

OUTER AUTOMORPHISMS OF ADJOINT GROUPS OF TYPE D AND NON-RATIONAL ADJOINT GROUPS OF OUTER TYPE A

DEMBA BARRY AND JEAN-PIERRE TIGNOL

ABSTRACT. For a classical group G of type D_n over a field k of characteristic different from 2, we show the existence of a finitely generated regular extension of k over which G admits outer automorphisms. Using this result and a construction of groups of type A from groups of type D, we construct new examples of groups of type 2A_n with $n \equiv 3 \pmod{4}$ and the first examples of type 2A_n with $n \equiv 1 \pmod{4}$ ($n \geq 5$) that are not R -trivial, hence not rational (nor stably rational).

1. INTRODUCTION

Two questions concerning algebraic groups of classical type are addressed in this paper: the existence of outer automorphisms of adjoint groups of type D and the rationality of adjoint groups of outer type A. The two questions are related by a construction of groups of type A from groups of type D that we call unitary extension.

To describe our contribution to the first topic, recall that when a classical group G of adjoint type D_n over a field k of characteristic different from 2 is represented as the connected component of the identity $\mathbf{PGO}^+(A, \sigma)$ in the group of automorphisms of a central simple algebra with orthogonal involution (A, σ) of degree $2n$, then outer automorphisms of G are induced by improper similitudes of (A, σ) , i.e., elements $g \in A$ such that

$$\sigma(g)g \in k^\times \quad \text{and} \quad \text{Nrd}_A(g) = -(\sigma(g)g)^n,$$

see [QMT1, Prop. 2.5]. The existence of an improper similitude is a serious constraint on A and σ : the algebra must be split by the quadratic extension given by the discriminant of σ (see [KMRT98, (13.38)]), hence its index is at most 2; and if the discriminant is trivial then A must be split. Nevertheless, we show:

Proposition 1.1. *Let σ be an orthogonal involution on a central simple algebra A of degree $2n$ over a field k of characteristic different from 2. If A is not split and the discriminant of σ is not trivial, there exists a finitely generated regular extension R of k such that the algebra with involution (A_R, σ_R) obtained from (A, σ) by scalar extension to R admits improper similitudes and A_R is not split.*

The proof¹ shows that one can take for R the function field of the connected component of improper similitudes in the group of automorphisms of (A, σ) : see Section 2.

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In the second part of the paper, we investigate the rationality problem for the underlying variety of adjoint linear algebraic groups of type ${}^2\mathbf{A}_n$. Voskresenskiĭ and Klyachko [VK85, Cor. of Th. 8] have shown that this variety is rational if n is even. By contrast, examples of adjoint groups of type ${}^2\mathbf{A}_n$ for $n \equiv 3 \pmod{4}$ that are not rational have been given by Merkurjev [Mer96] and by Berhuy–Monsurro–Tignol [BMT04], using Manin’s R -equivalence and Merkurjev’s computation of the group of R -equivalence classes of adjoint classical groups [Mer96]. On one hand we will use Proposition 1.1 to expand the range of these examples, and on another hand we will provide the first examples of adjoint groups of type ${}^2\mathbf{A}_n$ with arbitrary $n \equiv 1 \pmod{4}$ ($n \geq 5$) that are not R -trivial, hence not rational (nor stably rational). These examples are based on the adjoint groups of type D that have outer automorphisms but no outer automorphisms of order 2 found by Quéguiner-Mathieu and Tignol [QMT1].

To explain our construction in more detail, recall that adjoint groups of outer type A over a field F can be represented as groups of automorphisms $\mathbf{PGU}(B, \tau)$ of central simple algebras with unitary involution (B, τ) over separable quadratic field extensions K/F . We consider in particular the case where (B, τ) is obtained from a central simple F -algebra with orthogonal or symplectic involution (A, σ) as

$$(B, \tau) = (A, \sigma) \otimes_F (K, \iota),$$

where ι is the nontrivial F -automorphism of K . We then say (B, τ) is a *unitary extension* of (A, σ) . In Section 3 we give a necessary and sufficient condition for (B, τ) to be hyperbolic (excluding one exceptional case); see Theorem 3.1.

Of special interest are *generic* unitary extensions, where $F = k(x)$ is a rational function field in one variable over a field k of characteristic different from 2, $K = F(\sqrt{x})$, and A is defined over the field k of constants. (Generic unitary extensions are also used in [QMT1, Sec. 4.3].) In Section 5 we show:

Theorem 1.2. *Let (A, σ) be a central simple algebra with orthogonal involution of degree multiple of 4 over a field k of characteristic 0. If A is not split and the discriminant of σ is not trivial, then for the generic unitary extension (B, τ) of (A, σ) the group $\mathbf{PGU}(B, \tau)$ is not R -trivial, i.e., there exists a field extension E of F such that the group of E -rational points $\mathbf{PGU}(B, \tau)(E)$ has more than one R -equivalence class.*

It follows that the group $\mathbf{PGU}(B, \tau)$ is not rational, nor even stably rational, see [CTS77, Sec. 4]. It is a group of adjoint type ${}^2\mathbf{A}_n$ with $n \equiv 3 \pmod{4}$. The proof of Theorem 1.2 is given in Subsection 5.1.

Section 3 of [QMT1] yields examples of central simple algebras with orthogonal involution (A, σ) of degree $\deg A \equiv 2 \pmod{4}$ that have improper similitudes, none of them being square-central. We show in Example 5.5 that for their generic unitary extension (B, τ) the group $\mathbf{PGU}(B, \tau)$ is not R -trivial. We thus obtain examples of adjoint groups of type ${}^2\mathbf{A}_n$ that are not rational nor stably rational for every integer $n \geq 5$ with $n \equiv 1 \pmod{4}$.

The proof of Theorem 1.2 is prepared in Section 4 by Theorem 4.6, which yields a computation in terms of (A, σ) of the group of R -equivalence classes in the group of rational points of $\mathbf{PGU}(B, \tau)$ over the completion of F for the x -adic valuation.

Notation. We generally follow the notation and terminology of [KMRT98]. The characteristic of the base field is always assumed to be different from 2. If A is a central simple algebra of even degree over a field k and σ is an orthogonal involution on A , we write $\mathrm{GO}(A, \sigma)$ for the group of similitudes of (A, σ) ,

$$\mathrm{GO}(A, \sigma) = \{g \in A \mid \sigma(g)g \in k^\times\}.$$

For $g \in \mathrm{GO}(A, \sigma)$ we let $\mu(g) = \sigma(g)g$ be the *multiplier* of g . The group of proper similitudes is

$$\mathrm{GO}^+(A, \sigma) = \{g \in \mathrm{GO}(A, \sigma) \mid \mathrm{Nrd}_A(g) = \mu(g)^{(\deg A)/2}\}$$

and we write

$$\mathrm{GO}^-(A, \sigma) = \mathrm{GO}(A, \sigma) \setminus \mathrm{GO}^+(A, \sigma) = \{g \in \mathrm{GO}(A, \sigma) \mid \mathrm{Nrd}_A(g) = -\mu(g)^{(\deg A)/2}\}$$

for the coset of improper similitudes (which may be empty). The corresponding sets of multipliers are denoted as follows:

$$G(A, \sigma) = \{\mu(g) \mid g \in \mathrm{GO}(A, \sigma)\}, \quad G^\pm(A, \sigma) = \{\mu(g) \mid g \in \mathrm{GO}^\pm(A, \sigma)\}.$$

If $\delta \in k^\times$ represents the discriminant of σ , the quaternion algebra $(\delta, \mu(g))_k$ is Brauer-equivalent to k if g is proper, and to A if g is improper; see [MT95, Th. A] or [KMRT98, (13.38)]. Therefore,

$$G^+(A, \sigma) \cap G^-(A, \sigma) = \emptyset \quad \text{if } A \text{ is not split.} \quad (1)$$

The group of projective proper similitudes of (A, σ) is $\mathrm{PGO}^+(A, \sigma) = \mathrm{GO}^+(A, \sigma)/k^\times$. It is the group of k -rational points of the algebraic group $\mathbf{PGO}^+(A, \sigma)$, which is the connected component of the identity in the group $\mathbf{PGO}(A, \sigma) = \mathbf{Aut}(A, \sigma)$ of automorphisms of (A, σ) . It is a group of adjoint type D_n if $\deg A = 2n$ with $n \geq 2$.

For every field ℓ containing k we write A_ℓ for the ℓ -algebra $A \otimes_k \ell$ and σ_ℓ for the involution $\sigma \otimes \mathrm{id}_\ell$ on A_ℓ , and we let $(A, \sigma)_\ell = (A_\ell, \sigma_\ell)$. If ℓ is a finite-degree extension of k , we let $N(\ell/k) = N_{\ell/k}(\ell^\times)$ be the group of norms. Let $\mathrm{Hyp}(A, \sigma) \subset k^\times$ be the subgroup generated by the norm groups $N(\ell/k)$ where ℓ runs over the finite-degree field extensions of k such that $(A, \sigma)_\ell$ is hyperbolic. The following canonical isomorphism due to Merkurjev [Mer96, Th. 1] yields a description of the group of R -equivalence classes of $\mathrm{PGO}^+(A, \sigma)$:

$$\mathrm{PGO}^+(A, \sigma)/R \simeq G^+(A, \sigma)/(k^{\times 2} \cdot \mathrm{Hyp}(A, \sigma)). \quad (2)$$

In particular, if (A, σ) is hyperbolic, then $\mathrm{Hyp}(A, \sigma) \supset N(k/k) = k^\times$, hence

$$G(A, \sigma) = G^+(A, \sigma) = \mathrm{Hyp}(A, \sigma) = k^\times \quad \text{and} \quad \mathrm{PGO}^+(A, \sigma)/R = 1 \quad \text{if } (A, \sigma) \text{ is hyperbolic.} \quad (3)$$

Corresponding notions are defined for unitary involutions: if B is a central simple algebra over a field K and τ is a unitary involution on B , i.e., an involution that does not leave K elementwise fixed, let $F \subset K$ be the subfield of fixed elements and

$$\mathrm{GU}(B, \tau) = \{g \in B \mid \tau(g)g \in F^\times\}, \quad G(B, \tau) = \{\tau(g)g \mid g \in \mathrm{GU}(B, \tau)\} \subset F^\times.$$

The group of projective similitudes $\mathrm{PGU}(B, \tau) = \mathrm{GU}(B, \tau)/K^\times$ is the group of F -rational points of the algebraic group $\mathbf{PGU}(B, \tau) = \mathbf{Aut}_K(B, \tau)$, which is a group of adjoint type ${}^2\mathbf{A}_{n-1}$ over F if $\deg B = n > 2$. The group $\mathrm{Hyp}(B, \tau) \subset F^\times$ is defined as in the orthogonal case, and Merkurjev's canonical isomorphism takes the form

$$\mathrm{PGU}(B, \tau)/R \simeq G(B, \tau)/(N(K/F) \cdot \mathrm{Hyp}(B, \tau)). \quad (4)$$

As in the orthogonal case, we have

$$G(B, \tau) = \mathrm{Hyp}(B, \tau) = F^\times \quad \text{and} \quad \mathrm{PGU}(B, \tau)/R = 1 \quad \text{if } (B, \tau) \text{ is hyperbolic.} \quad (5)$$

2. IMPROPER SIMILITUDES

Throughout this section, A is a central simple algebra of degree $2n$ over an arbitrary field k of characteristic different from 2 and σ is an orthogonal involution on A . If A is a quaternion algebra, then (A, σ) admits improper similitudes (see [KMRT98, (12.25)]), hence Proposition 1.1 holds with $R = k$. We may therefore assume throughout $n \geq 2$, so $\mathbf{PGO}^+(A, \sigma)$ is a semisimple linear algebraic group.

Recall from [KMRT98, §23.B] that $\mathbf{PGO}(A, \sigma)$ has two connected components. Write $X = \mathbf{PGO}^-(A, \sigma)$ for the non-identity component. It is a $\mathbf{PGO}^+(A, \sigma)$ -torsor whose rational points consist of inner automorphisms induced by improper similitudes of (A, σ) . Therefore, X is an affine, smooth, geometrically connected k -variety, and its function field $k(X)$ is a finitely generated

regular extension of k . By definition, X has rational points over $k(X)$, hence $(A_{k(X)}, \sigma_{k(X)})$ admits improper similitudes. To establish Proposition 1.1, it is therefore sufficient to show:

Proposition 2.1. *If the discriminant of σ is not trivial, then the Brauer group map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(k(X))$ induced by scalar extension is injective.*

Proof. Since X is smooth the map $\mathrm{Br}(X) \rightarrow \mathrm{Br}(k(X))$ is injective, hence it suffices to show that the map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(X)$ is injective.

Let k_s be a separable closure of k and $\Gamma = \mathrm{Gal}(k_s/k)$ the absolute Galois group of k . To simplify notations, write G for $\mathbf{PGO}^+(A, \sigma)$, and let G_s (resp. X_s) denote the algebraic group over k_s (resp. algebraic variety over k_s) obtained from G (resp. X) by base change from k to k_s . Since $X_s(k_s) \neq \emptyset$, the variety X_s is isomorphic to the underlying variety of G_s . It then follows from a theorem of Rosenlicht [Ros61, Th. 3] that every invertible regular function on X_s is constant. Therefore, the Hochschild–Serre spectral sequence $H^p(\Gamma, H^q(X_s, \mathbf{G}_m)) \Rightarrow H^n(X, \mathbf{G}_m)$ yields the following exact sequence of low-degree terms (see [San81, Lemme 6.3(i)]):

$$0 \rightarrow \mathrm{Pic}(X) \rightarrow (\mathrm{Pic} X_s)^\Gamma \rightarrow \mathrm{Br}(k) \rightarrow \mathrm{Br}(X).$$

Now, by [San81, Lemme 6.7] we have $(\mathrm{Pic} X_s)^\Gamma \simeq (\mathrm{Pic} G_s)^\Gamma$ and, by [San81, Lemme 6.9] (see also [KMRT98, (31.21)]), $\mathrm{Pic} G_s$ can be identified with the dual Z_s^* of the center Z_s of the simply connected cover $\mathbf{Spin}(A_s, \sigma_s)$ of G_s . Since the discriminant of σ is not trivial, Γ acts non-trivially on Z_s^* , and we have $(\mathrm{Pic} G_s)^\Gamma \simeq \mathbb{Z}/2\mathbb{Z}$. Therefore, to complete the proof it suffices to show that $\mathrm{Pic}(X) \neq 0$.

For this, consider the canonical map $\mathbf{GO}^-(A, \sigma) \rightarrow X$: it defines a torsor for $\mathbf{G}_m$ over X , hence an element of $H^1(X, \mathbf{G}_m) = \mathrm{Pic}(X)$. This element is not trivial because after scalar extension to k_s the torsor is isomorphic to $\mathbf{GO}^+(A_s, \sigma_s) \rightarrow G_s$. The proof is thus complete. $\square$

3. UNITARY EXTENSIONS OF INVOLUTIONS OF THE FIRST KIND

In this section, A is a central simple algebra over an arbitrary field F of characteristic different from 2 and σ is an F -linear involution on A (i.e., an involution that may be orthogonal or symplectic). Let K be a quadratic field extension of F and let ι denote its nontrivial automorphism. We consider the algebra with unitary involution

$$(B, \tau) = (A, \sigma) \otimes_F (K, \iota).$$

In preparation for the next section, where a special case of this construction will be analyzed, we determine a necessary and sufficient condition for (B, τ) to be hyperbolic.

Theorem 3.1. *If there is an embedding of F -algebras with involution $(K, \mathrm{id}) \hookrightarrow (A, \sigma)$, then (B, τ) is hyperbolic. The converse holds, except in the case where A is split of degree 2 mod 4 and σ is symplectic.*

The proof uses the Witt decomposition of involutions. Recall that A can be represented as $\mathrm{End}_D V$ for some vector space V over a division algebra D ; then σ is adjoint to a nondegenerate hermitian (or skew-hermitian) form h on V with respect to some involution of the first kind on D . The space (V, h) has a decomposition

$$(V, h) \simeq (V_0, h_0) \perp (V_1, h_1)$$

with h_0 anisotropic and h_1 hyperbolic, which is reflected in a so-called orthogonal sum decomposition of $(\mathrm{End}_D V, \mathrm{ad}_h)$ into $(\mathrm{End}_D V_0, \mathrm{ad}_{h_0}) \boxplus (\mathrm{End}_D V_1, \mathrm{ad}_{h_1})$, see [BFT07, Sec. 1.4]. Thus, we may find a decomposition

$$(A, \sigma) \simeq (A_0, \sigma_0) \boxplus (A_1, \sigma_1)$$

where A_0, A_1 are central simple F -algebras Brauer-equivalent to A (if they are not $\{0\}$), where σ_0, σ_1 are involutions of the same type as σ , and where σ_0 is anisotropic (which means that

$\sigma_0(a)a = 0$ implies $a = 0$) and σ_1 is hyperbolic (which means there is an idempotent $e \in A_1$ such that $\sigma_1(e) = 1 - e$).

Proof of Theorem 3.1. Let $K = F(u)$ where $u^2 = a \in F^\times$. If (K, id) embeds into (A, σ) , we may find $s \in A$ such that $\sigma(s) = s$ and $s^2 = a$. Consider then

$$e = \frac{1}{2}(1 \otimes 1 + s \otimes u^{-1}) \in B.$$

Computation shows that $e^2 = e$ and $\tau(e) = 1 - e$, hence (B, τ) is hyperbolic.

For the converse, suppose we are not in the situation where A is split of degree 2 mod 4 with σ symplectic, and consider a Witt decomposition $(A, \sigma) \simeq (A_0, \sigma_0) \boxplus (A_1, \sigma_1)$ with σ_0 anisotropic and σ_1 hyperbolic. If σ is symplectic and A is split, then $A = A_1$ because symplectic involutions on split algebras are hyperbolic, and $\deg A$ is assumed to be divisible by 4. Likewise, if σ is symplectic and A is not split, then $\deg A_1$ is divisible by 4 because σ_1 is adjoint to a hyperbolic form over a noncommutative division algebra. By [BST93, Th. 2.2] it follows that in all cases (including the case where σ is orthogonal) there is a central simple F -algebra with involution (A'_1, σ'_1) such that

$$(A_1, \sigma_1) \simeq (M_2(F), \theta) \otimes_F (A'_1, \sigma'_1)$$

with θ the hyperbolic orthogonal involution defined by

$$\theta \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{22} & x_{12} \\ x_{21} & x_{11} \end{pmatrix} \quad \text{for } x_{11}, x_{12}, x_{21}, x_{22} \in F.$$

Then A_1 contains $s_1 = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \otimes 1$, which satisfies $\sigma_1(s_1) = s_1$ and $s_1^2 = a$. We next show that if (B, τ) is hyperbolic then A_0 contains an element s_0 such that $\sigma_0(s_0) = s_0$ and $s_0^2 = a$; then $s = \begin{pmatrix} s_0 & 0 \\ 0 & s_1 \end{pmatrix} \in A_0 \boxplus A_1 = A$ satisfies $\sigma(s) = s$ and $s^2 = a$, hence mapping $u \mapsto s$ defines an embedding $(K, \text{id}) \hookrightarrow (A, \sigma)$.

Note that $(B, \tau) = [(A_0, \sigma_0) \otimes (K, \iota)] \boxplus [(A_1, \sigma_1) \otimes (K, \iota)]$, and the second term on the right side is hyperbolic because σ_1 is hyperbolic. Therefore, the hypothesis that (B, τ) is hyperbolic implies $(A_0, \sigma_0) \otimes (K, \iota)$ is hyperbolic. We may then find $e = e_1 \otimes 1 + e_2 \otimes u \in A_0 \otimes_F K$ such that $e^2 = e$ and $(\sigma_0 \otimes \iota)(e) = 1 - e$, or equivalently

$$(\sigma_0 \otimes \iota)(e) = 1 - e \quad \text{and} \quad (\sigma_0 \otimes \iota)(e)e = 0. \quad (6)$$

These conditions yield

$$\sigma_0(e_1) = 1 - e_1 \quad (7)$$

and

$$\sigma_0(e_1)e_1 = a\sigma_0(e_2)e_2. \quad (8)$$

Now, consider the right ideal $I = \{x \in A_0 \mid e_2x = 0\}$. By [BST93, Cor. 1.8] we may find $f \in A_0$ such that $\sigma_0(f) = f = f^2$ and $I = fA_0$, because σ_0 is anisotropic. Since $e_2f = 0$, multiplying (8) on the left and on the right by f yields $\sigma_0(e_1f)e_1f = 0$, hence $e_1f = 0$ because σ_0 is anisotropic. By (7) we have

$$fe_1\sigma_0(fe_1) = f(e_1 - e_1^2)f.$$

The right side is 0 since $e_1f = 0$, hence $\sigma_0(fe_1) = 0$ because σ_0 is anisotropic. By (7) again, it follows that $(1 - e_1)f = 0$, hence $f = 0$ since $e_1f = 0$. Therefore, $I = \{0\}$, hence e_2 is invertible and we may set $s_0 = e_1e_2^{-1} \in A_0$. From (8) it follows that $s_0^2 = a$. Now, (6) also yields $\sigma_0(e_1)e_2 = \sigma_0(e_2)e_1$, hence $\sigma_0(s_0) = s_0$.

We have thus proved the existence of an embedding $(K, \text{id}) \hookrightarrow (A, \sigma)$ when (B, τ) is hyperbolic, setting aside the case where A is split of degree 2 mod 4 and σ is symplectic. Note that in the exceptional case (B, τ) is hyperbolic since (A, σ) is hyperbolic; but every symmetric element in A is a root of an odd-degree ‘‘pfaffian’’ polynomial (see [KMRT98, (2.9)]), hence there is no $s \in A$ such that $\sigma(s) = s$ and $s^2 \in F^\times \setminus F^{\times 2}$. $\square$

Remark 3.2. The proof above is a slight modification of [BST93, Th. 3.3], where the existence of an embedding $(K, \iota) \hookrightarrow (A, \sigma)$ is shown to be equivalent to the hyperbolicity of $(A, \sigma) \otimes_F (K, \text{id})$, except when A is split, σ is orthogonal and its Witt index is odd.

4. R -EQUIVALENCE ON PROJECTIVE UNITARY GROUPS

In this section, we consider a special case of unitary extension. Throughout the section, (A, σ) is a central simple algebra with orthogonal or symplectic involution over a field k of characteristic zero. We let $\widehat{F} = k((x))$ be the field of formal Laurent series in one indeterminate over k , and $\widehat{K} = \widehat{F}(\xi)$ where $\xi^2 = x$, hence $\widehat{K} = k((\xi))$. We write ι for the nontrivial automorphism of $\widehat{K}/\widehat{F}$ and consider the algebra with involution

$$(\widehat{B}, \widehat{\tau}) = (A, \sigma) \otimes_k (\widehat{K}, \iota). \quad (9)$$

Thus, $\widehat{B} = A((\xi))$ (with ξ centralizing A) and

$$\widehat{\tau}\left(\sum_{i=r}^{\infty} a_i \xi^i\right) = \sum_{i=r}^{\infty} \sigma(a_i)(-\xi)^i \quad \text{for } a_i \in A, i = r, r+1, \dots$$

The $\widehat{K}$ -algebra $\widehat{B}$ is central simple, and $\widehat{\tau}$ is a unitary involution on $\widehat{B}$. Our goal is to compute $\text{PGU}(\widehat{B}, \widehat{\tau})/R$ in terms of (A, σ) , using Merkurjev's canonical isomorphism (4).

As a first step, we show that the trivial hyperbolic cases (see (3) and (5)) are related:

Proposition 4.1. *The statements (a) and (b) (resp. (a') and (b')) are equivalent:*

- | | |
|---|---|
| (a) (A, σ) is isotropic, | (a') (A, σ) is hyperbolic, |
| (b) $(\widehat{B}, \widehat{\tau})$ is isotropic, | (b') $(\widehat{B}, \widehat{\tau})$ is hyperbolic. |

Similarly, if C is a central simple algebra over a quadratic field extension ℓ of k and ρ is a unitary involution on C fixing k , the statements (c) and (d) (resp. (c') and (d')) are equivalent:

- | | |
|---|---|
| (c) (C, ρ) is isotropic, | (c') (C, ρ) is hyperbolic, |
| (d) $(C, \rho)_{\widehat{F}}$ is isotropic, | (d') $(C, \rho)_{\widehat{F}}$ is hyperbolic. |

Proof. Since $(A, \sigma) \subset (\widehat{B}, \widehat{\tau})$, it is clear that (a) $\Rightarrow$ (b) and (a') $\Rightarrow$ (b'). To see (b) $\Rightarrow$ (a), suppose $y \in \widehat{B}$ is nonzero and $\widehat{\tau}(y)y = 0$. Write y as a series $y = \sum_{i=r}^{\infty} a_i \xi^i$ with coefficients in A , with $a_r \neq 0$. The coefficient of x^r in $\widehat{\tau}(y)y$ is $(-1)^r \sigma(a_r)a_r$, hence $\sigma(a_r)a_r = 0$. It follows that σ is isotropic, proving (b) $\Rightarrow$ (a).

To establish (b') $\Rightarrow$ (a'), consider a Witt decomposition $(A, \sigma) \simeq (A_0, \sigma_0) \boxplus (A_1, \sigma_1)$ with σ_0 anisotropic and σ_1 hyperbolic, as in the proof of Theorem 3.1. Then $(A_1, \sigma_1) \otimes_k (\widehat{K}, \iota)$ is hyperbolic, hence the condition that $(\widehat{B}, \widehat{\tau})$ is hyperbolic implies $(A_0, \sigma_0) \otimes_k (\widehat{K}, \iota)$ is hyperbolic. But $(A_0, \sigma_0) \otimes_k (\widehat{K}, \iota)$ is anisotropic since (b) $\Rightarrow$ (a), hence $A_0 = \{0\}$ and therefore (A, σ) is hyperbolic.

The proof of the equivalence of (c) and (d) (resp. (c') and (d')) is similar; we omit it. $\square$

We next make some observations on the norm group $N(L/\widehat{F})$ of a finite-degree field extension L of $\widehat{F}$. Recall that the x -adic valuation on $\widehat{F}$ extends uniquely to a valuation on L . We let v denote this valuation. Let ℓ be the residue field of L and M be the unramified closure of $\widehat{F}$ in L , which is the unique unramified extension of $\widehat{F}$ in L with residue field ℓ (see [TW15, Prop. A.17]). Let also π be a uniformizer of L , and let

$$[L : M] = e \quad \text{and} \quad [M : \widehat{F}] = f.$$

By [Ser68, Ch. II, Th. 2], we may identify $L = \ell((\pi))$ and $M = \ell((x))$ since the characteristic of k is zero. If $e = 1$, we take $\pi = x$. If $e > 1$, let

$$u = N_{L/M}(\pi)x^{-1} \in M^\times.$$

Since $v(N_{L/M}(\pi)) = e v(\pi) = v(x)$, it follows that $v(u) = 0$. We may therefore consider the residue $\bar{u} \in \ell^\times \subset M^\times$.

Lemma 4.2. (a) *If $e = 1$ and f is even, then $N(L/\widehat{F}) \subset N(\widehat{K}/\widehat{F}) \cdot N(\ell/k)$.*
 (b) *If e is even, then $N(L/\widehat{F}) \subset N(\widehat{K}/\widehat{F}) \cdot \{1, N_{\ell/k}(-\bar{u})\}$ and $x \equiv -\bar{u} \pmod{L^{\times 2}}$.*

Proof. (a) Every nonzero element of L can be written in the form $ax^r(1+m)$ for some $a \in \ell^\times$, some $r \in \mathbb{Z}$ and some $m \in L$ such that $v(m) > 0$. We have

$$N_{L/\widehat{F}}(ax^r(1+m)) = N_{\ell/k}(a) x^{rf} N_{L/\widehat{F}}(1+m).$$

Since f is even and $N_{\widehat{K}/\widehat{F}}(\xi) = -x$ it follows that

$$x^{rf} = N_{\widehat{K}/\widehat{F}}(\xi^{rf}) \in N(\widehat{K}/\widehat{F}).$$

Moreover, Hensel's lemma shows that $1+m \in L^{\times 2}$, hence $N_{L/\widehat{F}}(1+m) \in \widehat{F}^{\times 2} \subset N(\widehat{K}/\widehat{F})$. Therefore, the norm of every nonzero element in L lies in $N(\widehat{K}/\widehat{F}) \cdot N(\ell/k)$.

(b) In this case L and M have the same residue field, hence every element $y_0 \in L$ such that $v(y_0) = 0$ can be written as $y_0 = z(1+m)$ for some $z \in M^\times$ and some $m \in L$ with $v(m) > 0$. Therefore, for every element $y \in L^\times$ there exist $z \in M^\times$, $m \in L$ with $v(m) > 0$ and $r \in \mathbb{Z}$ such that $y = z\pi^r(1+m)$. Then

$$N_{L/\widehat{F}}(y) = N_{M/\widehat{F}}(z)^e N_{M/\widehat{F}}(ux)^r N_{L/\widehat{F}}(1+m). \quad (10)$$

Since e is even, $N_{M/\widehat{F}}(z)^e \in \widehat{F}^{\times 2}$. Similarly, $N_{L/\widehat{F}}(1+m) \in \widehat{F}^{\times 2}$ because $1+m \in L^{\times 2}$ by Hensel's lemma. Moreover, $N_{M/\widehat{F}}(ux) = N_{M/\widehat{F}}(-u)(-x)^f = N_{M/\widehat{F}}(-u)N_{\widehat{K}/\widehat{F}}(\xi)^f$, hence from (10) it follows that

$$N_{L/\widehat{F}}(y) \in N_{M/\widehat{F}}(ux)^r \cdot \widehat{F}^{\times 2} \subset N_{M/\widehat{F}}(-u)^r \cdot N(\widehat{K}/\widehat{F}).$$

Since $\overline{u\bar{u}^{-1}} = 1$, Hensel's lemma shows that $u\bar{u}^{-1} \in M^{\times 2}$, hence $N_{M/\widehat{F}}(u) \equiv N_{\ell/k}(\bar{u}) \pmod{\widehat{F}^{\times 2}}$. The first statement in (b) is thus proved.

To prove the second part, consider the minimal polynomial of π over M :

$$X^e - a_1 X^{e-1} + a_2 X^{e-2} - \dots + a_e \in M[X].$$

Each coefficient a_i is a sum of products of i conjugates of π in an algebraic closure of L , hence $v(a_i) \geq i v(\pi)$. But $a_i \in M$ and $v(M^\times) = e v(\pi)\mathbb{Z}$, hence in fact $v(a_i) \geq e v(\pi)$. Moreover, $v(a_e) = e v(\pi)$ because $a_e = N_{L/M}(\pi)$, hence $v(a_i a_e^{-1}) \geq 0$ for $i = 1, \dots, e$. Therefore, taking residues in the equation

$$\frac{\pi^e}{a_e} - \frac{a_1}{a_e} \pi^{e-1} + \frac{a_2}{a_e} \pi^{e-2} - \dots + 1 = 0,$$

we obtain $\overline{\left(\frac{\pi^e}{a_e}\right)} = -1$. Note that $a_e = N_{L/M}(\pi) = xu$, hence $\overline{\left(\frac{\pi^e}{-xu}\right)} = 1$. By Hensel's lemma again, it follows that $\frac{\pi^e}{-xu} \in L^{\times 2}$, hence $-xu \in L^{\times 2}$ because e is even. Since $u \equiv \bar{u} \pmod{M^{\times 2}}$, we finally get $x \equiv -\bar{u} \pmod{L^{\times 2}}$. $\square$

We now turn to the problem mentioned at the beginning of this section, which is to compute $\text{PGU}(\widehat{B}, \widehat{\tau})/R$ in terms of (A, σ) . In view of Proposition 4.1, we assume (A, σ) and $(\widehat{B}, \widehat{\tau})$ are not hyperbolic for the rest of this section.

Lemma 4.3. $G(\widehat{B}, \widehat{\tau}) = N(\widehat{K}/\widehat{F}) \cdot G(A, \sigma)$.

Proof. Consider a Witt decomposition $(A, \sigma) \simeq (A_0, \sigma_0) \boxplus (A_1, \sigma_1)$ with σ_0 anisotropic and σ_1 hyperbolic. Then $(A_1, \sigma_1) \otimes_k (\widehat{K}, \iota)$ is hyperbolic, hence

$$G(\widehat{B}, \widehat{\tau}) = G((A_0, \sigma_0) \otimes_k (\widehat{K}, \iota)) \quad \text{and similarly} \quad G(A, \sigma) = G(A_0, \sigma_0).$$

Therefore, substituting (A_0, σ_0) for (A, σ) we may assume σ is anisotropic.

Let $g = \sum_{i=r}^{\infty} a_i \xi^i \in \text{GU}(\widehat{B}, \widehat{\tau})$, with $a_i \in A$ for all i , and $a_r \neq 0$. Because $\widehat{\tau}(g)g \in \widehat{F}^\times$ and σ is anisotropic, we have $\sigma(a_r)a_r \in k^\times$. Then $x^{-r}(\sigma(a_r)a_r)^{-1}\widehat{\tau}(g)g \in k[[x]]$, and

$$\widehat{\tau}(g)g = \sigma(a_r)a_r(-x)^r(1+m) \quad \text{for some } m \in xk[[x]].$$

Hensel's lemma yields $1+m \in \widehat{F}^{\times 2}$, hence $(-x)^r(1+m) \in N(\widehat{K}/\widehat{F})$. Since $\sigma(a_r)a_r \in G(A, \sigma)$, it follows that $G(\widehat{B}, \widehat{\tau}) \subset N(\widehat{K}/\widehat{F}) \cdot G(A, \sigma)$. The reverse inclusion is clear. $\square$

We next consider $\text{Hyp}(\widehat{B}, \widehat{\tau})$.

Lemma 4.4. *Let L be a finite-degree field extension of $\widehat{F}$ such that $(\widehat{B}, \widehat{\tau})_L$ is hyperbolic, and let ℓ be the residue field of L . The following properties hold:*

- (a) $[L : \widehat{F}]$ is even.
- (b) If L is unramified, then $N(L/\widehat{F}) \subset N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(A, \sigma)$.
- (c) If $N(L/\widehat{F}) \not\subset N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(A, \sigma)$, then there exist $\lambda \in \ell^\times$ and $g \in A_\ell$ such that $\sigma_\ell(g) = g$, $g^2 = \lambda$, and $N(L/\widehat{F}) \subset N(\widehat{K}/\widehat{F}) \cdot \{1, N_{\ell/k}(\lambda)\}$.

Proof. (a) Since $(\widehat{B}, \widehat{\tau})$ is not hyperbolic, it follows from a theorem of Bayer-Fluckiger and Lenstra [BL90, Prop. 1.2] that $(\widehat{B}, \widehat{\tau})$ remains non-hyperbolic over every odd-degree extension of $\widehat{F}$.

(b) If L is unramified, then we may identify $L = \ell((x))$. By applying Proposition 4.1 after extending scalars of A from k to ℓ , we see that $(A, \sigma)_\ell$ is hyperbolic. Therefore, $N(\ell/k) \subset \text{Hyp}(A, \sigma)$, and by Lemma 4.2(a) it follows that $N(L/\widehat{F}) \subset N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(A, \sigma)$.

(c) Let $M \subset L$ be the unramified closure of $\widehat{F}$ in L . If $(\widehat{B}, \widehat{\tau})_M$ is hyperbolic, then (b) yields $N(M/\widehat{F}) \subset N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(A, \sigma)$. But $N(L/\widehat{F}) \subset N(M/\widehat{F})$, hence this case does not arise when $N(L/\widehat{F}) \not\subset N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(A, \sigma)$. Therefore, the hypothesis implies $(\widehat{B}, \widehat{\tau})_M$ is not hyperbolic. From the theorem of Bayer-Fluckiger and Lenstra mentioned in (a), it follows that $[L : M]$ is even, hence we may apply Lemma 4.2(b) to obtain (with the notation of that lemma)

$$N(L/\widehat{F}) \subset N(\widehat{K}/\widehat{F}) \cdot \{1, N_{\ell/k}(-\bar{u})\}. \quad (11)$$

To complete the proof, we show that $\lambda = -\bar{u}$ satisfies the requirements.

First, note that $-\bar{u} \notin \ell^{\times 2}$ since otherwise (11) yields $N(L/\widehat{F}) \subset N(\widehat{K}/\widehat{F})$. Lemma 4.2(b) shows that $x \equiv -\bar{u} \pmod{L^{\times 2}}$, hence

$$\widehat{K}L \simeq L(\sqrt{-\bar{u}}) \simeq \ell(\sqrt{-\bar{u}})((\pi)).$$

After scalar extension to L , the automorphism ι of $\widehat{K}$ yields the nontrivial automorphism α of $\ell(\sqrt{-\bar{u}})((\pi))$ over $\ell((\pi))$. Therefore,

$$(\widehat{B}, \widehat{\tau})_L \simeq (A, \sigma)_\ell \otimes_\ell (\ell(\sqrt{-\bar{u}}), \alpha) \otimes_\ell \ell((\pi)).$$

Since $(\widehat{B}, \widehat{\tau})_L$ is hyperbolic, it follows from the equivalence of (c') and (d') in Proposition 4.1 that $(A, \sigma)_\ell \otimes_\ell (\ell(\sqrt{-\bar{u}}), \alpha)$ is hyperbolic. Note that we are not in the exceptional case of Theorem 3.1, for if A_ℓ is split and σ_ℓ is symplectic then $(A, \sigma)_\ell$ is hyperbolic, hence $(\widehat{B}, \widehat{\tau})_M$ is hyperbolic. Therefore, Theorem 3.1 yields an element $g \in A_\ell$ such that $\sigma_\ell(g) = g$ and $g^2 = -\bar{u}$, which completes the proof. $\square$

In order to account for case (c) of Lemma 4.4, we introduce the following group $S(A, \sigma)$:

Definition 4.5. $S(A, \sigma) \subset k^\times$ is the subgroup generated by the elements $N_{\ell/k}(\lambda)$, where ℓ is a finite-degree field extension of k and $\lambda \in \ell^\times$ is such that there exists $g \in A_\ell$ satisfying $\sigma_\ell(g) = g$ and $g^2 = \lambda$.

Note that the conditions $\sigma_\ell(g) = g$ and $g^2 = \lambda$ imply $\sigma_\ell(g)g = \lambda$, hence $\lambda \in G(A_\ell, \sigma_\ell)$. By [KMRT98, (12.21)] it follows that $N_{\ell/k}(\lambda) \in G(A, \sigma)$, hence $S(A, \sigma) \subset G(A, \sigma)$.

Theorem 4.6. *There is a canonical group isomorphism*

$$G(\widehat{B}, \widehat{\tau}) / (N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(\widehat{B}, \widehat{\tau})) \simeq G(A, \sigma) / (k^{\times 2} \cdot \text{Hyp}(A, \sigma) \cdot S(A, \sigma)).$$

Proof. Lemma 4.4 shows that $N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(\widehat{B}, \widehat{\tau}) \subset N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(A, \sigma) \cdot S(A, \sigma)$. We show that this inclusion is an equality.

Every field extension that makes (A, σ) hyperbolic also makes $(\widehat{B}, \widehat{\tau})$ hyperbolic, hence

$$\text{Hyp}(A, \sigma) \subset \text{Hyp}(\widehat{B}, \widehat{\tau}).$$

Now, assume ℓ is a field extension of k of finite degree f , and $\lambda \in \ell^\times$, $g \in A_\ell$ satisfy $\sigma_\ell(g) = g$ and $g^2 = \lambda$. If $\lambda \in \ell^{\times 2}$, then $N_{\ell/k}(\lambda) \in k^{\times 2} \subset N(\widehat{K}/\widehat{F})$. If $\lambda \notin \ell^{\times 2}$, then Theorem 3.1 shows that $(A, \sigma)_\ell \otimes (\ell(\sqrt{\lambda}), \alpha)$ is hyperbolic, where α is the nontrivial automorphism of $\ell(\sqrt{\lambda})$ over ℓ . Let $L = \ell((\pi))$ where $\pi^2 = \lambda x$. Then

$$(A, \sigma)_\ell \otimes (\ell(\sqrt{\lambda}), \alpha) \otimes_\ell L \simeq (\widehat{B}, \widehat{\tau})_L,$$

hence $(\widehat{B}, \widehat{\tau})_L$ is hyperbolic. Moreover, $N_{L/\widehat{F}}(\pi) = N_{\ell((x))/\widehat{F}}(-\lambda x) = (-x)^f N_{\ell/k}(\lambda)$. Since $N_{\widehat{K}/\widehat{F}}(\xi) = -x$, it follows that $N_{\ell/k}(\lambda) \in N(\widehat{K}/\widehat{F}) \cdot N(L/\widehat{F}) \subset N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(\widehat{B}, \widehat{\tau})$. We have thus shown $S(A, \sigma) \subset N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(\widehat{B}, \widehat{\tau})$, hence

$$N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(\widehat{B}, \widehat{\tau}) = N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(A, \sigma) \cdot S(A, \sigma).$$

On the other hand, Lemma 4.3 yields $G(\widehat{B}, \widehat{\tau}) = N(\widehat{K}/\widehat{F}) \cdot G(A, \sigma)$. Since $N(\widehat{K}/\widehat{F}) \cap k^\times = k^{\times 2}$, it follows that the inclusion $G(A, \sigma) \subset N(\widehat{K}/\widehat{F}) \cdot G(A, \sigma)$ induces an isomorphism

$$\frac{G(A, \sigma)}{k^{\times 2} \cdot \text{Hyp}(A, \sigma) \cdot S(A, \sigma)} \xrightarrow{\sim} \frac{N(\widehat{K}/\widehat{F}) \cdot G(A, \sigma)}{N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(A, \sigma) \cdot S(A, \sigma)} = \frac{G(\widehat{B}, \widehat{\tau})}{N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(\widehat{B}, \widehat{\tau})}. \quad \square$$

We conclude this section with two special cases:

Corollary 4.7. *Suppose $\deg A \equiv 2 \pmod{4}$ and σ is symplectic. Then*

$$S(A, \sigma) \subset k^{\times 2} \quad \text{and} \quad \text{PGU}(\widehat{B}, \widehat{\tau})/R = 1.$$

Proof. Let ℓ be a finite-degree field extension of k and $\lambda \in \ell^\times$ be such that $g^2 = \lambda$ for some σ_ℓ -symmetric element $g \in A_\ell$. Since σ_ℓ is symplectic and $\deg A_\ell \equiv 2 \pmod{4}$, the reduced Pfaffian characteristic polynomial of g (see [KMRT98, (2.9)]) has odd degree, hence $\lambda \in \ell^{\times 2}$. Therefore, $S(A, \sigma) \subset k^{\times 2}$, and Theorem 4.6 yields

$$G(\widehat{B}, \widehat{\tau}) / (N(\widehat{K}/\widehat{F}) \cdot \text{Hyp}(\widehat{B}, \widehat{\tau})) \simeq G(A, \sigma) / (k^{\times 2} \cdot \text{Hyp}(A, \sigma)).$$

The right side is trivial because Merkurjev has shown [Mer96, Prop. 4] that $\mathbf{PGSp}(A, \sigma)$ is stably rational. By (4), it follows that $\text{PGU}(\widehat{B}, \widehat{\tau})/R = 1$. $\square$

Recall that when the involution σ is orthogonal, we let $G^+(A, \sigma)$ denote the group of multipliers of proper similitudes, and $G^-(A, \sigma)$ the coset of multipliers of improper similitudes (if any).

Lemma 4.8. *Let ℓ be a finite-degree field extension of k . Then $N_{\ell/k}(G^+(A_\ell, \sigma_\ell)) \subset G^+(A, \sigma)$. Moreover,*

$$N_{\ell/k}(G^-(A_\ell, \sigma_\ell)) \subset \begin{cases} G^+(A, \sigma) & \text{if } [\ell : k] \text{ is even,} \\ G^-(A, \sigma) & \text{if } [\ell : k] \text{ is odd.} \end{cases}$$

Proof. Let $\mu \in G(A_\ell, \sigma_\ell)$ and let $\delta \in k^\times$ be a representative of the square class disc σ . Recall from [MT95, Th. A] or [KMRT98, (13.38)] that

$$\mu \in G^+(A_\ell, \sigma_\ell) \text{ if and only if } [(\delta, \mu)_\ell] = 0 \text{ in } \text{Br}(\ell),$$

and

$$\mu \in G^-(A_\ell, \sigma_\ell) \text{ if and only if } [(\delta, \mu)_\ell] = [A_\ell] \text{ in } \text{Br}(\ell).$$

Taking the corestriction from ℓ to k , we obtain $[(\delta, N_{\ell/k}(\mu))_k] = 0$ if $\mu \in G^+(A_\ell, \sigma_\ell)$, and $[(\delta, N_{\ell/k}(\mu))_k] = [\ell : k] \cdot [A]$ if $\mu \in G^-(A_\ell, \sigma_\ell)$. By [KMRT98, (12.21)] we already know $N_{\ell/k}(\mu) \in G(A, \sigma)$; the lemma follows. $\square$

Corollary 4.9. *Suppose $\deg A \equiv 0 \pmod{4}$ and σ is orthogonal. Then $S(A, \sigma) \subset G^+(A, \sigma)$ and there is a canonical surjective map $\varphi: \text{PGU}(\widehat{B}, \widehat{\tau})/R \rightarrow G(A, \sigma)/G^+(A, \sigma)$.*

Proof. Let ℓ be a finite-degree field extension of k and $\lambda \in \ell^\times$ be such that $g^2 = \lambda$ for some σ_ℓ -symmetric $g \in A_\ell$. Then $\lambda = \sigma_\ell(g)g$ and $\text{Nrd}_{A_\ell}(g) = (-\lambda)^{\frac{1}{2} \deg A}$. Since $\frac{1}{2} \deg A$ is even, it follows that g is a proper similitude, hence $\lambda \in G^+(A_\ell, \sigma_\ell)$. Lemma 4.8 then yields $N_{\ell/k}(\lambda) \in G^+(A, \sigma)$, hence $S(A, \sigma) \subset G^+(A, \sigma)$. On the other hand, we have $\text{Hyp}(A, \sigma) \subset G^+(A, \sigma)$ (see (2)), hence there is a canonical surjective map

$$G(A, \sigma)/(k^{\times 2} \cdot \text{Hyp}(A, \sigma) \cdot S(A, \sigma)) \rightarrow G(A, \sigma)/G^+(A, \sigma).$$

The corollary follows from Theorem 4.6. $\square$

Remark 4.10. The cohomological invariant $\theta_2: \text{PGU}(\widehat{B}, \widehat{\tau}) \rightarrow H^3(\bullet, \mu_2)$ defined in [BMT04, Prop. 11] yields a map $\theta_{2\widehat{F}}: \text{PGU}(\widehat{B}, \widehat{\tau}) \rightarrow H^3(\widehat{F}, \mu_2)$ that factors through φ . If $G^-(A, \sigma) \neq \emptyset$, its image is $\{0, (x) \cup [A]\}$, where $(x) \in H^1(\widehat{F}, \mu_2)$ is the square class of $x \in \widehat{F}^\times$, see [BMT04, Prop. 13]. Therefore, the map φ is nontrivial if and only if the map $\text{PGU}(\widehat{B}, \widehat{\tau})/R \rightarrow H^3(\widehat{F}, \mu_2)$ induced by $\theta_{2\widehat{F}}$ is nontrivial.

5. EXAMPLES OF NON-RATIONAL ADJOINT GROUPS OF TYPE ${}^2\mathbf{A}_n$

5.1. Case $n \equiv 3 \pmod{4}$. In this subsection, we prove Theorem 1.2. Let (A, σ) be a central simple algebra with orthogonal involution of degree multiple of 4 over a field k of characteristic 0, and let $F = k(x)$ be the rational function field in one indeterminate over k . We let $K = F(\sqrt{x})$, write ι for the nontrivial automorphism of K/F , and let

$$(B, \tau) = (A, \sigma) \otimes_k (K, \iota).$$

Thus, (B, τ) is a generic unitary extension of (A, σ) . We assume A is not split and disc σ is not trivial. Proposition 1.1 yields a finitely generated extension $k_1 = k(y_1, \dots, y_r)$ of k such that A_{k_1} is not split and $(A, \sigma)_{k_1}$ admits improper similitudes. We may assume k_1 and K both lie in some field extension of k and are linearly disjoint over k , so we may consider the composite field extensions

$$F_1 = k_1(x) = F(y_1, \dots, y_r) \subset K_1 = F_1(\sqrt{x}) = K(y_1, \dots, y_r).$$

Let also $\widehat{F}_1 = k_1((x))$ and $\widehat{K}_1 = \widehat{F}_1(\sqrt{x})$. Because $\deg A \equiv 0 \pmod{4}$, Corollary 4.9 yields a surjective map

$$\text{PGU}(B_{\widehat{F}_1}, \tau_{\widehat{F}_1})/R \rightarrow G(A_{k_1}, \sigma_{k_1})/G^+(A_{k_1}, \sigma_{k_1}).$$

Since A_{k_1} is not split and $(A, \sigma)_{k_1}$ admits improper similitudes, the right side is not trivial (see (1)). We have thus found an extension $\widehat{F}_1$ of F such that $\text{PGU}(B_{\widehat{F}_1}, \tau_{\widehat{F}_1})/R \neq 1$, which means that $\text{PGU}(B, \tau)$ is not R -trivial. The proof of Theorem 1.2 is thus complete.

5.2. **Case** $n \equiv 1 \pmod{4}$. We start with the following construction, which will be iterated in the sequel: Q is a central quaternion division algebra over an arbitrary field E of characteristic zero. Let $(V, h), (V', h')$ be nondegenerate skew-hermitian spaces over Q (with respect to the conjugation involution on Q). Consider the field of Laurent series in one indeterminate over E ,

$$\widehat{E} = E((t)),$$

and let $\widehat{Q} = Q_{\widehat{E}}$, $(\widehat{V}, \widehat{h}) = (V, h)_{\widehat{E}}$, $(\widehat{V}', \widehat{h}') = (V', h')_{\widehat{E}}$ be the division algebra and skew-hermitian spaces obtained by extending scalars from E to $\widehat{E}$. We may then form the following nondegenerate skew-hermitian space over $\widehat{Q}$:

$$(W, h_W) = (\widehat{V} \oplus \widehat{V}', \widehat{h} \perp \langle t \rangle \widehat{h}').$$

Proposition 5.1. (1) *If h and h' are anisotropic, then h_W is anisotropic.*

(2) *Assume h and h' are anisotropic and not similar. If there exists $\widehat{g} \in \text{End}_{\widehat{Q}} W$ such that $\text{ad}_{h_W}(\widehat{g}) = \widehat{g}$ and $\widehat{g}^2 = \lambda$ for some $\lambda \in \widehat{E}^\times$, then there exist $\lambda_0 \in E^\times$ and $g \in \text{End}_Q V$, $g' \in \text{End}_Q V'$ such that $\lambda \equiv \lambda_0 \pmod{\widehat{E}^{\times 2}}$, $\text{ad}_h(g) = g$, $\text{ad}_{h'}(g') = g'$, and $g^2 = g'^2 = \lambda_0$.*

Proof. Throughout the proof, we assume h and h' are anisotropic. Let v be the t -adic valuation on $\widehat{E}$. We also write v for the valuation on $\widehat{Q}$ extending v . Observe that every vector $x \in \widehat{V}$ can be written as a series $x = \sum_{i=r}^\infty x_i t^i$ with $x_i \in V$ for all i . For such nonzero x , define $\nu(x) = \inf\{i \mid x_i \neq 0\}$, and let $\nu(0) = \infty$. Similarly, for $x' = \sum_{j=s}^\infty x'_j t^j \in \widehat{V}'$, let $\nu'(x') = \inf\{j \mid x'_j \neq 0\}$, and $\nu'(0) = \infty$. Finally, for $x \in \widehat{V}$ and $x' \in \widehat{V}'$, let

$$\nu_W(x + x') = \min\{\nu(x), \tfrac{1}{2} + \nu'(x')\} \in \tfrac{1}{2}\mathbb{Z} \cup \{\infty\}.$$

The map $\nu_W: W \rightarrow \tfrac{1}{2}\mathbb{Z} \cup \{\infty\}$ is a v -norm on W (see [TW15, p. 83]). Since h and h' are anisotropic, it follows that $\nu(x) = \tfrac{1}{2}v(\widehat{h}(x, x))$ for all $x \in \widehat{V}$ and $\nu'(x') = \tfrac{1}{2}v(\widehat{h}'(x', x'))$ for all $x' \in \widehat{V}'$. Therefore,

$$\nu_W(w) = \tfrac{1}{2}v(h_W(w, w)) \quad \text{for all } w \in W. \quad (12)$$

It follows that h_W is anisotropic, proving (1). Moreover, it is easy to see that

$$v(h_W(w_1, w_2)) \geq \nu_W(w_1) + \nu_W(w_2) \quad \text{for all } w_1, w_2 \in W. \quad (13)$$

To prove (2), let $\widehat{g} \in \text{End}_{\widehat{Q}} W$ be such that $\text{ad}_{h_W}(\widehat{g}) = \widehat{g}$ and $\widehat{g}^2 = \lambda$. We then have $\lambda \in G(\text{End}_{\widehat{Q}} W, \text{ad}_{h_W})$, hence $v(\lambda) \in 2\mathbb{Z}$ by [QMT2, Prop. 2.3] because h and h' are not similar. We may then find $\lambda_0 \in E^\times$, $r \in \mathbb{Z}$, and $m \in \widehat{E}$ with $v(m) > 0$ such that $\lambda = \lambda_0 t^{2r}(1 + m)$. Hensel's lemma yields $1 + m \in \widehat{E}^{\times 2}$, hence $\lambda \equiv \lambda_0 \pmod{\widehat{E}^{\times 2}}$. If $\lambda_1 \in \widehat{E}^\times$ is such that $\lambda = \lambda_0 \lambda_1^2$, then substituting $\widehat{g} \lambda_1^{-1}$ for $\widehat{g}$ we may (and will) assume for the rest of the proof that $\widehat{g}^2 = \lambda_0 \in E^\times$.

The remaining claims can be established by the graded algebra technique of [QMT1, Sec. 3.2]. For the convenience of the reader, we give an alternative more elementary argument.

Since $\widehat{g} \in \text{End}_{\widehat{Q}} W$ satisfies $\text{ad}_{h_W}(\widehat{g}) = \widehat{g}$ and $\widehat{g}^2 = \lambda_0 \in E^\times$, it follows that

$$h_W(\widehat{g}(w), \widehat{g}(w)) = \lambda_0 h_W(w, w) \quad \text{for all } w \in W,$$

hence by (12) $\nu_W(\widehat{g}(w)) = \nu_W(w)$ for all $w \in W$. In particular, for $x \in V$ we have $\nu_W(\widehat{g}(x)) = 0$ unless $x = 0$, hence there exist $g(x) \in V$ and $g_+(x) \in W$ such that

$$\widehat{g}(x) = g(x) + g_+(x) \quad \text{and} \quad \nu_W(g_+(x)) > 0.$$

The elements $g(x)$ and $g_+(x)$ are uniquely determined by these conditions, and the map $g: V \rightarrow V$ is Q -linear. Similarly, for $x' \in V'$ we have $\nu_W(\widehat{g}(x')) = \tfrac{1}{2}$ if $x' \neq 0$, and we get uniquely determined elements $g'(x') \in V'$, $g'_+(x') \in W$ such that

$$\widehat{g}(x') = g'(x') + g'_+(x') \quad \text{and} \quad \nu_W(g'_+(x')) > \tfrac{1}{2}.$$

The map g' belongs to $\text{End}_Q V'$. For $x_1, x_2 \in V$ we have

$$h_W(\widehat{g}(x_1), x_2) = h(g(x_1), x_2) + h_W(g_+(x_1), x_2)$$

and (13) shows that $v(h_W(g_+(x_1), x_2)) > 0$. Therefore, letting $\mathcal{O}_{\widehat{Q}}$ denote the valuation ring of $\widehat{Q}$ and $\bar{\cdot}: \mathcal{O}_{\widehat{Q}} \rightarrow Q$ the residue map, we have for all $x_1, x_2 \in V$

$$h(g(x_1), x_2) = \overline{h_W(\widehat{g}(x_1), x_2)} \quad \text{and similarly} \quad h(x_1, g(x_2)) = \overline{h_W(x_1, \widehat{g}(x_2))}.$$

Since $\text{ad}_{h_W}(\widehat{g}) = \widehat{g}$ and $\widehat{g}^2 = \lambda_0$, it follows that $\text{ad}_h(g) = g$ and $g^2 = \lambda_0$.

Likewise, for $x'_1, x'_2 \in V'$ we have

$$h_W(\widehat{g}(x'_1), x'_2) = t h'(g'(x'_1), x'_2) + h_W(g'_+(x'_1), x'_2)$$

and (13) yields $v(h_W(g'_+(x'_1), x'_2)) > 1$, hence for $x'_1, x'_2 \in V'$,

$$h'(g'(x'_1), x'_2) = \overline{t^{-1} h_W(\widehat{g}(x'_1), x'_2)} \quad \text{and similarly} \quad h'(x'_1, g'(x'_2)) = \overline{t^{-1} h_W(x'_1, \widehat{g}(x'_2))}.$$

Since $\text{ad}_{h_W}(\widehat{g}) = \widehat{g}$ and $\widehat{g}^2 = \lambda_0$, it follows that $\text{ad}_{h'}(g') = g'$ and $g'^2 = \lambda_0$. $\square$

For the application in Theorem 5.3 below, we need to show that Proposition 5.1 still holds after an odd-degree scalar extension. Let L be an odd-degree field extension of $\widehat{E}$. Extending scalars from $\widehat{E}$ to L , we obtain the quaternion division algebra $\widehat{Q}_L$ over L , the $\widehat{Q}_L$ -vector space W_L and the skew-hermitian form $(h_W)_L$ on W_L .

Corollary 5.2. *Assume h and h' are not similar and anisotropic. If there exists $\widehat{g} \in \text{End}_{\widehat{Q}_L} W_L$ such that $\text{ad}_{(h_W)_L}(\widehat{g}) = \widehat{g}$ and $\widehat{g}^2 = \lambda$ for some $\lambda \in L^\times$, then there exists an odd-degree field extension L_0 of E contained in L , a scalar $\lambda_0 \in L_0^\times$, and maps $g \in \text{End}_{Q_{L_0}}(V)_{L_0}$, $g' \in \text{End}_{Q_{L_0}}(V')_{L_0}$ such that $\lambda \equiv \lambda_0 \pmod{L^{\times 2}}$, $\text{ad}_{h_{L_0}}(g) = g$, $\text{ad}_{h'_{L_0}}(g') = g'$, and $g^2 = g'^2 = \lambda_0$.*

Proof. The t -adic valuation on $\widehat{E}$ extends uniquely to L because $\widehat{E}$ is complete. Let L_0 be the residue field of L and $\pi \in L$ be a uniformizer. Since $\text{char}(E) = 0$ we may identify $L = L_0((\pi))$, see [Ser68, Ch. II, Th. 2]. Let $e = [L : L_0((t))]$ and $f = [L_0 : E]$ be the ramification index and residue degree. Since $ef = [L : \widehat{E}]$ is odd, both e and f are odd. We have $v(\pi^e t^{-1}) = 0$, hence there exist $u \in L_0^\times$ and $m \in L$ with $v(m) > 0$ such that

$$\pi^e = tu(1 + m).$$

Now, $1 + m \in L^{\times 2}$ by Hensel's lemma, and e is odd, hence the last equation yields $t \equiv \pi u \pmod{L^{\times 2}}$. Therefore,

$$(h_W)_L \simeq \widehat{h}_L \perp \langle \pi u \rangle \widehat{h}'_L.$$

Note $\widehat{h}_L = (h_{L_0})_{L_0((\pi))}$ and $\langle u \rangle \widehat{h}'_L = (\langle u \rangle h'_{L_0})_{L_0((\pi))}$. Since f is odd, the anisotropic forms h and h' remain anisotropic under scalar extension to L_0 by a theorem of Parimala–Sridharan–Suresh [PSS01, Th. 3.5]. Moreover, since h and h' are not similar, a transfer argument due to Lewis [Lew00, Prop. 10] shows that h_{L_0} and h'_{L_0} are not similar, and therefore h_{L_0} and $\langle u \rangle h'_{L_0}$ are not similar either. Thus, we are in a position to apply Proposition 5.1 with L instead of $\widehat{E}$: if there exists $\widehat{g} \in \text{End}_{\widehat{Q}_L} W_L$ as in the statement, then we may find $\lambda_0 \in L_0^\times$ and g, g' as required. (Note that h'_{L_0} and $\langle u \rangle h'_{L_0}$ have the same adjoint involution.) $\square$

Iterating the Laurent series construction, we apply Corollary 5.2 inductively to the following situation: let $n \geq 2$ be an arbitrary integer, let Q be a central quaternion division algebra over an arbitrary field k_0 of characteristic zero, and let $q_1, \dots, q_n \in Q$ be nonzero pure quaternions. Let $a_i = q_i^2 \in k_0^\times$. Consider the field of iterated Laurent series in n indeterminates

$$k = k_0((t_1)) \dots ((t_n))$$

and the orthogonal involution σ on $A = M_n(Q_k)$ adjoint to the skew-hermitian form

$$h = \langle t_1 q_1, \dots, t_n q_n \rangle.$$

Theorem 5.3. *Suppose the skew-hermitian forms $\langle q_1 \rangle$ and $\langle q_2 \rangle$ are not similar. If ℓ is an odd-degree field extension of k and $g \in A_\ell$ is such that $\sigma_\ell(g) = g$ and $g^2 = \lambda$ for some $\lambda \in \ell^\times \setminus \ell^{\times 2}$, then there exists $\mu \in k_0^\times$ such that*

$$Q \simeq (a_1, \mu)_{k_0} \simeq \dots \simeq (a_n, \mu)_{k_0}. \quad (14)$$

Proof. For $i = 1, \dots, n$, let $E_i = k_0((t_1)) \dots ((t_i))$. Consider the following skew-hermitian forms over Q_{E_i} :

$$h_i = \langle t_1 q_1, \dots, t_i q_i \rangle \quad \text{and} \quad h'_i = \langle q_{i+1} \rangle \quad \text{for } i = 1, \dots, n-1.$$

Let V_i and V'_i be the Q_{E_i} -vector spaces underlying h_i and h'_i respectively. The form h'_i is clearly anisotropic, and Proposition 5.1(1) applied inductively shows that h_i is anisotropic. If $i = 1$ the forms h_i and h'_i are not similar by hypothesis; if $i \geq 2$ they are not similar because they do not have the same dimension.

Suppose ℓ is an odd-degree field extension of $k = E_n$ and $g \in A_\ell$ is as in the statement of the theorem. Corollary 5.2 (with $L = \ell$, $E = E_{n-1}$, and $\widehat{E} = E_n$) yields an odd-degree field extension ℓ_{n-1} of E_{n-1} contained in ℓ , a scalar $\lambda_{n-1} \in \ell_{n-1}^\times$ such that $\lambda_{n-1} \equiv \lambda \pmod{\ell^{\times 2}}$ and maps $g_{n-1} \in \text{End}_{Q_{\ell_{n-1}}}(V_{n-1})_{\ell_{n-1}}$, $g'_{n-1} \in \text{End}_{Q_{\ell_{n-1}}}(V'_{n-1})_{\ell_{n-1}}$, symmetric under $\text{ad}(h_{n-1})$ and $\text{ad}(h'_{n-1})$ respectively, such that $g_{n-1}^2 = g'^2_{n-1} = \lambda_{n-1}$. Applying again Corollary 5.2 (with $L = \ell_{n-1}$ and $\widehat{g} = g_{n-1}$), we obtain an odd-degree field extension ℓ_{n-2} of E_{n-2} contained in ℓ_{n-1} , a scalar $\lambda_{n-2} \in \ell_{n-2}^\times$ such that $\lambda_{n-2} \equiv \lambda_{n-1} \pmod{\ell_{n-1}^{\times 2}}$ and $g_{n-2} \in \text{End}_{Q_{\ell_{n-2}}}(V_{n-2})_{\ell_{n-2}}$, $g'_{n-2} \in \text{End}_{Q_{\ell_{n-2}}}(V'_{n-2})_{\ell_{n-2}}$, symmetric under $\text{ad}(h_{n-2})$ and $\text{ad}(h'_{n-2})$, such that $g_{n-2}^2 = g'^2_{n-2} = \lambda_{n-2}$. Repeating the procedure as many times as needed, we finally have field extensions

$$\ell_1 \subset \ell_2 \subset \dots \subset \ell_{n-1} \subset \ell_n = \ell,$$

scalars $\lambda_i \in \ell_i^\times$ for $i = 1, \dots, n-1$ and $\lambda_n = \lambda$ such that

$$\lambda_i \equiv \lambda_{i+1} \pmod{\ell_{i+1}^{\times 2}} \quad \text{for } i = 1, \dots, n-1, \quad (15)$$

and maps

$$g_i \in \text{End}_{Q_{\ell_i}}(V_i)_{\ell_i}, \quad g'_i \in \text{End}_{Q_{\ell_i}}(V'_i)_{\ell_i} \quad \text{for } i = 1, \dots, n-1,$$

each symmetric under the adjoint involution of the corresponding skew-hermitian form, such that

$$g_1^2 = \lambda_1 \quad \text{and} \quad g_i'^2 = \lambda_i \quad \text{for } i = 1, \dots, n-1.$$

Note that $V_1, V'_1, V'_2, \dots, V'_{n-1}$ are 1-dimensional, hence using bases we may identify $\text{End}_{Q_{\ell_1}}(V_1)_{\ell_1} = Q_{\ell_1}$ and $\text{End}_{Q_{\ell_i}}(V'_i)_{\ell_i} = Q_{\ell_i}$ for $i = 1, \dots, n-1$. From (15) it follows that $\lambda_i \equiv \lambda \pmod{\ell^{\times 2}}$, hence $\lambda_i \notin \ell_i^{\times 2}$ for $i = 1, \dots, n-1$. Therefore, $g_1, g'_1, \dots, g'_{n-1}$ are pure quaternions. The condition that g_1 is symmetric under $\text{ad}(h_1)$ then means that $g_1 q_1 = -q_1 g_1$, hence q_1, g_1 are part of a quaternion base of Q_{ℓ_1} and therefore

$$Q_{\ell_1} \simeq (a_1, \lambda_1)_{\ell_1}.$$

Similarly, because g'_i is symmetric under $\text{ad}(h'_i)$ we have

$$Q_{\ell_i} \simeq (a_{i+1}, \lambda_i)_{\ell_i} \quad \text{for } i = 1, \dots, n-1.$$

Extending scalars to ℓ and using $\lambda_i \equiv \lambda \pmod{\ell^{\times 2}}$ for $i = 1, \dots, n-1$, we see that

$$Q_\ell \simeq (a_1, \lambda)_\ell \simeq \dots \simeq (a_n, \lambda)_\ell.$$

Taking the corestriction from ℓ to k , we obtain since $[\ell : k]$ is odd

$$Q_k \simeq (a_1, N_{\ell/k}(\lambda))_k \simeq \dots \simeq (a_n, N_{\ell/k}(\lambda))_k. \quad (16)$$

Recall from [Lam05, Ch. VI, Cor. 1.3] that each element in $k^\times$ is in the coset of some monomial $t_1^{\varepsilon_1} \dots t_n^{\varepsilon_n}$ with each $\varepsilon_i \in \{0, 1\}$ modulo $k_0^\times k^{\times 2}$, hence we may find $\mu \in k_0^\times$ and $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$ such that

$$N_{\ell/k}(\lambda) \equiv \mu t_1^{\varepsilon_1} \dots t_n^{\varepsilon_n} \pmod{k^{\times 2}}.$$

But the Brauer class of Q_k is unramified for the $(t_1, \dots, t_n)$ -adic valuation on k , hence we must have $\varepsilon_1 = \dots = \varepsilon_n = 0$. From (16), it follows that μ satisfies (14). $\square$

Corollary 5.4. *With the same hypotheses and notation as in Theorem 5.3, suppose there does not exist any $\mu \in k_0^\times$ satisfying (14). Then $S(A, \sigma) \subset G^+(A, \sigma)$.*

Proof. If $\deg A \equiv 0 \pmod{4}$, the inclusion holds without any hypothesis on Q by Corollary 4.9. For the rest of the proof, we may thus assume $\deg A \equiv 2 \pmod{4}$, which means that n is odd. Let ℓ be a finite-degree field extension of k and $\lambda \in \ell^\times$ such that $g^2 = \lambda$ for some σ_ℓ -symmetric $g \in A_\ell$. As $\lambda = \sigma_\ell(g)g$ and $\text{Nrd}_{A_\ell}(g) = (-\lambda)^n$, it follows that g is an improper similitude, hence $\lambda \in G^-(A_\ell, \sigma_\ell)$. Since A_ℓ is not split we must have $\lambda \notin \ell^{\times 2}$, hence Theorem 5.3 shows that $[\ell : k]$ is even because there is no $\mu \in k_0^\times$ satisfying (14). Lemma 4.8 then yields $N_{\ell/k}(\lambda) \in G^+(A, \sigma)$. $\square$

Example 5.5. Let $k_0 = k_*(a_1, a_2)$ be the field of rational functions in two indeterminates over an arbitrary field k_* of characteristic zero. The quaternion algebra $Q = (a_1, a_2)_{k_0}$ contains pure quaternions q_1, q_2, q_3 satisfying

$$q_1^2 = a_1, \quad q_2^2 = a_2, \quad q_3^2 = a_1((1 - a_1)^2(1 + a_2)^2 - 4(1 - a_1)a_2),$$

see [QMT1, Ex. 3.12]. Let $a_3 = q_3^2$. It is shown in [QMT1, Ex. 3.12] that there is no $\mu \in k_0^\times$ such that $Q \simeq (a_1, \mu)_{k_0} \simeq (a_2, \mu)_{k_0} \simeq (a_3, \mu)_{k_0}$. Note that the forms $\langle q_1 \rangle$ and $\langle q_2 \rangle$ are not similar since they do not have the same discriminant. Therefore, for arbitrary $n \geq 3$ the construction before Theorem 5.3 with $q_3 = q_4 = \dots = q_n$ yields by Corollary 5.4 an algebra with orthogonal involution (A, σ) of degree $2n$ such that $S(A, \sigma) \subset G^+(A, \sigma)$. For the completion $(\widehat{B}, \widehat{\tau})$ of the generic unitary extension (B, τ) as in (9) we then have by (4) and Theorem 4.6 a canonical surjective map $\varphi: \text{PGU}(\widehat{B}, \widehat{\tau})/R \rightarrow G(A, \sigma)/G^+(A, \sigma)$ as in Corollary 4.9. If n is odd and $-1 \in k_*^{\times 2}$, it is shown in [QMT1, Cor. 3.13] that (A, σ) admits improper similitudes. Since A is not split it follows that $G(A, \sigma) \neq G^+(A, \sigma)$ (see (1)), hence $\text{PGU}(\widehat{B}, \widehat{\tau})/R \neq 1$. Therefore, the group $\mathbf{PGU}(B, \tau)$ is not R -trivial since $\mathbf{PGU}(B, \tau)(\widehat{F}) = \text{PGU}(\widehat{B}, \widehat{\tau})$. Note that $\mathbf{PGU}(B, \tau)$ can be defined over the field $k_*(a_1, a_2, t_1, \dots, t_n, x)$ of rational functions in $n + 3$ variables over an arbitrary field of characteristic zero.

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FACULTÉ DES SCIENCES ET TECHNIQUES DE BAMAKO, BP: E3206 BAMAKO, MALI AND DEPARTEMENT WISKUNDE–INFORMATICA, UNIVERSITEIT ANTWERPEN, BELGIUM
E-mail address: Barry.Demba@gmail.com

UNIVERSITÉ CATHOLIQUE DE LOUVAIN, ICTEAM INSTITUTE, AVENUE G. LEMAÎTRE 4, BOX L4.05.01, B-1348 LOUVAIN-LA-NEUVE, BELGIUM.
E-mail address: Jean-Pierre.Tignol@uclouvain.be