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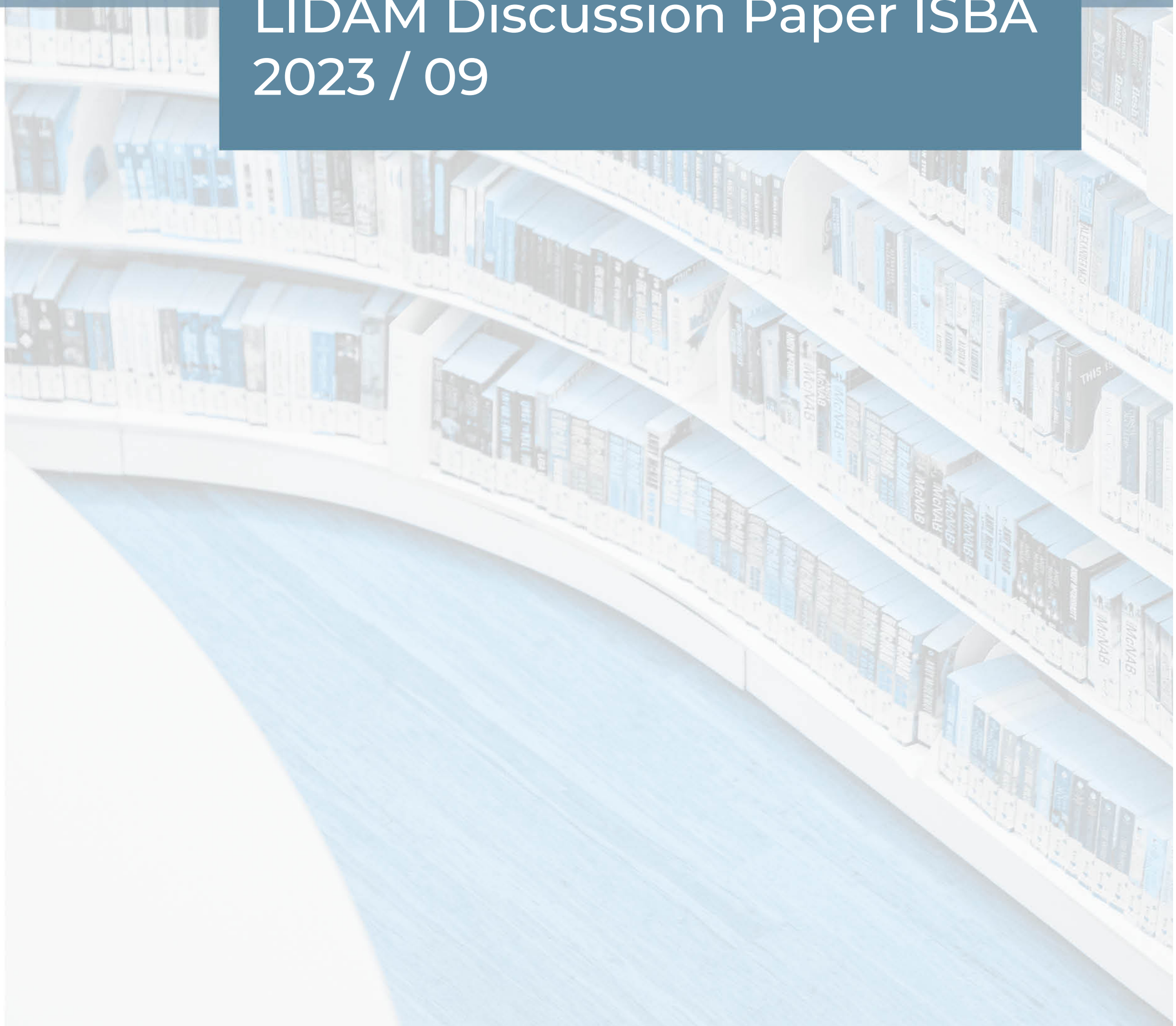
Michel Denuit, Christian Y. Robert

LIDAM Discussion Paper ISBA
2023 / 09



ISBA
Voie du Roman Pays 20 - L1.04.01
B-1348 Louvain-la-Neuve
Email : lidam-library@uclouvain.be
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ENDOWMENT CONTINGENCY FUNDS FOR MUTUAL AID AND PUBLIC FINANCING

Michel Denuit
Institute of Statistics, Biostatistics and Actuarial Science - ISBA
Louvain Institute of Data Analysis and Modeling - LIDAM
UCLouvain
Louvain-la-Neuve, Belgium
michel.denuit@uclouvain.be

Christian Y. Robert
Laboratory in Finance and Insurance - LFA
CREST - Center for Research in Economics and Statistics
ENSAE
Paris, France
chrobert@ensae.fr

February 12, 2023

Abstract

This paper explores a new risk-sharing vehicle, called endowment contingency fund. It targets groups of individuals exposed to the occurrence of a predefined event with adverse financial consequences such as death, survival or being diagnosed with a critical illness for instance. All members of the group agree to contribute in advance a fixed amount to a pool constituted over a reference period with the understanding that the sum of the contributions is shared in arrear among those participants having experienced the predefined event. This allocation is either uniform among claiming participants, each one receiving an equal share of total contributions, or participants are offered the choice to select a desired protection level. In the latter case, participants are free to subscribe one or several units of protection from the fund, and the total amount collected in advance is shared in arrear, equally among all units held by those participants who experienced the predefined event. Endowment contingency funds aim to provide participants with a cheap and effective protection compared to commercial insurance. The reason is that the proposed system is fully funded so that there is no risk borne by the organizer. The benefits in case the event occurs are therefore random but the volatility of the terminal payouts turns out to be limited when the number of participants gets large enough. Under independence, insurance at fair price is recovered at the limit, within infinitely large pools. As an application, the paper considers mutual aid funds and survivor funds. Several related issues are discussed, including a comparison with takaful insurance as well as the inclusion of minimum guarantees to make the system more attractive.

Keywords: Risk pooling, Conditional mean risk-sharing, Actuarial fairness, Mutual inheritance, Insurance at fair price.

1 Introduction and motivation

This paper considers a new, fully-funded risk-sharing mechanism: the endowment contingency funds targeting groups of individuals exposed to some adverse event. As the name indicates, participants to an endowment contingency fund contribute (or donate) money in advance to a mutual fund established over a given time period, with the understanding that the total amount collected is going to be distributed in arrear, among those participants who experienced a predefined event during that period. Benefits are therefore contingent to the occurrence of a random event bringing some adverse financial consequences. This corresponds to the mutual inheritance rule in survivor funds, where contributions paid by the decedents are lost and transferred to the survivors.

Events under consideration include death, survival or being diagnosed with a critical illness for instance. They can also correspond to more complex events, such as death of a reference individual (family's breadwinner, for instance) and survival of a dependent relative (like spouse or children). Any random event increasing financial needs when it occurs (to pay for medical treatments or to compensate lost income, for instance) is eligible. From a theoretical point of view, events under consideration may even differ among participants but we assume throughout the paper that the fund targets a specific peril for which insurance is offered, but is perceived as being expensive. Endowment contingency funds then purpose to provide a cheaper protection compared to regular insurance contracts, and to solve the conflict of interest inherent to commercial insurance since all money collected from participants is returned to claimants at the end of the period.

Participants select a desirable benefit target, that is, the amount that should be provided to any member who experiences the predefined event. Participants view joining an endowment contingency fund as a sound way to protect against adverse financial consequences, not as charity or subsidizing solidarity. Therefore, they require that none of them gain on average from this operation. As a consequence, the contributions paid in advance are set to match the expected benefit received at the end of the period, which also counteracts possible adverse selection effects. Contributions account for the heterogeneity in the occurrence probabilities and the fact that total contributions are shared equally among all participants having experienced the predefined event. If nobody experienced the event then contributions are reimbursed to participants. The latter situation is relevant for small communities but becomes unlikely for larger pools. The allocation of benefits to claiming participants is thus fully transparent.

Awarding equal benefits to all claiming participants may be regarded as being restrictive in some applications. One could then imagine a system where benefits paid to claimants are proportional to the amounts they initially contributed to the fund. However, it turns out that this sharing rule conflicts with actuarial fairness, in general. This is why we follow another route and let the participants free to subscribe one or several protection units from the fund. A targeted benefit is then specified per unit of protection and participants select the number of units appropriate to their situation. At the end of the period, total contributions are shared equally among all claiming units. By claiming units, we mean the units subscribed by participants who experience the predefined event. A targeted amount of benefit per claiming unit is announced, so that participants can select the appropriate number of units required to reach their desired level of protection.

The proposed endowment contingency funds are fully funded systems since contributions are received in advance and shared in arrear among claiming participants. Thus, amounts of contributions are known and paid with certainty, avoiding any counter-party risk whereas benefits remain unknown until the end of the coverage period. Participants have no guarantee about the actual amount of benefits awarded in case the predefined event occurs, but they are informed about the targeted amount of benefit per unit of protection. Also, information about the volatility inherent to the composition of the endowment contingency fund can be transmitted to participants so that they can adjust the number of units they subscribe. A partnering (re-)insurer may propose additional guarantees to participants, ensuring that benefits never fall below some minimum threshold for instance. This articulates endowment contingency funds to insurance, offering an integrated risk management solution to participants.

The way the proposed endowment contingency funds proceed differs from the dominant practice in risk sharing where participants contribute at the end of the period, by sharing total losses once they have been observed. Typically, a lump-sum benefit is specified and total losses correspond to the sum of the promised benefits to all claiming participants. The question is then to allocate to each participant the fair share of total losses, given their individual risk profile. Solutions exist, such as the conditional mean risk-sharing rule proposed by Denuit and Dhaene (2012) and axiomatized by Jiao et al. (2022). This rule has been successfully applied to peer-to-peer insurance by Denuit and Robert (2020). See also Denuit and Robert (2021, 2022) and Denuit et al. (2022), as well as the references therein. In this approach, contributions are random and paid in arrear whereas contributions are known and paid in advance with endowment contingency funds. The traditional approach also exposes participants to counter-party risk since some members may be unwilling or unable to assume the share of total losses allocated to them at the end of the period. On the contrary, endowment contingency funds are fully funded. But this comes at the cost of some randomness in the amount of benefit awarded per unit of protection, resulting from the unknown number of participants who experience the predefined event during the reference period. It seems reasonable to expect that the randomness in the benefits awarded in arrear to claimants in endowment contingency funds becomes acceptable when the number of participants is large enough. This question is rigorously addressed by studying diversification effects within large pools, under mutual independence. It is shown that insurance at fair price is recovered at the limit, when the number of participants tends to infinity. Interestingly, it is shown that the allocation of benefits in endowment contingency funds can be seen as an anticipated version of the conditional mean risk-sharing rule.

In organization theory, mutual aid is a voluntary reciprocal exchange of resources and services for mutual benefit. Endowment contingency funds thus fall within the scope of mutual aid, like mutual aid platforms operated in Asia. However, participants to mutual aid platforms generally contribute in arrear an amount corresponding to the total benefits awarded to claimants divided by the number of participants (supplemented with commissions and subscription fees). While participants generally contribute equally to the total benefits, some platforms let participants' contributions vary with age to recognize different loss probabilities. This is in contrast with the proposed endowment contingency funds and the respective advantages of the two approaches will be discussed.

The equal distribution of total contributions among claimants at the heart of endowment contingency funds is similar to the mechanism underlying life annuities with benefit of survivorship, or tontine. As pointed out by Li and Rothschild (2020), the provider in a prototypical tontine promises a periodic payment to the pool, to be divided among the members who are still alive at the time the payment is due. This is just a last-survivor life annuity paying constant benefits until all insured lives will have died. Since a decreasing number of survivors share a constant payment, this product comprises an element of gambling as demonstrated by the active participation of Genevan speculators in Irish tontines (see, e.g., Li and Rothschild, 2020). On the contrary, endowment contingency funds proposed in this paper are actuarially fair and cannot serve gambling objectives. They nevertheless proceed in a similar way when the predefined event corresponds to survival up to some given time horizon. But endowment contingency funds have added values, in particular they accommodate for heterogeneity in survival probabilities so that the pool can mix several generations and still respect actuarial fairness. This is important since tontines have been theorized in the homogeneous case; see e.g. Lange et al. (2007). Also, this prevents any gambling aspect and provides participants with a cheap and effective protection against longevity risk. In that respect, endowment contingency funds appear to be closely related to survivor funds where participants contribute a fixed amount in advance and agree that the accumulated contributions are shared among survivors, only.

The novelty of this paper compared to existing literature (reviewed in the subsequent sections) is as follows. Endowment contingency funds are proposed as fully funded systems eliminating all default risk, allowing individuals to mutually protect against a given peril at cheap price. Heterogeneity is allowed for among participants who may differ in their occurrence probabilities and desired protection levels. Contributions reflect individual risk profiles, under actuarial fairness. Since heterogeneity is taken into account, large pools can be constituted and limiting results are derived under mutual independence. These asymptotic results describe how diversification operates for large numbers of participants, as well as limiting values for individual contributions. It is shown that insurance at fair price can be recovered at the limit, under mild regularity conditions.

The remainder of this paper is organized as follows. Section 2 describes the individual life model considered in this paper and studies the case where total contributions are equally divided in arrear, among all members who experienced the predefined event. Amounts of contributions to be paid in advance reflect occurrence probabilities and the targeted amount of benefits per participant. Large pools are considered under mutual independence to assess how diversification operates when the number of participants increases, as well as limiting values for individual contributions. Section 3 then lets the benefits vary among participants, each of them selecting his or her own expected amount of benefits. This is achieved by allowing participants to subscribe the number of protection units they wish from the endowment contingency fund, each unit providing the claiming participant with the same amount of benefit, on average. The limiting case where the number of participants tends to infinity is considered under mutual independence. Section 4 considers two applications. First, a comparison with Asian mutual aid platforms is performed, where the predefined event corresponds to critical illness. Second, the case where the predefined event corresponds to survival is considered. Endowment contingency funds are called endowment longevity funds in that case. Endowment longevity funds are considered as a way to finance large infrastructure investments by public bodies. The final Section 5 briefly summarizes the results obtained in this paper and discusses some possible extensions. All proofs are gathered in Appendix A. Related issues are also addressed in appendix. Specifically, Appendix B discusses Poisson approximations for computing contributions in large pools when occurrence probabilities are small. Appendix C considers another system where benefits paid to claiming participants are proportional to the amounts they initially contributed to the fund. It is shown there that this sharing rule conflicts with actuarial fairness, in general. Appendix D compares the proposed endowment contingency funds with an example of takaful product, the Sharia-compliant alternative to conventional insurance.

2 Endowment contingency funds with equal protection levels

Consider a group of individuals exposed to the occurrence of a random event with adverse financial consequences. They decide to unite in order to mutually protect during a fixed period of time. To this end, they set up an endowment contingency fund and each participant contributes in advance a fixed amount to the fund. This contribution depends on the individual risk profile, being higher for participants who are more likely to experience the predefined event. In that respect, the system is designed to be fair from an actuarial point of view. This means that nobody gains nor lose on average by joining an endowment contingency fund, the latter being just meant as a risk protection instrument without any gambling or investment component. This section describes the mechanism underlying endowment contingency funds when claiming participants all receive the same amount of benefits.

2.1 Notation

Cash-flows associated to endowment contingency funds can be modeled in the framework of the individual life model of risk theory, where individual losses are binary random variables. Therefore, we first briefly recall the main elements of this approach.

Consider a group of individuals wishing to mutually protect against the financial consequences of the occurrence of a given predefined random, numbered $i = 1, 2, \ldots, n$. Assume that individual i contributes an amount c_i at the beginning of the reference period (time 0, say), $i = 1, 2, \ldots, n$. This contribution may be supplemented with fees covering running expenses (that are not considered in our analysis). The amount c_i may be invested and accumulate to a_i at the end of the reference period (time 1, say). Here, we work under zero interest rate so that $a_i = c_i$. Formulas are easily extended to the case $a_i \neq c_i$ accounting for (deterministic) financial return, if needed. Let $b = \sum_{j=1}^{n} c_j$ be the total contributions paid in advance by the n participants, to be allocated at time 1 among those participants who experienced the predefined event.

The respective probabilities to experience the event under consideration over the reference period are denoted as $q_1, q_2, \ldots, q_n$ for participants $1, 2, \ldots, n$. These values are assumed to be known and all participants agree about them. They are only used to compute individual contributions so that any error in $q_1, q_2, \ldots, q_n$ does not impact on solvency, the proposed system being fully funded and void of any default risk.

Let I_i denote the loss indicator

$$I_i = \begin{cases} 1 \text{ if individual } i \text{ experiences the predefined event before time 1,} \\ \\ 0 \text{ otherwise,} \end{cases}$$

with $\mathrm{P}[I_i = 1] = q_i = 1 - \mathrm{P}[I_i = 0]$. Also, let M_n be the number of participants who experienced the event during the reference period (0,1) and let $M_n^{(\backslash i)}$ be the same number, participant i excepted. Formally,

$$M_n = \sum_{j=1}^{n} I_j \text{ and } M_n^{(\backslash i)} = M_n - I_i = \sum_{j \neq i} I_j.$$

Depending on the situation, M_n can be the number of deaths recorded among the n participants during time interval (0,1), the number of survivors at time 1 or the number of individuals diagnosed with a critical illness during (0,1), for instance.

2.2 Terminal payout

Assume that participants decide that the total budget must be evenly distributed at time 1 among those M_n individuals who experienced the predefined event. Initial contributions under this uniform allocation rule are henceforth denoted as $c_{i,n}^{\text{uni}}$, with sum $b_n^{\text{uni}} = \sum_{i=1}^{n} c_{i,n}^{\text{uni}}$. Here, we make explicit the dependence of the contribution on the number n of participants since we aim to assess diversification benefits when n grows.

The cash-flows associated to the endowment contingency fund are as follows. At time 0, participant i transfers the amount $c_{i,n}^{\text{uni}}$ to the fund. The latter thus collects total contributions b_n^{uni}. At time 1, if at least one participant experienced the predefined event during time interval (0,1), that is, if $M_n \geq 1$ then b_n^{uni} is equally shared among all claimants, each of them receiving the amount b_n^{uni}/M_n. If nobody experienced the event under consideration, that is, if $M_n = 0$ then participants just get their initial contribution back so that participant i receives $c_{i,n}^{\text{uni}}$ at time 1.

The terminal payout $T_{i,n}^{\text{uni}}$ for participant i is thus random and given by

$$T_{i,n}^{\text{uni}} = \begin{cases} \frac{I_i}{M_n} b_n^{\text{uni}} \text{ if } M_n \geq 1, \\ \\ c_{i,n}^{\text{uni}} \text{ if } M_n = 0, \end{cases}$$

or equivalently

$$T_{i,n}^{\mathrm{uni}} = \frac{I_i}{1+M_n^{(\backslash i)}} b_n^{\mathrm{uni}} + c_{i,n}^{\mathrm{uni}} \mathbb{1}[M_n = 0],$$

where $\mathbb{1}[\cdot]$ denotes the indicator function (equal to 1 if the event appearing between the brackets is realized and to 0 otherwise). Notice that

$$\sum_{i=1}^{n} T_{i,n}^{\mathrm{uni}} = b_n^{\mathrm{uni}} \big(\mathbb{1}[M_n \geq 1] + \mathbb{1}[M_n = 0]\big) = b_n^{\mathrm{uni}}$$

so that the entire contribution is distributed among claimants, solving the conflict of interest inherent to commercial insurance.

2.3 Initial contributions

2.3.1 Actuarially fair contributions

To avoid systematic transfers among participants, initial contributions $c_{i,n}^{\mathrm{uni}}$ should reflect individual expected benefits. This is expressed by actuarial fairness which requires that no participant gains nor loses by joining the pool, on average. The initial contribution must then be given by

$$c_{i,n}^{\mathrm{uni}} = \mathrm{E}[T_{i,n}^{\mathrm{uni}}] = \mathrm{E}\left[\frac{I_i}{1+M_n^{(\backslash i)}}\right] b_n^{\mathrm{uni}} + c_{i,n}^{\mathrm{uni}} \mathrm{P}[M_n = 0], \quad i = 1, \dots, n,$$

which reduces to

$$c_{i,n}^{\mathrm{uni}} = \frac{b_n^{\mathrm{uni}}}{\mathrm{P}[M_n \geq 1]} \sum_{k=1}^{n} \frac{1}{k} \mathrm{P}\left[M_n^{(\backslash i)} = k-1, I_i = 1\right], \quad i = 1, \dots, n. \tag{2.1}$$

In particular,

$$I_1, I_2, \dots, I_n \text{ independent } \Rightarrow c_{i,n}^{\mathrm{uni}} = \frac{b_n^{\mathrm{uni}} q_i}{\mathrm{P}[M_n \geq 1]} \sum_{k=1}^{n} \frac{1}{k} \mathrm{P}\left[M_n^{(\backslash i)} = k-1\right]. \tag{2.2}$$

2.3.2 Link with conditional mean risk-sharing rule

Denuit (2019) proposed to adopt the conditional mean risk-sharing rule to allocate mortality credits within mutual inheritance schemes. Denuit et. (2022) implemented this idea in the individual life model. Transposed to the context of this paper, this would mean that claiming participants are promised to receive unit benefit (say) at the end of the period. The corresponding total benefits are shared among all participants, in arrear. Participant i

does not pay anything in advance but contributes in arrear an amount corresponding to the conditional expectation $\mathrm{E}[I_i|M_n]$. Clearly, $\sum_{i=1}^n \mathrm{E}[I_i|M_n] = M_n$ so that the total amount of benefit is covered by individual contributions in arrear.

At first sight, the uniform sharing rule within endowment contingency funds presented before does not seem to be related to the conditional mean risk-sharing rule applied in the individual life model by Denuit et al. (2022). It turns out that this risk-sharing rule is in fact used in relative terms, to allocate the available resources b_n^{uni}. Indeed, the conditional mean risk-sharing rule applied to losses $I_1, I_2, \ldots, I_n$ is given by

$$\mathrm{E}[I_i|M_n = k] = \mathrm{P}[I_i = 1|M_n = k] = \frac{\mathrm{P}[I_i = 1, M_n = k]}{\mathrm{P}[M_n = k]}.$$

Now, considering (2.1), we can write

$$\begin{aligned} c_{i,n}^{\mathrm{uni}} &= \frac{b_n^{\mathrm{uni}}}{\mathrm{P}[M_n \geq 1]} \sum_{k=1}^{n} \frac{1}{k} \mathrm{P}\left[M_n = k, I_i = 1\right] \\ &= \frac{b_n^{\mathrm{uni}}}{\mathrm{P}[M_n \geq 1]} \sum_{k=1}^{n} \frac{1}{k} \mathrm{E}[I_i|M_n = k] \mathrm{P}\left[M_n = k\right] \\ &= b_n^{\mathrm{uni}} \sum_{k=1}^{n} \frac{1}{k} \mathrm{E}[I_i|M_n = k] \mathrm{P}\left[M_n = k|M_n \geq 1\right] \\ &= b_n^{\mathrm{uni}} \mathrm{E}\left[\frac{\mathrm{E}[I_i|M_n]}{M_n} \Big| M_n \geq 1\right]. \end{aligned}$$

This shows that the uniform allocation rule in endowment contingency funds can be seen as an anticipated version of the conditional mean risk-sharing rule operating in arrear. This allows us to shift the randomness in the contributions paid in arrear under the conditional mean risk-sharing rule to the benefits paid in arrear with endowment contingency funds.

2.3.3 Targeted benefit

The initial contributions $c_{i,n}^{\mathrm{uni}}$ in (2.1) are only defined up to a constant factor. In mutual aid funds, the sponsor can for instance determine the targeted amount of benefit per claiming participant and then set b_n^{uni} to reach this target. More specifically, if the targeted benefit per claiming participant is β then b_n^{uni} can be obtained from

$$b_n^{\mathrm{uni}} \mathrm{E}\left[\frac{1}{M_n} \Big| M_n \geq 1\right] = \beta \Leftrightarrow b_n^{\mathrm{uni}} = \frac{\beta \mathrm{P}[M_n \geq 1]}{\sum_{k=1}^n k^{-1} \mathrm{P}[M_n = k]}. \tag{2.3}$$

Inserting this value in (2.1) gives the contributions

$$c_{i,n}^{\text{uni}} = \beta \frac{\sum_{k=1}^{n} k^{-1} \mathrm{P}\left[M_n = k, I_i = 1\right]}{\sum_{k=1}^{n} k^{-1} \mathrm{P}\left[M_n = k\right]}, \quad i = 1, \ldots, n. \tag{2.4}$$

Notice that the expected benefit for participant i is not exactly equal to β and depends on the joint distribution of the loss indicators $I_1, I_2, \ldots, I_n$.

2.3.4 Independent case

With the targeted benefit β in (2.3), (2.2) gives

$$I_1, I_2, \ldots, I_n \text{ independent } \Rightarrow c_{i,n}^{\text{uni}} = q_i \beta \frac{\sum_{k=1}^{n} k^{-1} \mathrm{P}\left[M_n^{(\setminus i)} = k - 1\right]}{\sum_{k=1}^{n} k^{-1} \mathrm{P}\left[M_n = k\right]}. \tag{2.5}$$

The next result shows that contributions (2.5) stabilize at level βq_i when the number of participants gets large.

Proposition 2.1. *Assume that $I_1, I_2, \ldots$ are mutually independent. Denote $q_{\min} = \inf_{i \geq 1} q_i$ and $\overline{q} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} q_i$, assuming that the limit exists. If $q_{\min} > 0$ and $0 < \overline{q} < 1$ then contributions* (2.5) *satisfy*

$$\lim_{n\to\infty} c_{i,n}^{\text{uni}} = \beta q_i.$$

The proof of Proposition 2.1 can be found in Appendix A.1. This result shows that within large pools, contributions correspond to pure premiums, or fair prices for an insurance contract providing the policyholder with a lump sum benefit β in case the predefined event occurs. Also, the composition of the pool (that is, the occurrence probabilities q_j for $j \neq i$) becomes irrelevant when $n \to \infty$ and only q_i matters.

2.4 Diversification effects and volatility of terminal payouts within large, independent pools

The terminal payout $T_{i,n}^{\text{uni}}$ is random in endowment contingency funds. The next result shows that risk can be fully eliminated when the number of participants tends to infinity, under mutual independence.

Proposition 2.2. *Assume that the assumptions of Proposition 2.1 are fulfilled. Then,*

$$T_{i,n}^{\text{uni}} \to I_i \beta \text{ as } n \to \infty, \text{ with probability 1.}$$

The proof of can be found in Appendix A.2. This result shows that within large pools, the terminal payout tends to the amount paid by an insurer for an insurance contract providing the policyholder with a lump sum benefit β in case the predefined event occurs. Also, the composition of the pool becomes irrelevant when $n \to \infty$ and only participant i matters. Propositions 2.1-2.2 show that insurance at fair price is recovered at the limit for endowment contingency funds under mutual independence.

Although the terminal payout for participant i in case he or she experiences the predefined event before time 1 is random, we can nevertheless control meaningful volatility indicators, including the variance of benefit awarded to a claiming participant, given by

$$\mathrm{Var}\left[T_{i,n}^{\mathrm{uni}}|I_i = 1\right] = \left(b_n^{\mathrm{uni}}\right)^2 \mathrm{Var}\left[\frac{1}{1 + M_n^{(\backslash i)}}\right].$$

The next result shows that the latter variance decreases to 0 at a rate proportional to the inverse of the number of participants.

Proposition 2.3. *Assume that the assumptions of Proposition 2.1 are fulfilled. Then,*

$$n\mathrm{Var}\left[T_{i,n}^{\mathrm{uni}}\middle| I_i = 1\right] \to \beta^2 \frac{\overline{q(1-q)}}{(\overline{q})^2} \quad \textit{as } n \to \infty,$$

where $\overline{q(1-q)} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} q_i(1-q_i)$.

The proof of Proposition 2.3 can be found in Appendix A.3. This result allows the actuary to determine the minimum number of participants to be attained to give individuals confidence in the variability of benefits.

2.5 Numerical example

Throughout this section, we assume that $I_1, I_2, \ldots$ are mutually independent. Figure 1 displays individual contributions $c_{i,n}^{\mathrm{uni}}$ when 50% of the participants (Group 1) have a probability 0.01 of experiencing the event under consideration, 30% of the participants (Group 2) have an occurrence probability 0.05 and the remaining 20% (Group 3) have an occurrence probability of 0.1. The results are given according to the number n of participants with n ranging from 10 to 1'000, for targeted benefits $\beta = 1$. All calculations are carried with the `distr` package available in `R`, see `distr.r-forge.r-project.org/`.

Figure 1 shows that $c_{i,n}^{\mathrm{uni}}$ stabilizes at level q_i as $n \to \infty$ as a consequence of Proposition 2.1 (remember that $\beta = 1$ here). Contributions $c_{i,n}^{\mathrm{uni}}$ in Figure 1 are below the limiting value from Proposition 2.1 when the number of participants gets large enough (this is true for $n \geq 40$

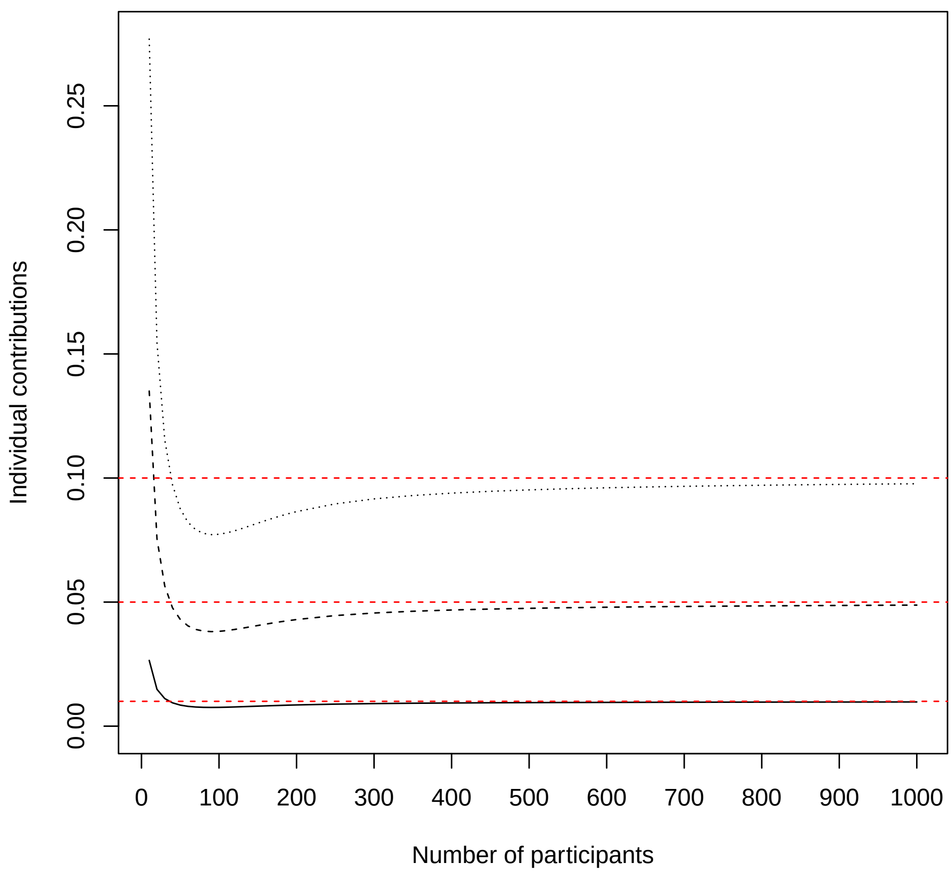


Figure 1: Individual contributions $c_{i,n}^{\text{uni}}$ in function of the number n of participants to the endowment contingency fund, for participant i in Group 1 (solid line, —), in Group 2 (broken line, - - -), and in Group 3 (dotted line, $\cdots$). Red horizontal lines corresponding to respective occurrence probabilities.

in this example). This comment is generally valid, as explained next. Define the function g as $g(k) = \min\{1, 1/k\}$ for $k = 0, 1, 2, \ldots$. Since g is non-increasing and $M_n \leq 1 + M_n^{(\backslash i)}$ holds with probability 1, we have

$$
\begin{aligned}
\mathrm{E}[g(M_n)] &= \mathrm{P}[M_n = 0] + \sum_{k=1}^{n} \frac{1}{k} \mathrm{P}[M_n = k] \\
&\geq \mathrm{E}[g(1 + M_n^{(\backslash i)})] \\
&= \sum_{k=0}^{n-1} \frac{1}{1+k} \mathrm{P}[M_n^{(\backslash i)} = k] \\
&= \sum_{k=1}^{n} \frac{1}{k} \mathrm{P}[M_n^{(\backslash i)} = k-1].
\end{aligned}
$$

Since $\mathrm{P}[M_n = 0]$ is negligible compared to $\mathrm{P}[M_n = 1]$ when n gets large, the inequality

$$
\sum_{k=1}^{n} \frac{1}{k} \mathrm{P}[M_n = k] \geq \sum_{k=1}^{n} \frac{1}{k} \mathrm{P}[M_n^{(\backslash i)} = k-1] \text{ holds for } n \text{ large enough,}
$$

from which we conclude that $c_{i,n}^{\mathrm{uni}} \leq \beta q_i$ holds for n large enough. In practice, charging the limiting contribution βq_i thus ensures that the endowment contingency fund collects enough money in advance, provided the number n of participants is large enough.

Let us now assess the rate of decrease of the volatility, in application of Proposition 2.3. Here, we have

$$
\overline{q} = 0.5 \times 0.01 + 0.3 \times 0.05 + 0.2 \times 0.1 = 0.04
$$

and

$$
\overline{q(1-q)} = 0.5 \times 0.01 \times 0.99 + 0.3 \times 0.05 \times 0.95 + 0.2 \times 0.1 \times 0.9 = 0.0372
$$

so that the rate of decrease established in Proposition 2.3 is equal to 46.5. The standard deviation of the terminal payout of a participant is then approximately equal to 0.215 when $n = 1'000$ and to 0.068 when $n = 10'000$.

Figure 2 displays $\mathrm{Var}\left[T_{i,n}^{\mathrm{uni}} | I_i = 1\right]$ for participants in each group, together with the approximation deduced from Proposition 2.3. We can see there that the approximation deduced from Proposition 2.3 becomes very accurate for $n \geq 200$. As an application, assume that we want to determine the number of participants required to limit the variation coefficient of benefits paid to a claimant to a certain acceptable level ℓ. We then solve

$$
\ell = \frac{\sqrt{\overline{q(1-q)}}}{n\overline{q}^2} \Leftrightarrow n = \frac{\sqrt{\overline{q(1-q)}}}{\ell\overline{q}^2}.
$$

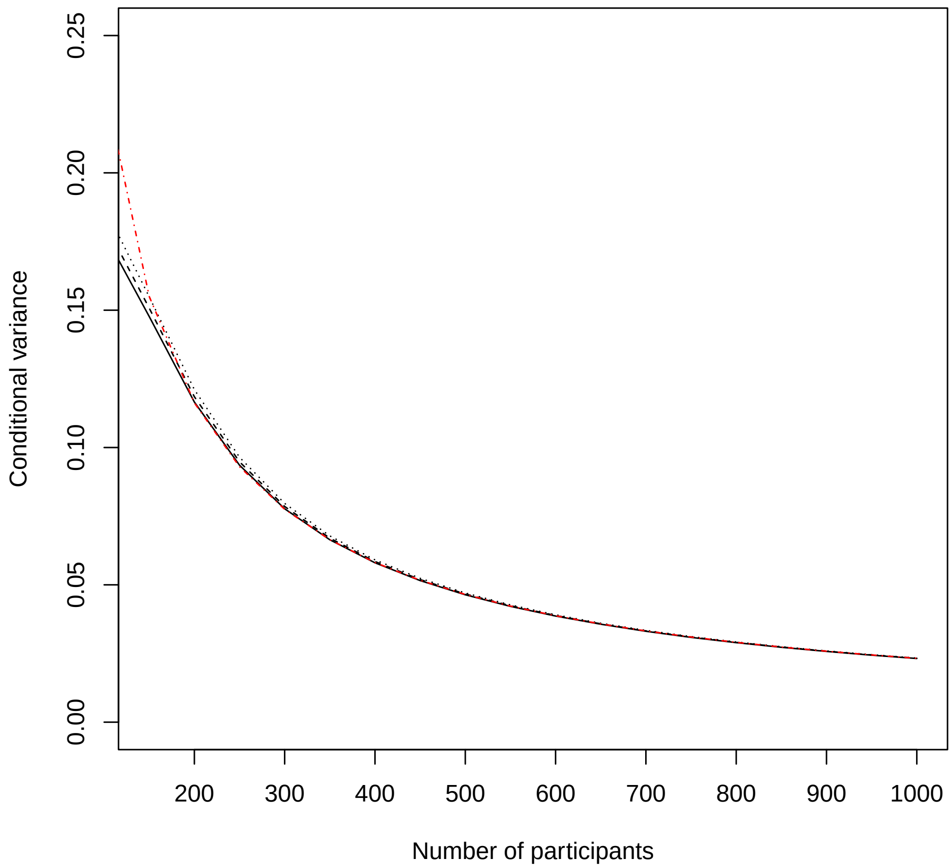


Figure 2: Conditional variance $\mathrm{Var}\left[T^{\mathrm{uni}}_{i,n}|I_i = 1\right]$ in function of the number n of participants to the endowment contingency fund, for participant i in Group 1 (solid line, —), in Group 2 (broken line, - - -), and in Group 3 (dotted line, $\cdots$). Red curve corresponding to the approximation deduced from Proposition 2.3.

Setting $\ell = 0.1$ for example, we get $n = 1'206$.

Remark 2.4. *When occurrence probabilities are small, Poisson approximations should appear as appealing tools. The idea is to approximate the distribution of M_n with the Poisson distribution with mean $\lambda = \sum_{j=1}^n q_j$ and the distribution of $M_n^{(\backslash i)}$ with the Poisson distribution with mean $\lambda_i = \sum_{j=1}^n q_j - q_i$. This is the classical approximation used to derive the collective counterpart to the individual risk model. It is shown in Appendix B that Poisson approximation applies to the calculation of individual contributions $c^{\mathrm{uni}}_{i,n}$ when occurrence probabilities get small, resulting in numerical instability for Binomial probabilities.*

2.6 Equitable terminal payouts and effect of heterogeneity

If nothing is returned to participants when $M_n = 0$ then the system cannot be actuarially fair. In order to demonstrate it formally, define the terminal payment as

$$T_{i,n}^{\text{eq}} = \frac{I_i}{1 + M_n^{(\backslash i)}} b_n^{\text{eq}}, \;\; i = 1, 2, \ldots, n,$$

where $b_n^{\text{eq}} = \sum_{i=1}^{n} c_{i,n}^{\text{eq}}$ is the sum of the contributions $c_{i,n}^{\text{eq}}$ paid in advance by the n participants. Notice that no contribution is returned to participants when $M_n = 0$. In that case, the amount b_n^{eq} may be donated to a charity, for instance. Here, the superscript "eq" refers to "equitable". Whereas actuarial fairness refers to zero expected gain, Milevsky and Salisbury (2016) introduced this concept because every tontine pool will have a little money left after the last member dies resulting in a slightly negative expected gain. According to these authors, the system can nevertheless be equitable in that no member is disadvantaged. In the present situation, equating $c_{i,n}^{\text{eq}}$ to $\mathrm{E}[T_{i,n}^{\text{eq}}]$ leads to an impossible relationship because

$$\sum_{i=1}^{n} \mathrm{E}[T_{i,n}^{\text{eq}}] = b_n^{\text{eq}} \mathrm{P}[M_n \geq 1]$$

cannot be equal to b_n^{eq} unless $\mathrm{P}[M_n \geq 1] = 1$. Hence, the system cannot be fair but it remains equitable. The probability $\mathrm{P}[M_n = 0]$ that participants lose their contributions rapidly tends to 0 when n increases so that this has no material effect in practice. Formally, it is shown in Appendix A.2 that $\mathbb{1}[M_n = 0]$ tends to 0 with probability 1 so that $T_{i,n}^{\text{uni}}$ can be accurately approximated by $T_{i,n}^{\text{eq}}$ when n gets large enough. Results derived for $T_{i,n}^{\text{eq}}$ thus approximately hold true for $T_{i,n}^{\text{uni}}$ in a sufficiently large pool.

Considering equitable terminal payouts, we can study the effects of heterogeneity with respect to occurrence probabilities. We can then deduce results which are expected to hold approximately true for actuarially fair contributions. Let J_i be independent random variables, Bernoulli distributed with mean r_i, $i = 1, 2, \ldots, n$. Occurrence probabilities q_i and r_i differ in terms of dispersion, in the sense that the probabilities r_i are more dispersed than the probabilities q_i, making the $I_1, I_2, \ldots, I_n$ more heterogeneous compared to $J_1, J_2, \ldots, J_n$. We also assume that $\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} r_i$, so that there is no difference on average. This situation can be formalized with the help of majorization.

To formalize the idea that occurrence probabilities q_i are "less spread out" or "more nearly equal" than occurrence probabilities r_i, let us first rank these two sets of probabilities in ascending order, resulting in

$$q_{(1:n)} \geq q_{(2:n)} \geq \ldots \geq q_{(n:n)} \text{ and } r_{(1:n)} \geq r_{(2:n)} \geq \ldots \geq r_{(n:n)}.$$

Then, $\boldsymbol{r}$ is said to majorize $\boldsymbol{q}$, which is denoted as $\boldsymbol{q} \preceq_{\text{maj}} \boldsymbol{r}$, if

$$\sum_{i=1}^{k} q_{(i:n)} \leq \sum_{i=1}^{k} r_{(i:n)} \text{ for } k = 1, 2, \ldots, n, \text{ and } \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} r_i.$$

Notice that $\overline{\boldsymbol{r}} \preceq_{\text{maj}} \boldsymbol{r}$ always holds, with

$$\overline{\boldsymbol{r}} = (\overline{r}, \overline{r}, \ldots, \overline{r}) \quad \text{where} \quad \overline{r} = \frac{1}{n} \sum_{i=1}^{n} r_i \tag{2.6}$$

so that homogeneity appears as the minimum in majorization order. Also,

$$\boldsymbol{r} \preceq_{\text{maj}} \left(\sum_{i=1}^{n} r_i, 0, \ldots, 0 \right) \tag{2.7}$$

so that concentrating all occurrence probabilities on a single participant (assuming that it stays below 1) is the maximum in majorization order.

Denuit and Frostig (2006) assessed the impact of the heterogeneity in claim occurrences probabilities in the individual risk model, using the concept of majorization and the closely related Schur-increasingness. The results derived in that paper turn out to be useful quantify the impact of an increase in heterogeneity within endowment contingency funds. Specifically,

$$\boldsymbol{q} \preceq_{\text{maj}} \boldsymbol{r} \Rightarrow \text{Var}\left[\sum_{i=1}^{n} I_i \right] \geq \text{Var}\left[\sum_{i=1}^{n} J_i \right].$$

Switching from $\boldsymbol{q}$ to $\boldsymbol{r}$ larger in the sense of the majorization, that is increasing the heterogeneity within the pool, thus leads to a decrease of the variance.

Property 2.5. *Assume that $I_1, I_2, \ldots, I_n$ are mutually independent, as well as $J_1, J_2, \ldots, J_n$, with $\text{P}[I_i = 1] = 1 - \text{P}[I_i = 0] = q_i$ and $\text{P}[J_i = 1] = 1 - \text{P}[J_i = 0] = r_i$. Let $M_n^{(\backslash i)} = \sum_{k \neq i} I_k$ and $Q_n^{(\backslash i)} = \sum_{k \neq i} J_k$. Then, the inequality*

$$\boldsymbol{q} \preceq_{\text{maj}} \boldsymbol{r} \Rightarrow \text{E}\left[\frac{I_i}{1 + M_n^{(\backslash i)}} \right] \geq \text{E}\left[\frac{J_i}{1 + Q_n^{(\backslash i)}} \right]$$

holds true for $i = 1, 2, \ldots, n$.

The proof of Property 2.5 can be found in Appendix A.4. It shows that the expected equitable payout decreases when occurrence probabilities become more homogeneous. The largest average payout is obtained when $r_i = \frac{1}{n}\sum_{j=1}^{n} q_j$ for all i.

3 Endowment contingency funds with varying protection levels

In order to offer each participant the opportunity to select the level of protection he or she desires, an actuarially fair system generalizing the uniform sharing rule studied in Section 2 is designed in this section. Now, each participant is free to subscribe one or several protection units from the fund and total contributions are divided equally among all claiming units. By claiming unit we mean a protection unit subscribed by a participant who experienced the predefined event. The sponsor determines the targeted benefit per claiming unit so that participants can select the number of units they need to reach the desired level of protection.

3.1 Mechanism

Milevsky and Salisbury (2015) let participants subscribe varying numbers of shares in their equitable retirement income tontines, to account for both participants' age and amounts invested. A similar idea is used here, to reflect individual levels of protection. Since no assumption has been made about the dependence structure of the loss indicators $I_1, I_2, \ldots, I_n$ in Section 2, allowing participant to freely subscribe the number of protection units they wish from the fund corresponds to the situation where I_i is repeated a number of times equal to the number of protection units subscribed by participant i, for each $i = 1, 2, \ldots, n$. This situation is similar to multi-detention in life insurance and it is known that it increases variability compared to the case where each unit is subscribed independently. We can nevertheless obtain more explicit formulas and exploit mutual independence between individuals, when it holds.

Let w_i be the number of protection units subscribed by participant i. We assume that $w_1, w_2, \ldots, w_n$ are positive integers and we denote as $w_\bullet = \sum_{i=1}^{n} w_i$ the total number of protection units for the entire fund. The number of claiming units is

$$O_n = \sum_{i=1}^{n} w_i I_i.$$

Total contributions are now evenly distributed at time 1 among those O_n units held by individuals who experienced the predefined event. Initial contributions per unit under this sharing rule are henceforth denoted as $c^{\mathrm{w}}_{i,n}$, with sum

$$b^{\mathrm{w}}_n = \sum_{i=1}^{n} w_i c^{\mathrm{w}}_{i,n}.$$

As previously, we assume that participants just get their initial contributions back when nobody experienced the event under consideration. Define

$$O_n^{(\backslash i)} = O_n - w_i I_i = \sum_{j\neq i} w_j I_j.$$

The terminal payout $T^{\mathrm{w}}_{i,n}$ per unit subscribed by participant i is then given by

$$T^{\mathrm{w}}_{i,n} = \frac{I_i}{w_i + O_n^{(\backslash i)}} b^{\mathrm{w}}_n + c^{\mathrm{w}}_{i,n} \mathbb{1}[O_n = 0].$$

The total payout for participant i is $w_i T^{\mathrm{w}}_{i,n}$.

Imposing actuarial fairness, the initial contribution per unit must satisfy

$$\begin{aligned} c^{\mathrm{w}}_{i,n} &= \mathrm{E}[T^{\mathrm{w}}_{i,n}] = \mathrm{E}\left[\frac{I_i}{w_i + O_n^{(\backslash i)}}\right] b^{\mathrm{w}}_n + c^{\mathrm{w}}_{i,n}\mathrm{P}[O_n = 0] \\ &= \frac{b^{\mathrm{w}}_n}{\mathrm{P}[O_n \geq 1]} \sum_{k=1}^{w_\bullet} \frac{1}{k} \mathrm{P}\left[O_n = k, I_i = 1\right] \end{aligned} \tag{3.1}$$

for $i = 1, 2, \ldots, n$. The joint probability appearing in (3.1) can be rewritten as

$$\mathrm{P}\left[O_n = k, I_i = 1\right] = \mathrm{P}\left[O_n^{(\backslash i)} = k - w_i, I_i = 1\right].$$

In particular,

$$I_1, I_2, \ldots, I_n \text{ independent } \Rightarrow c^{\mathrm{w}}_{i,n} = \frac{b^{\mathrm{w}}_n q_i}{\mathrm{P}[O_n \geq 1]} \sum_{k=1}^{w_\bullet} \frac{1}{k} \mathrm{P}\left[O_n^{(\backslash i)} = k - w_i\right]. \tag{3.2}$$

Again, the proposed allocation can be seen as an anticipated version of the conditional mean risk-sharing rule operating in arrear, since it is easily shown that

$$c^{\mathrm{w}}_{i,n} = b^{\mathrm{w}}_n \mathrm{E}\left[\frac{\mathrm{E}[I_i|O_n]]}{O_n}\middle| O_n \geq 1\right]$$

holds true for $i = 1, \ldots, n$ (the reasoning is similar to the one applying when $w_i = 1$ for all

i).

The initial contributions $c^{\mathrm{w}}_{i,n}$ in (3.1) are only defined up to a constant factor. Denoting as β_{w} the targeted average benefit per claiming unit specified by the sponsor, b^{w}_n can be obtained from

$$b^{\mathrm{w}}_n \mathrm{E}\left[\frac{1}{O_n}\middle| O_n \geq 1\right] = \beta_{\mathrm{w}} \Leftrightarrow b^{\mathrm{w}}_n = \frac{\beta_{\mathrm{w}} \mathrm{P}[O_n \geq 1]}{\sum_{k=1}^{w_\bullet} k^{-1} \mathrm{P}[O_n = k]}. \tag{3.3}$$

Inserting β_{w} appearing in (3.3) in (3.1) shows that the amount of contribution per unit subscribed by participant i is given by

$$c^{\mathrm{w}}_{i,n} = \beta_{\mathrm{w}} \frac{\sum_{k=1}^{w_\bullet} k^{-1} \mathrm{P}\left[O_n = k, I_i = 1\right]}{\sum_{k=1}^{w_\bullet} k^{-1} \mathrm{P}\left[O_n = k\right]}, \quad i = 1, \ldots, n. \tag{3.4}$$

With the targeted benefit β_{w} in (3.3), (3.2) can be rewritten as

$$I_1, I_2, \ldots, I_n \text{ independent } \Rightarrow c^{\mathrm{w}}_{i,n} = q_i \beta_{\mathrm{w}} \frac{\sum_{k=1}^{w_\bullet} k^{-1} \mathrm{P}\left[O_n^{(\backslash i)} = k - w_i\right]}{\sum_{k=1}^{w_\bullet} k^{-1} \mathrm{P}\left[O_n = k\right]}.$$

The next results extends Propositions 2.1-2.2 to the case where protection levels vary among participants.

Proposition 3.1. *Assume that $I_1, I_2, \ldots$ are mutually independent. Denote $q_{\min} = \inf_{i\geq 1} q_i$ and $\overline{wq} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} w_i q_i$, assuming that the limit exists. If $q_{\min} > 0$, $0 < \overline{wq} < 1$ and there exists a positive constant w_+ satisfying $sup_{i\geq 1} w_i \leq w_+$ then*

$$\lim_{n\to\infty} c^{\mathrm{w}}_{i,n} = \beta_{\mathrm{w}} q_i \textit{ and } T^{\mathrm{w}}_{i,n} \to I_i \beta_{\mathrm{w}} \textit{ as } n \to \infty \textit{ with probability 1.}$$

The proof of Proposition 3.1 is given in Appendix A.5. We can see that, at the limit, participant i pays an amount corresponding to the pure premium, or fair price of an insurance policy with lump sum benefit $\beta_{\mathrm{w}} w_i$ in case the predefined event occurs. The terminal payout tends to the payout of this contract and at the limit, only (w_i, q_i) matters and not (w_j, q_j) for $j \neq i$ anymore.

3.2 Numerical Example

Consider again the pool described in Section 2.5 with participants partitioned into three groups: occurrence probability equal to 0.01 in Group 1 gathering 50% of the participants, occurrence probability equal to 0.05 in Group 2 gathering 30% of the participants, and occurrence probability equal to 0.1 in Group 3 gathering 20% of the participants. In each group, let us now assume that half of the participants select $w_i = 1$ and the other ones

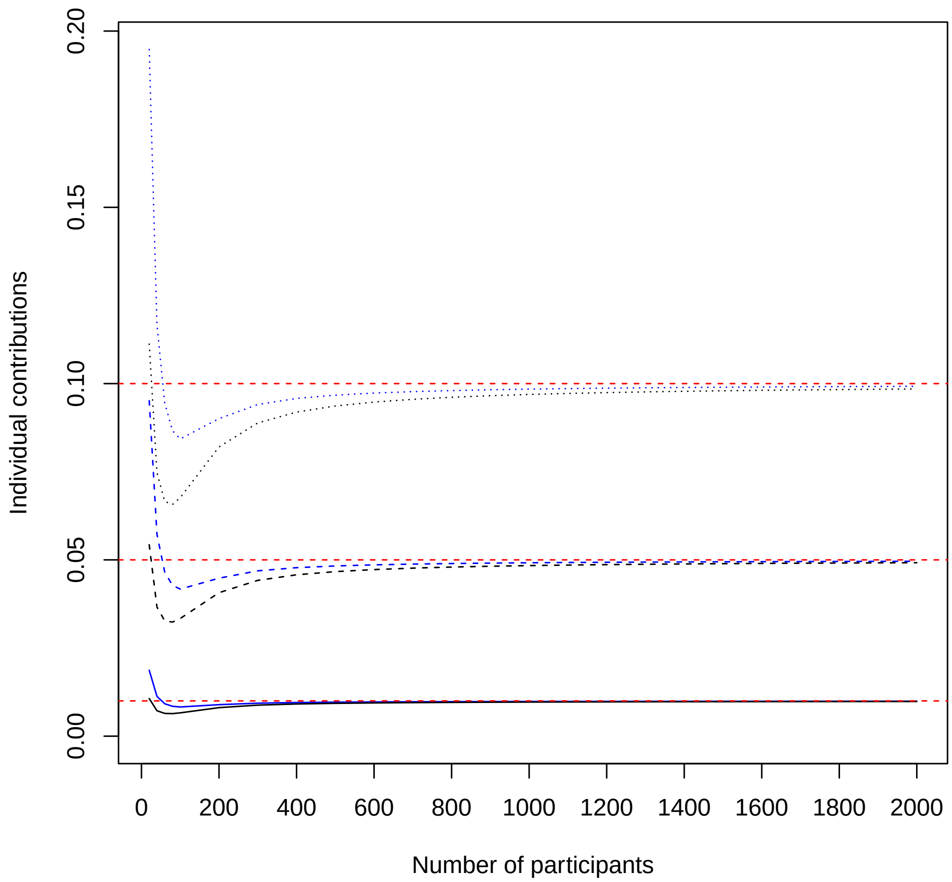


Figure 3: Individual contributions per unit in Group 1 (solid line, —), in Group 2 (broken line, - - -), and in Group 3 (dotted line, $\cdots$), blue curves correspond to participants with $w_i = 1$ and black curves to participants with $w_i = 2$, with red horizontal lines at occurrence probabilities.

$w_i = 2$. Figure 3 displays individual contributions $c^{\mathrm{w}}_{i,n}$ per unit. The results are given according to the number n of participants, for targeted benefits $\beta_{\mathrm{w}} = 1$ per unit. We can see there that individual contributions per protection unit again stabilize at level q_i as $n \to \infty$, in application of Proposition 3.1.

3.3 Effect of heterogeneity under equitable payouts

Denuit (2000) analyzed the impact of the presence of duplicates in life insurance portfolios. The results derived there help to assess the effect of allowing participants to subscribe one or several protection units, compared to the case where each unit is held independently.

For each i, let $J_1^{(i)}, \ldots, J_{w_i}^{(i)}$ be independent indicators distributed as I_i. Assume further that all $J_k^{(i)}$ are mutually independent. Formula (4.3) in Denuit (2000) then shows that $\sum_{i=1}^n \left(J_1^{(i)} + \ldots + J_{w_i}^{(i)}\right)$ is smaller than $\sum_{i=1}^n w_i I_i$ in the convex order. This means that the expectation of any convex function of $\sum_{i=1}^n w_i I_i$ is larger than the corresponding expectation of $\sum_{i=1}^n \left(J_1^{(i)} + \ldots + J_{w_i}^{(i)}\right)$. The same result holds true for the sums over all participants, except one. Since the inverse function is convex over $(0, \infty)$, we get

$$\mathrm{E}\left[\frac{I_i}{w_i + O_n^{(\backslash i)}}\right] \geq \mathrm{E}\left[\frac{J_1^{(i)}}{w_i + \sum_{k \neq i}^n \left(J_1^{(k)} + \ldots + J_{w_k}^{(k)}\right)}\right],$$

corresponding to expected equitable terminal payouts. See also Corollary 4.7 in Denuit and Frostig (2006), allowing for a comparison in the majorization order.

Proposition 4.3 in Denuit and Frostig (2006) allows us to analyze the combined effect of heterogeneity in the occurrence probabilities q_i and in the number of protection units w_i. Assumed that participants are numbered according to the magnitude of their respective occurrence probabilities, so that $q_1 \geq q_2 \geq \ldots \geq q_n$. Now, assume that $w_1 \geq w_2 \geq \ldots \geq w_n$, meaning that participants with higher occurrence probabilities subscribe more protection units. This assumption is somewhat restrictive but appears to be reasonable, at least as an approximation, in the case of survivor funds, for instance. The reason is that participants from upper socio-economic classes generally have higher survival probabilities. Then, assuming that $J_1, \ldots, J_n$ are numbered in a similar way, we have the expectation of any convex function of $\sum_{i=1}^n w_i I_i$ is larger than the corresponding expectation of $\sum_{i=1}^n w_i J_i$. In words, this means that $\sum_{i=1}^n w_i I_i$ tends to be smaller but more variable than $\sum_{i=1}^n w_i J_i$. The same result holds for sums over all participants, except one. Denoting

$$R_n^{(\backslash i)} = \sum_{k=1}^n w_k J_k - w_i J_i = \sum_{k \neq i} w_k J_k,$$

this implies that

$$\mathrm{E}\left[\frac{I_i}{w_i + O_n^{(\backslash i)}}\right] \geq \mathrm{E}\left[\frac{J_i}{w_i + R_n^{(\backslash i)}}\right],$$

corresponding to expected equitable terminal payouts.

4 Applications

4.1 Mutual aid funds

In organization theory, mutual aid is a voluntary reciprocal exchange of resources and services for mutual benefit. In our context, a mutual aid fund is a peer-to-peer risk sharing network joined by participants facing a common risk, generally related to health. Mutual aid funds are commonly encountered in Asia, particularly in China where they operate through online platforms. The latter are community-based risk-sharing networks connecting individuals exposed to given perils. Participants typically share the risk of being diagnosed with severe diseases.These platforms fill a gap in health systems, between the public cover with limited benefits and private insurance charging expensive premiums. Mutual aid platforms thus target low- and middle-income households needing cheap healthcare protection. We refer the reader e.g. to Abdikerimova and Feng (2022) for a more extensive presentation of these systems.

In China, mutual aid plans typically provide benefits to members who have been diagnosed with a critical illness, such as thyroid cancer, lung cancer or critical brain injury, for instance. Mutual aid started to be operated through online platforms in China, where Kangai launched the concept in 2011. In 2018, technology giants like Ant Financial and DiDi entered this market with huge capital injections, resulting in a rapid market expansion. See Chen et al. (2022) for more information about the development of mutual aid in China.

As documented by Fang et al. (2022), Xianghu Insurance was launched in 2018 as a peer-to-peer insurance company by Ant Financial, partnering with Trust Mutual Life Insurance. The insurance partner rapidly left the joint venture and Xianghu Insurance changed its name to Xiang Hu Bao (meaning “mutual treasure” or “protecting each other”, abbreviated as XHB in the remainder of the text). XHB thus became a pure online mutual aid platform and expanded rapidly, being supported by the popular Alipay App. By the end of 2019, it gathered about 100 millions members, becoming the largest mutual aid platform in China. Owing to regulatory pressure, XHB stopped operating in January 2022, following closures of more than ten mutual aid services in 2021. Chinese regulators now aim to issue new guidelines for internet platforms.

XHB mutual aid plan is structured as follows. During each coverage period, participants equally share aggregate medical claim payments supplemented with XHB’s operating expenses. In exchange, they receive lump-sum benefits equal to CNY300’000 for individuals under 40 years old and CNY100’000 for participants aged 40 to 59 if they are confirmed to have been diagnosed with one of the covered critical illnesses. In line with these benefits awarded in case of critical illness (three times higher for ages up to 39 compared to ages

Age group	Occurrence probability	Proportion of the pool	Contributions for $n = 1'000$	for $n = 10'000$
≤19	0.000269	0.232	0.000130	0.000241
20-29	0.000528	0.215	0.000255	0.000473
30-39	0.001444	0.208	0.000697	0.001294
40-49	0.003909	0.176	0.003693	0.003763
50-59	0.008553	0.102	0.008093	0.008235

Table 4.1: Mutual aid scheme.

40-59), we set benefits to 3 and 1, respectively. Participant i then pays at the end of the period an amount

$$B_{i,n} = \frac{1}{n}\sum_{i=1}^{n} w_i I_i \text{ with } w_i = \begin{cases} 3 \text{ if participant } i\text{'s age } \leq 39 \\ \\ 1 \text{ if participant } i\text{'s age } \geq 40 \end{cases}. \tag{4.1}$$

In this section, we compare the operating system adopted by XHB, where participants equally share losses at the end of the period (paying an amount $B_{i,n}$ to provide claimants with agreed benefits equal to w_i) to the alternative proposed in this paper, where participants contribute in advance a known amount $c^{\mathrm{w}}_{i,n}$ and total contributions are evenly shared among claiming participants at the end of the period. We specify a target benefit $\beta_{\mathrm{w}} = 1$ and assume that participants younger than 40 buy 3 units of protection while older participants only buy 1 unit. The endowment contingency fund then provides benefits hopefully close to w_i. Since mutual aid platforms also require participants to pay a minimum deposit on their personal account, the difference is whether all contributions are random but benefits are deterministic (as for XHB) or only the benefits paid to claiming participants are random whereas contributions are known with certainty as in the system proposed in this paper.

We perform numerical illustrations on the basis of information gathered in Table 6 from Li et al. (2022), displayed in the first columns of Table 4.1. Considering the small occurrence probabilities, we use the Poisson approximation described in Appendix B. Results are listed in Table 4.1.

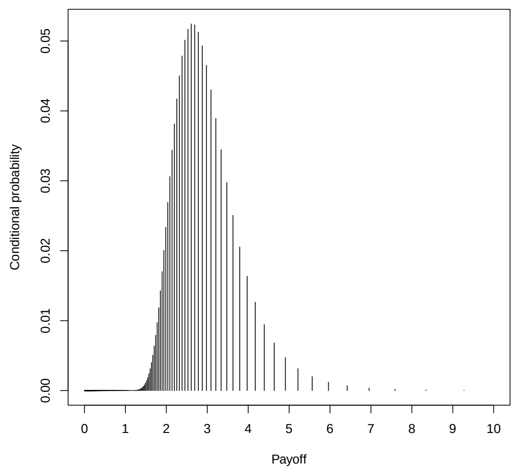
For $n = 1'000$, the sum of contributions is equal to 2.164906. This amount does not even allow to pay the targeted benefits to a single participant aged up to 39 diagnosed with a covered critical illness. Increasing the number of participants is thus required for the system to be effective. Let us now consider $n = 10'000$. We can see that contributions increase to become closer to the asymptotic level q_i. The probability mass function of the terminal payoff to a claiming participant is displayed in Figure 4. Precisely, this is the conditional probability mass function of the terminal payoff for participant i given that $I_i = 1$, depending on the age class to which individual i belongs. The bottom panel displays the probability mass function of the contribution paid in arrear, corresponding to total losses equally distributed among the participants, that is, $B_{i,n}$ defined in (4.1). We can see there that terminal payouts remain volatile despite the large number of participants enrolled with XHB. This is because the occurrence probabilities q_i in Table 4.1 are extremely small. The next example considers larger occurrence probabilities and shows that endowment contingency funds then become effective.

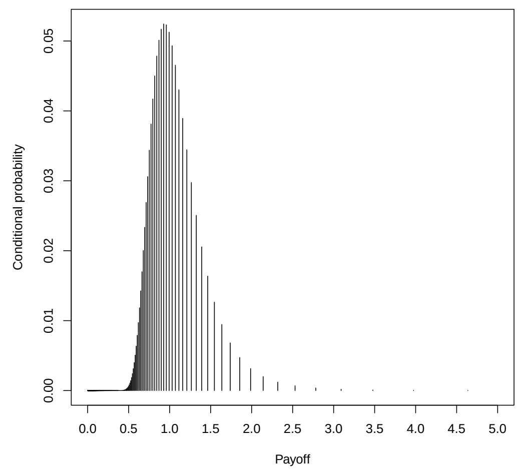
4.2 Endowment longevity funds

Survivor funds can be seen as a particular case of endowment contingency funds where the predefined event is survival until the end of the period. Survivor funds are named after Forman and Sabin (2016). Several variants exist and they are also known under various appellations, such as individual tontine accounts (Fullmer and Sabin, 2019) or pooled-survival fund (Newfield, 2014), for instance. A survivor fund refers to a fixed-term fund that boosts the yield of investments thanks to a mutual inheritance rule among participants. According to this rule agreed at inception, the contributions are lost in case of death during the reference period and the accumulated value of the fund is distributed among survivors.

It is easy to assemble survivor funds with increasing maturities to generate lifelong payments so that survivor funds appear to be building blocks to form tontine annuities. As pointed out by Olivieri and Pitacco (2020), these arrangements date back to the 17th century and are characterized by periodic or terminal payments increased thanks to the award of mortality credits in application of the mutual inheritance rule. Originally developed for speculative purposes, tontine annuities have been revised to become longevity risk management tools. See, for example, Sabin (2010), Milevsky and Salisbury (2015, 2016), Weinert and Gründl (2021), and Chen et al. (2019, 2021). Other proposals letting participants share mortality credits include group self-annuitisation pools (see, e.g., Piggot et al., 2005, and Qiao and Sherris, 2012), pooled annuity funds (see e.g. Donnelly et al., 2013, and Bernhardt and Donnelly, 2021), and annuity overlay funds (see e.g. Donnelly et al., 2014, and Donnelly, 2015). See also Winter and Planchet (2022) for a practical overview and Milevsky (2022) for a general introduction to the topic.

A large body of the literature discusses the appropriate way to allocate mortality credits, with special attention to the concept of actuarial fairness meaning that participants receive their contribution back, on average. Quite surprisingly, intuitive sharing rules generally fail to be actuarially fair (even though they are close to being so, as shown by Donnelly, 2015).





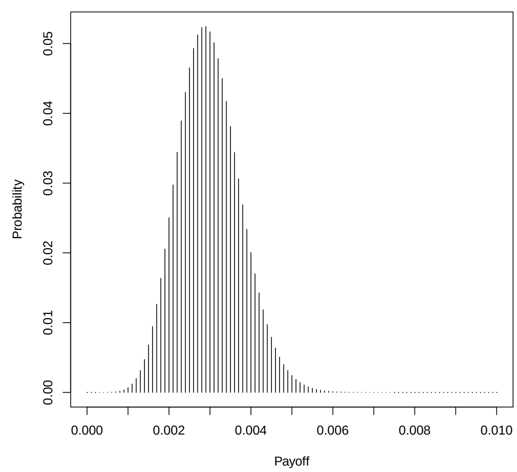


Figure 4: Probability mass function of the terminal payoff to a claiming participant aged up to 39 (upper left panel), aged 40 to 59 (upper right panel), and probability mass function of the share of total losses under XHB system (bottom panel) for $n = 10'000$ participants.

As pointed out by Bernhardt and Qu (2022), actuarially fair rules including the conditional mean risk-sharing rule appear to be very sensitive to heterogeneity in the sum insured. Denuit et al. (2022) provided several examples in that respect, and obtained full diversification provided sums insured are selected from a limited menu and corresponding numbers of participants all tend to infinity. See Bernhardt and Qu (2022) and the references therein for more details. On the contrary, full diversification is generally obtained at the limit with endowment contingency funds, under mutual independence. Henceforth, endowment contingency funds are called endowment longevity funds when the predefined event corresponds to survival.

Let us now illustrate the potential of endowment longevity funds. Assume that some public body sponsors such a scheme to borrow at zero interest rate in order to finance some large infrastructure. This offer is expected to attract older participants given the attractive returns corresponding to mortality credits at older ages. The idea is to set up an endowment longevity fund working as a tontine scheme. Retired people could lend money to the state on a voluntary basis, entering a survivor scheme in the sense that the debt would be repaid to survivors, only. The return on this operation for survivors corresponds to mortality credits. The latter are the amounts invested by those participants who died before maturity, which are inherited by survivors. These inheritance gains constitute the product's return, making the operation very attractive at older ages even under zero interest rate. In practice, a national register can be used to identify beneficiaries and the system would be costless to operate. The proposed approach avoids that public debt is traded on financial markets and may thus serve speculative purposes.

Let us assume that 10'000 participants join the endowment longevity fund. Within the pool, 30% are aged 65, 50% are aged 70 and the remaining 20% are aged 75. They all pay their contributions at time 0, with a targeted benefit of 1 at times 6, 7, 8, 9, and 10 (the 5-year deferred period is meant to built the infrastructure so that its revenues can then serve to repay the debt). Survival probabilities are taken from the regulatory XR life table enforced in Belgium.

Results are displayed in Table 4.2. We can see there that contributions are very close to the corresponding survival probabilities, as they should be according to Proposition 2.1, considering the large pool size. Mortality credits generate attractive returns, compared to the expected unit benefit at maturities 6 to 10. The volatility turns out to be very limited, as it can be seen from Figure 5. Results are displayed for a participant aged 65 at time 0, since there are no visible differences compared to ages 70 and 75. Terminal payoffs to survivors become more variable when time horizon gets more distant, but the volatility remains very small in all cases.

Time Horizon	Age 65 at issue		Age 70 at issue		Age 75 at issue	
	Survival probabilities	Contributions for $n = 10'000$	Survival probabilities	Contributions for $n = 10'000$	Survival probabilities	Contributions for $n = 10'000$
6	0.917909	0.917900	0.869513	0.869500	0.794999	0.794981
7	0.899896	0.899885	0.841736	0.841720	0.753652	0.753630
8	0.880432	0.880419	0.812089	0.812070	0.710456	0.710431
9	0.859441	0.859426	0.780556	0.780534	0.665587	0.665559
10	0.836851	0.836833	0.747145	0.747119	0.619277	0.619246

Table 4.2: Endowment longevity fund with uniform protection level.

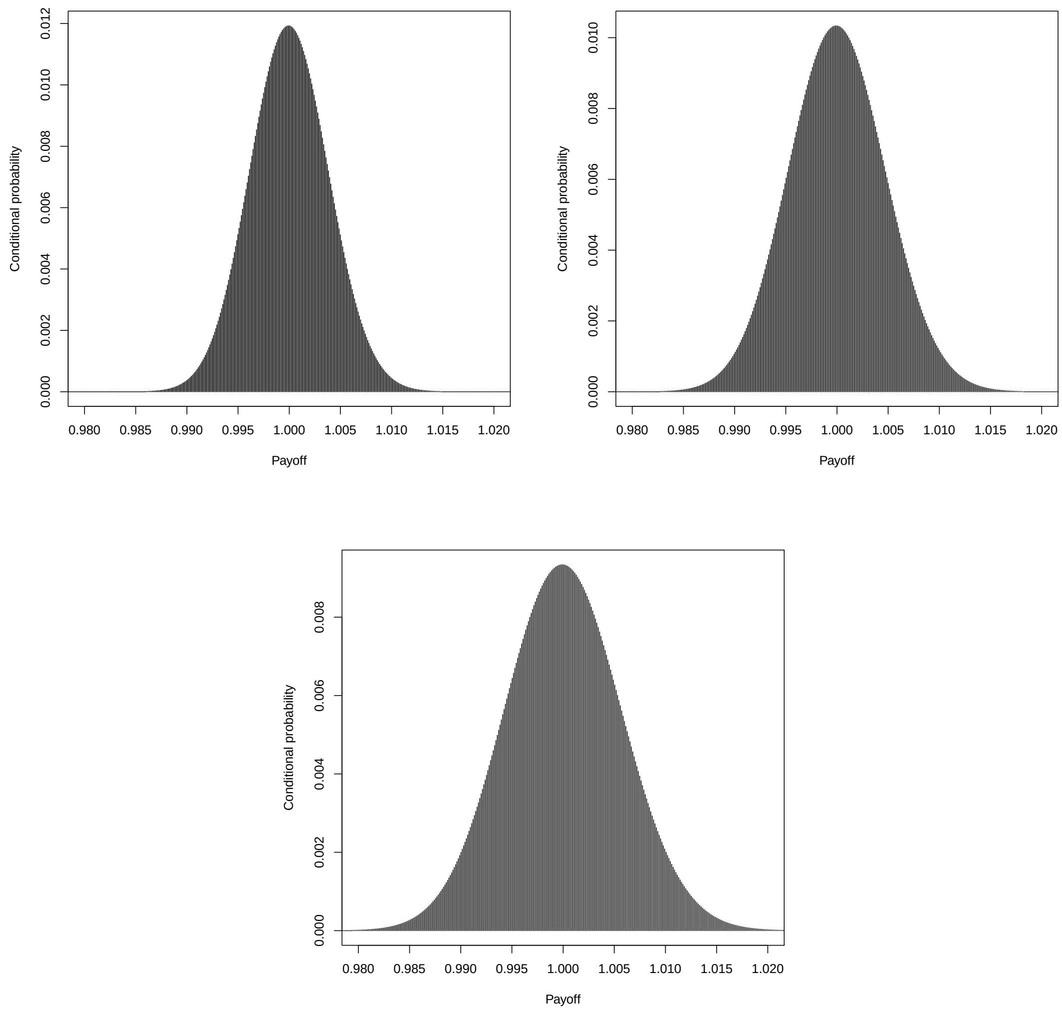


Figure 5: Probability mass function of the terminal payoff to a survivor for an endowment longevity fund gathering $n = 10'000$ participants with uniform protection level. Results corresponding to a participant aged 65 at policy issue, for payments at time 6 (upper left panel), 8 (upper right panel), and 10 (lower panel).

Time Horizon	Age 65 at issue Survival probabilities	Contribution per unit for $w_i = 1$	for $w_i = 2$
6	0.917909	0.917903	0.917897
7	0.899896	0.899889	0.899882
8	0.880432	0.880423	0.880415
9	0.859441	0.859431	0.859420
10	0.836851	0.836839	0.836827
Time Horizon	Age 70 at issue Survival probabilities	Contribution per unit for $w_i = 1$	for $w_i = 2$
6	0.869513	0.869505	0.869496
7	0.841736	0.841725	0.841715
8	0.812089	0.812077	0.812064
9	0.780556	0.780542	0.780527
10	0.747145	0.747128	0.747111
Time Horizon	Age 75 at issue Survival probabilities	Contribution per unit for $w_i = 1$	for $w_i = 2$
6	0.794999	0.794987	0.794974
7	0.753652	0.753638	0.753623
8	0.710456	0.710439	0.710422
9	0.665587	0.665568	0.665549
10	0.619277	0.619256	0.619235

Table 4.3: Endowment longevity fund with varying protection levels.

If we assume now that 50% of participants in each age group subscribe 1 protection unit and the remaining 50% subscribe 2 protection units, then we get the results displayed in Table 4.3. Again, the contributions are remarkably close to the corresponding survival probabilities, as a consequence of Proposition 3.1. As it was the case with uniform protection level, we can see on Figure 6 that the volatility in terminal payouts is very low but nevertheless larger when $w_i = 2$ compared to $w_i = 1$.

5 Discussion

5.1 Proportional sharing rule

Under the uniform sharing rule and as the pool size is large, all participants get on average the targeted amount β of benefit in case the event occurs. The initial contributions reflect the values of occurrence probabilities. Appendix C investigates the situation where participants

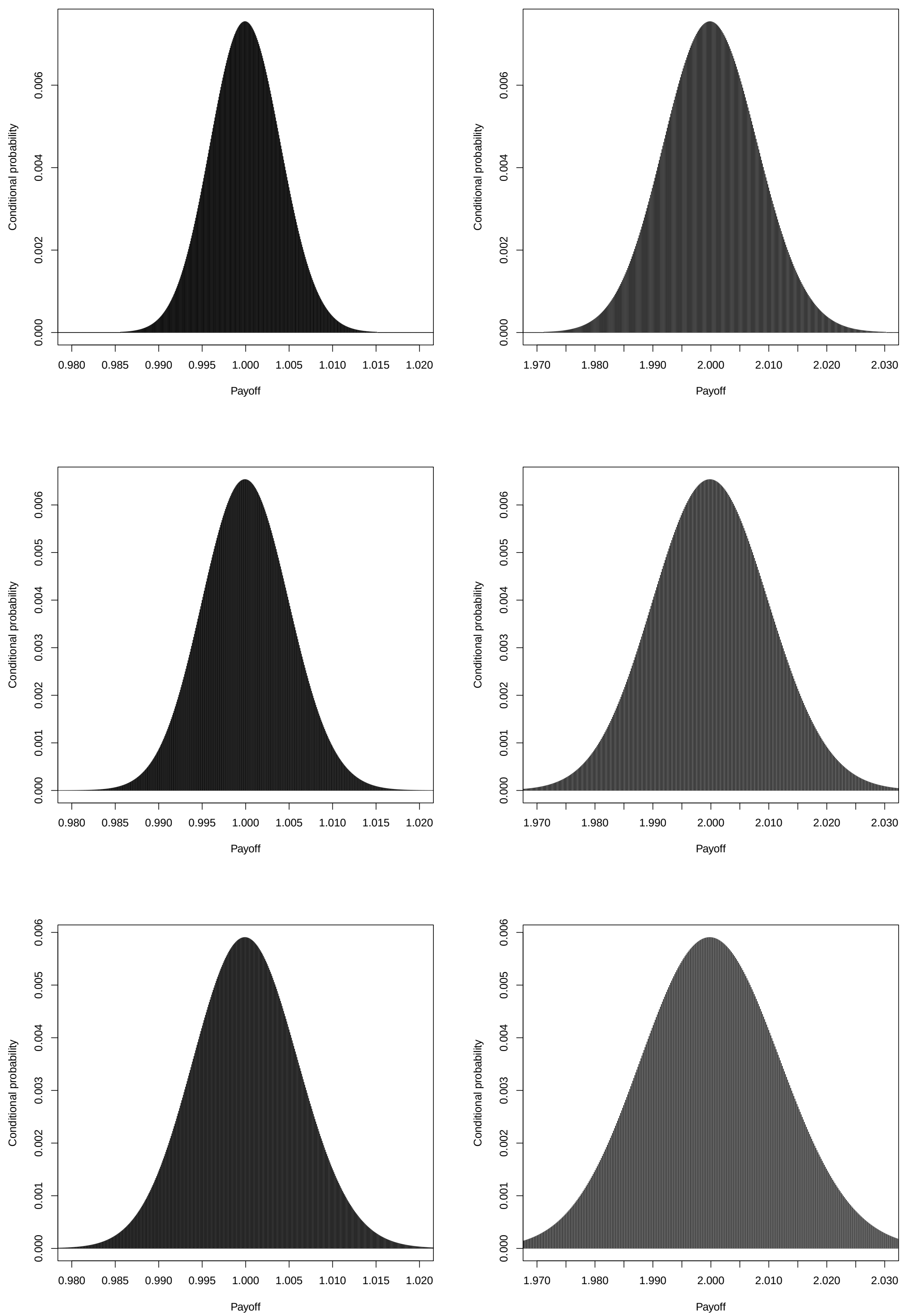


Figure 6: Probability mass function of the terminal payoff to a survivor for an endowment longevity fund gathering $n = 10'000$ participants with varying protection level. Results corresponding to a participant aged 65 at policy issue, for payments at time 6 (upper panels), 8 (middle panels), and 10 (lower panels). The left column corresponds to $w_i = 1$ and the right one to $w_i = 2$.

are willing to get benefits proportional to their initial contribution in case the predefined event occurs. It is shown there that the actuarial fairness condition can be satisfied only if $q_i = q$ for all i. This is why this approach has been abandoned and replaced with protection units in Section 3.

5.2 Takaful

The results derived in this paper appear to be useful in the context of takaful, the Sharia-compliant alternative to conventional insurance. Takaful can be translated as solidarity or mutual guarantee. We refer the reader to Malik and Ullah (2019) or Billah (2019) for an introduction to the topic. In a nutshell, conventional insurance has elements of riba (usury), maysir (gambling) and gharar (excessive uncertainty), which make it non-permissible from the Sharia perspective. Conventional insurance involves excessive uncertainty (gharar) because the transaction between the policyhoder and the insurer is uncertain and possibly unbalanced as the insurance contract may constitute a disproportionate loss in favor of one participant at the expense of the other.

Gharar being prohibited, pure risk transfer is impossible and takaful is therefore characterized by risk sharing. Islam indeed includes among its principles cooperation and mutual aid as well as the fair sharing of risks and benefits. To exclude gharar, the idea developed by jurists in Muslim law is to bring the insurance contribution closer to tabarru, which in Arabic means a donation. A donation does not require any counter-party and therefore cannot be associated with an uncertain return. When one of the participants is not a claimant, his or her contribution is considered as a donation since it benefits the claimants and is therefore similar to tabarru.

Compared with takaful insurance, the fundamental difference is that contributions are considered as irrevocable donations under takaful, to comply with Sharia precepts. Therefore, contributions cannot be returned back to participants if no one experiences the predefined event during the coverage period. As a consequence, actuarial fairness cannot hold (even if the difference is not material within large pools). This echoes Section 2.6. More information can be found in Appendix D where we describe a family takaful scheme providing participants with coverage in case of death as an illustration.

5.3 Guarantees

Endowment contingency funds expose participants to randomness in the benefits awarded in case the predefined event occurs during the reference period. In that respect, some Asian mutual aid platforms guarantee that the payments in arrear never exceed a specific upper limit. Transposed to endowment contingency funds, a partnering (re-)insurer may propose additional guarantees to participants, ensuring that the random benefits never fall below some minimum threshold for instance, as explained next.

Assume that the sponsor guarantees that the benefit paid to participant i never falls below a given threshold τ_i. Since this guarantee is risky, it can only be covered by an insurance company partnering with the fund. The cost of this guarantee must then be paid by participant i on the top of his or her initial contribution. Such guarantees parallel those included in Chinese mutual aid platforms. For example, XHB introduced a yearly upper limit to participants' contributions, with the excesses paid by Ant Financial. The cost of these guarantees is implicitly embedded in the management fee. Translated into our context, upper limits on individual contributions become guarantees that benefits paid to claiming individuals never fall below a selected threshold. Consider for instance the uniform sharing rule studied in Section 2. If participant i wishes a terminal payout at least equal to τ_i in case the predefined event occurs then his or her terminal payout $T_{i,n}^{\text{uni}}$ is supplemented with

$$T_{i,n}^{\text{uni},+} = I_i \left(\tau_i - \frac{b_n^{\text{uni}}}{M_n} \right)_+ \mathbb{1}[M_n \geq 1].$$

Then,

$$T_{i,n}^{\text{uni}} + T_{i,n}^{\text{uni},+} = I_i \max \left\{ \frac{b_n^{\text{uni}}}{M_n}, \tau_i \right\} \mathbb{1}[M_n \geq 1] + c_{i,n}^{\text{uni}} \mathbb{1}[M_n = 0].$$

The additional payout $T_{i,n}^{\text{uni},+}$ is then covered by an insurance policy sold by the partnering insurer.

5.4 Conclusion

In this paper, we have considered two different ways to allocate total contributions among claiming participants in an endowment contingency fund. The first one divides total contributions equally among all members who experienced the predefined event while the other one allocates benefits equally among protection units. The limiting case where the number of participants tends to infinity is considered for the different systems. Under mutual independence, insurance at fair price is recovered at the limit under mild technical conditions.

It is important to limit transfers among participants to pure mutualization, or random solidarity, preventing any subsidizing solidarity effect, or at least to clearly disentangle both types of transfers. If participants are willing to include subsidizing solidarity then such transfers can be transparently added to actuarially fair contributions. Actual contributions can be split into two parts: the actuarially fair component and the gift to some categories of participants.

Endowment contingency funds have great potential to help states raising funds to finance infrastructures. It turns out that numerous public buildings and other municipality projects throughout the Western world (and several wars) all have wholly, or partially, been funded by tontines. See e.g. Green (2019). Similarly, endowment contingency funds could serve that goal. For instance, they could play an important role to massively invest in the transition from the carbon economy to renewable energies. Now that interest rates started to rise, endowment contingency funds are able to raise capital at zero interest rate while providing retirees with an attractive investment helping them to decumulate their assets in a safe and adequate way. The mutual inheritance rule provides them with attractive returns corresponding to mortality credits. At a larger scale, endowment contingency funds can also be offered before retirement, subject to a deferred period.

The tontine concept is generally attributed to the Italian Lorenzo Tonti who proposed it to Cardinal Mazarin in 1653. It seems now that a similar scheme was proposed in Portugal as early as 1641 by Nicolas Bourey from Antwerp. Tontine operations are still permitted throughout EU. This constitutes a niche market in France where Le Conservateur sells this kind of product (registered as a "société à forme tontinière"). In Belgium, tontines can be sold by an insurer registered under insurance Branch 25 "opérations tontinières/tontineverrichtingen". Despite the fact that an insurer registered, we are not aware of any commercial offer. But this shows that the regulatory environment is ready for endowment longevity funds to be launched. It is interesting to realize that the tontine mechanism, proposed to the French royal court to finance wars at reduced cost, would be revived in this way to help modern societies to finance their fight against climate change, for instance.

An interesting question in practice is whether the operations under endowment contingency funds qualify as insurance contracts, and can then only be performed by registered insurance companies, or whether they escape this qualification and can thus be offered by actors in the sharing economy, through platforms or friendly associations. Considering endowment longevity funds, it seems that the answer would be negative, at least in Belgium and in France, because tontine operations fall within specific insurance branches, as explained before. In case events are not related to human lifetimes, endowment contingency funds could probably exist outside of the insurance market. Donation circles, including the popular broodfondsen in the Netherlands, are nice examples. Participants then donate contributions in advance to the fund, without receiving any guarantee. This question is often asked by Insurtech entrepreneurs considering to start their business and should be addressed by multidisciplinary teams joining actuaries with qualified insurance lawyers.

Acknowledgements

Michel Denuit gratefully acknowledges funding from the FWO and F.R.S.-FNRS under the Excellence of Science (EOS) programme, project ASTeRISK (40007517).

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Appendix

A Proofs of the main results

A.1 Proof of Proposition 2.1

The initial contribution (2.5) paid in advance by participant i can be rewritten as

$$c_{i,n}^{\text{uni}} = q_i \beta \frac{\sum_{k=0}^{n-1} \frac{n}{k+1} \mathrm{P}\left[M_n^{(\setminus i)} = k\right]}{\sum_{k=1}^{n} \frac{n}{k} \mathrm{P}\left[M_n = k\right]}.$$

Let us begin with the sum

$$s_n = \sum_{k=1}^{n} \frac{n}{k} \mathrm{P}\left[M_n = k\right],$$

appearing in the denominator. Henceforth, we denote the expectation of M_n as $m_n = \sum_{i=1}^{n} q_i$. Define $\gamma_n = \lfloor n^\gamma \rfloor$ for some $1/2 < \gamma < 1$. By assumption, $\lim_{n\to\infty} m_n/n = \overline{q}$. Let us now split the sum s_n into three parts as $s_n = s_{1,n} + s_{2,n} + s_{3,n}$ with

$$\begin{aligned}
s_{1,n} &= \sum_{k=1}^{\lfloor m_n \rfloor - \gamma_n} \frac{n}{k} \mathrm{P}\left[M_n = k\right] \\
s_{2,n} &= \sum_{k=\lfloor m_n \rfloor - \gamma_n + 1}^{\lfloor m_n \rfloor + \gamma_n} \frac{n}{k} \mathrm{P}\left[M_n = k\right] \\
s_{3,n} &= \sum_{k=\lfloor m_n \rfloor + \gamma_n + 1}^{n} \frac{n}{k} \mathrm{P}\left[M_n = k\right].
\end{aligned}$$

It turns out that when $n \to \infty$, only the middle sum remains, as shown next.

Hoeffding's inequality for $I_1, I_2, I_3, \ldots$ ensures that for every $t > 0$

$$\mathrm{P}\left[|M_n - m_n| > t\right] < 2 \exp\left(-2\frac{t^2}{n}\right).$$

Using this inequality, we deduce that $s_{1,n}$ and $s_{2,n}$ both tend to 0 as n tends to ∞. Indeed,

$$\begin{aligned}
s_{1,n} &\leq n\mathrm{P}\left[M_n < \lfloor m_n \rfloor - \gamma_n + 1\right] \\
&\leq n\mathrm{P}\left[M_n - m_n < -(m_n - \lfloor m_n \rfloor + \gamma_n - 1)\right] \\
&< 2n \exp\left(-2\frac{(\gamma_n - 1)^2}{n}\right) \to 0 \text{ as } n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
s_{3,n} &\leq nP\left[M_n > \lfloor m_n \rfloor - \gamma_n + 1\right] \\
&\leq nP\left[M_n - m_n > \lfloor m_n \rfloor - m_n + \gamma_n)\right] \\
&< 2n \exp\left(-2\frac{(\gamma_n - 1)^2}{n}\right) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

as announced. Considering the middle sum, we now have

$$\begin{aligned}
s_{2,n} &= \sum_{j=-\gamma_n+1}^{\gamma_n} \frac{n}{\lfloor m_n \rfloor + j} P\left[M_n = \lfloor m_n \rfloor + j\right] \\
&= \sum_{j=1}^{2\gamma_n} \frac{1}{\frac{\lfloor m_n \rfloor}{n} + \frac{j}{n}} P\left[M_n = \lfloor m_n \rfloor - \gamma_n + j\right] \\
&= \frac{1}{\frac{\lfloor m_n \rfloor}{n}} \sum_{j=1}^{2\gamma_n} \frac{1}{1 + \frac{j}{\lfloor m_n \rfloor}} P\left[M_n = \lfloor m_n \rfloor - \gamma_n + j\right] \\
&= \frac{1}{\frac{\lfloor m_n \rfloor}{n}} \sum_{j=1}^{2\gamma_n} \big(1 + o(1)\big) P\left[M_n = \lfloor m_n \rfloor - \gamma_n + j\right] \\
&= \frac{1}{\frac{\lfloor m_n \rfloor}{n}} \big(1 + o(1)\big) P\left[\lfloor m_n \rfloor - \gamma_n + 1 \leq M_n \leq \lfloor m_n \rfloor + \gamma_n\right].
\end{aligned}$$

Using once again Hoeffding's inequality, we can conclude that $s_{2,n} \to \frac{1}{q}$ as $n \to \infty$. Following the same lines, it is also possible to prove that

$$\sum_{k=0}^{n-1} \frac{n}{k+1} P\left[M_n^{(\setminus i)} = k\right] \to \frac{1}{q} \text{ as } n \to \infty$$

and therefore that $c_{i,n}^{\text{uni}} \to q_i \beta$ as $n \to \infty$. This ends the proof of Proposition 2.1.

A.2 Proof of Proposition 2.2

We have

$$T_{i,n}^{\text{uni}} = \beta I_i \frac{n}{M_n} \mathbb{1}[M_n \geq 1] \frac{P[M_n \geq 1]}{\sum_{k=1}^{n} \frac{n}{k} P[M_n = k]} + c_{i,n}^{\text{uni}} \mathbb{1}[M_n = 0].$$

From the proof of the first statement in Proposition 2.1, we know that

$$c_{i,n}^{\text{uni}} \to q_i \beta \text{ and } \sum_{k=1}^{n} \frac{n}{k} P[M_n = k] \to \frac{1}{q} \text{ as } n \to \infty.$$

Since $\mathrm{P}[M_n = 0] = \prod_{i=1}^{n}(1 - q_i) \leq (1 - q_{\min})^n$, we have that $\mathbb{1}[M_n = 0] \to 0$ with probability 1 as $n \to \infty$. Since

$$\sum_{i=1}^{\infty} \frac{q_i\,(1-q_i)}{i^2} < \infty,$$

we deduce from Kolmogorov's strong law of large numbers that $\frac{M_n}{n} \to \overline{q}$ with probability 1 as $n \to \infty$. It follows that $T_{i,n}^{\mathrm{uni}} \to I_i\beta$ with probability 1 as $n \to \infty$. This ends the proof.

A.3 Proof of Proposition 2.3

From the definition of $T_{i,n}^{\mathrm{uni}}$, we have

$$\mathrm{Var}\left[T_{i,n}^{\mathrm{uni}} \middle| I_i = 1\right] = \mathrm{Var}\left[\frac{1}{(1+M_n^{(\backslash i)})/n}\frac{b_n^{\mathrm{uni}}}{n}\right] = \left(\frac{b_n^{\mathrm{uni}}}{n}\right)^2 \mathrm{Var}\left[\frac{1}{(1+M_n^{(\backslash i)})/n}\right]$$

where, by Proposition 2.1,

$$\frac{b_n^{\mathrm{uni}}}{n} = \frac{\beta \mathrm{P}[M_n \geq 1]}{n\sum_{k=1}^{n} k^{-1}\mathrm{P}[M_n = k]} = \frac{\beta}{\mathrm{E}\left[\frac{n}{M_n}\middle| M_n \geq 1\right]} \to \beta\overline{q} \text{ as } n \to \infty.$$

Fuchs et al. (2002) derived useful expressions for moments of ratios of random variables. Proceeding as in their paper, we get

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{(1+M_n^{(\backslash i)})/n}\right] &= \int_0^{\infty} \mathrm{e}^{-s/n}\prod_{j\neq i}\left(1 + q_j\left(\mathrm{e}^{-s/n} - 1\right)\right)\mathrm{d}s \\
&= \int_0^{\infty} \mathrm{e}^{-s/n}\exp\left(\sum_{j\neq i}\ln\left(1 + q_j\left(\mathrm{e}^{-s/n} - 1\right)\right)\right)\mathrm{d}s.
\end{aligned}$$

We deduce from the following power series expansions

$$\begin{aligned}
\ln(1+u) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}u^k, \quad 0 < u < 1, \\
\mathrm{e}^{-u} - 1 &= \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}u^k, \quad u > 0,
\end{aligned}$$

that

$$\begin{aligned}\sum_{j\neq i}\ln\left(1+q_j\left(\mathrm{e}^{-s/n}-1\right)\right) &= \sum_{j\neq i}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}q_j^k\left(\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\left(\frac{s}{n}\right)^l\right)^k \\ &= -\frac{1}{n}\sum_{j\neq i}q_j s+\frac{1}{2}\frac{1}{n}\sum_{j\neq i}q_j\left(1-q_j\right)\frac{s^2}{n}+\sum_{k=3}^{\infty}d_{i,k}\frac{s^k}{n^{k-1}}\end{aligned}$$

where $|d_{i,k}|<1$. Then, we have

$$\begin{aligned}&\mathrm{e}^{-s/n}\prod_{j\neq i}\left(1+q_j\left(\mathrm{e}^{-s/n}-1\right)\right)\\ =\ &\left(1-\frac{s}{n}+\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\left(\frac{s}{n}\right)^k\right)\exp\left(-\frac{1}{n}\sum_{j\neq i}q_j s+\frac{1}{2}\frac{1}{n}\sum_{j\neq i}q_j\left(1-q_j\right)\frac{s^2}{n}+\sum_{k=3}^{\infty}d_{i,k}\frac{s^k}{n^{k-1}}\right)\\ =\ &\exp\left(-\left(\frac{1}{n}\sum_{j\neq i}q_j\right)s\right)\left(1-\frac{s}{n}+\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\left(\frac{s}{n}\right)^k\right)\\ &\times\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{2}\frac{1}{n}\sum_{j\neq i}q_j\left(1-q_j\right)\frac{s^2}{n}+\sum_{l=3}^{\infty}d_{i,l}\frac{s^l}{n^{l-1}}\right)^k\right)\\ =\ &\exp\left(-\left(\frac{1}{n}\sum_{j\neq i}q_j\right)s\right)\left(1+\frac{1}{n}\left(-s+\frac{1}{2}\left(\frac{1}{n}\sum_{j\neq i}q_j\left(1-q_j\right)\right)s^2\right)+\sum_{k=2}^{\infty}P_{i,k,n}\left(s\right)\frac{1}{n^k}\right)\end{aligned}$$

where $P_{i,k,n}\left(s\right)$ are polynomials of order at most $2k$. It follows that

$$\begin{aligned}&\mathrm{E}\left[\frac{1}{(1+M_n^{(\setminus i)})/n}\right]\\ =\ &\frac{1}{\frac{1}{n}\sum_{j\neq i}q_j}+\frac{1}{n}\left(-\frac{1}{\left(\frac{1}{n}\sum_{j\neq i}q_j\right)^2}+\left(\frac{1}{n}\sum_{j\neq i}q_j\left(1-q_j\right)\right)\frac{1}{\left(\frac{1}{n}\sum_{j\neq i}q_j\right)^3}\right)+o\left(\frac{1}{n}\right)\end{aligned}$$

Starting from

$$\mathrm{E}\left[\frac{1}{\left((1+M_n^{(\setminus i)})/n\right)^2}\right]=\int_0^{\infty}s\mathrm{e}^{-s/n}\prod_{j\neq i}\left(1+q_j\left(\mathrm{e}^{-s/n}-1\right)\right)\mathrm{d}s$$

and proceeding as before, we also have

$$
\begin{aligned}
& \mathrm{E}\left[\frac{1}{\left((1+M_n^{(\backslash i)})/n\right)^2}\right] \\
= \; & \frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^2} + \frac{1}{n}\left(-\frac{2}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^3} + 3\left(\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\right)\frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^4}\right) + o\left(\frac{1}{n}\right).
\end{aligned}
$$

We therefore deduce that

$$
\begin{aligned}
n\mathrm{Var}\left[\frac{1}{(1+M_n^{(\backslash i)})/n}\right] &= n\left(\mathrm{E}\left[\frac{1}{\left((1+M_n^{(\backslash i)})/n\right)^2}\right] - \left(\mathrm{E}\left[\frac{1}{(1+\sum_{j\neq i} I_j)/n}\right]\right)^2\right) \\
&= -\frac{2}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^3} + 3\left(\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\right)\frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^4} \\
&\quad + \frac{2}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^3} - 2\left(\frac{1}{n}\sum_{j\neq i} q_j\left(1-q_j\right)\right)\frac{1}{\left(\frac{1}{n}\sum_{j\neq i} q_j\right)^4} + o\left(1\right) \\
&= \frac{\overline{q(1-q)}}{\left(\overline{q}\right)^4} + o\left(1\right)
\end{aligned}
$$

and finally that $n\mathrm{Var}\left[T_{i,n}^{\mathrm{uni}}\,\middle|\, I_i=1\right] \to \beta^2\frac{\overline{q(1-q)}}{(\overline{q})^2}$ as $n\to\infty$, as announced.

A.4 Proof of Proposition 2.5

Property 4.1 in Denuit and Frostig (2006) shows that $\sum_{i=1}^n J_i$ is smaller than $\sum_{i=1}^n I_i$ in the convex order. This means that the expectation of any convex function of $\sum_{i=1}^n I_i$ is larger than the corresponding expectation of $\sum_{i=1}^n J_i$. The same result holds true for the sums over all participants, except one. Since the inverse function is convex over $(0,\infty)$, we the get

$$
\mathrm{E}\left[\frac{1}{1+M_n^{(\backslash i)}}\right] \geq \mathrm{E}\left[\frac{1}{1+Q_n^{(\backslash i)}}\right],
$$

The announced result then follows since

$$
\mathrm{E}\left[\frac{I_i}{1+M_n^{(\backslash i)}}\right] = q_i\mathrm{E}\left[\frac{1}{1+Q_n^{(\backslash i)}}\right].
$$

A.5 Proof of Proposition 3.1

The lines of this proof follow those of the proof of Proposition 2.1 where M_n and $M_n^{(\backslash i)}$ have to be replaced by O_n and $\sum_{j\neq i} w_j I_j$, respectively, and by noting that Hoeffding's inequality becomes:

$$\mathrm{P}\left[\left|O_n - \sum_{i=1}^{n} w_i q_i\right| > t\right] < 2\exp\left(-2\frac{t^2}{\sum_{i=1}^{n} w_i^2}\right) \leq 2\exp\left(-2\frac{t^2}{(n-1)w_+^2}\right), \quad t > 0.$$

B Poisson approximation

Poisson approximation consists in approximating the distribution of M_n with the Poisson distribution with mean $\lambda = \sum_{j=1}^{n} q_j$ and the distribution of $M_n^{(\backslash i)}$ with the Poisson distribution with mean $\lambda_i = \sum_{j=1}^{n} q_j - q_i$. Then,

$$c_{i,n}^{\mathrm{uni}} \approx q_i \beta \frac{\sum_{k=1}^{n} k^{-1} \exp(-\lambda_i) \frac{\lambda_i^{k-1}}{(k-1)!}}{\sum_{k=1}^{n} k^{-1} \exp(-\lambda) \frac{\lambda^k}{k!}}.$$

The next example suggests that the accuracy of the Poisson approximation may be enough for practical applications.

Example B.1 (Poisson approximation)**.** *Consider again the pool described in Section 2.5 except that occurrence probabilities are 10 times smaller. Specifically, participants in Group 1 have a probability 0.001 of experiencing the event under consideration, participants in Group 2 have an occurrence probability 0.005 and participants in Group 3 have an occurrence probability of 0.01. Figure 7 displays the relative differences between individual contributions* $c_{i,n}^{\mathrm{uni}}$ *and the values obtained under the Poisson approximation. The results are given according to the number* n *of participants with* $n \in \{50, 100, \ldots, 1'000\}$*. We can see there that the relative differences are very small and almost identical across the three groups (variations between groups are so small that they are not visible from Figure 7, in accordance with Proposition B.2 below).*

The next result provides us with a sufficient condition and a uniform approximation for large pool sizes.

Proposition B.2. *Assume that* $I_1, I_2, \ldots$ *are mutually independent. If* $\sup_{1\leq i\leq n} q_i \to 0$ *as* $n \to \infty$ *and* $\lim_{n\to\infty} \sum_{i=1}^{n} q_i = \lambda > 0$ *then*

$$\lim_{n\to\infty} c_{i,n}^{\mathrm{uni}} = \beta q_i \frac{\mathrm{E}\left[\frac{1}{1+N}\right]}{\mathrm{E}\left[\frac{1}{N}\middle| N \geq 1\right]},$$

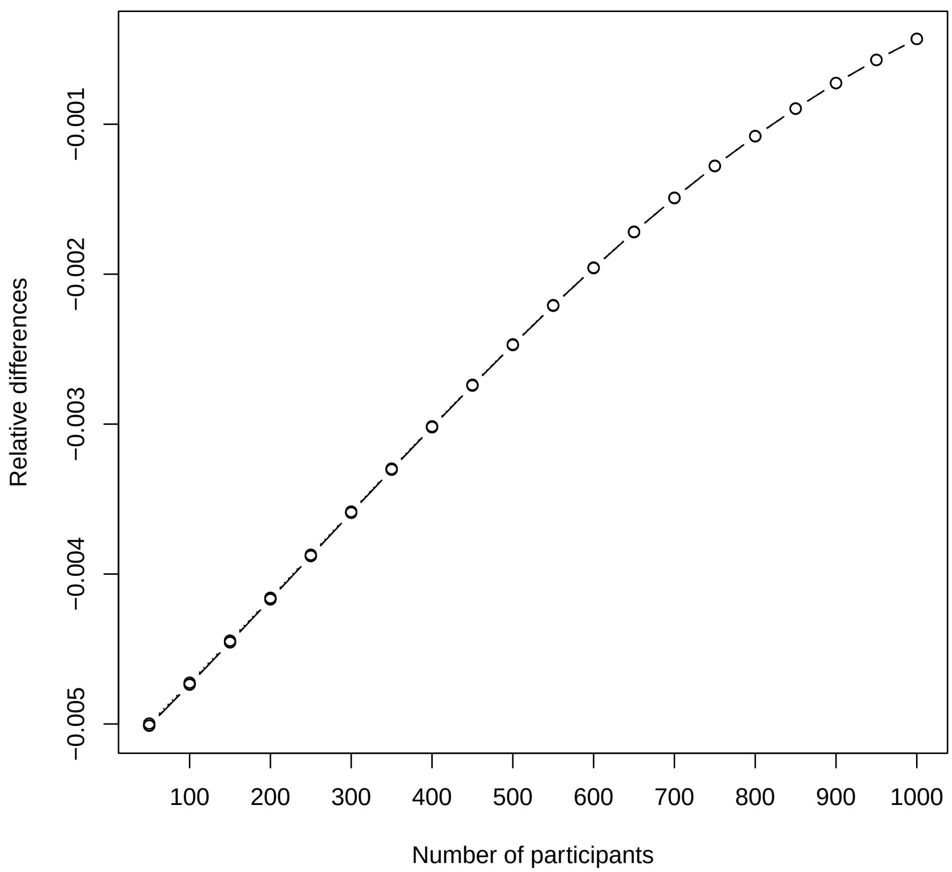

Figure 7: Relative differences for participants in Group 1 (solid line, —), in Group 2 (broken line, - - -), and in Group 3 (dotted line, $\cdots$).

where the random variable N obeys the Poisson distribution with parameter λ.

Proof. The Laplace transform of M_n is given by

$$\mathrm{L}_{M_n}(t) = \mathrm{E}[\exp(-tM_n)] = \prod_{i=1}^{n} \mathrm{L}_{X_i}(t) = \prod_{i=1}^{n} \left(1 + q_i \left(\mathrm{e}^{-t} - 1\right)\right).$$

As $n \to \infty$, since $\sup_{1 \le i \le n} q_i \to 0$ and $\lim_{n\to\infty} \sum_{i=1}^{n} q_i = \lambda > 0$, we have

$$\ln \mathrm{L}_{M_n}(t) = \sum_{i=1}^{n} \ln\left(1 + q_i \left(\mathrm{e}^{-t} - 1\right)\right) = \left(1 + o(1)\right) \sum_{i=1}^{n} q_i \left(\mathrm{e}^{-t} - 1\right) \to \lambda\left(\mathrm{e}^{-t} - 1\right),$$

and we conclude that M_n converges in distribution to a Poisson random variable N with parameter λ. Since, for all n and $\varepsilon > 0$,

$$\mathrm{E}\left[\left(\frac{1}{M_n}\right)^{1+\varepsilon} \middle| M_n \ge 1\right] \le 1,$$

the sequence $(M_n^{-1})_{n \ge 1}$ is uniformly integrable and we deduce that

$$\mathrm{E}\left[\frac{1}{M_n}\middle| M_n \ge 1\right] \to \mathrm{E}\left[\frac{1}{N}\middle| N \ge 1\right] \text{ as } n \to \infty.$$

With the same arguments, we can prove that $M_n^{(\backslash i)}$ also converges in distribution to a Poisson random variable N with parameter λ and that

$$\mathrm{E}\left[\frac{1}{1 + M_n^{(\backslash i)}}\right] \to \mathrm{E}\left[\frac{1}{1+N}\right] \text{ as } n \to \infty.$$

This ends the proof. □

Proposition B.2 explains why the relative differences in Example B.1 are so close for the three groups under consideration.

Let us now explain why the relative differences plotted on Figure 7 are negative. We know from the proof of Proposition 2.1 in Appendix A.1 that $\mathbb{1}[M_n = 0] \to 0$ with probability 1 as $n \to \infty$. Hence, $T_{i,n}^{\mathrm{uni}} \approx T_{i,n}^{\mathrm{eq}}$ for n large enough, where $T_{i,n}^{\mathrm{eq}}$ has been defined in Section 2.6. Let $K_1, K_2, \ldots, K_n$ be independent random variables obeying Poisson distributions with respective means $q_1, q_2, \ldots, q_n$. Then, M_n is smaller that $L_n = \sum_{i=1}^{n} K_i$ in the convex order, and the same result holds true for $M_n^{(\backslash i)}$ compared to $L_n^{(\backslash i)} = L_n - K_i$. The inequality

$$\mathrm{E}\left[\frac{1}{1 + M_n^{(\backslash i)}}\right] \le \mathrm{E}\left[\frac{1}{1 + L_n^{(\backslash i)}}\right]$$

is then valid because the function $t \mapsto (1+t)^{-1}$ is convex on $(0, \infty)$. This shows that assuming indicators I_i to be Poisson distributed with mean q_i is conservative when n gets large enough. Example B.1 suggests that this may even be the case for small n.

C Proportional sharing rule

C.1 Terminal payout

This section investigates the situation where participants are willing to get a level of compensation proportional to their initial contribution in case the predefined event occurs. Let $c_{i,n}^{\text{prop}}$ denote the amount of individual contribution for participant i and let $b_n^{\text{prop}} = \sum_{i=1}^n c_{i,n}^{\text{prop}}$ be the total contribution to be shared among the claiming participants. In this case, b_n^{prop} is no more evenly distributed among all participants who experienced the event under consideration but the sharing rule accounts for the individual targeted benefits reflected in their respective initial contributions $c_{i,n}^{\text{prop}}$.

Assuming that b_n^{prop} is distributed in arrear among those participants who experienced the predefined event in proportion to their initial contributions $c_{i,n}^{\text{prop}}$, the terminal payout $T_{i,n}^{\text{prop}}$ for participant i is then given by

$$T_{i,n}^{\text{prop}} = \begin{cases} \frac{c_{i,n}^{\text{prop}}}{c_{i,n}^{\text{prop}} + \sum_{j \neq i} c_{j,n}^{\text{prop}} I_j} b_n^{\text{prop}} \text{ if individual } i \text{ experiences the pre-defined event before time 1,} \\ \\ c_{i,n}^{\text{prop}} \text{ if nobody experiences the pre-defined event before time 1, that is, } M_n = 0, \\ \\ 0 \text{ otherwise,} \end{cases}$$

or equivalently

$$T_{i,n}^{\text{prop}} = \frac{c_{i,n}^{\text{prop}} I_i}{c_{i,n}^{\text{prop}} + \sum_{j \neq i} c_{j,n}^{\text{prop}} I_j} b_n^{\text{prop}} + c_{i,n}^{\text{prop}} \mathbb{1}[M_n = 0].$$

As in the uniform sharing rule, participants just get their initial contributions back when nobody experienced the event under consideration.

C.2 Initial contribution and lack of actuarial fairness

The condition for actuarial fairness which requires that no participant gains nor loses by joining the pool, on average, is now written

$$c_{i,n}^{\text{prop}} = \mathrm{E}[T_{i,n}^{\text{prop}}] = q_i \mathrm{E}\left[\frac{c_{i,n}^{\text{prop}}}{\sum_{j=1}^{n} c_{j,n}^{\text{prop}} I_j}\Big| I_i = 1\right] b_n^{\text{prop}} + c_{i,n}^{\text{prop}} \mathrm{P}[M_n = 0], \quad i = 1, \ldots, n,$$

or equivalently

$$\mathrm{P}[M_n \geq 1] = q_i h_{i,n}((c_{1,n}^{\text{prop}}, \ldots, c_{n,n}^{\text{prop}})), \quad i = 1, \ldots, n, \tag{C.1}$$

where, for $\boldsymbol{c}_n = (c_1, \ldots, c_n)$, the functions $h_{i,n} : \mathbb{R}_+^n \mapsto \mathbb{R}$ are defined as

$$h_{i,n}(\boldsymbol{c}_n) = \mathrm{E}\left[\frac{\sum_{j=1}^{n} c_j}{\sum_{j=1}^{n} c_j I_j}\Big| I_i = 1\right], \quad i = 1, \ldots, n.$$

The initial contributions $c_{i,n}^{\text{prop}}$ are still only defined up to a constant factor. Notice however that the system of equations (C.1) imposes n different constraints on the set of initial contributions $\left(c_{i,n}^{\text{prop}}\right)_{i=1,\ldots,n}$. Let us now explain how to obtain the conditional expectations appearing in the system of equations defining individual contributions. To this end, we use results derived in Fuchs et al. (2002) providing expressions for moments of ratios of random variables. The arguments developed in that paper allow us to state the following result.

Property C.1. *The homogeneous function of order* 0*,* $g_{i,n}$*, defined as*

$$g_{i,n}(\boldsymbol{c}_n) = \mathrm{E}\left[\frac{c_i I_i}{\sum_{j=1}^{n} c_j I_j}\Big| I_i = 1\right]$$

can be computed from

$$g_{i,n}(\boldsymbol{c}_n) = c_i \int_0^\infty \mathrm{e}^{-tc_i} \mathrm{E}\left[\mathrm{e}^{-tM_{n,\boldsymbol{c}}^{(\setminus i)}}\right] \mathrm{d}t,$$

where $M_{n,\boldsymbol{c}}^{(\setminus i)}$ *is distributed as* $\sum_{j \neq i} c_j I_j$ *given* $I_i = 1$*. If* $I_1, I_2, \ldots, I_n$ *are mutually independent then*

$$g_{i,n}(\boldsymbol{c}) = c_i \int_0^\infty \mathrm{e}^{-tc_i} \prod_{j \neq i} \left(1 - q_j + q_j \mathrm{e}^{-c_j t}\right) \mathrm{d}t.$$

Notice that the result in the independent case directly follows from the general expression derived in Property C.1 since the expectation of the product over participants factors out into the product of expectations and $\mathrm{E}\left[\exp\left(-tI_j\right)\right] = 1 - q_j + q_j \mathrm{e}^{-t}$.

Unfortunately, (C.1) does not necessarily possess a solution. This is formally shown next when $I_1, I_2, \ldots, I_n$ are mutually independent and when the number of participants gets large.

To this end, we need the following result giving the limits of the functions $h_{i,n}$ as $n \to \infty$.

Proposition C.2. *Assume that* $I_1, I_2, \ldots$ *are mutually independent. Let* $q_{\min} = \inf_{i\geq 1} q_i$, $\overline{c} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n c_i$ *and* $\overline{q} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n q_i$, *assumed to exist. Let* $\boldsymbol{c} = (c_1, c_2, \ldots)$ *be such that there exists a positive constant* c_+ *satisfying* $\sup_{i\geq 1} c_i \leq c_+$*. If* $q_{\min} > 0$, $0 < \overline{q} < 1$ *and* $\overline{c} > 0$ *then*

$$\lim_{n\to\infty} h_{i,n}(\boldsymbol{c}_n) = \frac{\overline{c}}{\overline{cq}}$$

where $\overline{cq} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n c_i q_i$.

Proof. We have

$$h_{i,n}(\boldsymbol{c}_n) = \frac{1}{n}\left(\sum_{j=1}^n c_j\right) \mathrm{E}\left[\frac{n}{c_i + \sum_{j\neq i} c_j I_j}\right], \quad i = 1, \ldots, n.$$

Since $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n c_i = \overline{c} > 0$, we only consider the following quantity

$$\mathrm{E}\left[\frac{1}{\frac{c_i}{n} + \frac{1}{n}\sum_{j\neq i} J_j}\right]$$

where the random variables $J_j = c_j I_j$ are independent, bounded ($\sup_{i\geq 1} c_i \leq c_+$) and with mean $c_j q_j$. Hoeffding's inequality for $\sum_{j\neq i} J_j$ states that

$$\mathrm{P}\left[\left|\sum_{j\neq i} J_j - \sum_{j\neq i} c_j q_j\right| > t\right] < 2\exp\left(-2\frac{t^2}{\sum_{j\neq i} c_j^2}\right) \leq 2\exp\left(-2\frac{t^2}{(n-1)c_+^2}\right).$$

Therefore we can use the same approach as in the proof of Proposition 2.1 to get

$$\mathrm{E}\left[\frac{1}{\frac{c_i}{n} + \frac{1}{n}\sum_{j\neq i} J_j}\right] \to \frac{1}{\overline{cq}} \text{ as } n \to \infty.$$

This ends the proof. □

As $n \to \infty$, the equations in (C.1) become

$$1 = q_i \frac{\overline{c}}{\overline{cq}}, \quad i = 1, 2, \ldots$$

and we clearly see that they cannot be satisfied in general. The actuarial fairness condition can be satisfied only if $q_i = q$ for all i.

Since actuarial fairness is generally considered as an important requirement, this approach is abandoned and replaced with the one described in Section 3, allowing for individualized protection levels. The results derived in this section are nevertheless relevant for the comparison with takaful insurance in Appendix D.

D Comparison with takaful

The fact that the contribution in takaful is considered as a donation in order to eliminate gharar constitutes the main distinctive feature compared to endowment contingency funds considered in this paper. In particular, the fact that participants get their initial contributions back when nobody experienced the event under consideration (that is, $I_i = 0$ for all i) is not in line with the irrevocable donation required by takaful which imposes that if $M_n = 0$ then all contributions are donated to a charity selected by the participants.

If contributions $c_{i,n}^{\text{taka}}$ are donated to a charity when $M_n = 0$ then actuarial fairness is no more possible, as shown in Section 2.6. This is inherent to takaful since contributions are donated by participants who can only get poorer on average when joining the pool. A practical way to solve this issue is to consider that participants indeed get their contributions back but engage to donate it to a charity. Since the probability $\mathrm{P}[M_n = 0]$ rapidly tends to 0 when n increases, this has no material effect in practice within sufficiently large pools.

Let us now illustrate the design of a family takaful product where the insured peril corresponds to participant's death. We assume that the system obeys the following rules:

Rule 1 survivors' contributions proportionally increase those of the beneficiaries of the deceased at time 1; survivors' contributions are functions of death probabilities;

Rule 2 if all participants die before time 1 then the beneficiaries of each participant receive a share of the fund corresponding to the proportion of the contributions paid at time 0;

Rule 3 if no one dies before time 1, no participant benefits from the distribution of the common fund which is entirely transferred to a charity.

Rules 1-2 implicitly assume that no reserves are made for the survivors and that the contributions have an irrevocable character to beneficiaries, in accordance with the gift character of tabarru. Rules 1-3 imply that the beneficiaries of participant i receive the terminal payout

$$T_{i,n}^{\text{prop-taka}} = \begin{cases} \frac{c_i^{\text{prop-taka}}}{c_i^{\text{prop-taka}} + \sum_{j \neq i} c_j^{\text{prop-taka}} I_j} b_n^{\text{prop-taka}} \text{ if } I_i = 1, \\ \\ 0 \text{ otherwise,} \end{cases}$$

where $c_i^{\text{prop-taka}} = h(q_i)$, with h a positive function that all participants agree, and $b_n^{\text{prop-taka}} = \sum_{j=1}^{n} c_j^{\text{prop-taka}}$. If no participant dies then $T_{i,n}^{\text{prop-taka}} = 0$ for all i and total contributions $b_n^{\text{prop-taka}}$ are transferred to a charity according to Rule 3.

For large pools, we deduce from Proposition C.2 that the ratio of the expected beneficiaries of participant i's terminal payout over participant i's contribution converges to a positive constant given by

$$\lim_{n\to\infty} \frac{\mathrm{E}[T_{i,n}^{\text{prop-taka}}]}{c_i^{\text{prop-taka}}} = \frac{q_i \overline{h(q)}}{\overline{qh(q)}},$$

where $\overline{h(q)} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} h(q_i)$ and $\overline{qh(q)} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} q_i h(q_i)$. If

$$q_i \geq \gamma \text{ where } \gamma = \frac{\overline{qh(q)}}{\overline{h(q)}}$$

then beneficiaries of participant i receive more on average than participant i contribute. Participants with the highest probabilities of death therefore give more to their beneficiaries relative to their donation.

Let us now consider the following two examples of interest and compare them with conventional insurance.

Case 1: In term life insurance, the pure premium is proportional to the insured's death probability. It is therefore natural to specify that participants' initial contributions are proportional to their death probabilities. Therefore, let us assume that $c_i^{\text{prop-taka}} = q_i\omega$ for $i = 1, \ldots, n$, for some positive factor ω, i.e. $h(q_i) = q_i\omega$. Then we obtain

$$\gamma = \frac{\overline{q^2}}{\overline{q}}.$$

In the case of a term life insurance contract with unit death benefit, policyholder pays a pure premium equal to q_i and receives nothing in case of survival while the beneficiary receives 1 in case of death. For the takaful product, participant i contributes q_i, and receives nothing in case of survival while the beneficiary receives on average $\overline{q}q_i/\overline{q^2}$ in case of death (for large pools).

Case 2: Another relevant choice is to require that the contributions to the beneficiaries are all equal. Specifically, let us assume that $c_i^{\text{prop-taka}} = \omega$ for $i = 1, \ldots, n$, for some positive factor ω, i.e. $h(q_i) = \omega$. Then we obtain

$$\gamma = \overline{q}.$$

In the case of a term life insurance contract with unit death benefit, policyholder pays a pure premium equal to 1 and receives nothing in case of survival while the beneficiary receives $1/q_i$ in case of death. For the takaful product, participant i contributes 1, and receives nothing in case of survival while the beneficiary receives on average $1/\overline{q}$ in case of death (for large pools).