

# DETERMINISTIC LIFESTYLE INVESTMENT STRATEGY IN MIXED LIFE INSURANCE CONTRACTS

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# Deterministic lifestyle investment strategy in mixed life insurance contracts

Vanessa Hanna\*

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## Abstract

Hybrid life insurance contracts have recently gained prominence in various markets. In this context, our interest lies in products that combine participating and unit-linked contracts, delving into the dynamic interplay between financial guarantees and investment strategy. This paper aims to ascertain if the equilibrium between participating and unit-linked life insurance policies aligns with the lifestyle investment strategy, thereby maximizing the expected utility of the policyholder at the contract's maturity while adhering to the fair pricing relation of the contract. In particular, it looks at the pricing of the fair guaranteed rates and the optimal investment strategy using a specified deterministic investment style. Our analysis reveals that a stable investment strategy often dominates across different levels of risk aversion, within the assumption of the classical utility function. However, to capture policyholders' changing risk preferences as they age, we extend the study to incorporate periodic premium payments and time-varying risk aversion. The results emphasize the effectiveness of this approach in tailoring the lifestyle investment strategy.

**Keywords** Hybrid products; Investment guarantees; Lifestyle investment strategy; Expected utility; Time-varying risk aversion.

## 1 Introduction

In an ever-evolving landscape of financial guarantees, the world of insurance stands as a steady pillar, offering individuals safety against life's uncertainties. At its core, traditional life insurance has long been synonymous with reliability, providing policyholders with a sense of security and predictability. Participating contracts, for instance, provide policyholders with return guarantees in addition to profit sharing in years of good performance (see, e.g., Bacinello 2001). These insurance products, including guarantees, have historically offered protection for those seeking stability in an uncertain world. However, the financial market, which is dynamic and often unpredictable, has brought challenges to the insurance industry. The persistent fluctuations in interest rates, the volatile stock markets, and the unanticipated figures of inflation have all contributed to reshaping the insurance landscape. In response to this uncertain environment, insurers have found themselves forced to explore alternative

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trends, moving away from renowned guarantees toward products that offer less certainty but potentially greater returns. For example, Reuß et al. (2015) have studied alternative product designs and assess how they would affect the insurer's financial situation. One prominent example of such a product is the unit-linked insurance policy. As stated by Bacinello & Persson (2002), this innovative product linked the returns to the investment funds, which are subject to fluctuations in the financial market. In some instances, these products may offer additional guarantees, which have been considered and studied in the literature (Boyle & Schwartz 1977 and Hürlimann 2010). While these policies do not offer the same level of guaranteed returns as traditional products, they provide the potential for higher growth by allowing individuals to participate in the market's performance. However, research repeatedly shows that consumers have risk-averse behaviors and confirms the significant demand for guarantees (Knoller 2016 and Døskeland & Nordahl 2008). Despite the shifting environment, customers' desire for protection is consistent.

Amidst this struggle between the desire for guarantees and the appeal of higher returns, a need for compromise has arisen. Mixed or hybrid insurance products have emerged as a popular option that represents a dynamic approach to closing the gap between guarantee-oriented and return-focused insurance offerings, enhancing the industry's capacity to meet evolving customer needs and regulatory expectations. In this area of development, several product designs have been examined by researchers and practitioners with different types of guarantees, as observed by Bohnert & Gatzert (2014) and Hambardzumyan & Korn (2019).

Hanna et al. (2022) combine the benefits of traditional participating life insurance contracts with the flexibility and potential for higher returns offered by unit-linked contracts in order to achieve a compromise between a guarantee from the participating part and no guarantee from the pure unit-linked part. A link is created between the two components of the contract by allowing for a transfer from the unit-linked account value to the participating part. In this regard, a constant fee is deducted from the investment fund associated with the pure unit-linked contract. This guarantee fee plays a crucial role in ensuring the fulfillment of guaranteed rates within the participating contract, as it increases the guaranteed return in the participating fund. So, this product has been designed and priced by considering a constant allocation between the two parts of the contract and studying the viewpoints of both the insurer and the policyholder.

Rather than considering constant allocations between the two components of this mixed contract, we can assume age-dependent allocations, which is mainly known as a lifestyle investment strategy.

In the realm of real-world finance, the lifestyle investment strategy offers a dynamic approach to pension and life insurance investments, tailoring long-term strategies based on individuals' concepts of lifestyle. This strategy aligns investment portfolios with policyholders' ages and risk tolerance, enabling them to benefit from equities' growth potential when young while transitioning to more stable assets as retirement or contract maturity approaches. Historically, equities have exhibited higher returns and risk. As policyholders age, the strategy shifts allocation towards bonds and cash, which are less volatile and more conservative assets. This empowers policyholders to adapt investment strategies through life stages and changing risk appetites. Typically, these investment strategies are deterministic in nature, with a predefined allocation stated at the inception of the contract. Policyholders agree upon a specific investment allocation for the entire duration of the contract. For instance, an investment plan might involve constant risky asset investments in the first half, followed by a linear decrease.



There is an important amount of research on the lifestyle subject that determines the optimal allocation in different types of contracts. For instance, Zhaoxun (2018) introduces a new pension contract that has a transparent structure and clear rule for bonus distribution that smoothes the investment return and shows that its outcome is consistent with traditional pension investment advice, which is the lifestyle strategy. On the other hand, Blake et al. (2014) argue that the optimal strategy is age-dependent and stochastic lifestyling, as opposed to the more traditional deterministic lifestyling. In this regard, Antolín et al. (2010) point out that dynamic investment strategies can offer slightly higher replacement rates for a given degree of risk than more deterministic strategies. Nevertheless, they indicate that there is no standard approach to investing. In every scenario, none of the investment strategies is the clear winner. Additionally, they show that the deterministic life cycle strategy that maintains a steady exposure to risky assets throughout the majority of the accumulation phase while fast-moving to bonds in the final ten years before retirement seems to generate appropriate returns. Moreover, several studies highlight that lifestyle investment strategies are not always beneficial (see, e.g., Arts & Vigna (2003), Booth & Yacoubov (2013) and Cairns et al. (2006)).

The purpose of this paper lies in proposing a predefined deterministic investment strategy that determines the fraction of the premium invested in the participating life insurance contract at each time. Under this framework, we design, price, and optimize this innovative contract with an annual guarantee, the most popular type on the market, as highlighted by Hieber et al. (2019). Following Hanna et al. (2022), we focus solely on the financial aspect and adopt the classical Black & Scholes model to achieve a fair valuation approach that ensures that the mixed contract is justly priced, satisfying both policyholders and investors, and adhering to principles of financial arbitrage-free pricing. More precisely, the objective of this paper is to evaluate whether the balancing between participating and unit-linked contracts aligns with the lifestyle investment strategy, using the Expected Utility Theory developed by von Neumann & Morgenstern (1944) to explain people's behavior under uncertainty, as it is frequently applied in designing insurance contracts when customers make ambiguous decisions. Hence, this study delves into pricing and the optimal investment strategy based on a predetermined deterministic investment style. The final goal is to fairly price the guaranteed rate within the lifestyle strategy and maximize the expected utility of the policyholder at the contract's maturity while ensuring the fair pricing relation of the contract is satisfied. We point out that the analysis in this paper is experimental and purely numerical based on classical Monte Carlo techniques in order to systematically choose the optimal proportion between participating and unit-linked contracts that maximizes the expected utility of the policyholder. Several sensitivity analyses confirm the general results, and the term "optimal" signifies the best outcome within our parameters and framework.

In this study, we explore various families of investment strategies under the lifestyle strategy framework. These include linear, step-wise, and piece-wise linear strategies, and we find that an optimal strategy is a stable allocation for a policyholder who pays a single premium and maintains the same attitude over the duration of the contract, which typically corresponds to a classical utility function. This outcome is consistent with Merton's optimal stock allocation fraction (Merton (1969)). However, incorporating periodic premiums can interact with investment strategies on one side, and on the other side, several studies have discovered that a person's lifetime risk preferences might vary. They proved that by incorporating age-dependent risk preferences can greatly enhance strategies for life-cycle pension funds and pension insurance, optimizing investment behavior and aligning with

evolving risk attitudes over time (see, for instance, Tang et al. (2023), Lichtenstern et al. (2021) and Steffensen (2011)). Campbell & Viceira (1999) were the first to raise concern about the uncertainty regarding the risk aversion coefficient. They highlighted that putting investing advice on a scientific basis has been one of the main goals of modern financial economics, and it has been accomplished for investors with short time horizons and constant investment strategies but not for those with long time horizons and facing time-varying expected returns on risky assets. Accordingly, we extend the analysis in two directions: the first is to consider periodic premium payments, and the second is to consider policyholders' risk preferences in a more dynamic manner by introducing time-varying risk aversion, and we explore how this coefficient changes over time to capture policyholders' evolving risk preferences as they age. This approach allows us to design optimal investment strategies that align with policyholders' changing risk attitudes and reflect lifestyle investment strategies.

The paper is organized as follows. In Section 2, we present the financial assumptions and the structure of the policy, in addition to the families of investment strategies to consider in this study. On the other hand, we state the fairness relation for each investment strategy with some numerical illustrations. In Section 3, we look at the optimal allocation for each investment strategy by considering first a classical utility function and then extending to consider periodic premiums on one side and a time-varying risk aversion on the other. Numerical results are illustrated as well. Section 4 concludes.

## 2 Mixed contract and investment strategy

### 2.1 Structure and financial assumptions

Our analysis focuses purely on the financial aspects of these contracts, disregarding mortality risk and concentrating solely on the uncertainties associated with the evolution of financial markets.

We recall the financial assumptions from Hanna et al. (2022), where they assume a Black & Scholes model with three assets in the market that take over the time horizon  $[0, T]$ . The risk-free asset  $B$  earning a constant interest rate  $r \in \mathbb{R}$ , and two risky funds  $G$  and  $H$ . The fund  $G$  is linked to the participating policy and has generally a low volatility, while the fund  $H$  is linked to the unit-linked policy with a high volatility.

The risk-free asset denoted by  $B = \{B_t\}_{t \in [0, T]}$  follows a deterministic dynamics:

$$dB_t = r B_t dt, \quad B_0 = 1.$$

The dynamics of fund  $G = \{G_t\}_{t \in [0, T]}$  follow a geometric Brownian motion:

$$dG_t = \mu_G G_t dt + \sigma_G G_t dW_t^G, \quad G_0 = 1, t \in [0, T], \quad (1)$$

where  $W^G$  is a standard Brownian motion under the real probability measure  $\mathbb{P}$ ,  $\mu_G$  and  $\sigma_G$  are the drift and the volatility of  $G$ , respectively, and  $\sigma_G > 0$ .

The dynamics of fund  $H = \{H_t\}_{t \in [0, T]}$  also follow a geometric Brownian motion:

$$dH_t = \mu_H H_t dt + \sigma_H \rho H_t dW_t^G + \sigma_H \sqrt{1 - \rho^2} H_t dW_t, \quad H_0 = 1, t \in [0, T], \quad (2)$$

where  $W$  is a standard Brownian motion under the real probability measure  $\mathbb{P}$ , independent of  $W^G$ ,  $\mu_H$  and  $\sigma_H$  are the drift and the volatility of  $H$ , respectively, and  $\sigma_H > 0$ . We define  $\rho \in ]-1, 1[$  as the correlation parameter between the log-returns of the two funds  $G$  and  $H$  such that  $dW_t^H = \rho \cdot dW_t^G + \sqrt{1 - \rho^2} \cdot dW_t$ .

In the framework of this paper, we design the mixed life insurance contract initially proposed by Hanna et al. (2022). These contracts combine features of participating and pure unit-linked life insurance contracts, aiming to strike a balance between the security provided by the participating component and the absence of guarantees in the pure unit-linked component. To ensure the fulfillment of guaranteed rates within the participating contract, a constant fee is deducted from the pure unit-linked account value to increase the guaranteed return in the participating fund.

We assume that the contract pays a lump sum at the policy's maturity  $T > 0$ . The allocation of the single premium  $P$ , paid at the inception of the contract, between the two mentioned contracts follows a predefined strategy, where  $\pi_t$  is the proportion of the premium defined at time  $t \in [0, T - 1]$  and invested in the participating life insurance contract, and the remaining fraction of the premium  $(1 - \pi_t)$  is allocated to the pure unit-linked contract. We note that the allocation of the premium includes various predefined strategies, which could be constant over the contract's maturity, deterministic allocations, as well as dynamic or stochastic strategies. In this paper, the allocation is no longer constant as in Hanna et al. (2022), but we move to a deterministic strategy.

The investment strategy  $\pi_t$  invested in the participating life insurance contract is directed towards the insurer's general fund  $G$ , also known as the participating fund. This fund typically carries a lower level of risk compared to other investment options. One of the key features of this contract is the guarantee of asset protection by providing an annual minimum guaranteed interest rate  $i$ . Moreover, the participating fund entitles the policyholder to a surplus return in addition to the guaranteed benefit. The annual surplus return is derived from the financial profits generated by the investments made by the insurance company as follows:

$$\delta_t = \max(\beta \cdot g_t - i, 0) = (\beta \cdot g_t - i, 0)^+,$$

where  $\beta \in [0, 1]$  is called the participation level and it is assumed to be constant over time, and  $g_t$  is the annual return of the fund  $G$ . The total return on the participating fund comprises two components: the guaranteed interest rate and the surplus return.

The remaining strategy  $(1 - \pi_t)$  is allocated to the pure unit-linked contract. Unlike the participating life insurance contract, the pure unit-linked contract does not provide any guarantees regarding investment returns or asset protection. Instead, the policyholder's contributions are directed towards the investment fund  $H$ , commonly referred to as the risky fund. The risky fund presents higher investment risks compared to the participating fund. The returns generated by this fund are subject to market fluctuations and can vary significantly over time. Policyholders who choose the pure unit-

linked contract bear the risk associated with the investment performance of the underlying assets. In addition, we introduce a continuous guarantee fee  $\varepsilon$  that is deducted from the investment fund  $H$  to enhance the guarantee of the participating part in the mixed product. The investment in the unit-linked part is represented by  $F = \{F_t\}_{t \in [0, T]}$  where  $F_t = e^{-\varepsilon t} H_t$ , and its dynamics are given by:

$$dF_t = (\mu_H - \varepsilon) F_t dt + \sigma_H \rho F_t dW_t^G + \sigma_H \sqrt{1 - \rho^2} F_t dW_t, \quad F_0 = 1, \quad t \in [0, T]. \quad (3)$$

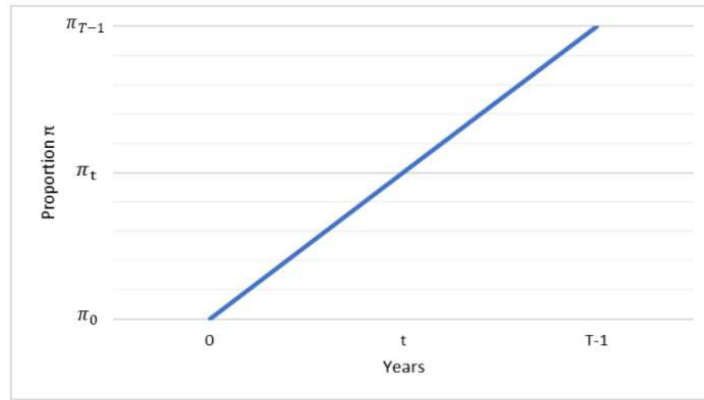
The guarantee fee  $\varepsilon$  is constant and used to boost the guarantee inside the contract.

## 2.2 Investment strategy

In this paper, we consider different families of investment strategies under the presented framework known as lifestyle strategies, including linear forms, step-wise, and piece-wise linear strategies, in order to check their impact on the guaranteed rate  $i$  and to determine, based on a predefined allocation form, the optimal investment strategy that maximizes the policyholder's expected utility:

- Linear strategy: it establishes a linear relationship between the initial share at the inception of the contract,  $\pi_0$ , and the final share,  $\pi_{T-1}$ , at the final allocation time, both invested in the participating contract, and it is expected to be a linear form as the time approaches maturity, such that

$$\pi_t = \frac{\pi_{T-1} - \pi_0}{T - 1} \cdot t + \pi_0, \quad t \text{ is discrete, } t = 0, 1, \dots, T - 1. \quad (4)$$



**Fig. 1:** Shape of a linear investment strategy.

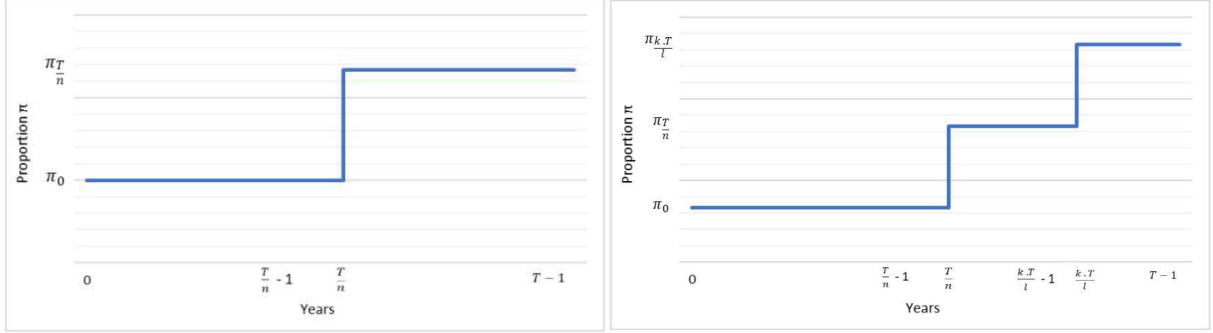
- Step-wise linear strategy: The investment strategy varies according to a step function where the analysis is examined on two and three paths as follows

$$\text{Two-step function: } \pi_t = \begin{cases} \pi_0 & \text{for } t \in [0, \frac{T}{n} - 1], \\ \pi_{\frac{T}{n}} & \text{for } t \in [\frac{T}{n}, T - 1], \end{cases} \quad n > 1, \quad (5)$$

or

$$\text{Three-step function: } \pi_t = \begin{cases} \pi_0 & \text{for } t \in [0, \frac{T}{n} - 1], \quad n > 1, \\ \pi_{\frac{T}{n}} & \text{for } t \in [\frac{T}{n}, \frac{k \cdot T}{l} - 1], \quad 1 \leq k < l, \\ \pi_{\frac{k \cdot T}{l}} & \text{for } t \in [\frac{k \cdot T}{l}, T - 1], \quad \frac{k \cdot T}{l} + 1 \leq T - 1, \end{cases} \quad (6)$$

where  $t$  is a natural number, and  $n$ ,  $k$ , and  $l$  are natural numbers.



**Fig. 2:** Shapes of two- and three-step investment strategies.

- Piece-wise linear strategy: This investment approach follows a segmented pattern, allowing for different allocation strategies over specific periods of time. The analysis considers both two- and three-path scenarios.

In the two-path strategy, the investment allocation begins with a fixed percentage and remains constant for all time steps within the first segment. Subsequently, there is a transition to a linear increase or decrease in allocation, which is maintained until contract maturity.

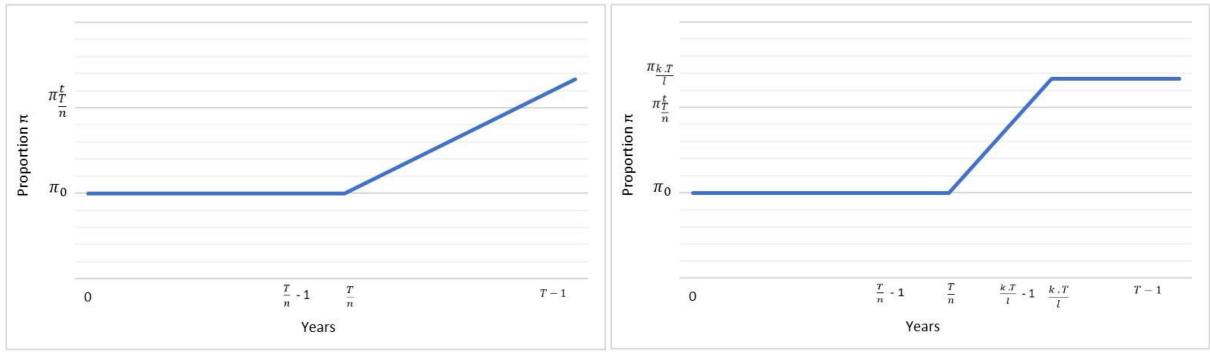
In the three-path strategy, three distinct segments are observed. The initial segment features a constant allocation. This is followed by a period of linear increase or decrease in allocation. Finally, the strategy reverts back to a constant allocation leading up to the contract's maturity. This strategy is mathematically expressed as follows:

$$\pi_t = \begin{cases} \pi_0 & \text{for } t \in [0, \frac{T}{n} - 1], \quad n > 1, \\ \pi_{\frac{T}{n}}^t = \frac{\pi_{\frac{k \cdot T}{l}} - \pi_0}{\frac{k \cdot T}{l} - 1 - \frac{T}{n}} \cdot t + \frac{(\frac{k \cdot T}{l} - 1) \cdot \pi_0 - \frac{T}{n} \cdot \pi_{\frac{k \cdot T}{l}}}{\frac{k \cdot T}{l} - 1 - \frac{T}{n}} & \text{for } t \in [\frac{T}{n}, \frac{k \cdot T}{l} - 1], \quad 1 \leq k < l, \\ \pi_{\frac{k \cdot T}{l}} = \frac{(\frac{k \cdot T}{l} - \frac{T}{n}) \cdot \pi_{\frac{k \cdot T}{l}} - \pi_0}{\frac{k \cdot T}{l} - 1 - \frac{T}{n}} & \text{for } t \in [\frac{k \cdot T}{l}, T - 1], \quad \frac{k \cdot T}{l} + 1 \leq T - 1, \end{cases} \quad (7)$$

where  $t$  is a natural number, and  $n$ ,  $k$ , and  $l$  are natural numbers. By taking  $l = k = 1$ , we examine the two-path strategy.

Figs. 1, 2, and 3 are just examples of such behaviors and do not present the definitive form.

Since the policyholder is meant to invest more in the less risky fund over time, if the lifestyle investment is used in practice, we assume that the investment strategy  $\pi_t$  is a non-decreasing function.



**Fig. 3:** Shapes of two- and three-path investment strategies.

It is worth noting that, in our analysis, the investment strategy  $\pi_t$  for the whole duration of the contract is predetermined at  $t = 0$ , and accordingly, the shares  $\pi_0, \dots, \pi_{T-1}$  are real positive numbers, taking values in  $]0, 1]$ , chosen at the issuance of the contract.

### 2.3 Fair valuation of the mixed contract

In this section, we price and evaluate theoretically the mixed contract at the first stage. Then, we numerically solve the fairness relation obtained with respect to the guaranteed interest rate  $i$  for the different families of investment strategies introduced in Section 2.2.

The pricing in this section is done using the risk-neutral measure  $\mathbb{Q}$ , obtained through Girsanov's theorem.

In our investment framework, we adopt a strategy where the investment portfolio is rebalanced each year to maintain the predefined allocation in the mixed contract. Apart from the initial single premium paid at the contract's inception, no further payments occur until the maturity of the contract. As a result, the benefit at maturity for the mixed product, which is an annual guarantee contract with rebalancing, is determined based on the performance of the participating contract and the unit-linked fund, and it is expressed as:

$$V_T = P \cdot \prod_{t=0}^{T-1} \left[ \pi_t \cdot ((1+i) + (\beta \cdot g_{t+1} - i)^+) + (1 - \pi_t) \cdot (1 + f_{t+1}) \right], \quad (8)$$

where  $g_{t+1}$  and  $f_{t+1}$  denote the annual returns of  $G$  and  $F$ , respectively, and defined, under  $\mathbb{Q}$ , as follows:

$$1 + g_{t+1} = \frac{G_{t+1}}{G_t} = \exp \left( \left( r - \frac{\sigma_G^2}{2} \right) + \sigma_G (W_{t+1}^G - W_t^G) \right), \quad t = 0, 1, \dots, T-1,$$

$$\begin{aligned}
1 + f_{t+1} &= \frac{F_{t+1}}{F_t} \\
&= \exp \left( \left( r - \varepsilon - \frac{\sigma_H^2}{2} \right) + \sigma_H \cdot \rho \cdot (W_{t+1}^G - W_t^G) + \sigma_H \cdot \sqrt{1 - \rho^2} \cdot (W_{t+1} - W_t) \right), \\
t &= 0, 1, \dots, T-1.
\end{aligned}$$

These returns are independent and log-normally distributed (i.i.d) for all  $t$ , and their logarithms are all independently distributed and have a normal distribution with mean  $r - \frac{\sigma_G^2}{2}$  and variance  $\sigma_G^2$  for  $\ln \left( \frac{G_{t+1}}{G_t} \right)$ , and mean  $r - \varepsilon - \frac{\sigma_H^2}{2}$  and variance  $\sigma_H^2$  for  $\ln \left( \frac{F_{t+1}}{F_t} \right)$ . As a result, the surplus rates  $\delta_t$  defined in Section 2.1 also prove to be i.i.d.

It is important to highlight that the parameters of the mixed contract are chosen fairly, ensuring that the contract is fair. The fairness of the contract is achieved by equalizing the discounted risk-neutral expectation of the final benefit, which is the initial fair value of the contract denoted by  $FV_0$ , to the initial premium  $P$ :

$$FV_0 := \mathbb{E}^{\mathbb{Q}} \left[ e^{-rT} \cdot V_T \right] = P. \quad (9)$$

Solving this equation leads to the following fairness relation:

$$\prod_{t=0}^{T-1} \left( \pi_t \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_t) \cdot e^{-\varepsilon} \right) = 1, \quad (10)$$

where  $c_1$  is the time 0 value of the one-year European call option given by the classical Black & Scholes (1973):

$$\begin{aligned}
c_1 &= \Phi(d_1) - \left( 1 + \frac{i}{\beta} \right) \cdot e^{-r} \cdot \Phi(d_2), \\
d_1 &= \frac{r + \frac{\sigma_G^2}{2} - \ln \left( 1 + \frac{i}{\beta} \right)}{\sigma_G}, \\
d_2 &= d_1 - \sigma_G,
\end{aligned}$$

where  $\Phi$  is the cumulative distribution function of a standard normal variable.

For a comprehensive understanding of the process used to derive this fairness relation in Equation (10), detailed steps and explanations can be found in the work of Hanna et al. (2022), where the premium equivalence of the annual guarantee contract under the same framework is treated. The only difference is the investment strategy, which is not constant but time-dependent in this study.

Equation (10) establishes a meaningful connection between various parameters, revealing a feedback mechanism between the guaranteed rate  $i$  and the guarantee fee  $\varepsilon$ , depending on the investment strategy  $\pi_t$  between the two parts of the contract. This relationship provides a valuable tool for policyholders and investors to comprehend how modifying one aspect affects other components, leading to a fair contract structure. In particular, Equation (10) helps the insurer assess the effect of the lifestyle strategy on boosting the fair guaranteed rate for a given guarantee fee.

In the following, we numerically solve the fairness relation of Equation (10) with respect to the

guaranteed interest rate  $i$ . The aim is to explore how the minimum guaranteed rate can be enhanced through the mixed product, as by increasing the guarantee fee  $\varepsilon$ , we strengthen the guaranteed rate  $i$ , and to compare these results to the constant investment strategy case presented in Hanna et al. (2022). To achieve this, we will present various solutions and sensitivity analyses for the fairness relation based on different investment strategies.

We consider a life insurance contract with a maturity  $T$  of 20 years. We recall the parameter values of the contract introduced in Hanna et al. (2022) for which the interest rate  $r$  is equal to 1.5%, and the volatility of the general asset  $G$  is  $\sigma_G = 3\%$ . Moreover, for the participating contract, we fix a participation level  $\beta$  of 70%, and for the unit-linked contract, we impose a guarantee fee  $\varepsilon$  of 0.25%.

### 2.3.1 Linear strategy

Under this strategy, the fair condition, using Equations (10) and (4), is expressed by the following equation:

$$\prod_{t=0}^{T-1} \left( \left( \frac{\pi_{T-1} - \pi_0}{T-1} \cdot t + \pi_0 \right) \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + \left( 1 - \frac{\pi_{T-1} - \pi_0}{T-1} \cdot t - \pi_0 \right) \cdot e^{-\varepsilon} \right) = 1. \quad (11)$$

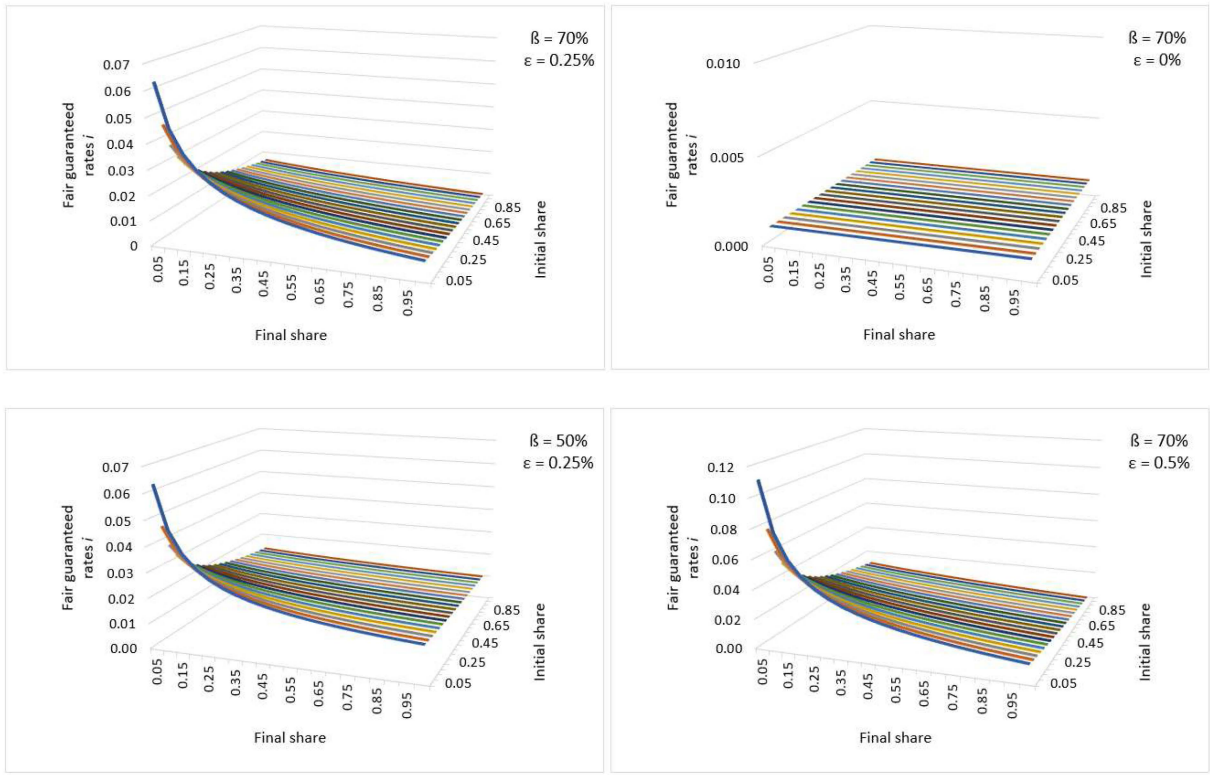
In all the subsequent numerical illustrations, we impose that the initial and final shares ( $\pi_0$  and  $\pi_{T-1}$ ) vary between 5%<sup>1</sup> and 100%, equidistantly by 5%. The fair guaranteed rates are presented in Fig; 4, and some examples are displayed in Table 1.

| $\pi_0$ | $\pi_{T-1}$ | $\beta = 70\%$<br>$\varepsilon = 0\%$ | $\beta = 70\%$<br>$\varepsilon = 0.25\%$ | $\beta = 50\%$<br>$\varepsilon = 0.25\%$ | $\beta = 70\%$<br>$\varepsilon = 0.5\%$ |
|---------|-------------|---------------------------------------|------------------------------------------|------------------------------------------|-----------------------------------------|
| 0.05    | 0.05        | 0.11                                  | 6.32                                     | 6.33                                     | 11.13                                   |
| 0.15    | 0.65        | 0.11                                  | 1.02                                     | 1.62                                     | 1.70                                    |
| 0.20    | 0.70        | 0.11                                  | 0.88                                     | 1.52                                     | 1.47                                    |
| 0.30    | 0.55        | 0.11                                  | 0.95                                     | 1.56                                     | 1.58                                    |
| 0.50    | 0.05        | 0.11                                  | 1.55                                     | 2.00                                     | 2.53                                    |
| 0.50    | 0.50        | 0.11                                  | 0.76                                     | 1.43                                     | 1.27                                    |
| 0.65    | 0.65        | 0.11                                  | 0.48                                     | 1.25                                     | 0.80                                    |
| 0.75    | 0.40        | 0.11                                  | 0.61                                     | 1.33                                     | 1.01                                    |
| 0.85    | 0.50        | 0.11                                  | 0.45                                     | 1.23                                     | 0.74                                    |
| 1.00    | 1.00        | 0.11                                  | 0.11                                     | 0.10                                     | 0.11                                    |

**Table 1:** Fair guaranteed rates  $i$  (expressed in %) with respect to a linear investment strategy for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

<sup>1</sup>We skip 0% in order to keep some form of guarantee inside the contract through the guaranteed interest rate of the participating contract.





**Fig. 4:** Fair guaranteed rates  $i$  with respect to a linear investment strategy for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

### 2.3.2 Step-wise linear strategy

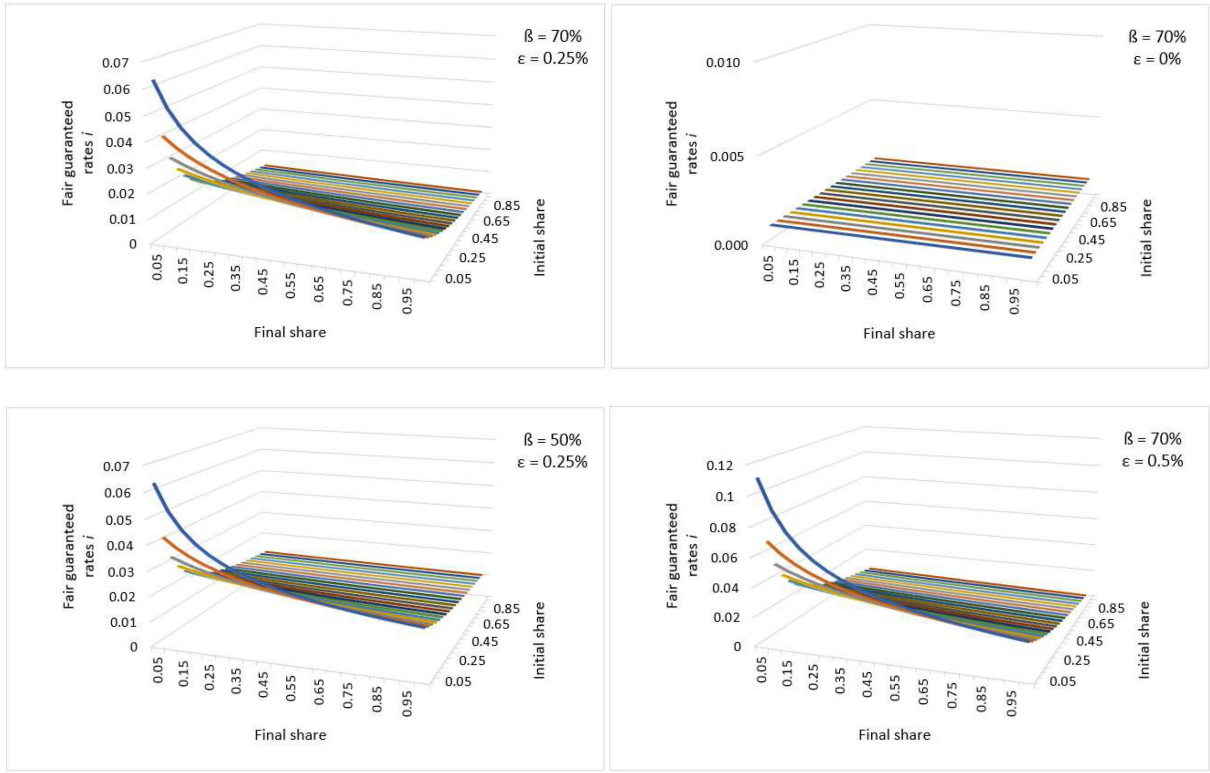
The step-wise strategy involves making gradual shifts in the investment portfolio at predefined time points. This strategy allows for more precise adjustments to the asset allocation as the policyholder ages or approaches contract maturity. We will provide the numerical results for both the two-step function and the three-step function.

In this part, we present two examples. The first example is the two-step strategy by taking  $n = \frac{4}{3}$  in Equation (5). In this setting, we assume that the policyholder would like to invest a constant proportion in the participating contract for the first three quarters of the time, then invest the same or another constant proportion for the last quarter of the time. In this regard, the fair condition is expressed as follows:

$$\left( \pi_0 \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-\pi_0) \cdot e^{-\varepsilon} \right)^{\frac{3}{4}} \cdot \left( \pi_{\frac{3}{4}T} \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-\pi_{\frac{3}{4}T}) \cdot e^{-\varepsilon} \right)^{\frac{1}{4}} = 1. \quad (12)$$

Again, we impose that the shares  $\pi_0$  and  $\pi_{\frac{3}{4}T}$  vary between 5% and 100% equidistantly by 5% in all the subsequent numerical figures. The results of the two-step function are illustrated in Fig. 5, and some examples are displayed in Table 2.

Another example is the three-step strategy by taking  $n = 2$ ,  $k = 3$ , and  $l = 4$  in Equation (6). In this setting, we assume that the policyholder invests a constant proportion  $\pi_0$  in the participating



**Fig. 5:** Fair guaranteed rates  $i$  with respect to a step-wise investment strategy (two-step function with  $n = \frac{4}{3}$ ) for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

| $\pi_0$ | $\pi_{\frac{3-T}{4}}$ | $\beta = 70\%$<br>$\varepsilon = 0\%$ | $\beta = 70\%$<br>$\varepsilon = 0.25\%$ | $\beta = 50\%$<br>$\varepsilon = 0.25\%$ | $\beta = 70\%$<br>$\varepsilon = 0.5\%$ |
|---------|-----------------------|---------------------------------------|------------------------------------------|------------------------------------------|-----------------------------------------|
| 0.05    | 0.05                  | 0.11                                  | 6.32                                     | 6.33                                     | 11.13                                   |
| 0.05    | 0.65                  | 0.11                                  | 2.09                                     | 2.42                                     | 3.39                                    |
| 0.20    | 0.70                  | 0.11                                  | 1.30                                     | 1.81                                     | 2.14                                    |
| 0.30    | 0.55                  | 0.11                                  | 1.15                                     | 1.71                                     | 1.90                                    |
| 0.45    | 0.65                  | 0.11                                  | 0.76                                     | 1.43                                     | 1.27                                    |
| 0.50    | 0.50                  | 0.11                                  | 0.76                                     | 1.43                                     | 1.27                                    |
| 0.65    | 0.65                  | 0.11                                  | 0.48                                     | 1.25                                     | 0.80                                    |
| 0.75    | 0.40                  | 0.11                                  | 0.46                                     | 1.24                                     | 0.77                                    |
| 0.90    | 0.50                  | 0.11                                  | 0.29                                     | 1.14                                     | 0.46                                    |
| 1.00    | 1.00                  | 0.11                                  | 0.11                                     | 0.10                                     | 0.11                                    |

**Table 2:** Fair guaranteed rates  $i$  (expressed in %) with respect to a step-wise investment strategy (two-step function with  $n = \frac{4}{3}$ ) for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

contract for the first half of the time, then invests the same or another constant proportion  $\pi_{\frac{T}{2}}$  for the remaining quarter of the time, and eventually invests the same or another constant proportion  $\pi_{\frac{3-T}{4}}$  in

the last quarter of the time. In this regard, the fair condition is expressed as follows:

$$\left( \pi_0 \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_0) \cdot e^{-\varepsilon} \right)^{\frac{1}{2}} \cdot \left( \pi_{\frac{T}{2}} \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_{\frac{T}{2}}) \cdot e^{-\varepsilon} \right)^{\frac{1}{4}} \cdot \left( \pi_{\frac{3T}{4}} \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_{\frac{3T}{4}}) \cdot e^{-\varepsilon} \right)^{\frac{1}{4}} = 1. \quad (13)$$

Also, we impose that the shares  $\pi_0$ ,  $\pi_{\frac{T}{2}}$  and  $\pi_{\frac{3T}{4}}$  vary between 5% and 100% equidistantly by 5% in all the subsequent numerical figures. The results of the three-step function are displayed in Table 3.

| $\pi_0$ | $\pi_{\frac{T}{2}}$ | $\pi_{\frac{3T}{4}}$ | $\beta = 70\%$<br>$\varepsilon = 0\%$ | $\beta = 70\%$<br>$\varepsilon = 0.25\%$ | $\beta = 50\%$<br>$\varepsilon = 0.25\%$ | $\beta = 70\%$<br>$\varepsilon = 0.5\%$ |
|---------|---------------------|----------------------|---------------------------------------|------------------------------------------|------------------------------------------|-----------------------------------------|
| 0.05    | 0.05                | 0.05                 | 0.11                                  | 6.32                                     | 6.33                                     | 11.13                                   |
| 0.05    | 0.45                | 0.65                 | 0.11                                  | 1.42                                     | 1.90                                     | 2.33                                    |
| 0.20    | 0.50                | 0.70                 | 0.11                                  | 1.02                                     | 1.62                                     | 1.70                                    |
| 0.35    | 0.55                | 0.65                 | 0.11                                  | 0.82                                     | 1.47                                     | 1.36                                    |
| 0.50    | 0.50                | 0.50                 | 0.11                                  | 0.76                                     | 1.43                                     | 1.27                                    |
| 0.65    | 0.65                | 0.65                 | 0.11                                  | 0.48                                     | 1.25                                     | 0.80                                    |
| 0.75    | 0.80                | 0.85                 | 0.11                                  | 0.30                                     | 1.14                                     | 0.48                                    |
| 0.75    | 0.80                | 1.00                 | 0.11                                  | 0.26                                     | 1.12                                     | 0.41                                    |
| 0.75    | 0.90                | 1.00                 | 0.11                                  | 0.24                                     | 1.11                                     | 0.36                                    |
| 1.00    | 1.00                | 1.00                 | 0.11                                  | 0.11                                     | 0.10                                     | 0.11                                    |

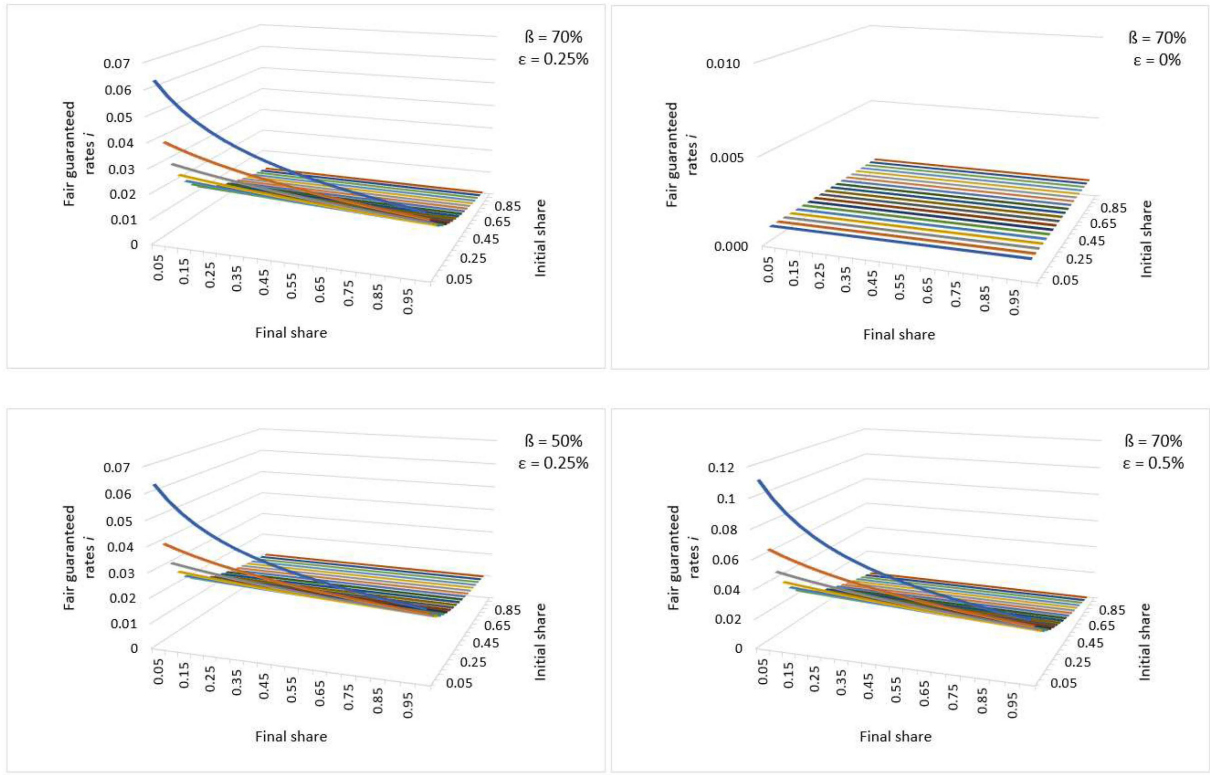
**Table 3:** Fair guaranteed rates  $i$  (expressed in %) with respect to a step-wise investment strategy (three-step function with  $n = 2$ ,  $k = 3$  and  $l = 4$ ) for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

### 2.3.3 Piece-wise linear strategy

The piece-wise linear strategy combines elements of both the linear and step-wise strategies. This strategy segments the investment horizon into multiple stages, with each stage characterized by a distinct linear relationship between the initial and final investment shares. In this strategy, we distinguish two different strategies and present an example for each. For the two-path strategy, we assume an investment strategy that is a step function for the first three quarters, followed by a linear increase in the last quarter. This is obtained by replacing  $n$  by  $\frac{4}{3}$  and taking  $k = l = 1$  in Equation (7). In this respect, we obtain the following fairness relation:

$$\left( \pi_0 \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_0) \cdot e^{-\varepsilon} \right)^{\frac{3T}{4}} \cdot \prod_{t=\frac{3T}{4}}^{T-1} \left( \pi_{\frac{3T}{4}}^t \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_{\frac{3T}{4}}^t) \cdot e^{-\varepsilon} \right) = 1. \quad (14)$$

The results of the two-path strategy are illustrated in Fig. 6, and some examples are displayed in Table 4.



**Fig. 6:** Fair guaranteed rates  $i$  with respect to a piece-wise investment strategy (two-path strategy with  $n = \frac{4}{3}$ ) for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

| $\pi_0$ | $\pi_{T-1}$ | $\beta = 70\%$      | $\beta = 70\%$         | $\beta = 50\%$         | $\beta = 70\%$        |
|---------|-------------|---------------------|------------------------|------------------------|-----------------------|
|         |             | $\varepsilon = 0\%$ | $\varepsilon = 0.25\%$ | $\varepsilon = 0.25\%$ | $\varepsilon = 0.5\%$ |
| 0.05    | 0.05        | 0.11                | 6.32                   | 6.33                   | 11.13                 |
| 0.05    | 0.65        | 0.11                | 3.09                   | 3.25                   | 5.04                  |
| 0.20    | 0.70        | 0.11                | 1.62                   | 2.05                   | 2.65                  |
| 0.35    | 0.75        | 0.11                | 1.02                   | 1.65                   | 1.70                  |
| 0.50    | 0.50        | 0.11                | 0.76                   | 1.43                   | 1.27                  |
| 0.50    | 0.70        | 0.11                | 0.70                   | 1.40                   | 1.18                  |
| 0.65    | 0.65        | 0.11                | 0.48                   | 1.25                   | 0.80                  |
| 0.75    | 0.40        | 0.11                | 0.40                   | 1.21                   | 0.66                  |
| 0.90    | 0.50        | 0.11                | 0.24                   | 1.11                   | 0.36                  |
| 1.00    | 1.00        | 0.11                | 0.11                   | 0.10                   | 0.11                  |

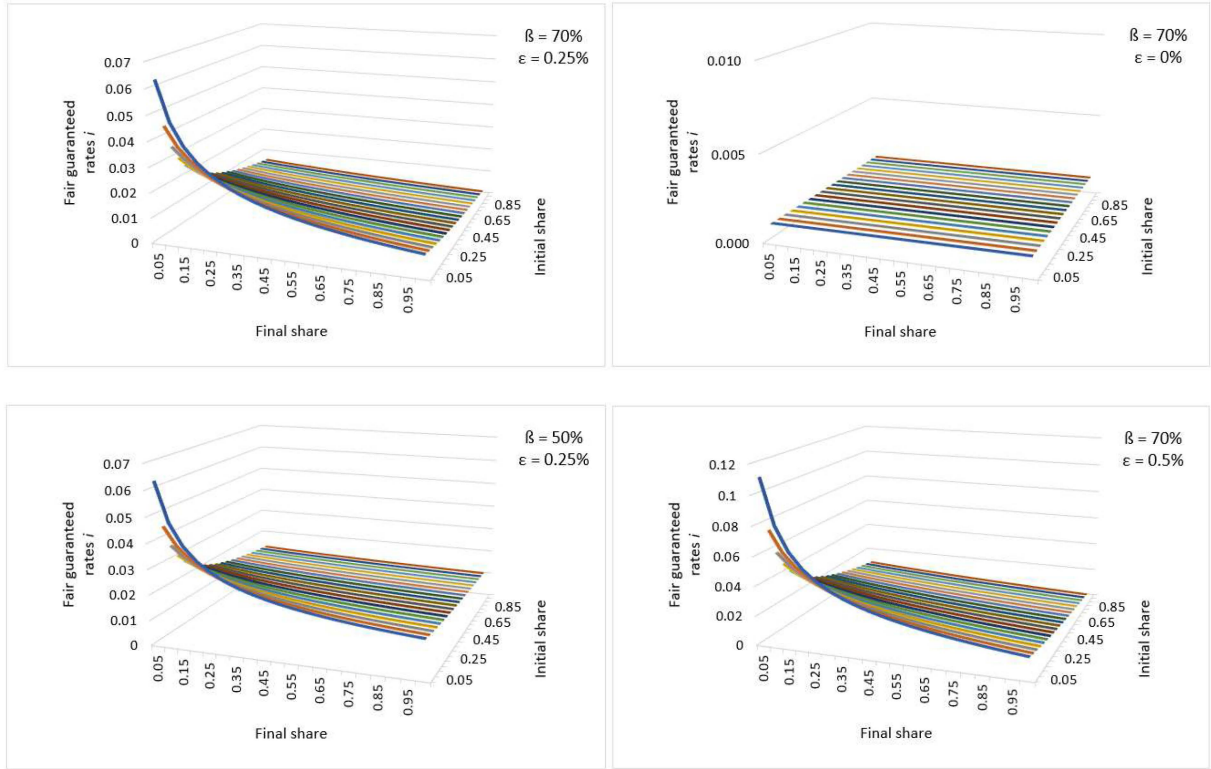
**Table 4:** Fair guaranteed rates  $i$  (expressed in %) with respect to a piece-wise investment strategy (two-path strategy with  $n = \frac{4}{3}$ ) for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

For the three-path strategy, we assume an investment strategy that is a step function for the first half, followed by a linear increase in the subsequent quarter, and then a step function for the last quarter of the time. This is obtained by replacing  $n$  by 2,  $k$  by 3, and  $l$  by 4 in Equation (7). In this respect,

we obtain the following fairness relation:

$$\left( \pi_0 \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_0) \cdot e^{-\varepsilon} \right)^{\frac{T}{2}} \cdot \prod_{t=\frac{T}{2}}^{\frac{3T}{4}-1} \left( \pi_{\frac{T}{2}}^t \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_{\frac{T}{2}}^t) \cdot e^{-\varepsilon} \right) \cdot \left( \pi_{\frac{3T}{4}} \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_{\frac{3T}{4}}) \cdot e^{-\varepsilon} \right)^{\frac{T}{4}} = 1. \quad (15)$$

The results of the three-path strategy are illustrated in Fig. 7, and some examples are displayed in Table 5.



**Fig. 7:** Fair guaranteed rates  $i$  with respect to a piece-wise investment strategy (three-path strategy with  $n = 2$ ,  $k = 3$  and  $l = 4$ ) for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

All the results exhibited in Figs. 4, 5, 6, and 7, and in Tables 1, 2, 3, 4, and 5, for the different examples of investment strategies, draw attention to the fact that for a constant proportion invested in the participating contract throughout the duration of the contract, the fair guaranteed rates are in line with the results presented in Hanna et al. (2022), where the allocation level was assumed to be constant; as well, we notice in this framework that these rates increase the more we invest in the unit-linked part, either by investing initially a high percentage in the unit-linked part and increasing this percentage until maturity or by investing a lower initial share and increasing it gradually until

| $\pi_0$ | $\pi_{\frac{3T}{4}-1}$ | $\beta = 70\%$      | $\beta = 70\%$         | $\beta = 50\%$         | $\beta = 70\%$        |
|---------|------------------------|---------------------|------------------------|------------------------|-----------------------|
|         |                        | $\varepsilon = 0\%$ | $\varepsilon = 0.25\%$ | $\varepsilon = 0.25\%$ | $\varepsilon = 0.5\%$ |
| 0.05    | 0.05                   | 0.11                | 6.32                   | 6.33                   | 11.13                 |
| 0.05    | 0.55                   | 0.11                | 1.59                   | 2.02                   | 2.59                  |
| 0.20    | 0.70                   | 0.11                | 0.97                   | 1.58                   | 1.61                  |
| 0.40    | 0.60                   | 0.11                | 0.79                   | 1.45                   | 1.31                  |
| 0.40    | 0.65                   | 0.11                | 0.74                   | 1.42                   | 1.23                  |
| 0.50    | 0.50                   | 0.11                | 0.76                   | 1.43                   | 1.27                  |
| 0.65    | 0.65                   | 0.11                | 0.48                   | 1.25                   | 0.80                  |
| 0.75    | 0.40                   | 0.11                | 0.57                   | 1.31                   | 0.95                  |
| 0.90    | 0.50                   | 0.11                | 0.38                   | 1.19                   | 0.62                  |
| 1.00    | 1.00                   | 0.11                | 0.11                   | 0.10                   | 0.11                  |

**Table 5:** Fair guaranteed rates  $i$  (expressed in %) with respect to a piece-wise investment strategy (three-path strategy with  $n = 2$ ,  $k = 3$  and  $l = 4$ ) for different participation levels  $\beta$  and guarantee fees  $\varepsilon$ .

maturity. In fact, the insurance company is less committed to the policyholder the more we invest in the unit-linked part, which allows it to offer a higher guarantee. For instance, Table 1 shows that if we initially invest 50% in the participating contract ( $\pi_0 = 50\%$ ) and decrease this proportion to  $\pi_{T-1} = 5\%$ , the insurance company grants a higher guaranteed rate than if the investment share stays constant. Additionally, we point out that, for a fixed guarantee fee  $\varepsilon$ , the lower the participation in the profits of the insurer ( $\beta \searrow$ ), the higher the guaranteed rates granted to the policyholder. This is also true if the policyholder accepts a larger guarantee fee ( $\varepsilon \nearrow$ ) for a fixed  $\beta$ . Furthermore, we highlight the scenario of the zero guarantee fee ( $\varepsilon = 0\%$ ), in which the guaranteed rates stay constant at any stage of the investment strategy. In this case, there is no transfer from the unit-linked to the participating part of the mixed contract, which results in the same guaranteed rate, regardless of the investment strategy. The insurer actually offers higher guaranteed rates in exchange for the guarantee fee. This instance demonstrates how the guarantee fee contributed to the mixed contract. Eventually, it is important to highlight the case of  $\pi_t = 100\% \forall t$ . In this case, the mixed contract is a traditional life insurance policy, and therefore, the guarantee fee has no impact at all, which leads to a low guaranteed rate that remains the same for all  $\varepsilon$ .

Consequently, the figures confirm that the guaranteed rate is boosted by an increase in the guarantee fee, a decrease in the participation level, and a higher allocation in the unit-linked contract over time. These results, with the incorporation of fluctuating proportions over time, are completely in line with the classical results when the proportion is constant. We can state that, indeed, the mixed contract increases the guaranteed rates significantly and that the shape of the strategy has an important impact on the fair guaranteed rates as well.



### 3 Lifestyle strategy and classical utility function

#### 3.1 Optimal allocation for classical utility function and single premium

By analyzing the fair condition of Equation (10), investors can make informed decisions on the optimal allocation strategy and adjust the parameters to achieve the desired risk and return profiles for the mixed product. Their preferences and attitude towards risk are vital considerations in designing this optimal investment strategy. Hence, we model the preferences using a constant relative risk aversion (CRRA) utility function, which exhibits risk aversion. The level of risk aversion is determined by the risk aversion coefficient. Power utility functions are commonly employed to represent CRRA utility functions. The power utility function captures the trade-off between risk and return and enables us to quantify the policyholder's utility based on their risk preferences:

$$U(z) = \frac{z^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1, \quad (16)$$

where  $U$  is the utility function of the affiliate satisfying  $U' > 0$  and  $U'' < 0$ , and  $\gamma$  is the risk aversion coefficient that is assumed to be constant over time, and its increase in value reflects the unwillingness of the policyholder to take a risk. The goal is to find the optimal deterministic investment strategy that maximizes the expected utility of the terminal benefit. Hence, the optimization problem is formulated as follows

$$\max_{\pi_t} \mathbb{E}^{\mathbb{P}} [U(V_T)], \quad t = 0, 1, \dots, T-1. \quad (17)$$

This optimization problem is subject to the constraint that ensures the fairness of the pricing relation of the mixed contract, which depends on  $\pi_t$ , as displayed:

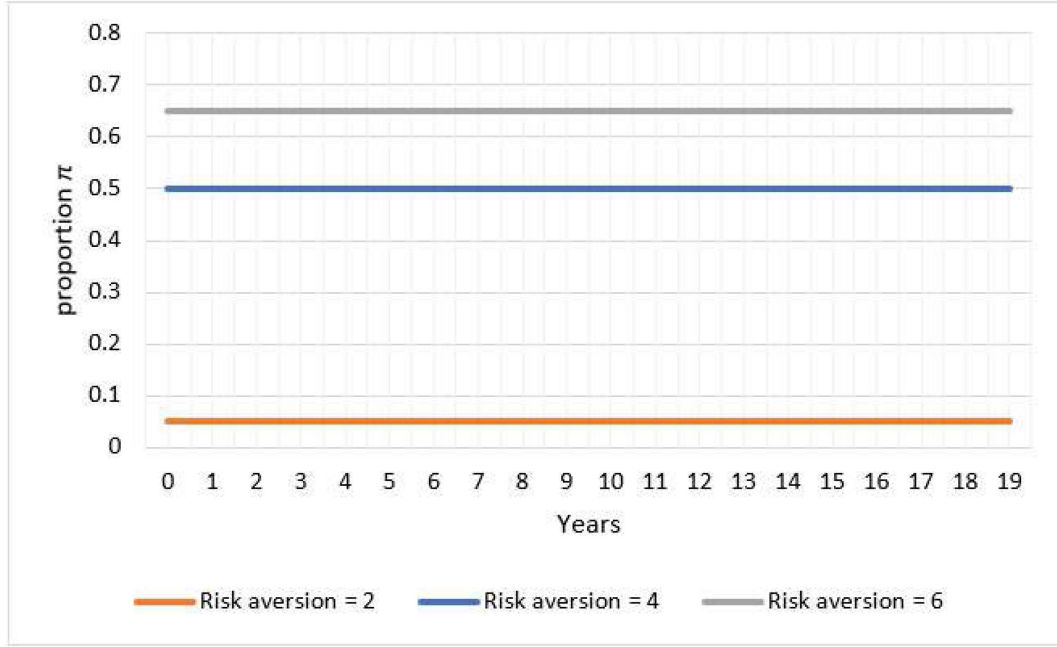
$$\mathbb{E}^{\mathbb{Q}} [e^{-rT} \cdot V_T] = P.$$

Under this framework, we present numerical illustrations of three investment strategies within the framework of the lifestyle investment strategy: the linear, the step-wise, and the piece-wise linear strategies. These illustrations provide insights into the optimal investment strategies based on a predetermined deterministic investment style that maximize the expected utility of the policyholder at the maturity of the contract.

We refer to the same contract introduced before, and we assume that the premium is paid in a single payment at inception ( $P = 1000$ ). We suppose the following values for the asset parameters:  $\mu_G = 3\%$ ,  $\mu_H = 7\%$ , and  $\sigma_H = 15\%$ , and the correlation between the funds,  $\rho$ , is supposed to be 10%. The results are based on Monte Carlo simulations. We simulate the benefit at maturity for the mixed product for a wide range of  $\pi_t$ -values,  $t \in [0, T-1]$ . For each Monte Carlo estimate, we compute the expected utility in order to select the optimal proportion  $\pi$  within our framework.

In order to present different policyholder preferences, we assume three different values of the risk aversion coefficient ( $\gamma = 2, 4$ , and 6). To examine the linear strategy, we apply the form of the investment strategy as indicated in Equation (4). Next, to explore the step-wise strategy, we provide the numerical results for both the two-step function (Equation (5)) and the three-step function (Equation (6)). For these functions, we refer to the same examples presented in the previous section. Lastly, we

examine the piece-wise linear strategy, in which we consider the two- and three-path strategies and refer to the same examples presented in the previous section. The results, illustrated in Fig. 8, show the optimal investment strategy for each risk aversion coefficient. The numerical results for the three strategies under the classical utility function and single premium are given in Table A.1 in Appendix A. The results are presented in terms of certainty equivalent concept defined as  $U(CEQ) = \mathbb{E}^{\mathbb{P}} [U(V_T)]$ .



**Fig. 8:** Optimal proportion in the participating contract using a classical utility function for different risk aversion values.

In the different investment strategies, the optimal investment strategy varies with the risk aversion coefficient, but it remains stable over time for each specific coefficient. Fig. 8 shows that as the risk aversion increases, the policyholder becomes more risk-averse and allocates a higher proportion of the premium to the participating contract. They maintain the same allocation in the unit-linked contract at each moment if their risk aversion remains unchanged. Particularly, we see from Fig. 8 that the optimal investment strategy for an investor with a risk aversion of  $\gamma = 2$  is  $\pi_t = 0.5\%$ , for  $\gamma = 4$  is  $\pi_t = 50\%$ , and for  $\gamma = 6$  is  $\pi_t = 65\% \forall t \in [0, T - 1]$ . Therefore, investors with the same risk appetite at all points in the contract prefer a stable investment strategy that aligns with their behavior throughout the contract.

In conclusion, the results obtained for the constant risk aversion case across different investment strategies demonstrate that the lifestyle investment strategy does not hold in this setting. Regardless of the strategy considered—linear, step-wise, or piece-wise linear—the optimal investment strategy remains stable for policyholders using a classical utility function. This behavior has been confirmed numerically by sensitivity analysis (Appendix D). Therefore, the findings suggest that under the assumption of constant risk aversion, policyholders tend to choose an investment strategy that remains consistent throughout the contract, regardless of their age or time to maturity. If we want to justify and generate some form of lifestyle strategy, we can generalize this section in two directions, either



considering periodic premiums or time-varying risk aversion. The latter suggests to consider a variable attitude towards risk with ageing.

### 3.2 Optimal allocation for classical utility function and periodic premiums

The presence of constant periodic premiums introduces an additional layer of complexity to the optimization problem. It requires a thorough examination of how premium payments interact with investment strategies and their impact on the policyholder's overall utility. When considering the impact of periodic premiums on the optimal allocation strategy, it is essential to account for the additional cash flows associated with these payments. Unlike the single premium scenario discussed in Section 3.1, periodic premiums introduce a time-varying cash inflow into the investment portfolio. This dynamic element necessitates adjustments in the optimization framework.

To incorporate periodic premiums, the benefit at maturity of Equation (8) becomes:

$$\tilde{V}_T = \tilde{P} \cdot \sum_{k=0}^{T-1} \left( \prod_{t=k}^{T-1} \left( \pi_t \cdot ((1+i) + (\beta \cdot g_{t+1} - i)^+) + (1 - \pi_t) \cdot (1 + f_{t+1}) \right) \right), \quad (18)$$

where  $\tilde{P}$  is the periodic premium, and  $\pi_t$  is the part of the total reserve invested at time  $t$  in the participating contract. To determine the fair guaranteed interest rate, we ensured fair pricing of the contract by equating the fair value of Equation (18) with the fair value of the premiums in the following manner:

$$\mathbb{E}^{\mathbb{Q}} [e^{-rT} \cdot \tilde{V}_T] = \mathbb{E}^{\mathbb{Q}} \left[ \sum_{k=0}^{T-1} \tilde{P} \cdot e^{-rk} \right],$$

which gives the following fairness relation:

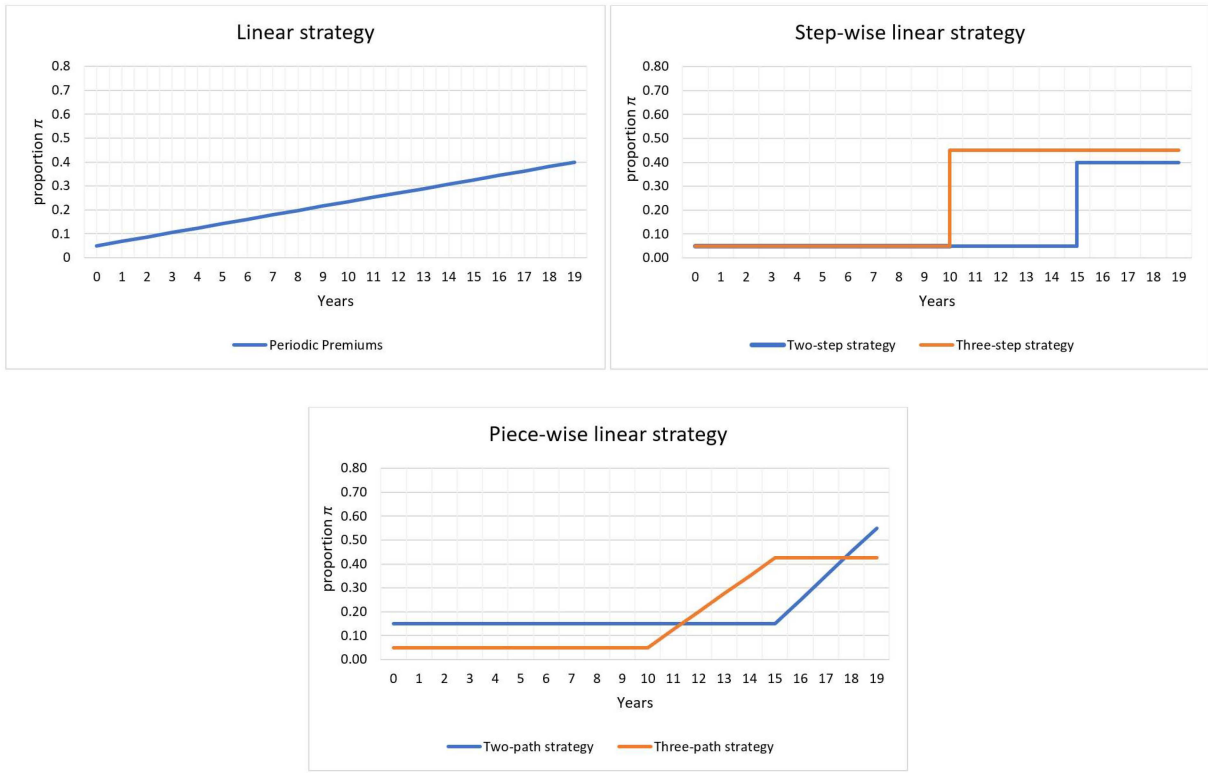
$$\tilde{P} \cdot \sum_{k=0}^{T-1} \left( e^{-rk} \cdot \prod_{t=k}^{T-1} \left( \pi_t \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1 - \pi_t) \cdot e^{-\varepsilon} \right) \right) = \tilde{P} \cdot \sum_{k=0}^{T-1} e^{-rk}. \quad (19)$$

Formally, the modified optimization problem can be stated as:

$$\max_{\pi_t} \quad \mathbb{E}^{\mathbb{P}} [U(\tilde{V}_T)], \quad t = 0, 1, \dots, T-1, \quad (20)$$

considering a fair guaranteed rate through the satisfaction of Equation (19). This formulation acknowledges the dual influence of investment decisions and premium payments on the policyholder's utility.

In the following, we will explore various investment strategies tailored for scenarios involving periodic premiums. Specifically, we will delve into numerical illustrations for strategies such as linear, step-wise, and piece-wise linear approaches. These illustrations will shed light on the optimal investment strategies that maximize the expected utility of the policyholder, considering periodic premium payments. We fix the risk aversion parameter  $\gamma$  to 4, and we divided the single premium  $P$  by the expected number of annuity payments over a 20-year period to determine the amount of each periodic premium.



**Fig. 9:** Optimal proportion in the participating contract for periodic premium payments.

Incorporating periodic premiums, we observe in Fig. 9 that the optimal proportion invested in the participating contract varies over time for each of the three investment strategies considered: linear, step-wise, and piece-wise. The numerical results for the three strategies under the classical utility function and periodic premiums are given in Table B.1 in Appendix B.

For the linear strategy, the allocation to the participating contract starts at 5% and gradually increases to 40% over the 20-year period, reflecting a steady progression towards a more conservative investment approach.

The step-wise linear strategy employs two distinct phases: an initial 5% allocation to the participating contract for the first 15 years, followed by a substantial increase to 40% for the remaining period. Analogously, the three-step strategy follows a similar pattern, but with a transition at year 10 to an allocation of 45% in the participating contract, and this proportion stays the same for the remaining period of time.

In addition, the piece-wise linear strategy offers a more interesting approach. The two-path strategy maintains a 15% allocation for the first three quarters of time and increases linearly to 55%. Conversely, the three-path strategy undergoes gradual shifts in allocation, starting at 5% and progressively increasing to 43% by the contract's maturity.

Incorporating periodic premiums has revealed a notable shift in the application of the lifestyle investment strategy. Each investment strategy offers policyholders unique risk-return profiles, enabling them to customize their investments to align with their specific preferences and goals throughout the contract's duration. This dynamic approach ensures that the investment strategy reflects the lifestyle strategy concept, which is in contrast to the scenario with a single premium payment. In the next section, we move on to consider time-varying risk aversion with a single premium payment to check

its impact on the optimal strategy for the different investment strategy shapes and, accordingly, check if it aligns with the lifestyle investment strategy.

### 3.3 Optimal allocation for time-varying risk aversion utility function

Tang et al. (2023) conducted research that challenges the assumption that investors' risk aversion remains constant over time, highlighting the unrealistic nature of such an assumption. Consequently, it becomes crucial to incorporate time variation in the risk aversion coefficient to capture the dynamic nature of investors' risk preferences. Typically, investors tend to undergo a shift in their investment style, transitioning from a certain level of risk-loving preference in their youth to risk aversion as they age. In other words, changes in risk preferences over an investor's lifetime are reflected by a risk aversion that increases as age advances. Consequently, it is plausible to suggest that investors become more risk-averse as they grow older.

Returning to the single premium case, the classical utility framework defined in Equation (16) is the common way to define aggregate utility, which is the utility function based on the final wealth. However, the final wealth of Equation (8) can be written equivalently as

$$V_T = P \cdot \prod_{t=0}^{T-1} R_{t+1}, \quad (21)$$

where

$$R_{t+1} = \pi_t \cdot (1 + i + (\beta \cdot g_{t+1} - i)^+) + (1 - \pi_t) \cdot (1 + f_{t+1}).$$

Using Equation (21), the utility function can be seen as a function of successive return factors and can be written as

$$\begin{aligned} U(V_T) &= \frac{(P \cdot R_1 \cdot \dots \cdot R_T)^{1-\gamma}}{1-\gamma} \\ &= \frac{P^{1-\gamma}}{1-\gamma} \cdot R_1^{1-\gamma} \cdot \dots \cdot R_T^{1-\gamma}. \end{aligned}$$

Based on that, we could generalize this approach by introducing time-varying risk attitudes. This allows us to write a new aggregate utility function:

$$\tilde{U}(R_1, \dots, R_T) = \frac{P^{1-\gamma_0}}{1-\gamma_0} \cdot R_1^{1-\gamma_1} \cdot \dots \cdot R_T^{1-\gamma_T}, \quad \gamma_0, \dots, \gamma_T > 1, \quad (22)$$

where  $\tilde{U}$  is the new utility function satisfying the following conditions for  $\gamma_0, \dots, \gamma_T > 1$  and  $i = 1, \dots, T$ :

$$\begin{aligned} \frac{\partial \tilde{U}}{\partial R_i} &= \frac{P^{1-\gamma_0}}{1-\gamma_0} \cdot R_1^{1-\gamma_1} \cdot \dots \cdot (1-\gamma_i) \cdot R_i^{-\gamma_i} \cdot \dots \cdot R_T^{1-\gamma_T} > 0, \quad \text{and} \\ \frac{\partial^2 \tilde{U}}{\partial R_i^2} &= \frac{P^{1-\gamma_0}}{1-\gamma_0} \cdot R_1^{1-\gamma_1} \cdot \dots \cdot (1-\gamma_i) \cdot (-\gamma_i) \cdot R_i^{-1-\gamma_i} \cdot \dots \cdot R_T^{1-\gamma_T} < 0. \end{aligned}$$

$(\gamma_0, \dots, \gamma_T)$  is a vector of successive time-varying risk aversions with  $\gamma_0 < \dots < \gamma_T$ . Investors' risk appetites change each year as they become more conservative with age, and based on this, we have annual decisions reflected by the time-varying risk aversions in Equation (22).

As a result, the new optimization problem is defined as

$$\max_{\pi_t} \mathbb{E}^{\mathbb{P}} [\tilde{U}(R_1, \dots, R_T)], \quad t = 0, 1, \dots, T-1, \quad (23)$$

subject to a fair guaranteed rate.

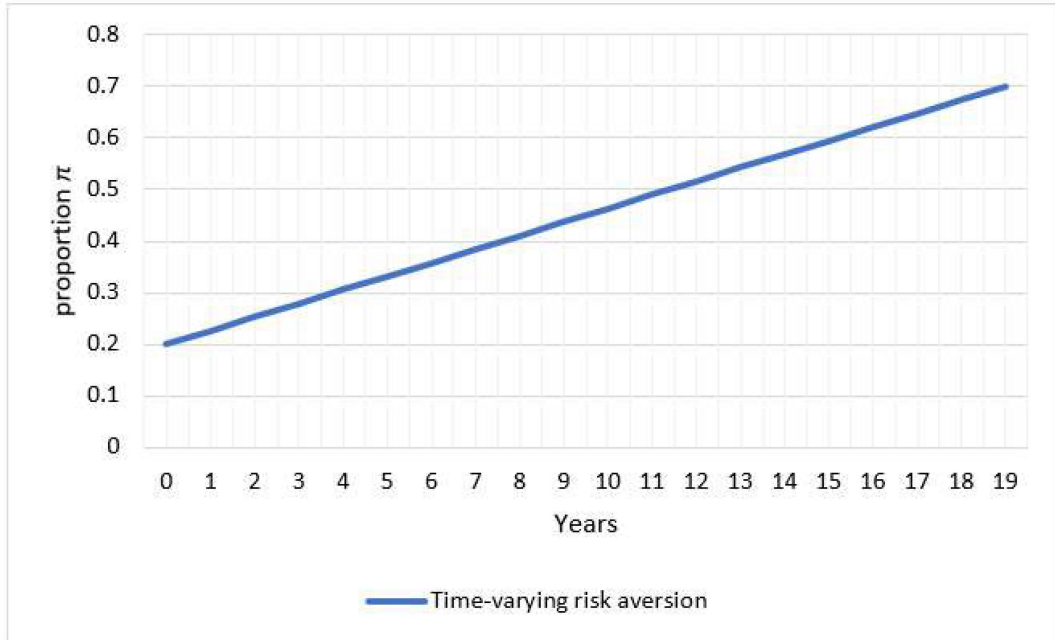
In the following part, we will focus on the numerical results of the time-varying risk aversion cases for each of the three investment strategies: the linear, the step-wise, and the piece-wise linear strategies.

### 3.3.1 Linear strategy

In this subsection, we present numerical illustrations of the linear strategy in the time-varying risk aversion case. We consider the policyholder's behavior as risk aversion increases with age from  $\gamma_0 = 2$  to  $\gamma_T = 6$ , equidistantly by 0.2 as follows

$$\gamma_t = 0.2 \cdot t + 2, \quad t = 0, 1, \dots, T. \quad (24)$$

The results, illustrated in Fig. 10, show the optimal investment strategy for each age-dependent risk aversion coefficient. The numerical results of the utility are displayed in Table C.4 in Appendix C.



**Fig. 10:** Optimal proportion in the participating contract based on a linear strategy for increasing values of risk aversion.

Fig. 10 highlights that by assuming a change in the preference of the policyholder at each point in the contract, the lifestyle strategy performs better than the case of constant risk aversion. More

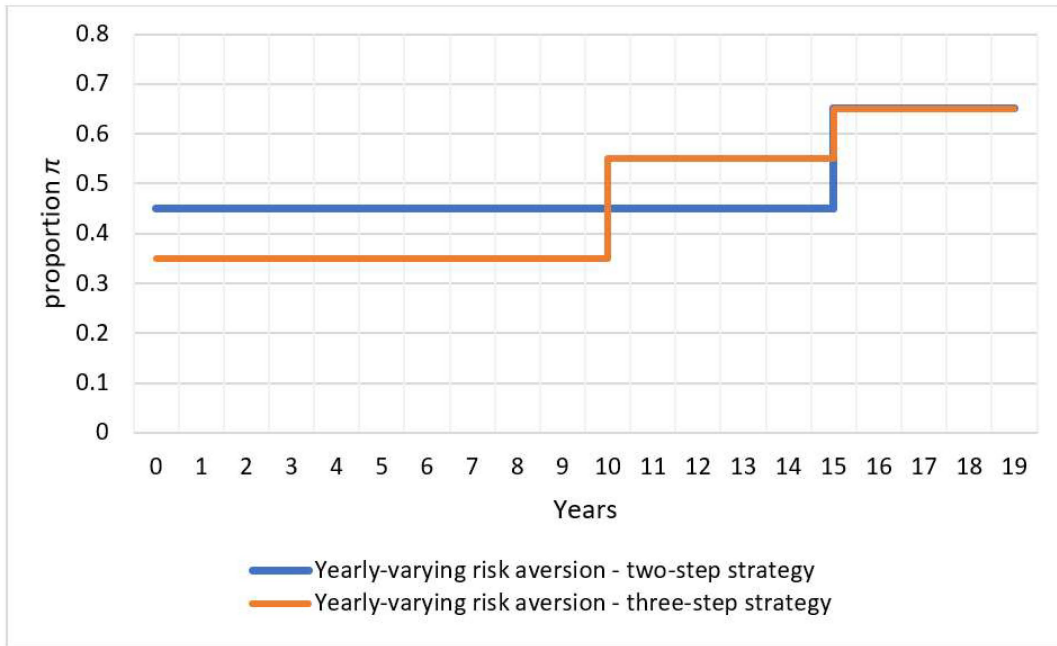
precisely, the policyholder becomes more risk-averse as he ages than when he is young. Therefore, the increase in the risk aversion parameter results in a lower investment in the unit-linked contract over time. A policyholder who starts with a risk aversion of  $\gamma = 2$  at the inception of the contract invests 20% of the premium in the participating part of the contract. This allocation increases to 70% at contract maturity, which satisfies the lifestyle strategy concept.

### 3.3.2 Step-wise linear strategy

Next, we explore the step-wise strategy in the time-varying risk aversion case. We will provide the numerical results for both the yearly-varying risk aversion example and the step-varying risk aversion example. For each example, we analyze the policyholder's behavior and their optimal investment strategy over time.

#### Yearly-varying risk aversion:

Recalling the two examples of two- and three-step strategies presented above, we assume that the behavior of the policyholder changes each year as risk aversion increases with age from  $\gamma_0 = 2$  to  $\gamma_T = 6$ , equidistantly by 0.2, see Equation (24).



**Fig. 11:** Optimal proportion in the participating contract based on two and three-step strategies for increasing values of risk aversion.

Fig. 11 emphasizes that by increasing their risk aversion each year, the policyholder tends to invest less in the risky part of the contract with time. For instance, we notice that the investment in the unit-linked contract moves from 55% to 35% for the last quarter of the time in the two-step strategy. Equivalently, the investment in the participating contract starts at 45% in the first quarter and increases to 65% in the last quarter of the time. Whereas, in the three-step strategy, the investment in

the unit-linked contract moves from 65% between  $t = 0$  and  $t = \frac{T}{2}$ , to 45% for the next quarter of the time, then to 35% for the last quarter of the time. This is analogously equivalent to an increase from 35% to 55%, then to 65% in the participating part of the contract, as shown in Fig. 11.

We note that the results of the utility are exhibited in Table C.4 in Appendix C.

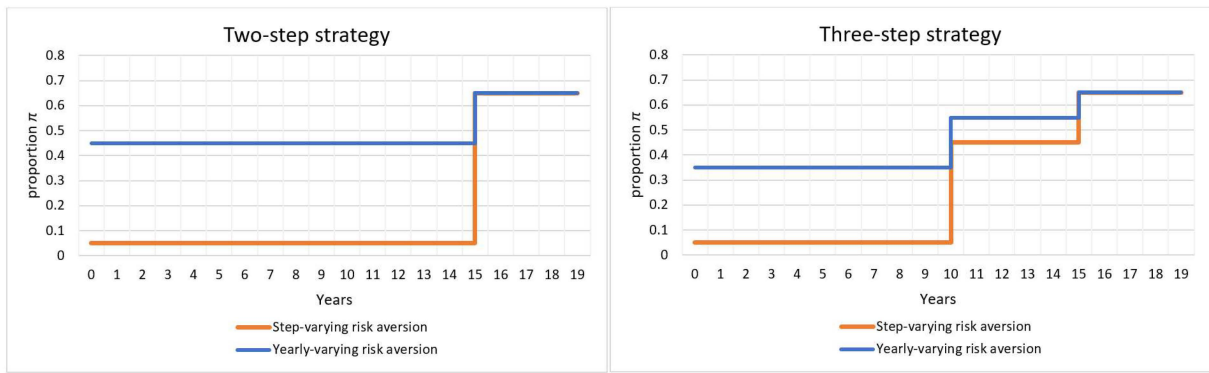
### Step-varying risk aversion:

This part highlights a special case of time-varying risk aversion, that is, instead of varying the risk aversion on a yearly basis, increase it just when moving from one step to another. More precisely, risk aversion increases with the investment strategy of this approach. Recalling the two examples of two- and three-step strategies presented above, for the two-step function, the policyholder has a constant risk aversion  $\gamma_0 = 2$  between  $t = 0$  and  $t = \frac{3T}{4} - 1$ , and another higher constant risk aversion  $\gamma_{\frac{3T}{4}} = 6$  for the remaining time, displayed as follows

$$\gamma_t = \begin{cases} 2 & \text{for } t \in [0, \frac{3T}{4} - 1], \\ 6 & \text{for } t \in [\frac{3T}{4}, T]. \end{cases} \quad (25)$$

Analogously, for the three-step function, the policyholder has a constant risk aversion  $\gamma_0 = 2$  between  $t = 0$  and  $t = \frac{T}{2} - 1$ , another higher but also constant risk aversion  $\gamma_{\frac{T}{2}} = 4$  between  $t = \frac{T}{2}$  and  $t = \frac{3T}{4} - 1$ , and eventually a higher constant risk aversion  $\gamma_{\frac{3T}{4}} = 6$  for the remaining time, represented as

$$\gamma_t = \begin{cases} 2 & \text{for } t \in [0, \frac{T}{2} - 1], \\ 4 & \text{for } t \in [\frac{T}{2}, \frac{3T}{4} - 1], \\ 6 & \text{for } t \in [\frac{3T}{4}, T]. \end{cases} \quad (26)$$



**Fig. 12:** Optimal proportion in the participating contract based on two and three-step strategies for step-varying vs yearly-varying risk aversion.

In Fig. 12, we point out that the investment in the participating contract is lower in the step-varying than in the yearly-varying risk aversion cases for both the two- and three-step strategies. This is intuitive, as in the yearly-varying risk aversion case, the policyholder becomes more conservative from one year to another. However, his attitude toward risk stays the same for a longer period in the step-varying risk aversion case. On the other side, the optimal constant investment strategy for the last



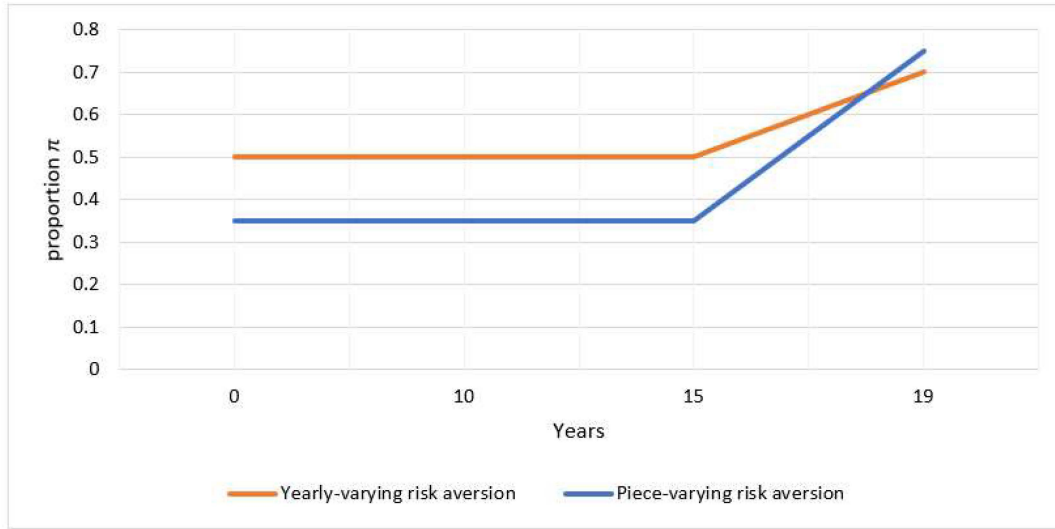
quarter is the same because the risk aversion levels are close in both cases at this stage of time. The results of the utilities for the two- and three-step functions are exhibited in Table C.5 in the Appendix C.

### 3.3.3 Piece-wise linear strategy

Lastly, we consider the piece-wise linear strategy under the time-varying risk aversion case. We will present the numerical results for both the yearly-varying risk aversion example and the piece-varying risk aversion example. For each example, we analyze the policyholder's changing risk aversion and its impact on the optimal investment allocation over time.

#### Two-path strategy:

Recalling the two-step strategy defined above, we present the yearly-varying risk aversion case, which states that the risk aversion increases from  $\gamma_0 = 2$  to  $\gamma_T = 6$  equidistantly by 0.2 (see Equation (24)), and the piece-varying risk aversion case, which alters the risk aversion at  $t = \frac{3 \cdot T}{4}$  only to be  $\gamma_0 = 2$  before and  $\gamma_{\frac{3 \cdot T}{4}} = 6$  after (see Equation (25)). For the utility results, please refer to Tables C.4 and C.5 in Appendix C for yearly-varying and step-varying risk aversions, respectively.



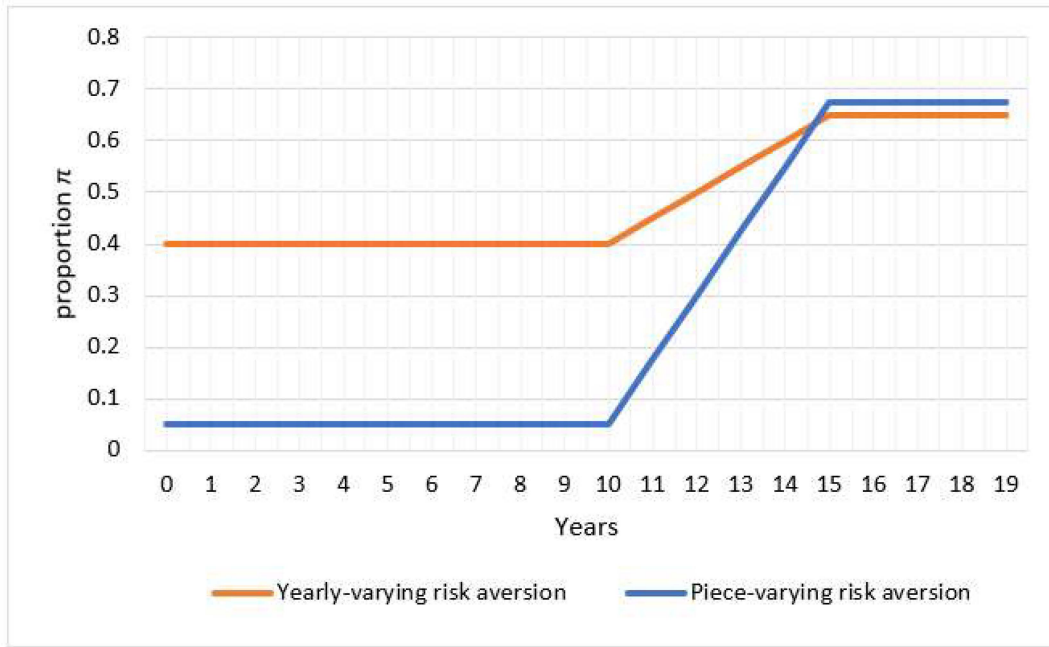
**Fig. 13:** Optimal proportion in the participating contract based on a piece-wise linear strategy for different values of risk aversion - Two-path strategy.

Fig. 13 points out that, by increasing the risk aversion, either on a yearly basis or just when moving from one piece to the next, the lifestyle investment strategy concept is fulfilled. It is intuitive to see that the policyholder who maintains a low level of risk aversion for the first three quarters of the time invests in the risky fund for a longer period (which is equivalent to a lower investment in the participating part of the contract as shown in Fig. 13) compared to the one who becomes more risk averse from one year to the next. However, as the optimal investment strategy is a linear form

in the last quarter of time, the strategy of a policyholder with a risk aversion of 6 (in the piece-varying risk aversion case) would intersect with another policyholder with a risk aversion between 5 and 6 in this interval of time (in the yearly-varying risk aversion case) since the latter is less risk averse.

Three-path strategy:

The last segment covers the three-path strategy stated in Equation (7) for  $n = 2$ ,  $k = 3$ , and  $l = 4$ . We note that the risk aversion increases from  $\gamma_0 = 2$  to  $\gamma_T = 6$  equidistantly by 0.2 for the yearly-varying risk aversion case (see Equation (24)) and changes from  $\gamma_0 = 2$ ,  $\gamma_{\frac{T}{2}} = 4$ , and  $\gamma_{\frac{3T}{4}} = 6$  for the piece-varying risk aversion case (see Equation (26)). The results are presented in Fig. 14. For the utility results, please refer to Tables C.4 and C.5 in Appendix C for yearly-varying and step-varying risk aversions, respectively.



**Fig. 14:** Optimal proportion in the participating contract based on a piece-wise linear strategy for different values of risk aversion - Three-path strategy.

It is interesting to point out that the lifestyle strategy concept holds for the time-varying risk aversion case. In fact, we observe in Fig. 14 that an investor with a risk aversion that changes each year prefers to invest 40% of his initial wealth in the participating contract for the first half of the time, then increases this allocation linearly to 60% at the third quarter  $t = \frac{3T}{4} - 1$  and continue with a constant allocation of 65% for the last quarter of the time. On the other hand, an investor with a risk aversion that changes only at  $t = \frac{T}{2}$  and  $t = \frac{3T}{4}$  is satisfied by investing 5% of his initial wealth in the participating contract for the first half of the time, analogously increasing this allocation linearly to 55% at  $t = \frac{3T}{4} - 1$ , and continuing with a constant allocation of 68% till maturity. In fact, the risk aversion level in the piece-varying risk aversion case is below the risk aversion level in the yearly-varying risk aversion until  $t = \frac{3T}{4}$ , and that's why the optimal strategy in the last quarter is to



invest more in the safe contract in the piece-varying risk aversion case.

In practice, considering time-varying risk aversion in designing investment strategies for mixed participating and unit-linked life insurance contracts may better capture the dynamic nature of an individual's risk preferences and lead to more tailored and suitable investment strategies that align with the policyholder's changing needs and risk appetite over the contract duration.

In the linear strategy, as illustrated in Fig. 10, policyholders with increasing risk aversion tend to adjust their investment strategy over time. Starting with a higher allocation to the risky asset when they are younger and less risk-averse, they gradually reduce their exposure to risk as they approach contract maturity. This behavior aligns with the lifestyle investment strategy concept, indicating that policyholders are willing to take on more risk when they have a longer investment horizon and gradually become more conservative as they age.

The step-wise and piece-wise linear strategies, as depicted in Figs. 11, 13, and 14, further emphasize the impact of time-varying risk aversion in shaping the optimal investment strategy. Policyholders are observed to allocate more to the unit-linked contract in the early stages of the contract and gradually decrease this allocation over time. This dynamic risk management approach allows policyholders to tailor their investment strategies to their evolving risk preferences as they progress through different life stages.

In conclusion, the findings suggest that by taking time-varying risk aversion into account in mixed life insurance contracts, the lifestyle investment strategy may be reflected as it takes into account the policyholders' evolving risk preferences over time. Insurers can accommodate clients with changing risk attitudes by including time-varying risk aversion into the design of their investment strategies.

## 4 Conclusion

In this paper, we have explored the optimal investment strategy within the framework of a mixed insurance contract, focusing on a predetermined deterministic investment style. By combining participating and unit-linked features, mixed contracts offer policyholders a balance between a guarantee and investment growth by deducting a continuous guarantee fee from the unit-linked part to increase, in exchange, the guarantee on the participating part. This work analyzes different families of investment strategies under the lifestyle strategy framework, including linear, step-wise, and piece-wise linear strategies.

In this context, we have considered an annual guarantee contract with rebalancing, striving to create a product that offers advantageous fair guaranteed rates in the participating part, and examining the impact of various investment strategies on the attractiveness of the fair guaranteed rates. We have found that the guaranteed rate increases as the investment in the unit-linked part increases over time, the guarantee fee's value rises, and the participation in the insurer's profits decreases. The incorporation of the guarantee fee into the model significantly enhances the guarantee on the participating part, making it more promising.

Moreover, we have maximized the policyholder's expected utility at the contract's maturity for the different investment strategies proposed while ensuring the contract's fair pricing relation is upheld. The results have clearly indicated that the lifestyle investment strategy does not hold under

the classical utility function. Regardless of the chosen strategy—linear, step-wise, or piece-wise linear—policyholders with the same risk aversion coefficient tend to adopt a stable investment strategy throughout the contract’s duration. Notably, the lifestyle investment strategy has been highlighted by considering either periodic premium payments or time-varying risk aversion in designing investment strategies for mixed life insurance contracts. As policyholders age and their needs change, the optimal investment strategy can be tailored accordingly, and so the lifestyle strategy is captured. These approaches lead to more personalized investment strategies that align with the policyholder’s changing risk appetite over the contract duration. In summary, the way premiums are paid, whether in a lump sum or periodically, along with the risk aversion coefficient, whether it is constant or changes over time, significantly influence the optimization of investment strategies for mixed participating and unit-linked life insurance contracts. They shape the policyholder’s allocation decisions within various investment strategies. Finally, we acknowledge the numerical experiments conducted in this paper, and for future work, we can attempt to obtain explicit results for the optimal proportion using classical methodologies, such as dynamic programming tools.

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**Competing interests** The authors have no relevant financial or non-financial interests to disclose.

**Authors contribution** The original draft of the manuscript has been written by V.H. under P.D. supervision. The results, conclusions and main contributions were discussed and agreed by the two authors. All authors have read and agreed to the final manuscript.

## Compliance with Ethical Standards

**Disclosure of potential conflicts of interest** No potential conflict of interest was reported by the author(s).

**Informed consent** Not applicable.

## Appendices

### A. Classical utility function and single premium

Table A.1 shows the CEQs of different possibilities of investment strategies under the three defined families: linear, step-wise, and piece-wise strategies, given a fair guaranteed rate. These results are based on risk preferences that remain the same throughout the contract period (the case of classical utility function).

| $\pi_0$ | $\pi_{T-1}$ | <i>Linear strategy</i> |              |              |              | <i>Step-wise linear strategy</i><br>(two-step function with $n = \frac{4}{3}$ ) |              |              |              |
|---------|-------------|------------------------|--------------|--------------|--------------|---------------------------------------------------------------------------------|--------------|--------------|--------------|
|         |             | $i$ (%)                | <i>CEQ</i>   |              |              | $i$ (%)                                                                         | <i>CEQ</i>   |              |              |
|         |             |                        | $\gamma = 2$ | $\gamma = 4$ | $\gamma = 6$ |                                                                                 | $\gamma = 2$ | $\gamma = 4$ | $\gamma = 6$ |
| 0.05    | 0.05        | 6.32                   | 2555         | 1679         | 1055         | 6.32                                                                            | 2555         | 1679         | 1055         |
| 0.10    | 0.55        | 1.30                   | 2374         | 1907         | 1513         | 1.98                                                                            | 2429         | 1787         | 1219         |
| 0.35    | 0.70        | 0.70                   | 2195         | 1961         | 1752         | 0.91                                                                            | 2272         | 1939         | 1641         |
| 0.50    | 0.50        | 0.76                   | 2233         | 1983         | 1760         | 0.76                                                                            | 2233         | 1983         | 1760         |
| 0.50    | 0.70        | 0.56                   | 2117         | 1954         | 1804         | 0.65                                                                            | 2171         | 1963         | 1773         |
| 0.65    | 0.65        | 0.48                   | 2060         | 1936         | 1821         | 0.48                                                                            | 2060         | 1936         | 1821         |
| 0.85    | 0.35        | 0.56                   | 2099         | 1922         | 1760         | 0.38                                                                            | 1923         | 1809         | 1704         |
| 0.90    | 1.00        | 0.15                   | 1647         | 1634         | 1621         | 0.17                                                                            | 1682         | 1666         | 1650         |
| 1.00    | 1.00        | 0.11                   | 1574         | 1564         | 1554         | 0.11                                                                            | 1574         | 1564         | 1554         |

| $\pi_0$ | $\pi_{T-1}$ | <i>Piece-wise linear strategy</i> |              |              |              |
|---------|-------------|-----------------------------------|--------------|--------------|--------------|
|         |             | $i$ (%)                           | <i>CEQ</i>   |              |              |
|         |             |                                   | $\gamma = 2$ | $\gamma = 4$ | $\gamma = 6$ |
| 0.05    | 0.05        | 6.32                              | 2555         | 1679         | 1055         |
| 0.10    | 0.55        | 2.58                              | 2484         | 1790         | 1301         |
| 0.35    | 0.70        | 1.04                              | 2327         | 1957         | 1651         |
| 0.50    | 0.50        | 0.76                              | 2233         | 1983         | 1760         |
| 0.50    | 0.70        | 0.70                              | 2203         | 1977         | 1778         |
| 0.65    | 0.65        | 0.48                              | 2060         | 1936         | 1821         |
| 0.85    | 0.35        | 0.30                              | 1862         | 1798         | 1736         |
| 0.90    | 1.00        | 0.18                              | 1702         | 1685         | 1668         |
| 1.00    | 1.00        | 0.11                              | 1574         | 1564         | 1554         |

**Table A.1:** CEQ with respect to constant risk aversion  $\gamma$ —single premium case.

## B. Classical utility function and periodic premiums

Table B.1 shows the CEQs of different possibilities of investment strategies under the three defined families: linear, step-wise, and piece-wise strategies, given a fair guaranteed rate. These results are based on a risk aversion  $\gamma$  of 4 that remains the same throughout the contract period (the case of the classical utility function).

| $\pi_0$ | $\pi_{T-1}$ | <i>Linear strategy</i> |      | <i>Step-wise linear strategy</i><br>(two-step function with $n = \frac{4}{3}$ ) |      |
|---------|-------------|------------------------|------|---------------------------------------------------------------------------------|------|
|         |             | $i$ (%)                | CEQ  | $i$ (%)                                                                         | CEQ  |
| 0.05    | 0.05        | 6.32                   | 1760 | 6.32                                                                            | 1760 |
| 0.05    | 0.40        | 1.52                   | 1801 | 2.13                                                                            | 1798 |
| 0.10    | 0.55        | 1.04                   | 1790 | 1.49                                                                            | 1793 |
| 0.35    | 0.70        | 0.60                   | 1730 | 0.77                                                                            | 1754 |
| 0.50    | 0.50        | 0.76                   | 1745 | 0.76                                                                            | 1745 |
| 0.50    | 0.70        | 0.51                   | 1704 | 0.59                                                                            | 1724 |
| 0.65    | 0.65        | 0.48                   | 1688 | 0.48                                                                            | 1688 |
| 0.85    | 0.35        | 0.71                   | 1704 | 0.49                                                                            | 1623 |
| 0.90    | 1.00        | 0.13                   | 1493 | 0.15                                                                            | 1511 |
| 1.00    | 1.00        | 0.11                   | 1467 | 0.11                                                                            | 1467 |

| $\pi_0$ | $\pi_{T-1}$ | <i>Piece-wise linear strategy</i><br>(two-path function with $n = \frac{4}{3}$ ) |      |
|---------|-------------|----------------------------------------------------------------------------------|------|
|         |             | $i$ (%)                                                                          | CEQ  |
| 0.05    | 0.05        | 6.32                                                                             | 1760 |
| 0.10    | 0.55        | 2.11                                                                             | 1791 |
| 0.15    | 0.55        | 1.79                                                                             | 1792 |
| 0.35    | 0.70        | 0.95                                                                             | 1770 |
| 0.50    | 0.50        | 0.76                                                                             | 1745 |
| 0.50    | 0.70        | 0.67                                                                             | 1736 |
| 0.65    | 0.65        | 0.48                                                                             | 1688 |
| 0.90    | 1.00        | 0.17                                                                             | 1526 |
| 1.00    | 1.00        | 0.11                                                                             | 1467 |

**Table B.1:** CEQ with respect to constant risk aversion  $\gamma$ —periodic premiums case.

## C. Time-varying risk aversion utility function

Table C.4 displays the expected utilities of different possibilities of investment strategies under the three defined families: linear, step-wise, and piece-wise strategies, given a fair guaranteed rate. These results are based on the yearly increasing risk aversion parameters, as presented in Table C.1 (the case of a yearly-varying risk aversion utility function).

| Years ( $t$ )                | 0 | 1   | 2   | ... | 10 | ... | 18  | 19  | 20 |
|------------------------------|---|-----|-----|-----|----|-----|-----|-----|----|
| Risk aversion ( $\gamma_t$ ) | 2 | 2.2 | 2.4 | ... | 4  | ... | 5.6 | 5.8 | 6  |

**Table C.1:** Values of risk aversion parameters in a yearly-varying risk aversion case.

Table C.5 displays the expected utilities of different possibilities of investment strategies under the two defined families: step-wise, and piece-wise strategies, given a fair guaranteed rate. These results are based on risk aversion parameters that increase with each step, as presented in Table C.2 for two-step or two-path strategies and Table C.3 for three-step or three-path strategies (the case of step- and piece-varying risk aversion utility function).

| Years ( $t$ )                | $[0, \frac{3T}{4} - 1]$ | $[\frac{3T}{4}, T]$ |
|------------------------------|-------------------------|---------------------|
| Risk aversion ( $\gamma_t$ ) | 2                       | 6                   |

**Table C.2:** Values of risk aversion parameters in a two-step/path varying risk aversion case.

| Years ( $t$ )                | $[0, \frac{T}{2} - 1]$ | $[\frac{T}{2}, \frac{3T}{4} - 1]$ | $[\frac{3T}{4}, T]$ |
|------------------------------|------------------------|-----------------------------------|---------------------|
| Risk aversion ( $\gamma_t$ ) | 2                      | 4                                 | 6                   |

**Table C.3:** Values of risk aversion parameters in a three-step/path varying risk aversion case.

| <i>Linear strategy</i> |             |                         | <i>Step-wise linear strategy</i>                                 |                       |                         |                                                                                        |                     |                       |                         |
|------------------------|-------------|-------------------------|------------------------------------------------------------------|-----------------------|-------------------------|----------------------------------------------------------------------------------------|---------------------|-----------------------|-------------------------|
|                        |             |                         | <i>(two-step function<br/>with <math>n = \frac{4}{3}</math>)</i> |                       |                         | <i>(three-step function<br/>with <math>n = 2, k = 3</math> and <math>l = 4</math>)</i> |                     |                       |                         |
| $\pi_0$                | $\pi_{T-1}$ | <i>Expected Utility</i> | $\pi_0$                                                          | $\pi_{\frac{3-T}{4}}$ | <i>Expected Utility</i> | $\pi_0$                                                                                | $\pi_{\frac{T}{2}}$ | $\pi_{\frac{3-T}{4}}$ | <i>Expected Utility</i> |
| 0.05                   | 0.55        | -0.0001326              | 0.05                                                             | 0.55                  | -0.0001695              | 0.05                                                                                   | 0.45                | 0.65                  | -0.0001341              |
| 0.05                   | 0.65        | -0.0001266              | 0.05                                                             | 0.65                  | -0.0001679              | 0.20                                                                                   | 0.35                | 0.70                  | -0.0001328              |
| 0.20                   | 0.70        | -0.0001240              | 0.20                                                             | 0.70                  | -0.0001435              | 0.35                                                                                   | 0.55                | 0.65                  | -0.0001253              |
| 0.35                   | 0.75        | -0.0001269              | 0.35                                                             | 0.75                  | -0.0001326              | 0.50                                                                                   | 0.55                | 0.65                  | -0.0001277              |
| 0.40                   | 0.60        | -0.0001265              | 0.40                                                             | 0.60                  | -0.0001288              | 0.60                                                                                   | 0.60                | 0.95                  | -0.0001518              |
| 0.45                   | 0.65        | -0.0001273              | 0.45                                                             | 0.65                  | -0.0001278              | 0.65                                                                                   | 0.75                | 0.85                  | -0.0001483              |
| 0.50                   | 0.70        | -0.0001300              | 0.50                                                             | 0.70                  | -0.0001288              | 0.75                                                                                   | 0.75                | 0.50                  | -0.0001502              |
| 0.65                   | 0.80        | -0.0001458              | 0.65                                                             | 0.80                  | -0.0001407              | 0.80                                                                                   | 0.30                | 0.75                  | -0.0001581              |
| 0.80                   | 0.95        | -0.0001901              | 0.80                                                             | 0.95                  | -0.0001778              | 0.95                                                                                   | 0.05                | 0.55                  | -0.0002107              |
| 0.90                   | 1.00        | -0.0002256              | 0.90                                                             | 1.00                  | -0.0002129              | 0.95                                                                                   | 0.95                | 1.00                  | -0.0002303              |

| <i>Piece-wise linear strategy</i>                                |             |                         |                                                                                        |                         |                         |
|------------------------------------------------------------------|-------------|-------------------------|----------------------------------------------------------------------------------------|-------------------------|-------------------------|
| <i>(two-path strategy<br/>with <math>n = \frac{4}{3}</math>)</i> |             |                         | <i>(three-path strategy<br/>with <math>n = 2, k = 3</math> and <math>l = 4</math>)</i> |                         |                         |
| $\pi_0$                                                          | $\pi_{T-1}$ | <i>Expected Utility</i> | $\pi_0$                                                                                | $\pi_{\frac{3-T}{4}-1}$ | <i>Expected Utility</i> |
| 0.05                                                             | 0.55        | -0.0001977              | 0.05                                                                                   | 0.55                    | -0.0001423              |
| 0.05                                                             | 0.65        | -0.0001927              | 0.05                                                                                   | 0.65                    | -0.0001461              |
| 0.20                                                             | 0.70        | -0.0001535              | 0.20                                                                                   | 0.70                    | -0.0001370              |
| 0.35                                                             | 0.75        | -0.0001339              | 0.35                                                                                   | 0.75                    | -0.0001352              |
| 0.40                                                             | 0.60        | -0.0001320              | 0.40                                                                                   | 0.60                    | -0.0001261              |
| 0.45                                                             | 0.65        | -0.0001293              | 0.45                                                                                   | 0.65                    | -0.0001273              |
| 0.50                                                             | 0.70        | -0.0001286              | 0.50                                                                                   | 0.70                    | -0.0001305              |
| 0.65                                                             | 0.80        | -0.0001373              | 0.65                                                                                   | 0.80                    | -0.0001463              |
| 0.80                                                             | 0.95        | -0.0001683              | 0.80                                                                                   | 0.95                    | -0.0001926              |
| 0.90                                                             | 1.00        | -0.0002033              | 0.90                                                                                   | 1.00                    | -0.0002274              |

**Table C.4:** Power utility with respect to yearly-varying risk aversion.

| <i>Step-wise linear strategy</i>                                 |                       |                         |                                                                                        |                     |                       |                         | <i>Piece-wise linear strategy</i>                                |             |                         |                                                                                        |                         |                         |
|------------------------------------------------------------------|-----------------------|-------------------------|----------------------------------------------------------------------------------------|---------------------|-----------------------|-------------------------|------------------------------------------------------------------|-------------|-------------------------|----------------------------------------------------------------------------------------|-------------------------|-------------------------|
| <i>(two-step function<br/>with <math>n = \frac{4}{3}</math>)</i> |                       |                         | <i>(three-step function<br/>with <math>n = 2, k = 3</math> and <math>l = 4</math>)</i> |                     |                       |                         | <i>(two-path strategy<br/>with <math>n = \frac{4}{3}</math>)</i> |             |                         | <i>(three-path strategy<br/>with <math>n = 2, k = 3</math> and <math>l = 4</math>)</i> |                         |                         |
| $\pi_0$                                                          | $\pi_{\frac{3-T}{4}}$ | <i>Expected Utility</i> | $\pi_0$                                                                                | $\pi_{\frac{T}{2}}$ | $\pi_{\frac{3-T}{4}}$ | <i>Expected Utility</i> | $\pi_0$                                                          | $\pi_{T-1}$ | <i>Expected Utility</i> | $\pi_0$                                                                                | $\pi_{\frac{3-T}{4}-1}$ | <i>Expected Utility</i> |
| 0.05                                                             | 0.55                  | -0.00007363             | 0.05                                                                                   | 0.45                | 0.65                  | -0.0001695              | 0.05                                                             | 0.55        | -0.0002817              | 0.05                                                                                   | 0.55                    | -0.0001761              |
| 0.05                                                             | 0.65                  | -0.00007237             | 0.20                                                                                   | 0.35                | 0.70                  | -0.0001770              | 0.05                                                             | 0.65        | -0.0002726              | 0.05                                                                                   | 0.65                    | -0.0001820              |
| 0.20                                                             | 0.70                  | -0.00007701             | 0.35                                                                                   | 0.55                | 0.65                  | -0.0001812              | 0.20                                                             | 0.70        | -0.0002617              | 0.20                                                                                   | 0.70                    | -0.0001866              |
| 0.35                                                             | 0.75                  | -0.00008177             | 0.50                                                                                   | 0.55                | 0.65                  | -0.0001884              | 0.35                                                             | 0.75        | -0.0002557              | 0.35                                                                                   | 0.75                    | -0.0001947              |
| 0.40                                                             | 0.60                  | -0.00008280             | 0.60                                                                                   | 0.60                | 0.95                  | -0.0002244              | 0.40                                                             | 0.60        | -0.0002599              | 0.40                                                                                   | 0.60                    | -0.0001836              |
| 0.45                                                             | 0.65                  | -0.00008382             | 0.65                                                                                   | 0.75                | 0.85                  | -0.0002184              | 0.45                                                             | 0.65        | -0.0002588              | 0.45                                                                                   | 0.65                    | -0.0001869              |
| 0.50                                                             | 0.70                  | -0.00008565             | 0.75                                                                                   | 0.75                | 0.50                  | -0.0002224              | 0.50                                                             | 0.70        | -0.0002602              | 0.50                                                                                   | 0.70                    | -0.0001926              |
| 0.65                                                             | 0.80                  | -0.00009427             | 0.80                                                                                   | 0.30                | 0.75                  | -0.0002158              | 0.65                                                             | 0.80        | -0.0002772              | 0.65                                                                                   | 0.80                    | -0.0002154              |
| 0.80                                                             | 0.95                  | -0.00011374             | 0.95                                                                                   | 0.05                | 0.55                  | -0.0002576              | 0.80                                                             | 0.95        | -0.0003218              | 0.80                                                                                   | 0.95                    | -0.0002753              |
| 0.90                                                             | 1.00                  | -0.00012768             | 0.95                                                                                   | 0.95                | 1.00                  | -0.0003104              | 0.90                                                             | 1.00        | -0.0003639              | 0.90                                                                                   | 1.00                    | -0.0003131              |

**Table C.5:** Power utility with respect to step- and piece-varying risk aversions.



## D. Classical utility function and single premium - Sensitivity analyses

Table D.1 shows some sensitivity analyses of various parameters under the single premium case and the classical utility function with a risk aversion  $\lambda$  of 4. Each sensitivity analysis is performed compared to the Basic scenario, which refers to the scenario studied in Section 3.1. The results stated in Table D.1 provide the first and the last values of the investment strategy and can be compared to Fig. 8, confirming that the optimal investment strategy stays stable over time.

| Parameters            | $\pi_0$ | $\pi_{T-1}$ |
|-----------------------|---------|-------------|
| <i>Basic scenario</i> | 0.50    | 0.50        |
| $r = 2.5\%$           | 0.50    | 0.55        |
| $\sigma_G = 7\%$      | 0.45    | 0.45        |
| $\sigma_G = 1\%$      | 0.45    | 0.50        |
| $\sigma_H = 20\%$     | 0.70    | 0.70        |
| $\varepsilon = 0.5\%$ | 0.50    | 0.50        |
| $\beta = 50\%$        | 0.45    | 0.45        |
| $\rho = 50\%$         | 0.45    | 0.50        |
| $\rho = -10\%$        | 0.50    | 0.50        |
| $T = 10$              | 0.45    | 0.50        |
| $T = 40$              | 0.50    | 0.50        |

**Table D.1:** Sensitivity analyses under classical utility function with  $\lambda = 4$ —single premium case.

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