Trade Fragmentation and Coordination in Bilateral Oligopolies

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Abstract

This paper studies a strategic market game where agents fragment their bids on different markets. Simple conditions for existence of an interior equilibrium point are provided. In equilibrium, all agents are active on the same markets and prices are identical across markets, so that all equilibria are equivalent to an equilibrium where all agents trade on a single market.

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1 Introduction

Models of exchange economies typically assume that all agents trade on a single, anonymous market. If markets are perfectly competitive, this assumption is easily justified. Since the competitive equilibrium belongs to the core, no group of agents would benefit from trading on smaller submarkets. If markets are imperfectly competitive, the agents' incentives to trade on a single market are less transparent. In fact, the classical Cournot oligopoly analysis suggests that some agents could benefit from forming smaller submarkets in order to increase their market power. In an earlier paper (Bloch and Ghosal (1997)), we have studied a simple symmetric strategic market game with two types of traders and shown that the only stable trading structure is the grand coalition, where all agents trade on the same market. The intuition underlying this result is easily grasped. When traders on both sides of the market behave strategically, agents benefit from the presence of more traders on the market, since an increase in the number of traders results in a higher level of trade.

In this short note, we extend the analysis of Bloch and Ghosal (1997) in two directions. First, we relax the assumption of symmetry among traders, and consider a strategic market game where traders on the two sides of the market have general endowments and utility functions. Second, and most importantly, we do not assume that traders choose ex ante the market on which they trade. Instead, we suppose that agents can fragment their trade on different markets. We show that, in equilibrium, even though agents cannot coordinate their actions ex ante by choosing to trade on the same market, they coordinate their actions ex post by all trading on the same markets. Furthermore, while many markets may be active in equilibrium, all market equilibria are equivalent to the equilibrium obtained when all agents trade on a single market.

Our results thus provide an additional justification to the assumption that there exists a single market on which all agents trade. In imperfectly competitive markets, the traders' ability to influence prices creates an "agglomeration" effect which leads all agents to trade on the same markets.When all agents are active on the same markets, arbitrage opportunities preclude the emergence of different prices on different markets, so that the equilibrium allocations are independent of the number of active markets.[1].

[1] However, it should be noted that this last result may depend on the particular version

This note is organized as follows. After introducing the model, we first show that there exists an interior equilibrium point, where all traders put positive bids on the market. This existence result is based on Dubey and Shubik (1978)'s original study of equilibrium points of strategic market games and shows which assumptions are needed to prevent the emergence of trivial equilibrium points as in the example of Cordella and Gabszewicz (1998). We then prove our main result, showing that all agents trade on the same markets and that prices are identical across markets.

2 The Model

We consider a simple version of Shapley-Shubik market games (Shapley (1976), Shubik (1973) and Shapley and Shubik (1977)), which was introduced by Gabszewicz and Michel (1997) as "bilateral oligopoly". In this market game, there are two commodities labeled x and y and two types of agents. Agents of type I, indexed by $i = 1, 2, ..., I$ are endowed with the first commodity. Agents of type II, indexed by $j = 1, 2, ..., J$ are endowed with the second commodity. We let α^i denote the initial endowment of a trader i of type I, and $u^i : \Re^2 \to \Re$ her utility function. Similarly, we denote by β^j and v^j the initial endowment and utility function of a trader j of type II. For any variable t, and any function f, we let f_t denote the partial derivative of f with respect to t. We make the following assumptions on the utility functions.

1. The utility functions are strictly increasing and strictly concave.
2. The two goods are strict complements: $u^i_{xy} > 0$ and $v^j_{xy} > 0$.
3. The utility functions satisfy the following boundary conditions $\lim_{x\to 0} u^i_x = \lim_{y\to 0} u^i_y = \infty$ and $\lim_{x\to 0} v^j_x = \lim_{y\to 0} v^j_y = \infty$.

We suppose that there are M markets on which the agents can trade. We denote by b^i_m the bid of trader i on market m and by g^j_m the bid of trader

of strategic market games that we are using. Using a more general version of strategic market games, where traders can both buy and sell on the same market, Koutsougeras (1998a) and (1998b) provides examples where different prices emerge on different trading posts. The robustness of these examples to variations on the structure of the strategic market games remains an open question.

j on market m. Each trader of type I selects a vector of bids on the M markets, $b^i = (b_1^i, b_2^i, ..., b_M^i)$ and each trader of type II chooses a vector of bids $g^j = (g_1^j, g_2^j, ..., g_M^j)$. The strategy sets of the traders of type I and II are thus given by

$$S^i = \{b^i \in \Re^M | b_m^i \geq 0, \sum_m b_m^i \leq \alpha^i\}$$
$$S^j = \{g^j \in \Re^M | g_m^j \geq 0, \sum_m g_m^j \leq \beta^j\}$$

On any market m, the total amount of bids of traders of type I is given by $B_m = \sum_i b_m^i$ and the bids of traders II by $G_m = \sum_j g_m^j$. The relative price of good x on market m is obtained as $p_m = \frac{G_m}{B_m}$ if $B_m > 0$ and $p_m = 0$ if $B_m = 0$. Hence, the final allocations of the two types of traders are

$$(x^i, y^i) = \left(\alpha^i - \sum_m b_m^i, \sum_m b_m^i \frac{G_m}{B_m}\right),$$
$$(x^j, y^j) = \left(\sum_m g_m^j \frac{B_m}{G_m}, \beta^j - \sum_m g_m^j\right).$$

Definition 1 *A market equilibrium is a strategy profile* (b^{i*}, g^{j*}) *such that*

$$u^i\left(\alpha^i - \sum_m b_m^{i*}, \sum_m b_m^{i*} \frac{\sum_j g_m^{j*}}{\sum_i b_m^{i*}}\right) \geq u^i\left(\alpha^i - \sum_m b_m^i, \sum_m b_m^i \frac{\sum_j g_m^{j*}}{\sum_{k \neq i} b_m^{k*} + b_m^i}\right) \forall b^i \in S^i,$$

$$v^j\left(\sum_m g_m^{j*} \frac{\sum_i b_m^{i*}}{\sum_j g_m^{j*}}, \beta^j - \sum_m g_m^{j*}\right) \geq v^j\left(\sum_m g_m^j \frac{\sum_i b_m^{i*}}{\sum_{k \neq j} g_m^{k*} + g_m^j}, \beta^j - \sum_m g_m^j\right) \forall g^j \in S^j.$$

3 Market Equilibria

It is well known that the strategic market game we analyze always admits a trivial equilibrium, where all traders put zero bids on all the markets. As a first step in the analysis, we show that there exists an equilibrium where all traders make positive bids on a single trading post. To this end, we first define an *equilibrium point*, as in Dubey and Shubik (1978) and Amir et al. (1990), as the limit of a sequence of equilibria of perturbed games.

A perturbed game Γ^ε is a game where an outside agency puts a fixed quantity $\varepsilon > 0$ of the two goods on trading post $m = 1$. Hence, the price of good x on trading post 1 in the perturbed game Γ^{ε} is $p_1^{\varepsilon} = \frac{G_1+\varepsilon}{B_1+\varepsilon}$. We let $(b^{i*,\varepsilon}, g^{j*,\varepsilon})$ denote a market equilibrium of the perturbed game Γ^{ε}.

Definition 2 *An equilibrium point is a market equilibrium* (b^{i*}, g^{j*}) *such that there exists a sequence* ε_n*, with* $\lim_{n\to\infty}\varepsilon_n = 0$ *and market equilibria of the perturbed games* Γ^{ε_n} *with* $\lim_{n\to\infty}(b^{i*,\varepsilon_n}, g^{j*,\varepsilon_n}) = (b^{i*}, g^{j*})$.

In words, equilibrium points are market equilibria which can be approached by a sequence of equilibria of perturbed games. We recall the following result from Dubey and Shubik (1978) which will prove helpful in the proof of existence of an interior equilibrium.

Lemma 3 *(Dubey and Shubik, 1978). In a game with more than two traders of the two types, there exist positive constants* C *and* D *such that* $C < p_1^{*,\varepsilon} = \frac{G_1^{*,\varepsilon}+\varepsilon}{B_1^{*,\varepsilon}+\varepsilon} < D$ *for all* $\varepsilon > 0$.

Notice that this Lemma shows that there exist *uniform bounds* on the relative prices of the two commodities. However, it does not guarantee that equilibrium bids are bounded away from zero. In fact, examples can be constructed to show that all bids can be equal to zero at an equilibrium point. (See Cordella and Gabszewicz (1998) for a simple example with linear utilities.) In our model, the boundary assumptions on traders' utilities guarantee the existence of an equilibrium point where all traders put positive bids on market 1.

Theorem 4 *There exists an equilibrium point where all traders put positive bids on market* 1.

Proof: Consider the perturbed game Γ^{ε}. We consider an equilibrium where all traders put zero bids on all markets $m = 2, .., M$. For simplicity, we drop the market subscript on all variables. Consider the behavior of a trader of type I. She solves the following problem.

$$\max_{b^i\in[0,\,\alpha^i]} u^i\left(\alpha^i - b^i, b^i\,\frac{G+\varepsilon}{b^i + \sum_{k\neq i} b^k + \varepsilon}\right)$$

Taking derivatives we obtain

$$\frac{\partial u^i}{\partial b^i} = -u_x^i + u_y^i \frac{(G+\varepsilon)(\sum_{k\neq i} b^k + \varepsilon)}{\left(b^i + \sum_{k\neq i} b^k + \varepsilon\right)^2}.$$

The second derivative is given by

$$\begin{aligned}\frac{\partial^2 u^i}{\partial b^{i2}} &= u_{xx}^i - 2u_{xy}^i \frac{(G+\varepsilon)(\sum_{k\neq i} b^k + \varepsilon)}{\left(b^i + \sum_{k\neq i} b^k + \varepsilon\right)^2} + u_{yy}^i \left(\frac{(G+\varepsilon)(\sum_{k\neq i} b^k + \varepsilon)}{\left(b^i + \sum_{k\neq i} b^k + \varepsilon\right)^2}\right)^2 \\ &\quad -2u_y^i \frac{(G+\varepsilon)(\sum_{k\neq i} b^k + \varepsilon)}{\left(b^i + \sum_{k\neq i} b^k + \varepsilon\right)^3}\end{aligned}$$

Since the utility function is strictly increasing, strictly concave, and the two goods are strict complements, $\frac{\partial^2 u^i}{\partial b^{i2}} < 0$. The boundary conditions on the utility function guarantee that $\lim_{b^i \to 0} \frac{\partial u^i}{\partial b^i} > 0$ and $\lim_{b^i \to \alpha^i} \frac{\partial u^i}{\partial b^i} < 0$. Hence there exists a unique interior solution to trader i's maximization problem. Define this solution as $\phi^i \left(\underset{k\neq i}{b^k}, g^j\right)$ and construct the function $\phi : \times_i [0, \alpha^i] \times_j [0, \beta^j]$, $\phi = \times_i \phi^i \times_j \phi^j$. The function ϕ is a continuous function over a compact, convex subset of a Euclidean space and admits a fixed point by Brouwer's fixed point theorem. We denote this fixed point by $(b^{i*,\varepsilon}, g^{j*,\varepsilon})$.

In the second step of the proof, we show that for any trader i there exist constants D^i, E^i such that

$$0 < D^i \leq b^{i*,\varepsilon} \leq E^i < \alpha^i.$$

A similar argument can be used to show that the bids of traders of type II are uniformly bounded as well. We first show $b^{i*,\varepsilon} < \alpha^i$. Fix the strategy choices of the other players in equilibrium, and consider trader i's marginal utility,

$$\frac{\partial u^i}{\partial b^i} = -u_x^i + u_y^i \frac{(G^{*,\varepsilon}+\varepsilon)(\sum_{k\neq i} b^{k*,\varepsilon} + \varepsilon)}{\left(b^i + \sum_{k\neq i} b^{k*,\varepsilon} + \varepsilon\right)^2}$$

Note that

$$\frac{(G^{*,\varepsilon}+\varepsilon)(\sum_{k\neq i} b^{k*,\varepsilon} + \varepsilon)}{\left(b^i + \sum_{k\neq i} b^{k*,\varepsilon} + \varepsilon\right)^2} < \frac{(G^{*,\varepsilon}+\varepsilon)}{\left(b^i + \sum_{k\neq i} b^{k*,\varepsilon} + \varepsilon\right)}$$

Now consider a fixed lower bound z^i on trader i's bid, with $z^i > 0$ independent of ε. Then

$$\frac{(G^{*,\varepsilon}+\varepsilon)}{\left(b^i+\sum_{k\neq i} b^{k*,\varepsilon}+\varepsilon\right)} \leq \frac{(G^{*,\varepsilon}+\varepsilon)}{\left(z^i+\sum_{k\neq i} b^{k*,\varepsilon}+\varepsilon\right)} \leq \frac{(G^{*,\varepsilon}+\varepsilon)}{(z^i+\varepsilon)}$$
$$< \frac{G^{*,\varepsilon}}{z^i}+1 < \frac{\sum_j \beta^j}{z^i}+1$$

Hence, for all $b^i > z^i$ and for all ε,

$$\frac{\partial u^i}{\partial b^i} < -u^i_x + u^i_y\left(\frac{\sum_j \beta^j}{z^i}+1\right).$$

Next observe that since $C < p_1^{*,\varepsilon} < D$, $b^i p_1^{*,\varepsilon}$ and u^i_y are uniformly bounded. Furthermore, $\lim_{b^i\to\alpha^i} u^i_x = \infty$, so that $\lim_{b^i\to\alpha^i} \frac{\partial u^i}{\partial b^i} < 0$ for all $\varepsilon > 0$, showing that there exists a uniform bound E^i such that $b^{i*,\varepsilon} \leq E^i < \alpha^i$.

Now since $b^{i*,\varepsilon} \leq E^i < \alpha^i$, in equilibrium

$$\frac{\partial u^i}{\partial b^i} = -u^i_x + u^i_y \frac{(G^{*,\varepsilon}+\varepsilon)(\sum_{k\neq i} b^{k*,\varepsilon}+\varepsilon)}{(B^{*,\varepsilon}+\varepsilon)^2} \leq 0$$

Since $C < p_1^{*,\varepsilon} < D$, $b^i p_1^{*,\varepsilon}$ and u^i_x are uniformly bounded. Furthermore, $\frac{B^{*,\varepsilon}+\varepsilon}{G^{*,\varepsilon}+\varepsilon} < \frac{1}{C}$. Hence there exists a uniform bound F such that

$$u^i_y \frac{(\sum_{k\neq i} b^{k*,\varepsilon}+\varepsilon)}{(B^{*,\varepsilon}+\varepsilon)} \leq F$$

Now consider
$$\tilde{U}_y = \max_i u^i_y$$

Summing over i, we obtain:

$$\tilde{U}_y \frac{(I-1)\,B^{*,\varepsilon}+I\varepsilon}{(B^{*,\varepsilon}+\varepsilon)} \leq IF$$

Implying
$$u^i_y \leq \tilde{U}_y \leq (I/(I-1))F.$$

But since $\lim_{b^i\to 0} u^i_y = \infty$, and $C < p_1^{*,\varepsilon} < D$, there must exist a uniform lower bound D^i on trader i's bid, such that $0 < D^i \leq b^i$ for all $\varepsilon > 0$.

To complete the proof, consider a sequence ε_n converging to 0. We have just shown that $\forall i$, $b^{i*,\varepsilon_n} \in [D^i, E^i]$. Similarly, $\forall j$, $g^{j*,\varepsilon_n} \in [D^j, E^j]$. The sequence $(b^{i*,\varepsilon_n}, g^{j*,\varepsilon_n})$ is thus defined over a compact set. Taking subsequences if necessary, it converges to a limit point (b^{i*}, g^{j*}) where $b^{i*} \in [D^i, E^i]$ and $g^{j*} \in [D^j, E^j]$. By continuity of the utility functions, it is clear that (b^{i*}, g^{j*}) is a Nash equilibrium of the game Γ.■

The preceding Theorem shows that there exists an interior market equilibrium, where all traders put positive bids on market 1. We now characterize *all* the equilibria of the game, and show that they are equivalent to an equilibrium with a single trading post.

Theorem 5 *In any market equilibrium, all agents are active on the same markets. Prices are identical across markets, so that any market equilibrium is equivalent to an equilibrium with a single trading post.*

Proof: We first show that all agents are active on the same markets. As a first step, we prove that the ranking of bids on different markets has to be identical. In other words, if $b^i_\mu > b^{i'}_\mu$ for some market μ and some traders i,i' of type I, then $b^i_m \geq b^{i'}_m$ for all markets $m = 1, 2, .., M$.

To prove this statement, consider market μ. We immediately obtain that

$$\sum_{k \neq i} b^k_\mu < \sum_{k \neq i'} b^k_\mu.$$

Hence

$$\frac{u^{i'}_x}{u^{i'}_y} \geq \frac{G_\mu}{B_\mu} \frac{\sum_{k \neq i'} b^k_\mu}{B_\mu} > \frac{G_\mu}{B_\mu} \frac{\sum_{k \neq i} b^k_\mu}{B_\mu} = \frac{u^i_x}{u^i_y}.$$

If there exists a market μ' on which $b^{i'}_{\mu'} > b^i_{\mu'} \geq 0$, the preceding inequality is reversed, yielding a contradiction. The above argument shows that the sets of agents participating to different markets are monotone: if $I(m)$ denotes the set of agents active at trading post m, then there exists an ordering of trading posts for which $I(1) = I$ and $I(m) \subseteq I(m-1)$ for all m. We will show that $I(m) = I(m-1)$ for all m such that $I(m) \neq \emptyset$.

Suppose to the contrary that there exist two markets m and m' such that $I(m') \supset I(m) \neq \emptyset$. Let i be a trader in $I(m)$ and i' a trader in $I(m') \backslash I(m)$.

Since $\dfrac{\partial\, u^{i'}}{\partial\, b^{i'}_{m'}} = 0$,

$$\frac{u^{i'}_x}{u^{i'}_y} = \frac{G_{m'}}{B_{m'}} \frac{\sum_{k \neq i'} b^k_{m'}}{B_{m'}}$$

Since $\dfrac{\partial\, u^{i'}}{\partial\, b^{i'}_{m}} < 0$

$$\frac{u^{i'}_x}{u^{i'}_y} > \frac{G_m}{B_m} \frac{\sum_{k \neq i'} b^k_m}{B_m} = \frac{G_m}{B_m}$$

Hence, we have

$$\frac{G_{m'}}{B_{m'}} > \frac{G_m}{B_m}$$

$>$From the equations characterizing the optimal behavior of trader i, we obtain

$$\begin{aligned}
\frac{u^i_x}{u^i_y} &= \frac{G_m}{B_m} \frac{\sum_{k \neq i} b^k_m}{B_m} \\
\frac{u^i_x}{u^i_y} &= \frac{G_{m'}}{B_{m'}} \frac{\sum_{k \neq i} b^k_{m'}}{B_{m'}}
\end{aligned}$$

Since $\dfrac{G_{m'}}{B_{m'}} > \dfrac{G_m}{B_m}$,

$$\frac{\sum_{k \neq i} b^k_m}{B_m} > \frac{\sum_{k \neq i} b^k_{m'}}{B_{m'}}$$

Summing up over all agents in $I(m')\backslash I(m)$ and letting $\iota = \#(I(m')\backslash I(m))$,

$$\sum_{i' \in I(m')\backslash I(m)} \frac{\sum_{k \neq i} b^k_{m'}}{B_{m'}} < \sum_{i' \in I(m')\backslash I(m)} \frac{\sum_{k \neq i} b^k_m}{B_m} = \iota - 1.$$

Letting $B^1_{m'} = \sum_{k \in I(m')\backslash I(m)} b^k_{m'}$ and $B^2_{m'} = \sum_{k \in I(m)} b^k_{m'}$,

$$\sum_{i' \in I(m')\backslash I(m)} \frac{\sum_{k \neq i} b^k_{m'}}{B_{m'}} = \frac{(\iota - 1)B^1_{m'} + \iota B^2_{m'}}{B^1_{m'} + B^2_{m'}} > \iota - 1,$$

yielding a contradiction.

We now prove that on any two active trading posts m and $m^{'}$, prices are identical. In equilibrium, we have

$$\frac{G_m}{B_m}\frac{\sum_{k\neq i} b_m^k}{B_m} = \frac{G_{m'}}{B_{m'}}\frac{\sum_{k\neq i} b_{m'}^k}{B_{m'}}$$

for all agents i. Summing up over the agents, we find

$$\frac{G_m}{B_m}(I-1) = \frac{G_{m'}}{B_{m'}}(I-1)$$

so that the prices are equal on the two markets.

Finally, observe that this last statement shows that any equilibrium with multiple markets is equivalent to an equilibrium with a single market. If (b^i, g^j) is an equilibrium with a single trading post, any fragmentation of the trade, where all agents put the same fractions $\lambda^m b^i$ and $\lambda^m g^j$ on market m is an equilibrium of the game with multiple trading posts. Conversely, starting from an equilibrium with multiple trading posts, strategies where all agents put on a single trading post the bids $\sum_m b_m^i, \sum_m g_m^j$ form an equilibrium of the game.■

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