An Intensity Model for Credit Risk with Switching Lévy Processes.

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Abstract

We develop a switching regime version of the intensity model for credit risk pricing. The default event is specified by a Poisson process whose intensity is modeled by a switching Lévy process. This model presents several interesting features. Firstly, as Lévy processes encompass numerous jump processes, our model can duplicate sudden jumps observed in credit spreads. Also, due to the presence of jumps, probabilities do not vanish at very short maturities, contrary to models based on Brownian dynamics. Furthermore, as parameters of the Lévy process are modulated by a hidden Markov chain, our approach is well suited to model changes of volatility trends in credit spreads, related to modifications of unobservable economic factors.

Keywords. Regime-switching model, Markov chain, Lévy process.

1 Introduction.

Assessing correctly credit risk is a matter of concerns for all institutional lenders. There exists two main approaches to price the risk of default. The first category of models is called structural and prices a defaultable debt by an option theoretic approach wherein the debt raised is an option on the firm value. Merton (1974) pioneered this approach by pricing debt assuming both a constant interest rate and a constant volatility of the firm assets. Intensity models are an efficient alternative to structural models. In an intensity model, the time of default is modeled directly as the time of the first jump of a Poisson process with random intensity (a Cox Process). In this group of models, a striking similarity to default-free interest rate modeling can be found. The first models of this type were developed by Jarrow and Turnbull (1995), Madan and Unal (1998) and Duffie and Singleton (1999). Lando (1998) developed the Cox-process methodology with iterated conditional expectations.

Most of mathematical credit risk models use Brownian motion as a source of uncertainty. It ensures a certain analytical tractability but also displays serious drawbacks. In structural models for instance, credit spreads vanish for short term corporate bonds when asset dynamics are lognormal: this is not realistic from an empirical viewpoint. An efficient way to avoid such features is to replace Brownian motion by a jump process, typically a compound Poisson or a Lévy process. In the case of structural models, we can cite the following contributions that use compound Poisson processes to compute credit spreads : Chen and Kou (2009), Dao and Jeanblanc (2012), Maenhout (2008), and Scherer (2005). Le Courtois and Quittard-Pinon (2006) developed a similar analysis for the computation of default probabilities. In a second paper published in 2008, they constructed a structural model relying on the use of stable Lévy processes. See also the PhD thesis of Cariboni

(2007). In the case of intensity models, we can cite the previously-mentioned PhD thesis and the contributions of Cariboni and Schoutens (2006, 2008) and the PhD thesis of Kluge (2005). For an overview, see the book of Cariboni and Schoutens (2009). For a recent contribution on point processes, see Giesecke, Kakavand and Mousavi (2011).

Even if credit risk models based on Lévy processes represent a significant advance in research, they are still partly unsatisfactory. In particular, as mentioned in Maalaoui et al. (2010), there exist evidence that credit spreads exhibit changes of trends that cannot be replicated by Lévy processes. In particular, the volatility of credit spreads can suddenly switch from a low to a high level, after a rating downgrade or in a phase of decline. Numerous theoretical papers use regime switches to capture state dependent movements in credit spread dynamics. The contribution of this paper is to explore the ability of switching Lévy processes to model credit risk, in an intensity framework. Switching Lévy processes are constructed from Lévy processes whose parameters are modulated by a hidden Markov chain. For a survey of properties of this category of processes, we refer the reader to the seminal papers of Buffington and Elliot (2002), and Elliott, Chan and Siu (2005).

The outline of this paper is as follows. In the first two sections, we develop an affine intensity based model and define the hidden Markov chain modulating the intensity of default. Next, we briefly describe the switching Lévy process driving the default rate. The following sections develop a numerical method to assess survival probabilities. Finally, after a review of switching versions of popular Lévy processes, we present an econometric calibration procedure and fit these processes to historical default intensities of four companies. Finally, we test the ability of this family of models to replicate survival probability curves, bootstrapped from CDS quotes.

2 Intensity Model

For a given time horizon T, the default of the firm is triggered by the first jump of a Poisson process N_t, whose intensity is a stochastic process λ_t. This latter quantity may be seen as the instantaneous failure rate and is assumed independent from N_t. The default time of the firm is a stopping time τ defined on a filtration noted $\mathcal{G}$. We also define $\mathcal{H}$ as the filtration carrying information about λ_t (see section 4 for details). $\left(\Omega,\, \mathcal{F},\, (\mathcal{F}_t)_{t\in[0,T]}\; P\right)$ is a filtered probability space such that $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$. As in Bielecki and Rutkowski (2002, chapter 8), we assume that conditionally on the path followed by the intensity until time $T > t$, the probability that a firm is still active at time t is given by:

$$P(\tau \geq t \,|\, \mathcal{H}_T) \quad = \quad e^{-\int_0^t \lambda_s ds},$$

whilst the survival probability from time $u < t$ to time t is given by:

$$P(\tau \geq t \,|\, \mathcal{F}_u) \quad = \quad 1_{\tau > u} E\left(e^{-\int_u^t \lambda_s ds} \,|\, \mathcal{H}_u\right).$$

Furthermore, the process defined for all t by $N_{t\wedge\tau} - \int_0^{t\wedge\tau} \lambda_s ds$ is a martingale under the considered measure.

In our approach, the intensity is stochastic and depends on the state of the economy. As mentioned in the introduction, there are evidences that credit spreads exhibit changes of trends that are directly related to the evolution of unobservable economic factors. In periods of economic recession, defaults are more likely and the volatility of credit spreads can be important. However, in periods of economic growth, spreads are smaller and less volatile. To model this phenomenon, we assume that the economic conjuncture can be categorized into a finite number of N states. We consider a Markov chain α_t that contains the information about the effective economic factors but that is not directly observable. This hidden process influences the dynamics of the failure rate. In the remainder of this work, we consider that λ admits the following dynamics:

$$d\lambda_t \quad = \quad a(\lambda_t, \alpha_t)dt + dX_t^{\alpha_t}, \tag{2.1}$$

where the process defined for all t by $X_t^{\alpha_t}$ is a switching Lévy process (the features of this category of processes are detailed in the next section) and where $a(\lambda_t, \alpha_t)$ is a linear function of λ_t given by the following expression:

$$a(\lambda_t, \alpha_t) \quad = \quad a_{1,\alpha_t} + a_2 \lambda_t.$$

Note that, if by convention $\tilde{a}_2 = -a_2$ and $\tilde{a}_{1,\alpha_t} = \frac{a_{1,\alpha_t}}{\tilde{a}_2}$, the intensity can be rewritten as a switching mean-reverting process:

$$d\lambda_t = \tilde{a}_2 \left(\tilde{a}_{1,\alpha_t} - \lambda_t\right) dt + dX_t^{\alpha_t}. \tag{2.2}$$

In this formulation, the long term mean of the failure rate $\frac{a_{1,\alpha_t}}{\tilde{a}_2}$ and the source of uncertainty $X_t^{\alpha_t}$ both depend on the state of the economy. However, the speed of mean reversion is assumed in our framework independent from α_t. This may be seen as an intrinsic feature of the firm.

3 The Markov chain

In this article, we model the source of noise X in the dynamics of default intensities by a Lévy process whose parameters depend on a certain state of a hidden Markov chain. The state indicator, denoted by α, is a Markov chain that is not directly observable. This approach allows us to model the eventual changes of trends exhibited by credit spreads.

Under the assumption that there exist N states, α takes its values in the set $\mathcal{N} = \{1, 2, ...N\}$ and admits an intensity matrix Q whose elements, denoted by $q_{i,j}$, satisfy the following conditions:

$$q_{i,j} \geq 0 \quad \forall i \neq j \qquad\qquad \sum_{j=1}^{N} q_{i,j} = 0 \quad \forall i \in \mathcal{N}. \tag{3.1}$$

The transition probabilities (under the real measure) between any two times t and $u \geq t$ are computed as the (matrix) exponential of Q :

$$P(t, u) = \exp\left(Q(u - t)\right). \tag{3.2}$$

The elements of the matrix $P(t, u)$ are denoted by $p_{i,j}(t, u)$ for all $i, j \in \mathcal{N}$. Indeed, $p_{i,j}(t, u)$ is the probability of jumping from state i at time t to state j at time u :

$$p_{i,j}(t, u) \quad = \quad P(\alpha_u = j \,|\, \alpha_t = i) \quad i, j \in \mathcal{N}. \tag{3.3}$$

The probability of being in state i at time t, denoted by $p_i(t)$, can be expressed as a function of the initial probabilities $p_{k=1..N}(0)$ at time $t = 0$ as follows:

$$p_i(t) \quad = P(\alpha_t = i) = \quad \sum_{k=1}^{N} p_k(0) p_{k,i}(0, t) \quad \forall i \in \mathcal{N}. \tag{3.4}$$

When the time tends to infinity, it can be shown that the asymptotic probability is independent from the initial state (steady-state probabilities):

$$\lim_{t \to +\infty} p_i(t) = p_i \qquad \forall i \in \mathcal{N}. \tag{3.5}$$

In this framework, we denote by τ_i the random time at which the Markov chain α changes of state for the i^{th} times.

Among the approaches chosen to model the Markov chain, we adopt the marked point process one for its simplicity. Following Landen (2000), we define a mark space E which includes all possible regime switches as:

$$E = \{z = (i, j) \,:\, i \in \{1, \ldots, N\}\; j \in \{1, \ldots, N\}\,,\, i \neq j\}.$$

The σ-algebra generated by E is denoted by $\mathcal{E}$. On $\mathcal{E}$ we define a marked point process $\mu(t, .)$. See Bremaud (1981) for an introduction to these processes. If A is a subset of E, $\mu(t, A)$ counts the cumulative number of regime shifts that belong to A during $(0; t]$. The compensator of $\mu(t; .)$ is given by:

$$\gamma(dt, dz) = \sum_{i \neq j} q_{i,j}\; I(\alpha_{t-} = i)\; \epsilon_{(i,j)}(dz)\; dt,$$

where $I(.)$ is the indicator function and $\epsilon_{(i,j)}$ denotes the Dirac measure at point $z = \{i, j\}$. The Markov chain α is equal to an integral on E of the function $\eta(z) = \eta(i, j) = j - i$ with respect to time and to the marked point process:

$$\alpha_t = \int_0^t \int_E \eta(z)\, \mu(ds, dz).$$

The natural filtration of α_t is noted $\mathcal{E}_t$. Furthermore, if we define $q(t, z) = \mu(t, z) - \gamma(t, z)$, then:

$$\begin{aligned} M_t &= \alpha_t - \int_0^t \int_E \eta(z)\, \gamma(ds, dz) \\ &= \int_0^t \int_E \eta(z)\, q(ds, dz), \end{aligned}$$

is a local martingale under the real measure P.

4 Switching Lévy Processes

The dynamics of the failure rate λ_t is driven by a particular stochastic process X_t^{α} which is a Lévy process conditionally on the state of the economy α. Remember that the filtration of λ_t and then of X_t^{α} is noted $\mathcal{H}$. A Lévy process is a càdlàg stochastic process, continuous in probability, with independent and stationary increments. We refer the reader to Appelbaum (2004) for a detailed presentation. A switching Lévy process may be seen as a piecewise Lévy process. By piecewise, we mean that the process X^{α} is a Lévy process characterized by a set of parameters that depend on the state of α. If between two times $[\tau_1, \tau_2]$ of transition, the Markov chain α is in state j, the switching Lévy process is driven by the following SDE:

$$dX_t^{\alpha_t} = dX_t^j \quad \alpha_t = j \in \mathcal{N},$$

where each X_t^j is a Lévy process, independent from others, defined on filtrations noted by $\mathcal{H}^j$. Remark that X_t^j is only observable when $\alpha_t = j$ and then $\mathcal{H} \neq \mathcal{H}^1 \vee ... \vee \mathcal{H}^N$. Each X^j can be split into three components (according to the Lévy-Itō decomposition): a deterministic drift of parameter β_j, a Brownian motion of unit-time variance σ_j^2, and a jump process given by $J_{X^j}(t\,,\,z)$. The intensity of the latter component is $\nu(j\,,\,z)$. This is the Lévy measure of X in state j. By Lévy measure, we mean that the probability of observing k jumps of size included in a set $B \subset \mathbb{R}$ between $[t_1, t_2]$ is given by:

$$P\left(J_{X^j}([t_1, t_2] \times B) = k\right) = e^{-\int_{t_1}^{t_2} \int_B \nu(j, dz) dt} \frac{\left(\int_{t_1}^{t_2} \int_B \nu(j, dz) dt\right)^k}{k!}. \tag{4.1}$$

If W^j designates a standard Brownian motion, the Lévy-Itō decomposition of X^j is given by:

$$dX_t^j = \beta_j dt + \sigma_j dW_t^j + \int_{|z|>1} z\, J_{X^j}(dt\,,\,dz) + \int_{|z|\leq 1} z\, \left(J_{X^j}(dt\,,\,dz) - \nu(j\,,\,dz)dt\right). \tag{4.2}$$

The triplet $(\beta_j\,,\,\sigma_j\,,\,\nu(j,z))$ fully determines the characteristic function of X^j:

$$\begin{aligned}
\phi_t^j(u) &= \mathbb{E}\left(\exp\left(iuX_t^j\right)\,|\,\mathcal{H}_0\right) \\
&= \exp\left(t\left(i\,\beta_j\,u-\frac{1}{2}\sigma_j^2\,u^2+\int_{\mathbb{R}}\left(e^{iuz}-1-i\,u\,z\,1_{|z|\leq 1}\right)\nu(j,dz)\right)\right).
\end{aligned}$$

Hence, the dynamics defined in equation (2.1) can be rewritten as follows:

$$\begin{aligned}
d\lambda_t &= \left(a(\lambda_t,\alpha_t)+\,\beta_{\alpha_t}\right)dt+\sigma_{\alpha_t}\,dW_t \\
&\quad+\int_{|z|>1} z\,J_{X^{\alpha_t}}(dt\,,\,dz) \\
&\quad+\int_{|z|\leq 1} z\,\left(J_{X^{\alpha_t}}(dt\,,\,dz)-\nu(\alpha_t\,,\,dz)dt\right),
\end{aligned}$$

If we consider finite variation Lévy processes, such that:

$$\int_{|z|\leq 1} z\,\nu(j,dz)<+\infty \quad j=1...N,$$

the dynamics of λ can be simplified as follows:

$$d\lambda_t=\left(a(\lambda_t,\alpha_t)+\,\beta^{'}_{\alpha_t}\right)dt+\sigma_{\alpha_t}dW_t+\int z\,J_{X^{\alpha_t}}(dt\,,\,dz),$$

where $\beta^{'}_{\alpha_t}=\beta_{\alpha_t}-\int_{|z|\leq 1} z\,\nu(\alpha_t\,,\,dz)$.

We end this paragraph with a remark about filtrations. As mentioned earlier, $\mathcal{F}$ is the filtration on which the process N and the intensity λ are defined. We underline the fact that $\mathcal{F}$, is not the smallest filtration including both $\mathcal{E}$ and $\mathcal{H}$ simply because the process α (adapted to $\mathcal{E}$) is not visible. However, the relationship $\mathcal{F}_t=\mathcal{G}_t\vee\mathcal{H}_t\subset\mathcal{E}_t\vee\mathcal{G}_t\vee\mathcal{H}_t$ obviously holds for all t. This relationship will play an important role in the forthcoming developments.

5 Default Probabilities

This section presents a method to calculate corporate default probabilities when the dynamics of the failure rate is driven by a switching Lévy process. This is particularly useful for pricing defaultable claims such as defaultable zero coupon bonds or credit default swaps. For example, let us assume that a company issues a zero coupon bond of nominal L that is fully repaid at time T if no default has occurred. If the firm goes bankrupt before this time horizon, only a fraction R of the nominal is repaid at maturity. The price of this bond, noted $B(t,T)$, is the discounted expected cash-flow with respect to the available information:

$$\begin{aligned}
B(t,T) &= \mathbb{E}\left(e^{-r(T-t)}\left(L1_{\tau>T}+R\,L\,1_{\tau\leq T}\right)\,|\,\mathcal{F}_t\right) \\
&= e^{-r(T-t)}\,L\,R+e^{-r(T-t)}L\,(1-R)\,\mathbb{E}\left(\,1_{\tau>T}\,|\,\mathcal{F}_t\right),
\end{aligned} \tag{5.1}$$

where $\mathbb{E}\left(\,1_{\tau>T}\,|\,\mathcal{F}_t\right)$ is the survival probability of the firm till time T, conditionally to $\mathcal{F}_t$. By the tower property of conditional expectations, this is equivalent to:

$$\begin{aligned}
\mathbb{E}\left(\,1_{\tau>T}\,|\,\mathcal{F}_t\right) &= \mathbb{E}\left(\mathbb{E}\left(1_{\tau>T}\,|\,\mathcal{E}_t\vee\mathcal{G}_t\vee\mathcal{H}_t\right)\,|\,\mathcal{F}_t\right) \\
&= 1_{\tau>t}\mathbb{E}\left(\mathbb{E}\left(1_{\tau>T}\,|\,\mathcal{E}_t\vee\mathcal{H}_t,\,\tau\geq t\right)\,|\,\mathcal{F}_t\right),
\end{aligned} \tag{5.2}$$

where the expectation $\mathbb{E}\left(1_{\tau>T}\,|\,\mathcal{E}_t\vee\mathcal{H}_t,\,\tau\geq t\right)$ is the probability that the company survives till time T for given λ_t and α_t. We denote it by $P(t,T,\lambda_t,\alpha_t)$ and have that:

$$\begin{aligned}P(t,T,\lambda_t,\alpha_t) &= P(\tau\geq T|\,\mathcal{H}_t\vee\mathcal{E}_t,\,\tau\geq t) \\ &= \frac{P(\tau\geq T|\,\mathcal{H}_t\vee\mathcal{E}_t)}{P(\tau\geq t|\,\mathcal{H}_t\vee\mathcal{E}_t)} \\ &=\mathbb{E}\left(e^{-\int_t^T\lambda_s ds}\,|\,\mathcal{H}_t\vee\mathcal{E}_t\right).\end{aligned} \tag{5.3}$$

In addition, we have the following natural boundary conditions:

$$P(T,T,\lambda_T,\alpha)=1,$$

and:

$$\lim_{\lambda_t\to+\infty}P(t,T,\lambda_t,\alpha)=0.$$

The following result allows us to calculate the $P(t,T,\lambda_t,\alpha_t)$.

Proposition 5.1. *Let us denote Q the matrix of transition probabilities of the Markov chain α_t and $F(t)$ the $N-$vector of:*

$$f(t,j)=-a_{1,j}\,B(t)+\psi_j(i\,B(t)). \tag{5.4}$$

The survival probabilities $P(t,T,\lambda_t,\alpha_t)$ are given by the following expression:

$$P(t,T,\lambda,j)=\exp(A(t,j)-B(t)\lambda), \tag{5.5}$$

where $B(t)$ is a function of time:

$$B(t)=\frac{1}{a_2}\left(e^{a_2(T-t)}-1\right), \tag{5.6}$$

and where $\tilde{A}(t)=\left[e^{A(t,1)},\,...,\,e^{A(t,N)}\right]'$ is a vector, solution of the ODE system:

$$\frac{\partial\tilde{A}(t)}{\partial t}+(diag(F(t))+Q)\,\tilde{A}(t)=0, \tag{5.7}$$

under the terminal boundary condition:

$$\tilde{A}(T,j)=1\quad j=1...N.$$

Proof. By definition of $P(t,T,\lambda_t,\alpha_t)$, we have that for all $u\geq t$:

$$P(t,T,\lambda_t,\alpha_t)=\mathbb{E}\left(\mathbb{E}\left(e^{-\int_t^T\lambda_s ds}\,|\,\mathcal{H}_u\vee\mathcal{E}_u\right)\,|\,\mathcal{H}_t\vee\mathcal{E}_t\right),$$

yielding, thanks to the definition (5.3) of the quantity P:

$$P(t,T,\lambda_t,\alpha_t)=\mathbb{E}\left(e^{-\int_t^u\lambda_s ds}P(u,T,\lambda_u,\alpha_u)\,|\,\mathcal{H}_t\vee\mathcal{E}_t\right).$$

Then, by assuming enough regularity to allow one to take the limit within the expectation, the following limit converges to zero:

$$\lim_{u\to t}\frac{\mathbb{E}\left(e^{-\int_t^u\lambda_s ds}P(u,T,\lambda_u,\alpha_u)\,|\,\mathcal{H}_t\vee\mathcal{E}_t\right)-P(t,T,\lambda_t,\alpha_t)}{u-t}=0.$$

If we develop the exponential by its Taylor approximation of first order, we can rewrite this limit as:

$$\lim_{u\to t}\frac{\mathbb{E}\left(P(u,T,\lambda_u,\alpha_u)\,|\,\mathcal{H}_t\vee\mathcal{E}_t\right)-P(t,T,\lambda_t,\alpha_t)}{u-t}=\lambda_t P(t,T,\lambda_t,\alpha_t)\,.$$

The right hand term being calculable by the Itō formula for switching Lévy processes, we infer that $P(t,T,\lambda,\alpha_t)$ is the solution of a system of partial integro-differential equations:

$$\frac{\partial}{\partial t}P(t,T,\lambda,j)\ +\mathcal{L}P(t,T,\lambda,j)=\lambda P(t,T,\lambda,j)\quad j=1\ldots N, \tag{5.8}$$

where $\mathcal{L}P(t,T,\lambda,j)$ is the generator of the switching Lévy process:

$$\begin{aligned}\mathcal{L}P(t,T,\lambda,j)&=(a(\lambda,j)+\beta_j)\,\frac{\partial P}{\partial\lambda}+\frac{\sigma_j^2}{2}\frac{\partial^2 P}{\partial\lambda^2}\\&+\sum_{k\neq j}q_{j,k}\left(P(t,T,\lambda,k)-P(t,T,\lambda,j)\right)\\&+\int_{\mathbb{R}\backslash\{0\}}\left(P(t,T,\lambda+z,j)-P(t,T,\lambda,j)-z\,1_{|z|\leq 1}\,\frac{\partial P}{\partial\lambda}\right)\nu(j,dz)\,.\end{aligned} \tag{5.9}$$

As $\sum_{k\neq j}^{N}q_{j,k}=-q_{j,j}$, this last expression is also equivalent to:

$$\begin{aligned}\mathcal{L}P(t,T,\lambda,j)&=(a(\lambda,j)+\beta_j)\,\frac{\partial P}{\partial\lambda}+\frac{\sigma_j^2}{2}\frac{\partial^2 P}{\partial\lambda^2}\ +\sum_{k=1}^{N}q_{j,k}\,P(t,T,\lambda,k)\\&+\int_{\mathbb{R}\backslash\{0\}}\left(P(t,T,\lambda+z\,,j)-P(t,T,\lambda,j)-z\,1_{|z|\leq 1}\,\frac{\partial P}{\partial\lambda}\right)\nu(j,dz)\,.\end{aligned} \tag{5.10}$$

If we try a solution of the form:

$$P(t,T,\lambda,\alpha)=\exp(A(t,\alpha)-B(t)\lambda),$$

we get the following expressions for the derivatives of P:

$$\frac{\partial P}{\partial t}=P(t,T,\lambda,j)\left(\frac{\partial A(t,j)}{\partial t}-\frac{\partial B(t)}{\partial t}\lambda\right),$$

$$\frac{\partial P}{\partial\lambda}=-P(t,T,\lambda,j)B(t),$$

and:

$$\frac{\partial^2 P}{\partial\lambda^2}=P(t,T,\lambda,j)B(t)^2.$$

The other terms involved in equation (5.8) are:

$$P(t,T,\lambda+z,k)=P(t,T,\lambda,j)\,e^{-B(t)\,z},$$

and:

$$P(t,T,\lambda,k)=P(t,T,\lambda,j)\,e^{A(t,k)-A(t,j)}.$$

We infer from the previous relationships that equation (5.8) can be reformulated as follows:

$$\begin{aligned}&\frac{\partial A(t,j)}{\partial t}-\frac{\partial B(t)}{\partial t}\lambda-(a_{1,j}+\beta_j+a_2\lambda)\ B(t)+\frac{1}{2}\sigma_j^2B(t)^2\\&-\lambda+\sum_{k\neq j}q_{j,k}\left(\exp\left(A(t,k)-A(t,j)\right)\right)+\\&\int_{\mathbb{R}\backslash\{0\}}\left(e^{-B(t)\,z}-1+z\,B(t)\,1_{|z|\leq 1}\right)\nu(j,dz)\ =0.\end{aligned}$$

This equation can indeed be split into two ODEs. The first one groups all the terms that are multiplied by λ:

$$\frac{\partial B(t)}{\partial t}+a_2B(t)=-1,$$

with the terminal condition $B(T) = 0$. Solving, we obtain the expression of $B(t)$:

$$B(t) = \frac{1}{a_2}\left(e^{a_2(T-t)} - 1\right).$$

The second ODE is:

$$\begin{aligned}\frac{\partial A(t,j)}{\partial t} - (a_{1,j} + \beta_j)\, B(t) + \frac{1}{2}\,\sigma_j^2\, B(t)^2 + \sum_{k\neq j} q_{j,k}\left(\exp\left(A(t,k) - A(t,j)\right)\right)\\ + \int_{\mathbb{R}\setminus\{0\}} \left(e^{-B(t)\,z} - 1 + B(t)\,z\,1_{|z|\leq 1}\right)\nu(j,dz) = 0 \quad \forall\, j = 1\ldots N,\end{aligned} \tag{5.11}$$

with the following boundary conditions:

$$A(T,j) = 0 \qquad j = 1...N.$$

The integral term in equation (5.11) can be inferred from the characteristic function of the Lévy process. We know indeed that the characteristic function of X_t^j is given by:

$$\phi_t^j(u) = \exp\left(t\psi_j(u)\right) = \exp\left(t\left(i\,\beta_j\,u - \frac{1}{2}\sigma_j^2\,u^2 + \int_{\mathbb{R}}\left(e^{iuz} - 1 - i\,u\,z\,I(|z|\leq 1)\right)\nu(j,dz)\right)\right),$$

where $\psi_j(u)$ is called characteristic exponent. Therefore:

$$\int_{\mathbb{R}}\left(e^{iuz} - 1 - i\,u\,z\,I(|z|\leq 1)\right)\nu(j,dz) = \psi_j(u) - i\,\beta_j\,u + \frac{1}{2}\sigma_j^2\,u^2,$$

and if we set $u = iB(t)$, we get that:

$$\int_{\mathbb{R}}\left(e^{-B(t)\,z} - 1 + B(t)\,z\,I(|z|\leq 1)\right)\nu(j,dz) = \psi_j(i\,B(t)) + \beta_j\,B(t) - \frac{1}{2}\,\sigma_j^2\,B(t)^2.$$

Equation (5.11) can therefore be rewritten as:

$$\begin{aligned}\frac{\partial A(t,j)}{\partial t} - (a_{1,j} + \beta_j)\, B(t) + \frac{1}{2}\,\sigma_j^2\, B(t)^2 + \sum_{k\neq j} q_{j,k}\left(\exp\left(A(t,k) - A(t,j)\right)\right)\\ + \psi_j(i\,B(t)) + \beta_j\,B(t) - \frac{1}{2}\,\sigma_j^2\,B(t)^2 = 0 \quad \forall\, j = 1\ldots N,\end{aligned} \tag{5.12}$$

or, after simplifications, as:

$$\frac{\partial A(t,j)}{\partial t} - a_{1,j}\,B(t) + \psi_j(i\,B(t)) + \sum_{k\neq j} q_{j,k}\left(\exp\left(A(t,k) - A(t,j)\right)\right) = 0 \quad \forall\, j = 1\ldots N\,. \tag{5.13}$$

Choosing the convention:

$$f(t,j) = -a_{1,j}B(t) + \psi_j(i\,B(t)),$$

we can rewrite equation (5.13) as follows:

$$\frac{\partial A(t,j)}{\partial t}e^{A(t,j)} + f(t,j)e^{A(t,j)} + \sum_{k=1}^{N} q_{j,k}e^{A(t,k)} = 0 \quad \forall\, j = 1\ldots N,$$

or equivalently as:

$$\frac{\partial e^{A(t,j)}}{\partial t} + f(t,j)e^{A(t,j)} + \sum_{k=1}^{N} q_{j,k}e^{A(t,k)} = 0 \quad \forall\, j = 1\ldots N. \tag{5.14}$$

If we define $\tilde{A}(t,j) = e^{A(t,j)}$, the equation can finally be put in matrix form as:

$$\frac{\partial \tilde{A}(t)}{\partial t} + \left(\mathrm{diag}(F(t)) + Q\right)\tilde{A}(t) = 0, \tag{5.15}$$

where $\mathrm{diag}(F(t))$ is the diagonal matrix whose components are these of $F(t)$, and under the boundary condition $\tilde{A}(T,j) = 1 \quad j = 1...N$. □

In the following applications, the system of equations (5.15) is solved numerically by Euler's method. An alternative is however presented in appendix A. The Markov chain modulating the parameters of the intensity being hidden, the probability of survival from t to T, given a certain λ is calculated as a weighted sum:

$$\begin{aligned} P(t,T,\lambda) &= \mathbb{E}\left(1_{\tau>T} \mid \mathcal{F}_t\right) \\ &= 1_{\tau>t}\mathbb{E}\left(P(t,T,\lambda,\alpha_t) \mid \mathcal{F}_t\right) \\ &= 1_{\tau>t}\sum_{j=1}^{N} p_j(t)\, P(t,T,\lambda), \end{aligned}$$

where $p_j(t)$ is the probability of being in state N at time t. In the following numerical applications, the $p_j(t)$'s are replaced by stationary probabilities p_i as defined by equation (3.5). Once that $P(t,T,\lambda)$ is calculated, we can infer the price of the corporate bond by equation (5.1). Finally, we can infer from the previous proposition, the following corollary:

Corollary 5.2. *The survival probabilities $P(t,T,\lambda_t,\alpha_t)$ are driven by the following stochastic differential equation on the enlarged filtration $\mathcal{H}_t \vee \mathcal{E}_t$:*

$$\begin{aligned} dP(t,T,\lambda_t,\alpha_t) &= \lambda_t\, P(t,T,\lambda_t,\alpha_t)\, dt - \sigma_{\alpha_t}\, B(t)\, P(t,T,\lambda_t,\alpha_t)\, dW_t \\ &+ \int_{\mathbb{R}\setminus\{0\}} \left(P(t,T,\lambda+z\,,\alpha_t) - P(t,T,\lambda,\alpha_t) - z\, 1_{|z|\le 1}\frac{\partial P}{\partial \lambda}\right)\left(dJ_{X^{\alpha_t}} - \nu(\alpha_t,dz)dt\right) \\ &+ \int_E \left(P(t,T,\lambda\,,\alpha_t+\eta(z)) - P(t,T,\lambda\,,\alpha_t)\right)\left(\mu(ds,dz) - \gamma(ds,dz)\right) . \end{aligned}$$

and

$$\mathbb{E}\left(dP(t,T,\lambda_t,\alpha_t) \mid \mathcal{H}_t \vee \mathcal{E}_t\right) = \lambda_t\, P(t,T,\lambda_t,\alpha_t)\, dt$$

Proof. The proof is a direct consequence of relations (5.5), (5.8), and of Itō's lemma, which states that:

$$\begin{aligned} dP &= \left[\frac{\partial P}{\partial t} + \left(a(\lambda_t,\alpha_t) + \beta_{\alpha_t}\right)\frac{\partial P}{\partial \lambda} + \frac{1}{2}\sigma^2_{\alpha_t}\frac{\partial^2 P}{\partial \lambda^2}\right] dt + \sigma_{\alpha_t}\frac{\partial P}{\partial \lambda} dW_t \\ &+ \int_{\mathbb{R}\setminus\{0\}} \left(P(t,T,\lambda+z\,,\alpha_t) - P(t,T,\lambda,\alpha_t) - z\, 1_{|z|\le 1}\frac{\partial P}{\partial \lambda}\right) dJ_{X^{\alpha_t}} \\ & \int_E \left(P(t,T,\lambda\,,\alpha_t+\eta(z)) - P(t,T,\lambda\,,\alpha_t)\right)\mu(ds,dz) \end{aligned}$$

□

6 Lévy Processes

This section presents four examples of switching Lévy processes which will be used in the following numerical applications. The first of these processes is the switching Brownian motion. If W^j denotes a Brownian motion, the instantaneous return of the asset value is ruled by the following SDE:

$$dX_t^j = \theta_j dt + \sigma_j dW_t^j \quad \forall j \in \mathcal{N},$$

and its characteristic exponent is equal to:

$$\psi_j(u) = i\,\theta_j\,u - \frac{1}{2}\sigma_j^2 u^2 \quad \forall j \in \mathcal{N}.$$

In numerical applications, we will set $\theta_j = 0$, given that λ already admits a drift term, so that $\psi_j(iB(t)) = \frac{1}{2}\sigma_j^2 B(t)^2$. Note that the trajectory of a switching Brownian motion is continuous.

The second process is directly inspired by the popular jump diffusion model developed by Kou (2002). The number of jumps observed in the asset return X^j is a Poisson process N^j whose intensity is λ_j. The amplitude of jumps, denoted by Z_j, has a double exponential distribution. Upward or downward exponentially-distributed jumps are observed, with respective probabilities p_j and $q_j = 1 - p_j$. The parameters of the jump distribution are denoted by η_j^+ and η_j^-. The density function of Z_j is therefore:

$$f_{Z_j}(z) \quad = \quad p_j\,\eta_j^+ e^{-\eta_j^+\,z}\,1_{\{z\geq 0\}} + q_j\,\eta_j^- e^{\eta_j^-\,z}\,1_{\{z<0\}}. \tag{6.1}$$

The dynamics of the asset evolution are given by the following SDE:

$$dX_t^j = \theta_j dt + \sigma_j dW_t^j + Z_j dN_t^j. \tag{6.2}$$

The Lévy measure of X_t^j is in this case the product of the frequency of jumps and of the density function describing the amplitude of jumps. From there, the characteristic exponent can be deduced:

$$\psi_j(u) = i\,\theta_j\,u - \frac{\sigma_j^2\,u^2}{2} + i\,u\,\lambda_j\left(\frac{p_j}{\eta_j^+ - i\,u} - \frac{q_j}{\eta_j^- + i\,u}\right). \tag{6.3}$$

As with the switching Brownian motion, we will set $\theta_j = 0$ in numerical applications (because θ_j is redundant with the drift term of λ), yielding:

$$\psi_j(iB(t)) \quad = \quad \frac{1}{2}\sigma_j^2 B(t)^2 - B(t)\,\lambda_j\left(\frac{p_j}{\eta_j^+ + B(t)} - \frac{q_j}{\eta_j^- - B(t)}\right).$$

Note that marginal distributions of the Kou process do not admit closed-form-expressions. As we will see in the next section, this prevents us from fitting it to observed past intensities by an econometric approach. The next two Lévy processes are subordinated Brownian motions. These are Brownian motions observed in a new time scale (sometimes called business time) given by S, which is a increasing positive stochastic process. In financial models based upon subordinated Brownian motions, each economic agent assumes that the instantaneous asset return is normal but that trading time is randomly distributed according to S, which is referred to as the subordinator. More detailed information about subordinated Brownian motions can be found in Cont and Tankov (2004) or Applebaum (2004).

The Variance Gamma process (VG), used in financial modeling by Madan and Seneta (1990), is a Brownian motion subordinated by a Gamma random variable. If θ_j and σ_j are the drift and variance of the Brownian motion, X_t^j is defined as follows:

$$dX_t^j \quad = \quad \theta_j dS_t^j + \sigma_j dW_{S_t^j} \quad \forall j \in \mathcal{N}, \tag{6.4}$$

where $S_t^j \sim Gamma(\frac{t}{\kappa_j}, \frac{1}{\kappa_j})$. In this case, the expectation and variance of S_t^j are respectively equal to t and $\kappa_j t$. The characteristic exponent of this process is

$$\psi_j(u) = -\frac{1}{\kappa_j}\log\left(1 - i\,\theta_j\,\kappa_j u + \frac{1}{2}\,u^2\,\kappa_j\,\sigma_j^2\right) \quad \forall j \in \mathcal{N}, \tag{6.5}$$

where, setting $\theta_j = 0$ as in the Kou and Brownian motion models:

$$\psi_j(iB(t)) = -\frac{1}{\kappa_j} \log\left(1 - \frac{1}{2} B(t)^2 \kappa_j \sigma_j^2\right) \quad \forall j \in \mathcal{N}.$$

Another popular subordinated Brownian motion is the Normal Inverse Gaussian process (NIG), where the subordinator is an Inverse Gaussian (IG) process. In state j, the parameters of the Inverse Gaussian are chosen such that $\mathbb{E}(S_t^j) = t$ and $\mathbb{V}(S_t^j) = \kappa_j t$. The NIG process X^j is defined as in equation (6.4). This process exhibits important features such as leptokurticity and asymmetry. We refer the interested reader to the paper by Barndorff-Nielsen (1998) for a detailed analysis of this process. In this setting, the characteristic function of X^j is known analytically:

$$\psi_j(u) = \frac{1}{\kappa_j} - \frac{1}{\kappa_j}\sqrt{1 - 2\, i\, \kappa_j\, \theta_j u + u^2\, \sigma_j^2\, \kappa_j}, \tag{6.6}$$

where, setting $\theta_j = 0$:

$$\psi_j(iB(t)) = \frac{1}{\kappa_j} - \frac{1}{\kappa_j}\sqrt{1 - B(t)^2\, \sigma_j^2\, \kappa_j}. \tag{6.7}$$

The Variance Gamma and the Normal Inverse Gaussian processes both have closed-form probability density functions. These expressions are presented in the next section.

7 Econometric Calibration

In this section, we report the results of the fit of mean-reverting switching Lévy processes to historical default intensities. For the sake of illustration, we considered four companies: Volvo, Banco Bilbao, BNP Paribas, and Mittal. Default intensities were inferred from daily quotes (in Euros) of credit default swaps (CDS), of maturity 6 months. The 6 Month CDS premiums have been retrieved from Reuters and run from the 17/12/2007 to the 8/11/2011. In exchange of a premium, expressed as a percentage of the principal, the default swap seller promises to make a payment in the event of default of a reference obligation, which is usually a bond or a loan. In case of default, the CDS pays an amount of money equal to one minus the recovery rate (which is the rate of the company's debt that is redeemed to debtholders), times the principal. Usually, the recovery rate, noted R, is assumed to be 40%. There are 3 main seniorities/tiers: SECDOM Secured Debts, SNRFOR Senior Unsecured Debts, SUBLT2 Subordinated or Lower Tier2 Debts. As SNRFOR debts are the most actively traded, we considered CDS quotes on this category of debts. The CDS premium is calculated as the expected discounted cost of the claim. For a 6 month CDS, if the intensity of default is assumed constant, the premium at time t is given by the product of the default probability times the discounted cost:

$$CDS_{6M}(t) = \left(1 - e^{-\lambda_{6M}(t)\frac{1}{2}}\right) e^{-r_{6M}(t)\frac{1}{2}} (1 - R),$$

where $r_{6M}(t)$ and $\lambda_{6M}(t)$ are respectively the 6 months interest rate (in our case, the 6M Euribor) and the intensity of default at time t. From this relationship, we can infer λ_{6M}, used as a proxy for the instantaneous intensity λ. Figure 7.1 presents the evolution of these intensities.

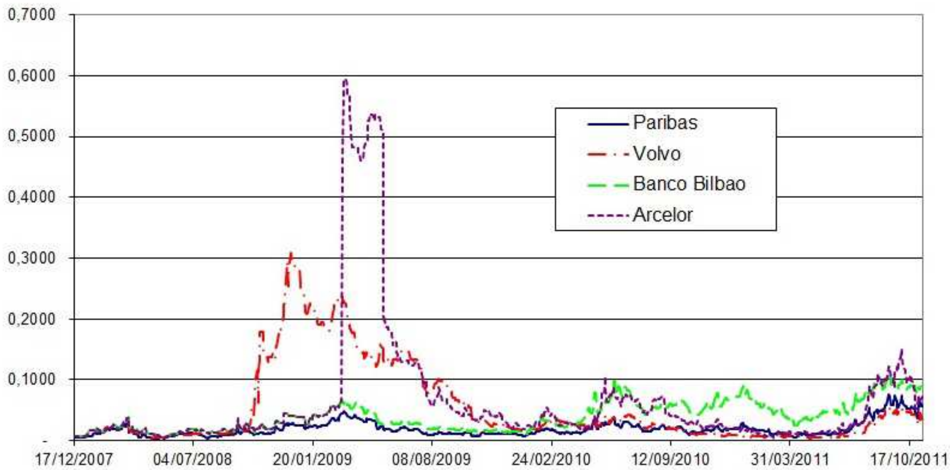

Figure 7.1: Intensities of Default, from the 17/12/2007 to the 8/11/2011.

We fitted to these four time series a discrete version of a mean reverting switching Lévy processes, such as introduced in section 2 equation (2.2):

$$\lambda_{6M}(t+\Delta t)-\lambda_{6M}(t)=\tilde{a}_2\left(\tilde{a}_{1,\alpha_t}-\lambda_{6M}(t)\right)\Delta t+\Delta X_t^{\alpha_t},$$

where α is a 2 states Markov chain ($\mathcal{N}=\{1,\,2\}$) whose daily transition matrix shall be denoted by $P=p_{i,j}(t\,,\,t+\Delta t)_{1\leq i,j\leq d}$ in the remainder of this section. Thus, we set $\Delta t=1/250$. The state of α is not directly observable, but the filtering technique developed by Hamilton (1989) and inspired by the Kalman filter (1960) allows us to retrieve the probabilities of being in a state given previous observations. We briefly summarize this filter. Let us define the probabilities of presence in state j as:

$$\Pi_t^j \quad = \quad P\left(\alpha_t=j\,|\,\lambda_{6M}(t),\ldots,\lambda_{6M}(1)\right).$$

Hamilton proved that the vector $\Pi_t=\left(\Pi_t^j\right)_{j=1\ldots m}$ can be calculated as a function of the probabilities of presence during the previous period. If we denote by $f_\lambda\left(t,\lambda_{6M}(t)\right)$ the vector of probability densities of $\lambda_{6M}(t)$, in state 1 and 2, the vector of presence probabilities is given by:

$$\Pi_t=\frac{f_\lambda\left(t,\lambda_{6M}(t)\right)*\left(\Pi_{t-\Delta t}P\right)}{\left\langle f_\lambda\left(t,\lambda_{6M}(t)\right)*\left(\Pi_{t-\Delta t}P\right)\,,\,\mathbf{1}\right\rangle},\tag{7.1}$$

where $\mathbf{1}=(1,\ldots,1)\in\mathbb{R}^d$ and $x*y$ is the Hadamard product $(x_1y_1,\ldots,x_dy_d)$. To start the recursion, we assume that the Markov chain has reached their stable distribution. Π_0 is then set to the ergodic distribution, which is the eigenvector of the matrix P, coupled to the eigenvalue equal to 1. If the intensity process is observed on T days, the loglikelihood is:

$$\ln L(\lambda_{6M}(1)\ldots\lambda_{6M}(T))=\sum_{t=1}^{T}\ln\left\langle f_\lambda\left(t,\lambda_{6M}(t)\right),\left(\Pi_{t-\Delta t}P\right)\right\rangle.\tag{7.2}$$

The most likely parameters are obtained by numerical maximization of (7.2). The Hamilton filter requires a closed form expression for the density of $\lambda_{6M}(t)$. As the Kou process does not have an analytical expression for its marginal distributions, we limit our study to Brownian, Variance Gamma and Normal Inverse Gaussian distributions, for $\Delta X_t^{\alpha_t}$. To simplify future calculations, we define the random variable as follows, when $\alpha_t=j$:

$$Y_t=\lambda_{6M}(t)-\lambda_{6M}(t-\Delta t)-\tilde{a}_2\left(\tilde{a}_{1,j}-\lambda_{6M}(t-\Delta t)\right)\Delta t.\tag{7.3}$$

This random variable has the same density function as λ_t: $f_\lambda(t, \lambda_{6M}(t)) = f_Y(t, y(t))$. In the Brownian case, Y_t is distributed as a normal random variable $N\left(\theta_j \Delta t, \sigma_j^2 \Delta t\right)$. If the intensity is driven by a Variance Gamma process, the density of Y_t (in state $\alpha_t = j$) is given by

$$f(t, y, j) = C_j\, |y|^{\frac{\Delta t}{\kappa} - \frac{1}{2}} \exp(A_j\, y) K_{\frac{\Delta t}{\kappa} - \frac{1}{2}}\left(B_j |y|\right), \tag{7.4}$$

where A_j , B_j and C_j are constants defined hereafter, and $K_{\frac{\Delta t}{\kappa} - \frac{1}{2}}(.)$ is the modified Bessel function of the second kind. Indeed:

$$A_j = \frac{\theta_j}{\sigma_j^2}, \qquad B_j = \frac{1}{\sigma_j^2} \sqrt{\theta_j^2 + 2\frac{\sigma_j^2}{\kappa_j}},$$

and:

$$C_j = \frac{\left(\theta_j^2 \kappa_j + 2\frac{\sigma_j^2}{\kappa_j}\right)^{\frac{1}{4} - \frac{\Delta t}{2\kappa_j}}}{\Gamma\left(\frac{\Delta t}{\kappa_j}\right)} \frac{2}{\sqrt{2\pi} \sigma_j \kappa_j^{\frac{\Delta t}{\kappa_j}}}.$$

Finally, if the intensity is driven by a Normal Inverse Gaussian process, the density of Y_t (in state $\alpha_t = j$) is given by:

$$f(t, y, j) = C_j \frac{1}{\sqrt{\delta_j^2 + y^2}} \exp(A_j\, y)\, K_1\left(B_j \sqrt{\delta_j^2 + y^2}\right), \tag{7.5}$$

where A_j, δ_j, B_j and C_j are constants defined by:

$$A_j = \frac{\theta_j}{\sigma_j^2}, \qquad \delta_j = \frac{\sigma_j \Delta t}{\sqrt{\kappa_j}}, \qquad B_j = \frac{1}{\sigma_j^2} \sqrt{\theta_j^2 + \delta_j^2},$$

and:

$$C_j = \Delta t \frac{1}{\pi \sigma_j \sqrt{\kappa_j}} \sqrt{\theta_j^2 + \frac{\delta_j^2}{\kappa_j}} \exp\left(\delta_j \sqrt{B_j^2 - A_j^2}\right).$$

Note that, as mentioned in the previous section, we set $\theta_j = 0$ in numerical applications for the Brownian, VG and NIG dynamics, given that it is redundant with the drift term of λ. The standard deviations of the VG and NIG processes are equal by construction to $\sigma_{j=1,2}\sqrt{\Delta t}$. The parameters $\kappa_{j=1,2}$ control the skew and the kurtosis of the process. Table 7.1 compares loglikelyhoods of mean reverting switching processes. Working with VG or NIG processes clearly improves the quality of the fit.

	Brownian	**VG**	**NIG**
BNP Paribas	5 031	6 643	9 663
Volvo	4 586	6 706	9 375
Banco Bilbao	5 372	6 487	9 267
Mittal	4 111	6 160	8 934

Table 7.1: Loglikelyhoods, 2 states models

All parameters are available in Appendix B. We see that the speed of mean reversion is quasi null with the Variance Gamma process. For the NIG process, the parameter κ_j is small whatever the state. The filter identifies for each model a state in which the failure rate has a low volatility and a state in which the volatility is significantly higher. Exhibit 7.2 emphasizes the influence of the state of the Markov chain on the shape of the one year probability density function of $X_t^{\alpha_t}$, involved in the dynamics of spreads in equation 6.4. The distribution plotted is a NIG and parameters are those obtained for Volvo. In state 1, the leptokurticity is accentuated and the default probability is higher than in state 2.

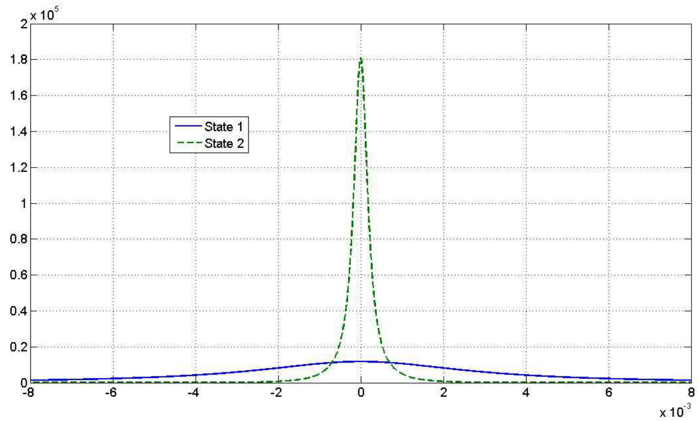

Figure 7.2: Distribution of $X_{\Delta t}^{\alpha_t}$, NIG model.

To conclude this paragraph, we draw the attention of the reader on the fact that the Hamilton filter also yields probabilities of sojourn in each state (the Π_t^j such as defined by equation (7.1)). This information could be used by traders to anticipate the evolution of CDS spreads (a similar approach has been developed in Hainaut and Macgilchrist, 2012). A daily fit of the model with the Hamilton filter will indeed reveal the probability of being in a period of high or low volatility for default intensities and help traders to take positions.

8 Application to Pricing

A default model should not only be justified from a econometric point of view but should also be able to replicate the curve of survival probabilities used by the market to price defaultable claims. If it is not the case, prices of defaultable claims computed with this model are not arbitrage free. This is why we test in this section the ability of the previously defined switching Lévy processes to fit survival probabilities extracted from CDS curves (source Reuters). We conduct this test for the four companies studied in the preceding section. Table 8.1 presents the CDS spreads in bps and the Euro swap curve on the 7/5/2011. The recovery rate chosen to bootstrap survival probabilities is set to 40%.

CDS quote in bps	**BNP Paribas**	**Volvo**	**Banco Bilbao**	**Mittal**	**Eur Swap curves**	**Rates**
0.5	32.1	13.97	113.95	30.61		
1	33.58	20.71	108.22	37.02	1	1.970%
2	51.88	40.36	140.21	93.19	2	2.307%
3	65.7	68.17	166.59	138.36	3	2.527%
4	85.21	81.21	188.98	169.99	4	2.742%
5	99.08	103.22	215.72	192.34	5	2.913%
7	108.5	120.11	221.68	210.24	7	3.169%
10	115.35	135.26	229.38	225.53	10	3.447%
20	122.47	140.02	232.08	244.48	20	3.828%

Table 8.1: CDS quotes, 17/5/11

Figure 8.1 presents survival probabilities inferred from CDS quotes (detailed figures are provided in table 8.2). According to market data, Mittal and Banco Bilbao can go bankrupt with a high probability, compared with Volvo and BNP Paribas. CDS quotes were linearly interpolated for missing maturities. From these quotes, we bootstrapped 20 default probabilities.

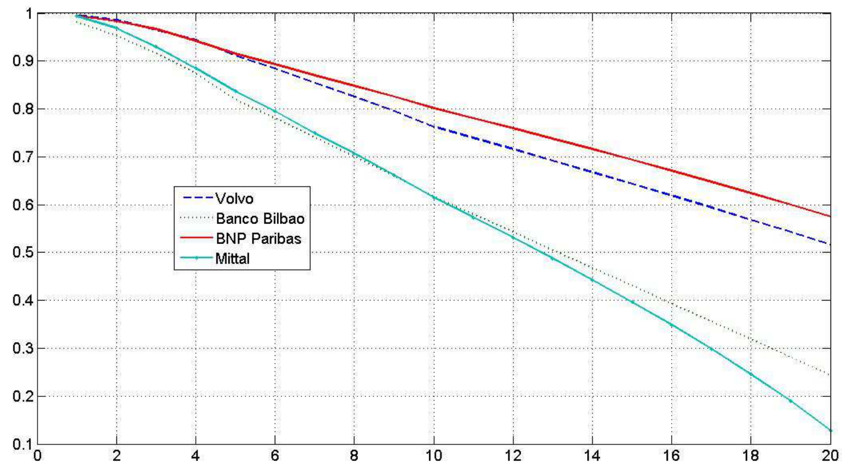

Figure 8.1: Survival probabilities

Maturity	**BNP Paribas**	**Volvo**	**Banco Bilbao**	**Mittal**
1	0.9943	0.9965	0.9820	0.9938
2	0.9825	0.9863	0.9534	0.9687
3	0.9667	0.9652	0.9169	0.9300
4	0.9419	0.9446	0.8739	0.8850
5	0.9151	0.9110	0.8187	0.8366
6	0.8933	0.8843	0.7803	0.7950
7	0.8697	0.8542	0.7408	0.7497
8	0.8480	0.8262	0.7008	0.7071
9	0.8254	0.7958	0.6596	0.6620
10	0.8017	0.7631	0.6173	0.6145
11	0.7808	0.7396	0.5800	0.5733
12	0.7597	0.7160	0.5428	0.5310
13	0.7381	0.6920	0.5054	0.4874
14	0.7161	0.6679	0.4681	0.4427
15	0.6938	0.6434	0.4308	0.3964
16	0.6711	0.6188	0.3935	0.3485
17	0.6478	0.5936	0.3561	0.2984
18	0.6241	0.5683	0.3188	0.2457
19	0.5998	0.5426	0.2815	0.1894
20	0.5750	0.5165	0.2442	0.1278

Table 8.2: Estimated survival probabilities

Next, we fitted switching Lévy models to survival probabilities. Having at our disposal only 20 survival probabilities, we limited the number of states of the Markov chain to $N = 2$. The mean error after calibration, defined as:

$$\epsilon = \frac{1}{20}\sqrt{\sum_{i=1}^{20}\left(Modeled\,DP(i) - Market\,DP(i)\right)^2}\,,$$

is presented in table 8.3. Given that the Kou model is overparametrized (2 times 7 parameters), we assumed that $\eta_j^+ = \eta_j^-$. Even with this assumption, the calibration remains unstable and errors are high. For the considered curves, the most efficient model seems to be the Variance Gamma. The calibrated parameters are provided in Appendix C. For a given model, we note that they are well-behaved and consistent in so much that all the parameters exhibit stability. Whatever the dynamics, the calibration procedure identifies a state with a low volatility and one with a high volatility. And for most of models, the probabilities of transition between states are low.

	Brownian	**Kou**	**VG**	**NIG**
BNP Paribas	0.0004	0.0003	0.0003	0.0002
Volvo	0.0007	0.0024	0.0006	0.0009
Banco Bilbao	0.0004	0.0011	0.0004	0.0004
Mittal	0.0017	0.0053	0.0004	0.0004

Table 8.3: Average errors, 2 states models

9 Conclusions

This paper explores an extension of the intensity model for credit risk pricing. The default event is specified by a Poisson process whose intensity is modeled by a switching Lévy process. A switching Lévy process is constructed from a Lévy process whose parameters are modulated by a hidden Markov chain. This category of models is well suited to duplicate the change of credit spread dynamics, observed in markets. In this setting, we show that probabilities of default can easily be retrieved by solving a system of ordinary differential equations.

Furthermore, if the probability density function of the Lévy process has a closed form expression, we can fit with the Hamilton filter the switching Lévy processes to historical time series. The performed econometric tests show that this category of models, and in particular a mean reverting 2D NIG processes, explains relatively well the evolution of past default intensities. Finally, it seems that an intensity model based on 2D VG or NIG processes is well suited for pricing purposes, given that they fit relatively well survival probabilities, bootstrapped from the CDS market.

Appendix A

We recall Equation (5.15):

$$\frac{\partial \tilde{A}(t)}{\partial t} + \text{diag}F(t)\tilde{A}(t) + Q\tilde{A}(t) = 0 \tag{9.1}$$

where $\tilde{A}(t) = (\tilde{A}(t,1), ..., \tilde{A}(t,N))'$, $\text{diag}(F(t))$ is a diagonal matrix components $(f(t,j))_{j=1...N}$, $G(t)$ is a diagonal matrix of components $\left(\int_T^t f(s,j)ds\right)_{j=1...N}$, and $Q = (q_{j,k})_{j=1...N,k=1...N}$.

We modify Equation (9.1) as follows:

$$e^{G(t)}\frac{\partial \tilde{A}(t)}{\partial t} + e^{G(t)}\text{diag}(F(t))\tilde{A}(t) + e^{G(t)}Q\tilde{A}(t) = 0 \tag{9.2}$$

so that:

$$\frac{\partial e^{G(t)}\tilde{A}(t)}{\partial t} + e^{G(t)}Q\tilde{A}(t) = 0$$

Note that $e^{G(t)}$ and Q may not necessarily commute, preventing us from solving the above equation readily. However, it is possible to write:

$$\frac{\partial e^{G(t)}\tilde{A}(t)}{\partial t} + e^{G(t)}Qe^{-G(t)}e^{G(t)}\tilde{A}(t) = 0$$

Defining $\tilde{V}(t) = e^{G(t)}\tilde{A}(t)$ and $D(t) = e^{G(t)}Qe^{-G(t)}$, we can write:

$$\frac{\partial \tilde{V}(t)}{\partial t} + D(t)\tilde{V}(t) = 0$$

This equation does not admit a closed-form solution because D is a function of time. However, it admits a semi-closed-form formula, the so-called Magnus expansion. This corresponds to writing:

$$\tilde{V}(t) = \tilde{V}(T)e^{-M(t)}$$

where:

$$M(t) = \sum_{k=1}^{+\infty} M_k(t)$$

and where:

$$M_1(t) = \int_T^t D(u)du$$

$$M_2(t) = \frac{1}{2}\int_T^t \int_T^u [D(u), D(v)]dudv$$

with $[D(u), D(v)] = D(u)D(v) - D(v)D(u)$, and:

$$M_3(t) = \frac{1}{6}\int_T^t \int_T^u \int_T^v \left([D(u),[D(v),D(w)]] + [D(w),[D(v),D(u)]]\right) dudvdw$$

and so on.

Appendix B

	BNP Paribas	**Volvo**	**Banco Bilbao**	**Mittal**
$p_{12}(0, \Delta t)$	0.0348	0.0849	0.6463	0.0547
$p_{21}(0, \Delta t)$	0.1063	0.2082	0.2238	0.5116
σ_1	0.0153	0.0152	0.0001	0.0373
σ_2	0.0652	0.1577	0.0577	0.8388
$a_{1,1}$	0.0158	0.0001	0.0436	0.0081
$a_{1,2}$	0.0430	0.2881	2.0000	0.9371
a_2	4.7156	0.2861	0.0100	0.4746

Table 9.1: Parameters Brownian motion

	BNP Paribas	**Volvo**	**Banco Bilbao**	**Mittal**
$p_{12}(0, \Delta t)$	0.1803	0.0398	0.2269	0.1995
$p_{21}(0, \Delta t)$	0.7468	0.0092	0.6414	0.4300
σ_1	0.0416	0.0394	0.0667	0.0781
σ_2	0.0001	0.1002	0.0001	0.1878
κ_1	0.0100	0.0806	0.0100	0.0100
κ_2	0.0655	0.0318	0.0547	0.0984
$a_{1,1}$	0.0000	0.0070	0.0909	0.0482
$a_{1,2}$	0.0186	0.0115	0.0199	0.0203
a_2	0.0001	0.0001	0.0001	0.0000

Table 9.2: Parameters Variance Gamma

	BNP Paribas	**Volvo**	**Banco Bilbao**	**Mittal**
$p_{12}(0, \Delta t)$	0.0373	0.2600	0.1032	0.0714
$p_{21}(0, \Delta t)$	0.0142	0.2786	0.1510	0.0308
σ_1	0.0048	0.0077	0.0050	0.0121
σ_2	0.0014	0.0005	0.0005	0.0023
κ_1	0.0001	0.0001	0.0001	0.0001
κ_2	0.0001	0.0001	0.0001	0.0001
$a_{1,1}$	0.0663	0.1065	0.9731	0.0000
$a_{1,2}$	0.0178	0.0000	0.0475	0.0335
a_2	2.5188	0.0386	0.0431	0.0642

Table 9.3: Parameters Normal inverse Gaussian

Appendix C

	BNP Paribas	**Volvo**	**Banco Bilbao**	**Mittal**
a_2	-0.2913	-0.0043	-0.2689	-0.1171
λ_0	0.0000	0.0251	0.0059	0.0104
$a_{1,2}$	-0.0095	-0.0035	-0.0100	-0.0460
$a_{1,2}$	-0.0110	-0.0024	-0.0024	0.0664
σ_1	0.0348	0.0122	0.0002	0.0376
σ_2	0.0001	0.2003	0.0420	0.1598
$p_{11}(0,1)$	0.7996	0.9997	0.9732	0.9358
$p_{22}(0,1)$	0.9909	0.9967	0.9501	0.9998

Table 9.4: Parameters Brownian motion

	BNP Paribas	**Volvo**	**Banco Bilbao**	**Mittal**
a_2	-0.3765	-0.0674	-0.7369	-0.0374
λ_0	0.0199	0.0509	0.0109	0.0657
$a_{1,1}$	-0.0300	-0.0132	-0.0459	-0.0103
$a_{1,2}$	-0.0281	-0.0049	-0.0516	-0.0013
σ_1	0.0156	0.0025	0.0278	0.0126
σ_2	0.0236	0.0000	0.0371	0.0550
λ_1	0.1041	0.0396	0.1372	0.1005
λ_2	0.2614	0.0518	0.2620	0.3687
p_1	0.2716	0.2708	0.3403	0.2159
p_2	0.5244	0.4367	0.2631	0.1844
η_1	22.1097	22.0925	22.1176	21.9651
η_2	21.0373	21.0207	21.0115	20.9393
$p_{11}(0,1)$	0.9733	0.9999	0.9037	0.8297
$p_{22}(0,1)$	0.7524	0.9999	0.8042	0.9999

Table 9.5: Parameters, Kou's model.

	BNP Paribas	**Volvo**	**Banco Bilbao**	**Mittal**
a_2	-0.3138	-2.9088	-1.9300	-0.3008
λ_0	0.0456	0.0000	0.0013	0.0165
$a_{1,2}$	-0.0489	-0.3260	-0.2392	-0.0452
$a_{1,2}$	-0.0119	-0.1546	-0.0018	-0.0096
σ_1	5.7660	5.6625	5.7014	5.7040
σ_2	0.1378	0.4668	0.0157	0.0605
κ_1	61.1049	57.6398	64.3565	187.5381
κ_2	3.2855	3.2928	3.2947	3.3198
$p_{11}(0,1)$	0.8747	0.9824	0.9999	0.9999
$p_{22}(0,1)$	0.9996	0.9997	0.8367	0.6364

Table 9.6: Parameters Variance Gamma

	BNP Paribas	**Volvo**	**Banco Bilbao**	**Mittal**
a_2	-0.3150	-3.1781	-0.9957	-4.6992
λ_0	0.0038	0.0000	0.0000	0.0000
$a_{1,1}$	-0.0028	-0.0952	-0.0305	-0.0000
$a_{1,2}$	-0.0048	-0.0023	-0.0061	-0.2392
σ_1	0.0892	2.0697	0.6506	5.2050
σ_2	0.0000	0.0379	0.0185	6.2577
κ_1	25.2348	122.2019	222.1992	222.1901
κ_2	3.1581	3.2656	3.2663	33.2658
$p_{11}(0,1)$	0.9159	0.9838	0.9907	0.9999
$p_{22}(0,1)$	0.9651	0.9996	0.8245	0.9851

Table 9.7: Parameters Normal Inverse Gaussian

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