Lattices of pretorsion classes

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joint work in progress with Francesca Fedele and Emine Yıldırım



Torsion theories in abelian categories [Dickson]

Definition: Let $\mathbb{C}$ be an abelian category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T},\ F \in \mathcal{F}$;
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Example:
$(\mathcal{T}, \mathcal{F})$ in the category Ab of abelian groups, where

- $\mathcal{T}$ = torsion groups;
- $\mathcal{F}$ = torsionfree groups

$$0 \longrightarrow t(G) \longrightarrow G \longrightarrow G/t(G) \longrightarrow 0 \qquad s.e.s$$

with $t(G)$ = torsion subgroup of G.

Torsion theories in pointed categories [Many authors in several contexts]

Definition: Let $\mathbb{C}$ be any pointed category.
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Torsion theories in pointed categories [Many authors in several contexts]

- (Div, Red) in the category Ab of abelian groups, where
  - Div = divisible groups; • Red = reduced groups

$$0 \longrightarrow D(G) \longrightarrow G \longrightarrow G/D(G) \longrightarrow 0 \qquad s.e.s$$

- (NilCRng, RedCRng) in the category CRngs of commutative rings, where
  - NilCRng = nilpotent rings • RedCRng = reduced rings

$$0 \longrightarrow Nil(R) \longrightarrow R \longrightarrow R/Nil(R) \longrightarrow 0 \qquad s.e.s.$$

- (GrpInd, GrpHaus) in the category GrpTop of topological groups, where
  - GrpInd = groups with the indiscrete topology • GrpHauss = Hausdorff groups

$$0 \longrightarrow \overline{\{1\}} \longrightarrow G \longrightarrow G \,/\, \overline{\{1\}} \longrightarrow 0 \qquad s.e.s.$$

- $(\mathrm{PrimHopf}_K, \mathrm{GrpHopf}_K)$ in the category $\mathrm{Hopf}_{K,coc}$ of cocommutative Hopf algebras, where
  - $\mathrm{PrimHopf}_K$ = primitive Hopf algebras • $\mathrm{GrpHopf}_K$ = group Hopf algebras

$$0 \longrightarrow \mathcal{U}(L_H) \longrightarrow H \cong \mathcal{U}(L_H) \rtimes K[G_H] \longrightarrow K[G_H] \longrightarrow 0 \qquad s.e.s.$$

A different example: PreOrd = category of preordered sets [Facchini, Finocchiaro]

Objects: sets endowed with a preorder (A, ρ) (reflexive + transitive relation)

Morphisms: monotone maps between (preordered) sets.

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There is a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in PreOrd given by:

- $\mathcal{T}$ = equivalence relations (symmetric preorders)
- $\mathcal{F}$ = partial orders (antysimmetric preorders)
- $\mathcal{Z} := \mathcal{T} \cap \mathcal{F}$ = discrete relations (the "equality" relations).

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The short $\mathcal{Z}$-exact sequence of an object (A, ρ) is of the form

$$(A, \equiv) \xrightarrow{Id_A} (A, \rho) \xrightarrow{\pi} (A/\equiv, \leq)$$

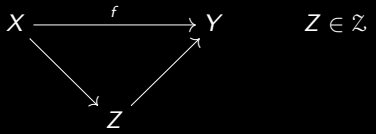
where

· $a \equiv b$ if and only if $a \rho b$ and $b \rho a$; · $[a] \leq [b]$ if and only if $a \rho b$.

Replace the zero object and the zero morphisms

Let $\mathbb{C}$ be any category.

- Consider two full replete subcategories $\mathcal{T}$ and $\mathcal{F}$ of $\mathbb{C}$.
- Define $\mathcal{Z} := \mathcal{T} \cap \mathcal{F}$, the class of trivial objects.
- We say that a morphism $X \xrightarrow{f} Y$ in $\mathbb{C}$ is $\mathcal{Z}$-trivial if it factors through an object in $\mathcal{Z}$:



$Z \in \mathcal{Z}$

- The class of trivial morphisms forms an ideal of morphisms (denoted by Triv) in $\mathbb{C}$: if $f \in \mathrm{Triv}(A, B)$ or $g \in \mathrm{Triv}(B, C)$, then $g \cdot f \in \mathrm{Triv}(A, C)$.

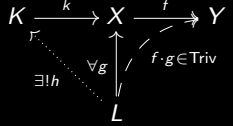
Remark: if $\mathbb{C}$ is pointed and $\mathcal{Z} = \mathcal{T} \cap \mathcal{F} = 0$, then the ideal of trivial morphisms is the ideal of zero morphisms of $\mathbb{C}$.

Kernels and cokernels with respect to an ideal of morphisms

A morphism $k\colon K \to X$ is a $\mathcal{Z}$-kernel of $f\colon X \to Y$ if

(i) $K \xrightarrow{k} X \xrightarrow{f} Y$ is $\mathcal{Z}$-trivial;

(ii) for any $g\colon L \to X$ such that $f \cdot g$ is trivial, there is a unique $h\colon L \to K$ such that $k \cdot h = g$



The notion of $\mathcal{Z}$-cokernel is defined dually. A sequence

$$W \xrightarrow{f} X \xrightarrow{g} Y$$

is a short $\mathcal{Z}$-exact sequence if f is the $\mathcal{Z}$-kernel of g and g is the $\mathcal{Z}$-cokernel of f.

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A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a pretorsion theory if

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$$T_X \to X \to F_X \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

Comparable torsion theories [—, Fedele]:

Let $\mathbb{C}$ be a pointed category and consider two torsion theories $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ in it.

The following conditions are equivalent:

(i) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ $(\mathcal{F}_1 \subseteq \mathcal{F}_2)$

(ii) $(\mathcal{T}_1, \mathcal{F}_2)$ is a pretorsion theory.

Moreover, if these conditions hold, then the $\mathcal{Z}$-short exact sequence of an object $X \in \mathbb{C}$ is given by

$$T_1X \longrightarrow X \longrightarrow F_2X$$

Notice: no hypothesis are required for $\mathbb{C}$ or the torsion theories.

Some remarks:

- If $\mathcal{T}$ is a torsion class, then $\mathcal{F}$ is uniquely determined

$$\mathcal{F} = \mathcal{T}^{\perp} := \{X \in \mathbb{C} \mid \hom(T, X) = 0 \text{ for all } T \in \mathcal{T}\}$$

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- Pretorsion classes in $\mathbb{C}$ are precisely the monocoreflective subcategories of $\mathbb{C}$ (strongly covering subcategories where all strong covers are monomorphisms).

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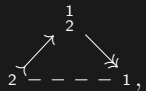
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- All the important information can be encoded into its Aulander-Reiten quiver.
- Torsion and pretorsion classes are quite easy to detect.

Example

Example: $Q = \mathbb{A}_2 : 1 \to 2$



Let me try to draw a picture...

Classification of distributive lattices for finite representation type (here $Q = A_n, D_n, E_6, E_7, E_8$)

The poset of pretorsion classes is a complete lattice, with meet and join given, for every $\mathcal{T}_1$ and $\mathcal{T}_2$, by

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2 \qquad \text{and} \qquad \mathcal{T}_1 \vee \mathcal{T}_2 = \langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle_t.$$

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Result 1 [— , Fedele, Yıldırım]

The lattice of pretorsion classes is distributive if and only if $\mathrm{add}\{\mathcal{T}_1 \cup \mathcal{T}_2\} = \langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle_t$ for every pair of pretorsion classes $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathrm{mod} kQ$.

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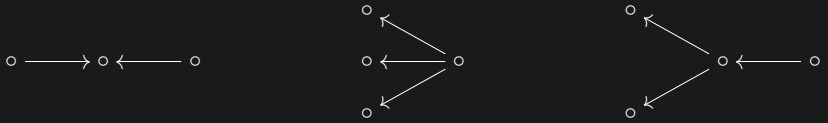
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Result 2 [— , Fedele, Yıldırım]

The lattice of pretorsion classes is distributive if and only if Q does not contain subquivers of the form



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Result 3 [— , Fedele, Yıldırım]

There is a bijection between the isomorphism classes of indecomposable modules and the join-irreducible elements of the lattice of pretorsion classes, given by $M \mapsto \langle M \rangle_t$. Moreover, the join-irreducible elements are torsion classes.

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Result 4 [— , Fedele, Yıldırım]

If the lattice of pretorsion classes is distributive, than it is the distributive completion of the lattice of torsion classes.

Thank you