t-1} \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} \\ \frac{N_{t-1}^2}{N_t^3} &= \frac{N_{t-1}^2}{N_t^3} \frac{N_{t-1}^3}{N_{t-1}^3} = z_{t-1} \frac{N_{t-1}^3}{N_t^3} = z_{t-1} \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} \end{aligned}$$

We can rewrite the economy's resource constraint as:

$$\begin{aligned} y_t &= x_t c_t^1 + x_{t-1} \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} d_t^1 + z_t c_t^2 \\ &\quad + z_{t-1} \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} d_t^2 + c_t^3 + \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} d_t^3 \\ &\quad + k_{t+1} [n^1 m^{1,3} x_t + n^2 m^{2,3} z_t + n^3 m^{3,3}] - (1-\delta)k_t \end{aligned}$$

Hence, at the stationary equilibrium with constant population structure (i.e.,

$x_t = x_{t-1} = x$ and $z_t = z_{t-1} = z$), the resource constraint is:

$$\begin{aligned} y = & xc^1 + x \frac{1}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3}} d^1 + zc^2 \\ & + z \frac{1}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3}} d^2 + c^3 + \frac{1}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3}} d^3 \\ & + k [n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3}] - (1 - \delta)k \end{aligned} \quad (22)$$

Consumption per capita at time t is defined as total consumption divided by the total population size at t :

$$c_t \equiv \frac{G(K_t, L_t^1, L_t^2, L_t^3) - K_{t+1} + (1 - \delta)K_t}{N_t^1 + N_t^2 + N_t^3 + N_{t-1}^1 + N_{t-1}^2 + N_{t-1}^3} \quad (23)$$

Dividing the numerator and the denominator by N_t^3 and using the resource constraint, consumption per capita can be rewritten as:

$$c_t = \frac{\frac{C_t}{N_t^3}}{\left[x_t + z_t + 1 + x_{t-1} \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} + z_{t-1} \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} + \frac{1}{n^1 m^{1,3} x_{t-1} + n^2 m^{2,3} z_{t-1} + n^3 m^{3,3}} \right]}$$

Note that

$$\frac{C_t}{N_t^3} = g(k_t, x_t, z_t) - k_{t+1} [n^1 m^{1,3} x_t + n^2 m^{2,3} z_t + n^3 m^{3,3}] + (1 - \delta)k_t$$

Hence, given that, at the stationary equilibrium, demographic ratios x and z are constants, the steady-state consumption per capita can be rewritten as the function $\varphi(k)$:

$$c = \frac{g(k, x, z) - k [n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3}] + (1 - \delta)k}{\left[x + z + 1 + \frac{x}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3}} + \frac{z}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3}} + \frac{1}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3}} \right]} \equiv \varphi(k) \quad (24)$$

As in the 2-type case, the expression $\varphi'(k) = 0$ defines an interior Golden Rule capital level only if $\varphi(k)$ is neither always decreasing in k (implying that the capital level maximizing $\varphi(k)$ is 0), nor always increasing in k (implying that the level of k maximizing $\varphi(k)$ is infinite). The interiority of the solution requires the following condition, which guarantees that $\varphi'(k)$ is positive when k tends to 0, but negative when it tends to $+\infty$.

Proposition 9 *Assume that:*

$$\lim_{k \rightarrow 0+} g_k(k, x, z) > n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta$$

and

$$\lim_{k \rightarrow +\infty} g_k(k, x, z) < n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta$$

where $x = \frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{(n^2 m^{2,2} - \lambda)(n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]}$ and $z = \frac{n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [n^1 m^{1,1} - \lambda]}{(n^2 m^{2,2} - \lambda)(n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]}$.

Then there exists a level of k^{GR} that maximizes steady-state consumption per capita. That level k^{GR} satisfies:

$$g_k(k, x, z) = n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta$$

Proof. See the Appendix. ■

Proposition 9 presents the extended Golden Rule of capital accumulation when the population is partitioned in three subpopulations with distinct fertility patterns. In comparison to the Golden Rule derived in the two-type case (Proposition 3), we can see that adopting a finer decomposition of the population into subpopulations modifies the form of the Golden Rule in a non-trivial manner. In particular, composition effects here concern the impact of fertility on not one, but on two demographic ratios x and z . Shifting towards a finer decomposition of the population into subpopulations modifies also the form of the relation between optimal capital and fertility, as we will now see.

Fertility and the Golden Rule capital In order to reexamine the relation between the Golden Rule capital and fertility in this extended setting, let us assume a Cobb-Douglas production technology:

$$g(k_t, x_t, z_t) = A k_t^\alpha x_t^\beta z_t^\gamma \quad (25)$$

where $A > 0$, $0 < \alpha < 1$, $0 < \beta < 1$ and $0 < \gamma < 1$. We also assume $\alpha + \beta + \gamma < 1$.

Hence, the Golden Rule capital k^{GR} is here:

$$k^{GR} = \left[\frac{A \alpha x^\beta z^\gamma}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta} \right]^{\frac{1}{1-\alpha}} \quad (26)$$

where $x = \frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{(n^2 m^{2,2} - \lambda)(n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]}$ and $z = \frac{n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [n^1 m^{1,1} - \lambda]}{(n^2 m^{2,2} - \lambda)(n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]}$.

Proposition 10 summarizes our results concerning the impact of a marginal change of a type-specific fertility rate on the Golden Rule capital k^{GR} .

Proposition 10 Consider an economy with a Cobb-Douglas technology: $g(k_t, x_t, z_t) = A k_t^\alpha x_t^\beta z_t^\gamma$. We have:

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^1} &\geq 0 \iff \left[\frac{\frac{\partial x}{\partial n^1} [\beta z [n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - (1 - \beta) z x n^1 m^{1,3}]}{+ \frac{\partial z}{\partial n^1} [\gamma x [n^1 m^{1,3} x + n^3 m^{3,3} - 1 + \delta] - (1 - \gamma) x z n^2 m^{2,3}]} \right] \geq x z m^{1,3} x \\ \frac{\partial k^{GR}}{\partial n^2} &\geq 0 \iff \left[\frac{\frac{\partial x}{\partial n^2} [\beta z [n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - (1 - \beta) x z n^1 m^{1,3}]}{+ \frac{\partial z}{\partial n^2} [x \gamma [n^1 m^{1,3} x + n^3 m^{3,3} - 1 + \delta] - (1 - \gamma) x z n^2 m^{2,3}]} \right] \geq x z m^{2,3} z \\ \frac{\partial k^{GR}}{\partial n^3} &\geq 0 \iff \left[\frac{\frac{\partial x}{\partial n^3} [\beta z [n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - (1 - \beta) x z n^1 m^{1,3}]}{+ \frac{\partial z}{\partial n^3} [x \gamma [n^1 m^{1,3} x + n^3 m^{3,3} - 1 + \delta] - (1 - \gamma) x z n^2 m^{2,3}]} \right] \geq x z m^{3,3} \end{aligned}$$

where $x = \frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{(n^2 m^{2,2} - \lambda)(n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]}$ and $z = \frac{n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [n^1 m^{1,1} - \lambda]}{(n^2 m^{2,2} - \lambda)(n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]}$.

Proof. See the Appendix. ■

The main result obtained in the 2-type model still holds under a finer partition of the population into subpopulations: a marginal rise in a type-specific fertility rate - and, hence, in the total fertility rate - does not necessarily reduce the Golden Rule capital k^{GR} , unlike what prevails under a homogenous population. The reason is that a variation of fertility implies here not only a capital dilution effect, but, also, a composition effect. In a 3-type economy, this composition effect is more complex, since it goes through the impact of fertility on demographic ratios x and z . Variations in ratios x and z affect not only the investment required to maintain capital intensity constant, but, also, the marginal productivity of capital, with an ambiguous net effect on k^{GR} . But despite the increased complexity of the composition effect, the overall result is qualitatively the same as in the 2-type framework: when the composition effect outweighs the standard capital dilution effect prevailing under a fixed composition of the population, a rise in fertility leads to a higher Golden Rule capital.

6 Concluding remarks

This paper reexamined the relationship between the Golden Rule capital and fertility in economies where the population is partitioned into subpopulations having distinct fertility patterns. We showed that the relation between the Golden Rule capital and fertility is not robust to partitioning the population into subpopulations with different fertility. Whereas a rise in fertility must necessarily reduce the Golden Rule capital under a homogeneous population, the same is not true under a heterogeneous population. In that general case, an increase in fertility can either decrease or increase the Golden Rule capital.

The intuition underlying that result is that, once the population is partitioned into subpopulations with distinct fertility behaviors, a change in a type-specific fertility rate modifies the long-run composition of the population (i.e., the composition effect). The resulting variation in the composition of the population modifies not only the marginal productivity of capital (through a change in the composition of the labour force), but, also, the level of the investment required to maintain capital intensity constant. Depending on its sign and strength, the composition effect can, in some cases, outweigh the standard capital dilution effect prevailing under a fixed partition of the population, leading to an increasing relation between the Golden Rule capital and the level of fertility.

The lack of robustness of the Golden Rule to the introduction of heterogeneity in fertility behaviors should be contrasted with its robustness to the introduction of heterogeneity in labour endowments and in preferences (Kuhle 2012). Given that the Golden Rule focuses on the capital that maximizes steady-state average consumption, heterogeneity along these two dimensions was shown to have no effect on the Golden Rule capital. But the present paper highlights that the same robustness does not prevail as soon as one considers heterogeneity in fertility. Heterogeneity in fertility patterns modifies the Golden Rule and, also, the form of the relation between the Golden Rule capital and fertility.

From a historical perspective, this result has important implications when studying Western economies's fertility transition in the 19th and 20th centuries. Using Phelps's Golden Rule, the decline of fertility supports a rise in capital intensity, on the ground of the negative capital dilution effect induced by the fall of fertility. However, from the perspective of the extended Golden Rule studied in this paper, the sign and the size of the change in the optimal capital intensity following a fertility decline depend on the effects of the fall of fertility on the long-run composition of the population and on its corollaries for production and consumption possibilities. Thus the composition effect associated to the fertility change must be studied before being able to discuss the optimality of the rise in capital intensity observed during the last two centuries.

Our findings have also implications for policy-making, in particular concerning developing countries in the fertility transition. On the basis of Phelps's Golden Rule, fertility declines justify a rise in capital intensity. However, this paper suggests that whether or not a lower fertility justifies a higher capital intensity depends on the effects of this fertility decline on the long-run partition of the population. To see this, suppose, for instance, that heterogeneity takes the form of a rural *versus* urban partition, and that the urban population contributes more substantially to production than the rural one. In that context, a decline in total fertility driven mainly by a fall of urban fertility may not necessarily lead to a higher optimal capital intensity. The reason is that the decline in total fertility is likely to lead to a composition of the labor force involving a lower urban/rural ratio, which is associated with lower production and consumption possibilities for a given capital intensity. Depending on the size of that effect, the fertility decline may not justify a higher capital intensity. Before concluding that a variation in total fertility justifies more or less accumulation, one must study the composition effect due to the fertility change.

7 References

- Cardano, G., 1545. *Ars Magna or the Rules of Algebra*, new edition, 1993. Dover, London.
- Cass, D., 1972. On capital overaccumulation in the aggregative neoclassical model of economic growth: a complete characterization. *Journal of Economic Theory*, 4: 200-223.
- Caswell, H., 2001. *Matrix Population Models. Construction, Analysis and Interpretation*. Sinauer, Massassuchets.
- Clark, G., 2007. *A Farewell to Alms. A Brief Economic History of the World*. Princeton University Press.
- Clark, G., Cummins, N., 2015. Malthus to modernity: wealth, status and fertility in England, 1500-1879. *Journal of Population Economics*, 28: 3-29.
- Cummins, N., 2020. The micro-evidence for the Malthusian system. France, 1670-1840, *European Economic Review*, 129(C): 103544.
- Davis, E., 1969. A Modified Golden Rule: The case with endogenous labor supply, *American Economic Review*, 59: 177-181.

- de la Croix, D., Michel, P., 2002. *A Theory of Economic Growth*. Cambridge University Press.
- de la Croix, D., Ponthiere, G., 2010. On the Golden Rule of capital accumulation under endogenous longevity. *Mathematical Social Sciences*, 59: 227-238.
- de la Croix, D., Schneider, E., Weisdorf, J., 2019. Childlessness, celibacy and net fertility in Pre-Industrial England. *Journal of Economic Growth*, 24: 223-256.
- Galor, O., 2011. *Unified Growth Theory*. Princeton University Press.
- Galor, O., Moav, O., 2002. Natural selection and the origin of economic growth. *Quarterly Journal of Economics*, 117: 1133-1191.
- Galor, O., Ryder, H., 1991. Dynamic efficiency of steady-state equilibria in an overlapping-generations model with productive capital. *Economics Letters*, 35: 385-390.
- Kuhle, W., 2012. Dynamic efficiency and the two-part golden rule with heterogeneous agents. *Journal of Macroeconomics*, 34: 992-1006.
- McFadden, D., 1963. Constant elasticity of substitution production functions. *The Review of Economic Studies*, 30: 73-83.
- Mertens, J-F., Rubinchik, A., 2015. Pareto optimality of the Golden Rule equilibrium in an overlapping generations model with production and transfers. *Macroeconomic Dynamics*, 19: 1780-1799.
- Mitra, T., Ray, D., 2012. On the Phelps-Koopmans theorem. *Journal of Economic Theory*, 147: 833-849.
- Pestieau, P., Ponthiere, G., 2014. Optimal fertility along the lifecycle. *Economic Theory*, 55: 185-224.
- Phelps, E., 1961. The golden rule of capital accumulation: a fable for growthmen. *American Economic Review*, 51: 638-643.
- Phelps, E., 1965. Second essay on the golden rule of accumulation. *American Economic Review*, 55: 793-814.
- Solow, R., 1956. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70: 65-94.
- Strulik, H. 2021. Hyperbolic discounting and the time-consistent solution of three canonical environmental problems, *Journal of Public Economic Theory*, 23: 462-486.
- Uzawa, H. 1962. Production functions with constant elasticities of substitution. *The Review of Economic Studies*, 29: 291-299.
- Zilcha, I., 1990. Dynamic efficiency in overlapping generations models with stochastic production. *Journal of Economic Theory*, 52: 364-379.

8 Appendix

8.1 Proof of Proposition 1

Irreducibility prevails when the life cycle graph associated to the matrix admits at least one path from each node and towards each node. This is the case for matrix $\mathbf{M}$. As shown in Figure 1, the life cycle graph associated to our model includes two distinct populations, i.e., type 1 and type 2, which contribute to each other through the social mobility process. Thus each population makes contributions to the other population, implying irreducibility of $\mathbf{M}$.

Primitivity arises when there exists a power p such that raising the matrix to that power makes it positive. This is clearly the case for matrix $\mathbf{M}$, which is a positive matrix.

8.2 Proof of Proposition 2

Let us characterize eigenvalues of the matrix $\mathbf{M}$. We look for solutions for the equation:

$$\mathbf{M}\mathbf{w} = \lambda\mathbf{w}$$

where λ is the eigenvalue (a scalar) while $\mathbf{w}$ is the associated eigen vector, a vector that makes matrix multiplication and scalar multiplication equivalents. From the definition of the eigen vectors, it follows that:

$$\begin{aligned}\mathbf{M}\mathbf{w} - \lambda\mathbf{w} &= \mathbf{0} \\ (\mathbf{M} - \lambda\mathbf{I})\mathbf{w} &= \mathbf{0}\end{aligned}$$

Non-zero solutions require $(\mathbf{M} - \lambda\mathbf{I})$ to be a singular matrix, that is, that it has a zero determinant.

Hence eigenvalues are solutions to:

$$\det \begin{pmatrix} n^1\bar{m}^1 - \lambda & n^2\bar{m}^2 \\ n^1\bar{m}^1 & n^2\bar{m}^2 - \lambda \end{pmatrix} = 0$$

Therefore we have:

$$(n^1\bar{m}^1 - \lambda)(n^2\bar{m}^2 - \lambda) - n^2\bar{m}^2 n^1\bar{m}^1 = 0$$

Hence, after some simplifications:

$$\lambda^2 - \lambda(n^1\bar{m}^1 + n^2\bar{m}^2) + n^1n^2(1 - m^2 - m^1) = 0$$

Eigenvalues can be found as the roots of this polynomial. We have:

$$\Delta = (n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)$$

Note that Δ can be rewritten as:

$$\begin{aligned}\Delta &= (n^1(1 - m^1))^2 + (n^2(1 - m^2))^2 + 2n^1(1 - m^1)n^2(1 - m^2) \\ &\quad - 4n^1n^2(1 - m^2 - m^1) \\ &= (n^1(1 - m^1))^2 + (n^2(1 - m^2))^2 - 2n^1n^2(1 - m^1 - m^2) + 2n^1n^2m^1m^2 \\ &= (n^1(1 - m^1))^2 + (n^2(1 - m^2))^2 - 2n^1n^2(1 - m^1 - m^2 - m^1m^2)\end{aligned}$$

Hence the two eigenvalues are:

$$\begin{aligned}\lambda_1 &= \frac{n^1\bar{m}^1 + n^2\bar{m}^2 + \sqrt{(n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)}}{2} \\ \lambda_2 &= \frac{n^1\bar{m}^1 + n^2\bar{m}^2 - \sqrt{(n^1\bar{m}^1 + n^2\bar{m}^2)^2 - 4n^1n^2(1 - m^2 - m^1)}}{2}\end{aligned}$$

We have $\lambda_1 > \lambda_2$, so that the dominant eigenvalue is λ_1 .

We can then derive the long-run young adult population structure by calculating the eigenvector $\mathbf{w}_1$ associated to the dominant eigenvalue λ_1 . The associated eigenvector is such that:

$$\begin{pmatrix} n^1 \bar{m}^1 & n^2 \bar{m}^2 \\ n^1 m^1 & n^2 \bar{m}^2 \end{pmatrix} \begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = \begin{pmatrix} n^1 \bar{m}^1 + n^2 \bar{m}^2 + \sqrt[2]{\frac{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}{2}} \\ \frac{n^1 \bar{m}^1 + n^2 \bar{m}^2 + \sqrt[2]{\frac{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}{2}}}{2} \end{pmatrix} \begin{pmatrix} N^1 \\ N^2 \end{pmatrix}$$

Hence we have

$$\begin{aligned} n^1 \bar{m}^1 N^1 + n^2 \bar{m}^2 N^2 &= \frac{n^1 \bar{m}^1 + n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2} N^1 \\ n^1 m^1 N^1 + n^2 \bar{m}^2 N^2 &= \frac{n^1 \bar{m}^1 + n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2} N^2 \end{aligned}$$

Two equations and two unknowns. Normalizing to $N^1 + N^2 = 1$, the second equation can be rewritten as:

$$n^1 m^1 N^1 + n^2 \bar{m}^2 (1 - N^1) = \frac{n^1 \bar{m}^1 + n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2} (1 - N^1)$$

From which it follows that

$$N^1 = \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2n^1 - (n^1 \bar{m}^1 + n^2 \bar{m}^2) + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}$$

Hence the eigen vector associated to λ_1 is

$$\mathbf{w}_1 = \begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = \begin{pmatrix} \frac{N^1}{1 - N^1} \end{pmatrix} = \begin{pmatrix} \frac{\frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2n^1 - (n^1 \bar{m}^1 + n^2 \bar{m}^2) + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}}{\frac{2n^1 (1 - \bar{m}^1)}{2n^1 - (n^1 \bar{m}^1 + n^2 \bar{m}^2) + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}} \end{pmatrix}$$

From the Strong Ergodic Theorem, we know that the asymptotic young adult population structure is given by the eigen vector $\mathbf{w}_1$, while the precise size of the different subpopulations can always be scaled as desired, since eigenvectors are defined up to a multiplicative constant. Hence the long-run young adult population composition x is given by:

$$x = \frac{N^1}{N^2} = \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2n^1 m^1}$$

8.3 Proof of Proposition 3

The conditions $\lim_{k \rightarrow 0+} \phi'(k) > 0$ and $\lim_{k \rightarrow +\infty} \phi'(k) < 0$ are sufficient to have an interior maximum.

The first limit can be written as:

$$\begin{aligned} \lim_{k \rightarrow 0+} \phi'(k) &= \lim_{k \rightarrow 0+} \left[\frac{f_k(k, x) - (n^2 \bar{m}^2 + x n^1 m^1) + (1 - \delta)}{x + 1 + x \frac{1}{n^2 \bar{m}^2 + x n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1}} \right] \\ &= \frac{1}{x + 1 + x \frac{1}{n^2 \bar{m}^2 + x n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1}} \lim_{k \rightarrow 0+} [f_k(k, x) - (n^2 \bar{m}^2 + x n^1 m^1) + (1 - \delta)] \end{aligned}$$

Given that the first factor is strictly positive, the conditions $\lim_{k \rightarrow 0+} \phi'(k) > 0$ is always true when:

$$\lim_{k \rightarrow 0+} [f_k(k, x) - (n^2 \bar{m}^2 + x n^1 m^1) + (1 - \delta)] > 0$$

that is, when

$$\begin{aligned} &\lim_{k \rightarrow 0+} \left[f_k \left(k, \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4 n^1 n^2 (1 - m^2 - m^1)}}{2 n^1 m^1} \right) \right] \\ &> \frac{n^1 \bar{m}^1 + n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4 n^1 n^2 (1 - m^2 - m^1)}}{2} - 1 + \delta \end{aligned}$$

which is the condition of the proposition.

Regarding the second condition, it can be written as:

$$\begin{aligned} \lim_{k \rightarrow +\infty} \phi'(k) &= \lim_{k \rightarrow +\infty} \left[\frac{f_k(k, x) - (n^2 \bar{m}^2 + x n^1 m^1) + (1 - \delta)}{x + 1 + x \frac{1}{n^2 \bar{m}^2 + x n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1}} \right] \\ &= \frac{1}{x + 1 + x \frac{1}{n^2 \bar{m}^2 + x n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1}} \lim_{k \rightarrow +\infty} [f_k(k, x) - (n^2 \bar{m}^2 + x n^1 m^1) + (1 - \delta)] \end{aligned}$$

Given that $\frac{1}{x+1+x \frac{1}{n^2 \bar{m}^2 + x n^1 m^1} + \frac{1}{n^2 \bar{m}^2 + x_{t-1} n^1 m^1}} > 0$, the condition $\lim_{k \rightarrow +\infty} \phi'(k) < 0$ is necessarily true when:

$$\lim_{k \rightarrow +\infty} f_k(k, x) < n^2 \bar{m}^2 + x n^1 m^1 + 1 - \delta$$

that is, when

$$\begin{aligned} &\lim_{k \rightarrow +\infty} f_k \left(k, \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4 n^1 n^2 (1 - m^2 - m^1)}}{2 n^1 m^1} \right) \\ &< \frac{n^1 \bar{m}^1 + n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4 n^1 n^2 (1 - m^2 - m^1)}}{2} - 1 + \delta \end{aligned}$$

as stated in the proposition.

8.4 Proof of Proposition 5

We have:

$$\begin{aligned}
 x &= \frac{n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}{2n^1 m^1} \\
 \frac{\partial x}{\partial n^1} &= \frac{\left[\bar{m}^1 + \frac{1}{2} \frac{[2(n^1 \bar{m}^1 + n^2 \bar{m}^2) \bar{m}^1 - 4n^2 (1 - m^2 - m^1)]}{\sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}} \right] 2n^1 m^1}{\left[n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)} \right] 2m^1} \\
 &\quad \frac{(2n^1 m^1)^2}{(2n^1 m^1)^2} \\
 &= \frac{\left[\bar{m}^1 + \frac{1}{2} \frac{[2(n^1 \bar{m}^1 + n^2 \bar{m}^2) \bar{m}^1 - 4n^2 (1 - m^2 - m^1)]}{[(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)]^{\frac{1}{2}}} \right] n^1}{\left[n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)} \right]} \\
 &\quad \frac{2m^1 (n^1)^2}{2m^1 (n^1)^2}
 \end{aligned}$$

We have thus:

$$\begin{aligned}
\frac{\partial x}{\partial n^1} &\geq 0 \\
\iff &\left[\bar{m}^1 + \frac{[(n^1 \bar{m}^1 + n^2 \bar{m}^2) \bar{m}^1 - 2n^2 (1 - m^2 - m^1)]}{\sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}} \right] n^1 \\
&\geq \left[n^1 \bar{m}^1 - n^2 \bar{m}^2 + \sqrt{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)} \right] \\
\iff &\frac{[(n^1 \bar{m}^1 + n^2 \bar{m}^2) n^1 \bar{m}^1 - 2n^1 n^2 (1 - m^2 - m^1)]}{\sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}} \\
&\geq \frac{-n^2 \bar{m}^2 \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)} + (n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}{\sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}} \\
\iff &[(n^1 \bar{m}^1 + n^2 \bar{m}^2) n^1 \bar{m}^1 - 2n^1 n^2 (1 - m^2 - m^1)] \\
&\geq -n^2 \bar{m}^2 \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)} + (n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1) \\
\iff &n^1 n^2 (1 - m^2 - m^1) + n^1 n^2 (1 - m^2 - m^1) - n^1 n^2 (1 - m^1) + n^1 n^2 (1 - m^1) m^2 \\
&\geq -n^2 \bar{m}^2 \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)} + (n^2 \bar{m}^2)^2 \\
\iff &n^1 n^2 (1 - m^1 - m^2 - m^1 m^2) + n^2 \bar{m}^2 \sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)} \geq (n^2 \bar{m}^2)^2 \\
\iff &n^1 n^2 (1 - m^1 - m^2 - m^1 m^2) + n^2 \bar{m}^2 [x 2n^1 m^1 - n^1 \bar{m}^1 + n^2 \bar{m}^2] \geq (n^2 \bar{m}^2)^2 \\
\iff &n^1 n^2 (\bar{m}^1 - m^2 - m^1 m^2) + n^1 n^2 \bar{m}^2 [x 2m^1 - \bar{m}^1] \geq 0 \\
\iff &n^1 n^2 \bar{m}^1 (m^2) - n^1 n^2 m^2 (1 + 1 - \bar{m}^1) + n^1 n^2 \bar{m}^2 x 2m^1 \geq 0 \\
\iff &2n^1 n^2 \bar{m}^1 (m^2) - 2n^1 n^2 m^2 + n^1 n^2 \bar{m}^2 x 2m^1 \geq 0 \\
\iff &(-m^1)(m^2) + (1 - m^2) x m^1 \geq 0 \\
\iff &-m^2 + (1 - m^2) x \geq 0 \\
\iff &\frac{x}{1 + x} \geq m^2
\end{aligned}$$

We also have:

$$\frac{\partial x}{\partial n^2} = \frac{-\bar{m}^2 + \frac{1}{2} \frac{[2(n^1 \bar{m}^1 + n^2 \bar{m}^2) \bar{m}^2 - 4n^1 (1 - m^2 - m^1)]}{\sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}}}{2n^1 m^1}$$

Hence we have:

$$\begin{aligned}
\frac{\partial x}{\partial n^2} &\geq 0 \\
\iff -\bar{m}^2 + \frac{[(n^1 \bar{m}^1 + n^2 \bar{m}^2) \bar{m}^2 - 2n^1 (1 - m^2 - m^1)]}{\sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}} &\geq 0 \\
\iff \frac{(n^1 \bar{m}^1 + n^2 \bar{m}^2) \bar{m}^2 - 2n^1 (1 - m^2 - m^1)}{\sqrt[2]{(n^1 \bar{m}^1 + n^2 \bar{m}^2)^2 - 4n^1 n^2 (1 - m^2 - m^1)}} &\geq \bar{m}^2 \\
\iff \frac{(n^1 \bar{m}^1 + n^2 \bar{m}^2) \bar{m}^2 - 2n^1 (1 - m^2 - m^1)}{x2n^1 m^1 - n^1 \bar{m}^1 + n^2 \bar{m}^2} &\geq \bar{m}^2 \\
\iff \frac{n^1 (1 - m^2)(1 - m^1) + n^2 \bar{m}^2 \bar{m}^2 - 2n^1 \bar{m}^2 + 2n^1 m^1}{x2n^1 m^1 - n^1 \bar{m}^1 + n^2 \bar{m}^2} &\geq \bar{m}^2 \\
\iff \frac{n^1 \bar{m}^2 - n^1 \bar{m}^2 m^1 + n^2 \bar{m}^2 \bar{m}^2 - 2n^1 \bar{m}^2 + 2n^1 m^1}{x2n^1 m^1 - n^1 \bar{m}^1 + n^2 \bar{m}^2} &\geq \bar{m}^2 \\
\iff \frac{n^1 m^1 m^2 + n^2 \bar{m}^2 \bar{m}^2 - n^1 \bar{m}^2 + n^1 m^1}{x2n^1 m^1 - n^1 (1 - m^1) + n^2 \bar{m}^2} &\geq \bar{m}^2 \\
\iff \frac{n^1 m^1 m^2 + n^2 \bar{m}^2 \bar{m}^2 - n^1 \bar{m}^2 + n^1 m^1}{x2n^1 m^1 - n^1 (1 - m^1) + n^2 \bar{m}^2} &\geq \bar{m}^2 [n^1 (m^1 (2x + 1) - 1) + n^2 \bar{m}^2] \\
\iff m^2 + 1 &\geq 2x\bar{m}^2 + 1 - m^2 \\
\iff m^2 &\geq x(1 - m^2) \\
\iff m^2 &\geq \frac{x}{1 + x}
\end{aligned}$$

8.5 Proof of Proposition 6

The Golden Rule capital is:

$$k^{GR} = \left[\frac{A\alpha x^\beta}{(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta} \right]^{\frac{1}{1-\alpha}}$$

Differentiating k^{GR} with respect to n^1 yields:

$$\begin{aligned}
\frac{\partial k^{GR}}{\partial n^1} &= \frac{1}{1 - \alpha} \left[\frac{A\alpha x^\beta}{(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta} \right]^{\frac{1}{1-\alpha} - 1} \\
&\quad \left[\frac{A\alpha \beta x^{\beta-1} \frac{\partial x}{\partial n^1} [(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta] - A\alpha x^\beta \left[\frac{\partial x}{\partial n^1} n^1 m^1 + x m^1 \right]}{[(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta]^2} \right]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial k^{GR}}{\partial n^1} &= \frac{1}{1 - \alpha} \left[\frac{A\alpha x^\beta}{(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta} \right]^{\frac{1}{1-\alpha} - 1} \\
&\quad \left[\frac{A\alpha \beta x^{\beta-1} \frac{\partial x}{\partial n^1} (n^2 \bar{m}^2 + \delta - 1) + A\alpha x^\beta \left[\frac{\partial x}{\partial n^1} n^1 m^1 (\beta - 1) - x m^1 \right]}{[(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta]^2} \right]
\end{aligned}$$

The first two factors are strictly positive. Hence,

$$\begin{aligned}
\frac{\partial k^{GR}}{\partial n^1} &\geq 0 \\
&\iff A\alpha\beta x^{\beta-1} \frac{\partial x}{\partial n^1} (n^2 \bar{m}^2 + \delta - 1) + A\alpha x^\beta \left[\frac{\partial x}{\partial n^1} n^1 m^1 (\beta - 1) - x m^1 \right] \geq 0 \\
&\iff \frac{\partial x}{\partial n^1} [\beta (n^2 \bar{m}^2 + \delta - 1) - (1 - \beta) n^1 m^1 x] \geq (x)^2 m^1
\end{aligned}$$

Consider now the effect of a variation of the type-specific fertility rate n^2 .

Differentiating k^{GR} with respect to n^2 yields:

$$\begin{aligned}
\frac{\partial k^{GR}}{\partial n^2} &= \frac{1}{1 - \alpha} \left[\frac{A\alpha x^\beta}{(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta} \right]^{\frac{1}{1-\alpha} - 1} \\
&\quad \left[\frac{A\alpha\beta x^{\beta-1} \frac{\partial x}{\partial n^2} [(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta] - A\alpha x^\beta [\bar{m}^2 + \frac{\partial x}{\partial n^2} n^1 m^1]}{[(n^2 \bar{m}^2 + x n^1 m^1) - 1 + \delta]^2} \right]
\end{aligned}$$

Hence

$$\frac{\partial k^{GR}}{\partial n^2} \geq 0 \iff \frac{\partial x}{\partial n^2} [\beta (n^2 \bar{m}^2 - 1 + \delta) - (1 - \beta) x n^1 m^1] \geq x \bar{m}^2$$

8.6 Proof of Proposition 7

The Golden Rule capital is:

$$k^{GR} = \left[\frac{\left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1+\rho}} - \alpha}{(1 - \alpha) \left(\beta(x)^{-\sigma} + (1 - \beta) \right)^{\frac{\rho}{\sigma}}} \right]^{\frac{1}{\rho}}$$

Hence we have:

$$\begin{aligned}
\frac{\partial k^{GR}}{\partial n^1} &= \frac{1}{\rho} \left[\frac{\left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1-\rho}} - \alpha}{(1 - \alpha) \left(\beta(x)^{-\sigma} + (1 - \beta) \right)^{\frac{\rho}{\sigma}}} \right]^{\frac{1}{\rho} - 1} \\
&\quad \left[\frac{\left[\frac{-\rho}{-1-\rho} \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1-\rho} - 1} \left[\frac{\frac{\partial x}{\partial n^1} n^1 m^1 + x m^1}{A\alpha} \right] \right] \left[(1 - \alpha) \left(\beta(x)^{-\sigma} + (1 - \beta) \right)^{\frac{\rho}{\sigma}} \right]}{\left[\left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1-\rho}} - \alpha \right] \left[(1 - \alpha) \left(\beta(x)^{-\sigma} + (1 - \beta) \right)^{\frac{\rho}{\sigma} - 1} \beta(-\sigma) x^{-\sigma-1} \frac{\partial x}{\partial n^1} \right]} \right] \\
&\quad \left[\frac{\left[(1 - \alpha) \left(\beta(x)^{-\sigma} + (1 - \beta) \right)^{\frac{\rho}{\sigma}} \right]^2}{\left[(1 - \alpha) \left(\beta(x)^{-\sigma} + (1 - \beta) \right)^{\frac{\rho}{\sigma}} \right]^2} \right]
\end{aligned}$$

Given that the first two factors of the RHS are unambiguously positive, the condition can be simplified to:

$$\begin{aligned}
\frac{\partial k^{GR}}{\partial n^1} &\geq 0 \\
&\iff \\
&\frac{\partial x}{\partial n^1} \left[\begin{aligned} &\left[\left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1+\rho}} - \alpha \right] \frac{\rho \beta x^{-\sigma-1}}{(\beta(x)^{-\sigma} + (1-\beta))} \\ &- \frac{\rho}{1+\rho} \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1+\rho}-1} \frac{n^1 m^1}{A\alpha} \end{aligned} \right] \\
&\geq \left[\frac{\rho}{1+\rho} \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1+\rho}-1} \frac{x m^1}{A\alpha} \right]
\end{aligned}$$

We have also:

$$\begin{aligned}
\frac{\partial k^{GR}}{\partial n^2} &= \frac{1}{\rho} \left[\frac{\left[\left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{1-\rho}} - \alpha \right]^{\frac{1}{\rho}-1}}{(1-\alpha) \left(\beta(x)^{-\sigma} + (1-\beta) \right)^{\frac{\rho}{\sigma}}} \right] \\
&\quad \left[\frac{\begin{aligned} &\left[\frac{\rho}{-1-\rho} \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{-1-\rho}-1} \left[\frac{\frac{\partial x}{\partial n^2} n^1 m^1 + \bar{m}^2}{A\alpha} \right] \right] \left[(1-\alpha) \left(\beta(x)^{-\sigma} + (1-\beta) \right)^{\frac{\rho}{\sigma}} \right] \\ &- \left[\left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{-1-\rho}} - \alpha \right] \left[(1-\alpha)^{\frac{\rho}{\sigma}} \left(\beta(x)^{-\sigma} + (1-\beta) \right)^{\frac{\rho}{\sigma}-1} \beta(-\sigma) x^{-\sigma-1} \frac{\partial x}{\partial n^2} \right] \end{aligned}}{\left[(1-\alpha) \left(\beta(x)^{-\sigma} + (1-\beta) \right)^{\frac{\rho}{\sigma}} \right]^2} \right]
\end{aligned}$$

Hence we have the condition:

$$\begin{aligned}
\frac{\partial k^{GR}}{\partial n^2} &\geq 0 \\
&\iff \\
&\frac{\partial x}{\partial n^2} \left[\begin{aligned} &\left[\left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{-1-\rho}} - \alpha \right] \frac{\rho \beta x^{-\sigma-1}}{(\beta(x)^{-\sigma} + (1-\beta))} \\ &- \frac{\rho}{1+\rho} \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{-1-\rho}-1} \frac{n^1 m^1}{A\alpha} \end{aligned} \right] \\
&\geq \left[\frac{\rho}{1+\rho} \left[\frac{n^2 \bar{m}^2 + x n^1 m^1 - 1 + \delta}{A\alpha} \right]^{\frac{-\rho}{-1-\rho}-1} \frac{\bar{m}^2}{A\alpha} \right]
\end{aligned}$$

8.7 Proof of Proposition 8

Let us characterize eigenvalues of the matrix $\mathbf{M}$. We look for solutions for the equation:

$$\mathbf{M}\mathbf{w} = \lambda \mathbf{w}$$

where λ is the eigenvalue (a scalar) while $\mathbf{w}$ is the associated eigen vector, a vector that makes matrix multiplication and scalar multiplication equivalents.

From the definition of the eigen vectors, it follows that:

$$(\mathbf{M}-\lambda\mathbf{I}) \mathbf{w} = \mathbf{0}$$

Non-zero solutions require $(\mathbf{M}-\lambda\mathbf{I})$ to be a singular matrix, that is, that it has a zero determinant.

Hence eigenvalues are solutions to:

$$\det \begin{pmatrix} n^1 m^{1,1} - \lambda & n^2 m^{2,1} & n^3 m^{3,1} \\ n^1 m^{1,2} & n^2 m^{2,2} - \lambda & n^3 m^{3,2} \\ n^1 m^{1,3} & n^2 m^{2,3} & n^3 m^{3,3} - \lambda \end{pmatrix} = 0$$

Therefore we have:

$$\begin{aligned} & (n^1 m^{1,1} - \lambda) (n^2 m^{2,2} - \lambda) (n^3 m^{3,3} - \lambda) + (n^2 m^{2,1}) (n^3 m^{3,2}) (n^1 m^{1,3}) \\ & + (n^3 m^{3,1}) (n^1 m^{1,2}) (n^2 m^{2,3}) - (n^3 m^{3,1}) (n^2 m^{2,2} - \lambda) (n^1 m^{1,3}) \\ & - (n^2 m^{2,1}) (n^1 m^{1,2}) (n^3 m^{3,3} - \lambda) - (n^1 m^{1,1} - \lambda) (n^3 m^{3,2}) (n^2 m^{2,3}) \\ & = 0 \end{aligned}$$

Hence, after some simplifications, the condition is:

$$\begin{aligned} & -\lambda^3 + \lambda^2 [n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}] \\ & + \lambda \left[-n^1 m^{1,1} n^2 m^{2,2} - n^1 m^{1,1} n^3 m^{3,3} - n^2 m^{2,2} n^3 m^{3,3} \right] \\ & + n^1 n^2 n^3 \left(\begin{aligned} & m^{3,3} m^{1,1} m^{2,2} - m^{3,1} m^{1,3} m^{2,2} - m^{2,1} m^{1,2} m^{3,3} \\ & + m^{2,1} m^{3,2} m^{1,3} + m^{3,1} m^{1,2} m^{2,3} - m^{1,1} m^{2,3} m^{3,2} \end{aligned} \right) \\ & = 0 \end{aligned}$$

Let us denote:

$$\begin{aligned} B & \equiv -n^1 m^{1,1} n^2 m^{2,2} - n^1 m^{1,1} n^3 m^{3,3} - n^2 m^{2,2} n^3 m^{3,3} \\ & \quad + n^3 m^{3,1} n^1 m^{1,3} + n^2 m^{2,1} n^1 m^{1,2} + n^3 m^{3,2} n^2 m^{2,3} \\ D & \equiv n^1 n^2 n^3 \left(\begin{aligned} & m^{3,3} m^{1,1} m^{2,2} - m^{3,1} m^{1,3} m^{2,2} - m^{2,1} m^{1,2} m^{3,3} \\ & + m^{2,1} m^{3,2} m^{1,3} + m^{3,1} m^{1,2} m^{2,3} - m^{1,1} m^{2,3} m^{3,2} \end{aligned} \right) \end{aligned}$$

Note that

$$B = n^1 n^2 [m^{2,1} m^{1,2} - m^{1,1} m^{2,2}] + n^1 n^3 [m^{3,1} m^{1,3} - m^{1,1} m^{3,3}] + n^2 n^3 [m^{3,2} m^{2,3} - m^{2,2} m^{3,3}]$$

If there is, in general, limited mobility across types (relative to non-mobility), we have $B < 0$.

Eigenvalues can be found as the roots of the polynomial:

$$-\lambda^3 + \lambda^2 [n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}] + \lambda B + D = 0$$

This polynomial can be rewritten as:

$$\lambda^3 - \lambda^2 [n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}] - \lambda B - D = 0$$

Let us introduce the variable:

$$u \equiv \lambda - \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]}{3} \implies \lambda = u + \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]}{3}$$

We then have:

$$\begin{aligned} & \left(u + \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]}{3} \right)^3 \\ & - \left(u + \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]}{3} \right)^2 [n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}] \\ & - \left(u + \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]}{3} \right) B - D \\ & = 0 \end{aligned}$$

Hence

$$\begin{aligned} & u^3 + u \left(\frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^2}{3} - B \right) - \frac{2 [n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^3}{27} \\ & - \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}] B}{3} - D \\ & = 0 \end{aligned}$$

$$\text{Let us denote } p \equiv \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^2}{3} - B \text{ and } q \equiv -\frac{2 [n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^3}{27} - \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}] B}{3} - D.$$

We then obtain the depressed cubic equation:

$$u^3 + pu + q = 0$$

Note that, when there is little mobility across types, so that the condition:

$$B = n^1 n^2 [m^{2,1} m^{1,2} - m^{1,1} m^{2,2}] + n^1 n^3 [m^{3,1} m^{1,3} - m^{1,1} m^{3,3}] + n^2 n^3 [m^{3,2} m^{2,3} - m^{2,2} m^{3,3}] < 0$$

is satisfied, it is also the case that $p = \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^2}{3} - B > 0$.

As a consequence, under that assumption, the discriminant of depressed cubic is negative, since:

$$-(4p^3 + 27q^2) < 0$$

Thus our depressed cubic has one real root and two complex conjugates.

In order to find the real root of the depressed cubic, we follow Cardano (1545), we now introduce two new variables whose sum equals $u = s + t$. We substitute for these variables in the depressed cubic equation, and we obtain:

$$\begin{aligned} (s + t)^3 + p(s + t) + q &= 0 \\ s^3 + t^3 + (p + 3st)(s + t) + q &= 0 \end{aligned}$$

Then, imposing the constraint $p + 3st = 0$, we obtain:

$$\begin{aligned} s^3 + t^3 &= -q \\ st &= \frac{-p}{3} \implies s^3 t^3 = \frac{-p^3}{27} \end{aligned}$$

Thus s^3 and t^3 are the roots of the equation:

$$h^2 + hq - \frac{p^3}{27} = 0.$$

We have that:

$$\Delta = q^2 + 4\frac{p^3}{27} \geq 0$$

Note that, under the condition:

$$B = n^1 n^2 [m^{2,1} m^{1,2} - m^{1,1} m^{2,2}] + n^1 n^3 [m^{3,1} m^{1,3} - m^{1,1} m^{3,3}] + n^2 n^3 [m^{3,2} m^{2,3} - m^{2,2} m^{3,3}] < 0$$

it is also the case that $p = \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^2}{3} - B > 0$. As a consequence, we have also $\Delta > 0$.

In the case where $\Delta > 0$, we have two roots: $h_1 = s^3 = \frac{-q + \sqrt[2]{q^2 + \frac{4}{27}p^3}}{2}$ and $h_2 = t^3 = \frac{-q - \sqrt[2]{q^2 + \frac{4}{27}p^3}}{2}$.

Hence:

$$u = s + t = \sqrt[3]{\frac{-q + \sqrt[2]{q^2 + \frac{4}{27}p^3}}{2}} + \sqrt[3]{\frac{-q - \sqrt[2]{q^2 + \frac{4}{27}p^3}}{2}}$$

Moreover, since $\lambda = u + \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]}{3}$, we then obtain:

$$\lambda = \sqrt[3]{\frac{-q + \sqrt[2]{q^2 + \frac{4}{27}p^3}}{2}} + \sqrt[3]{\frac{-q - \sqrt[2]{q^2 + \frac{4}{27}p^3}}{2}} + \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]}{3}$$

where $p \equiv \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^2}{3} - B$ and $q \equiv -\frac{2[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]^3}{27} - \frac{[n^1 m^{1,1} + n^2 m^{2,2} + n^3 m^{3,3}]B}{3} - D$.

Taking λ as the dominant eigen value, we can derive the long-run young adult population structure by calculating the eigenvector associated to the dominant eigenvalue λ . The associated eigenvector is such that:

$$\begin{pmatrix} n^1 m^{1,1} & n^2 m^{2,1} & n^3 m^{3,1} \\ n^1 m^{1,2} & n^2 m^{2,2} & n^3 m^{3,2} \\ n^1 m^{1,3} & n^2 m^{2,3} & n^3 m^{3,3} \end{pmatrix} \begin{pmatrix} N^1 \\ N^2 \\ N^3 \end{pmatrix} = \lambda \begin{pmatrix} N^1 \\ N^2 \\ N^3 \end{pmatrix}$$

Assuming $N^3 = 1 - N^1 - N^2$, we have:

$$\begin{aligned} n^1 m^{1,1} N^1 + n^2 m^{2,1} N^2 + n^3 m^{3,1} (1 - N^1 - N^2) &= \lambda N^1 \\ n^1 m^{1,2} N^1 + n^2 m^{2,2} N^2 + n^3 m^{3,2} (1 - N^1 - N^2) &= \lambda N^2 \\ n^1 m^{1,3} N^1 + n^2 m^{2,3} N^2 + n^3 m^{3,3} (1 - N^1 - N^2) &= \lambda (1 - N^1 - N^2) \end{aligned}$$

Isolating N^2 in the second equation yields:

$$N^2 = \frac{(n^3 m^{3,2} - n^1 m^{1,2})}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda]} N^1 - \frac{n^3 m^{3,2}}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda]}$$

Substituting for this in the first equation yields:

$$N^1 = N^2 \frac{(n^3 m^{3,1} - n^2 m^{2,1})}{(n^1 m^{1,1} - n^3 m^{3,1} - \lambda)} - \frac{n^3 m^{3,1}}{(n^1 m^{1,1} - n^3 m^{3,1} - \lambda)}$$

Substituting for N^2 , one has:

$$N^1 = \left[\frac{(n^3 m^{3,2} - n^1 m^{1,2})}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda]} N^1 - \frac{n^3 m^{3,2}}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda]} \right] \frac{(n^3 m^{3,1} - n^2 m^{2,1})}{(n^1 m^{1,1} - n^3 m^{3,1} - \lambda)} - \frac{n^3 m^{3,1}}{(n^1 m^{1,1} - n^3 m^{3,1} - \lambda)}$$

Hence

$$\begin{aligned} & N^1 \left[1 - \frac{(n^3 m^{3,2} - n^1 m^{1,2})}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda]} \frac{(n^3 m^{3,1} - n^2 m^{2,1})}{(n^1 m^{1,1} - n^3 m^{3,1} - \lambda)} \right] \\ &= \frac{-n^3 m^{3,2} (n^3 m^{3,1} - n^2 m^{2,1})}{(n^1 m^{1,1} - n^3 m^{3,1} - \lambda) [n^2 m^{2,2} - n^3 m^{3,2} - \lambda]} - \frac{n^3 m^{3,1}}{(n^1 m^{1,1} - n^3 m^{3,1} - \lambda)} \end{aligned}$$

Hence:

$$\begin{aligned} N^1 &= \frac{\frac{-n^3 m^{3,2} (n^3 m^{3,1} - n^2 m^{2,1}) - n^3 m^{3,1} [n^2 m^{2,2} - n^3 m^{3,2} - \lambda]}{(n^1 m^{1,1} - n^3 m^{3,1} - \lambda) [n^2 m^{2,2} - n^3 m^{3,2} - \lambda]}}{\frac{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda)}} \\ &= \frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \end{aligned}$$

Then, substituting for this in N^2 , we have:

$$\begin{aligned} N^2 &= \frac{(n^3 m^{3,2} - n^1 m^{1,2}) \left[\frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \right]}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda]} \\ &\quad - \frac{\left[n^3 m^{3,2} \left[\frac{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda)}{-(n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \right] \right]}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] \left[\frac{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda)}{-(n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \right]} \end{aligned}$$

Hence

$$\begin{aligned} N^2 &= \left[\frac{\left[\frac{(n^3 m^{3,2} - n^1 m^{1,2}) [n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]]}{-n^3 m^{3,2} \left[\frac{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda)}{-(n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \right]} \right]}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] \left[\frac{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda)}{-(n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \right]} \right] \\ &= \left[\frac{[n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [(n^1 m^{1,1} - \lambda)]]}{[[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})]} \right] \end{aligned}$$

Then, given $N^3 = 1 - N^1 - N^2$,

$$N^3 = 1 - \frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} - \frac{n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [(n^1 m^{1,1} - \lambda)]}{[[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})]}$$

Hence

$$\begin{aligned} N^3 &= \frac{\left[\frac{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})}{- [n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]] - [n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [(n^1 m^{1,1} - \lambda)]]} \right]}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \\ &= \frac{(n^2 m^{2,2} - \lambda) (n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]}{[n^2 m^{2,2} - n^3 m^{3,2} - \lambda] (n^1 m^{1,1} - n^3 m^{3,1} - \lambda) - (n^3 m^{3,2} - n^1 m^{1,2}) (n^3 m^{3,1} - n^2 m^{2,1})} \end{aligned}$$

Hence the ratios x and z are:

$$\begin{aligned} x &= \frac{n^3 m^{3,2} n^2 m^{2,1} - n^3 m^{3,1} [n^2 m^{2,2} - \lambda]}{(n^2 m^{2,2} - \lambda) (n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]} \\ z &= \frac{n^1 m^{1,2} n^3 m^{3,1} - n^3 m^{3,2} [n^1 m^{1,1} - \lambda]}{(n^2 m^{2,2} - \lambda) (n^1 m^{1,1} - \lambda) - [n^1 m^{1,2} n^2 m^{2,1}]} \end{aligned}$$

8.8 Proof of Proposition 10

The Golden Rule capital k^{GR} is here:

$$k^{GR} = \left[\frac{A \alpha x^\beta z^\gamma}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta} \right]^{\frac{1}{1-\alpha}}$$

Differentiating k^{GR} with respect to n^1 yields:

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^1} &= \frac{1}{1-\alpha} \left[\frac{A \alpha x^\beta z^\gamma}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta} \right]^{\frac{1}{1-\alpha}-1} \\ &\quad \left[\frac{\left[A \alpha \left(\beta x^{\beta-1} \frac{\partial x}{\partial n^1} z^\gamma + x^\beta \gamma z^{\gamma-1} \frac{\partial z}{\partial n^1} \right) \right] [n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - [A \alpha x^\beta z^\gamma] [m^{1,3} x + n^1 m^{1,3} \frac{\partial x}{\partial n^1} + n^2 m^{2,3} \frac{\partial z}{\partial n^1}]}{[n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta]^2} \right] \end{aligned}$$

This simplifies to:

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^1} &= \frac{1}{1-\alpha} \left[\frac{A \alpha x^\beta z^\gamma}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta} \right]^{\frac{1}{1-\alpha}-1} \\ &\quad \left[\frac{\left[A \alpha \left(\beta x^{\beta-1} \frac{\partial x}{\partial n^1} z^\gamma + x^\beta \gamma z^{\gamma-1} \frac{\partial z}{\partial n^1} \right) \right] [n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - [A \alpha x^\beta z^\gamma] [m^{1,3} x + n^1 m^{1,3} \frac{\partial x}{\partial n^1} + n^2 m^{2,3} \frac{\partial z}{\partial n^1}]}{[n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta]^2} \right] \end{aligned}$$

The first factor is strictly positive, so that:

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^1} &\geq 0 \\ \iff &\left(\begin{aligned} &\frac{\partial x}{\partial n^1} [\beta z [n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - (1 - \beta) z x n^1 m^{1,3}] \\ &+ \frac{\partial z}{\partial n^1} [\gamma x [n^1 m^{1,3} x + n^3 m^{3,3} - 1 + \delta] - (1 - \gamma) x z n^2 m^{2,3}] \end{aligned} \right) \geq x z m^{1,3} x \end{aligned}$$

Differentiating k^{GR} with respect to n^2 yields:

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^2} &= \frac{1}{1 - \alpha} \left[\frac{A \alpha x^\beta z^\gamma}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta} \right]^{\frac{1}{1-\alpha}-1} \\ &\quad \left[\frac{[A \alpha (\beta x^{\beta-1} \frac{\partial x}{\partial n^2} z^\gamma + x^\beta \gamma z^{\gamma-1} \frac{\partial z}{\partial n^2})] [n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - [A \alpha x^\beta z^\gamma] [n^1 m^{1,3} \frac{\partial x}{\partial n^2} + m^{2,3} z + n^2 m^{2,3} \frac{\partial z}{\partial n^2}]}{[n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta]^2} \right] \end{aligned}$$

Hence:

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^2} &\geq 0 \\ \iff &\left[\begin{aligned} &[(\beta \frac{\partial x}{\partial n^2} z + x \gamma \frac{\partial z}{\partial n^2})] [n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] \\ &- x z [n^1 m^{1,3} \frac{\partial x}{\partial n^2} + m^{2,3} z + n^2 m^{2,3} \frac{\partial z}{\partial n^2}] \end{aligned} \right] \geq 0 \\ \iff &\left[\begin{aligned} &\frac{\partial x}{\partial n^2} [\beta z [n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - (1 - \beta) x z n^1 m^{1,3}] \\ &+ \frac{\partial z}{\partial n^2} [x \gamma [n^1 m^{1,3} x + n^3 m^{3,3} - 1 + \delta] - (1 - \gamma) x z n^2 m^{2,3}] \end{aligned} \right] \geq x z m^{2,3} z \end{aligned}$$

Differentiating k^{GR} with respect to n^3 yields:

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^3} &= \frac{1}{1 - \alpha} \left[\frac{A \alpha x^\beta z^\gamma}{n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta} \right]^{\frac{1}{1-\alpha}-1} \\ &\quad \left[\frac{[A \alpha (\beta x^{\beta-1} \frac{\partial x}{\partial n^3} z^\gamma + x^\beta \gamma z^{\gamma-1} \frac{\partial z}{\partial n^3})] [n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - [A \alpha x^\beta z^\gamma] [n^1 m^{1,3} \frac{\partial x}{\partial n^3} + n^2 m^{2,3} \frac{\partial z}{\partial n^3} + m^{3,3}]}{[n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta]^2} \right] \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial k^{GR}}{\partial n^3} &\geq 0 \\ \iff &\left[\begin{aligned} &[(\beta \frac{\partial x}{\partial n^3} z + x \gamma \frac{\partial z}{\partial n^3})] [n^1 m^{1,3} x + n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] \\ &- x z [n^1 m^{1,3} \frac{\partial x}{\partial n^3} + n^2 m^{2,3} \frac{\partial z}{\partial n^3}] \end{aligned} \right] \geq x z m^{3,3} \\ \iff &\left[\begin{aligned} &\frac{\partial x}{\partial n^3} [\beta z [n^2 m^{2,3} z + n^3 m^{3,3} - 1 + \delta] - (1 - \beta) x z n^1 m^{1,3}] \\ &+ \frac{\partial z}{\partial n^3} [x \gamma [n^1 m^{1,3} x + n^3 m^{3,3} - 1 + \delta] - (1 - \gamma) x z n^2 m^{2,3}] \end{aligned} \right] \geq x z m^{3,3} \end{aligned}$$