

TOPOLOGICAL DEGREE METHODS FOR SOME NONLINEAR PROBLEMS

Cristian Bereanu

TOPOLOGICAL DEGREE METHODS FOR SOME
NONLINEAR PROBLEMS
Cristian Bereanu

Dissertation présentée en séance publique, le 7 décembre 2006, au
Département de Mathématique de la Faculté des Sciences de l'Université
catholique de Louvain, à Louvain-la-Neuve, en vue de l'obtention du
grade de Docteur en Sciences.

PROMOTEUR: Jean Mawhin

JURY:

George Dină (Universitatea din București, Roumanie)

Christian Fabry (UCL)

Patrick Habets (UCL)

Jean Mawhin (UCL)

Michel Willem (UCL)

Fabio Zanolin (Università di Udine, Italie)

Contents

Acknowledgement	3
Summary	5
Introduction	7
Chapter I. Continuation theorems	21
1. Mawhin's continuation theorem in finite dimension	21
2. Mawhin's continuation theorem in infinite dimension	22
3. Capietto-Mawhin-Zanolin continuation theorem	26
Chapter II. Nonlinear difference equations	29
1. One-side bounded nonlinearity	29
2. Upper and lower solutions for difference equations	43
3. Ambrosetti-Prodi type multiplicity results	55
Chapter III. Second order differential equations with ϕ-Laplacian	63
1. Notation and preliminaries	63
2. Forced ϕ -Laplacian with various boundary conditions	67
3. Fixed point formulations	71
4. Homotopy to the averaged nonlinearity	74
5. Villari type nonlinearities	76
6. Upper and lower solutions and degree	89
7. Ambrosetti-Prodi type multiplicity results	95
8. Singular nonlinearities	101
Chapter IV. Delay-Lotka-Volterra systems	107
1. Competition systems with Zanolin condition	107
2. Competition systems with May-Leonard condition	112
3. Delay-prey-predator systems	120
Chapter V. Multiplicity results for superlinear planar systems	127
1. A first result	127

2. A second result	130
Bibliography	137

Acknowledgement

It is a pleasure for me to thank my thesis advisor, Professor Jean Mawhin, for his constant help and encouragement during this four years. The completion of this thesis was not possible without his invaluable support and his precious ideas.

I am grateful to Professor George Dincă for guiding my first steps into the beautiful world of Nonlinear Functional Analysis and for having confidence in me.

I am also grateful to Professors Patrick Habets and Michel Willem for their interest for my work and to the other members of the committee, Professors Christian Fabry and Fabio Zanolin. My thanks also go to Professor Bevan Thompson for interesting discussions.

I thank also to my colleagues for their warm hospitality and especially to my friends Vincent Bouchez, Julien Federinov and Jean van Schaftingen.

Finally, I thank my wife Dana for her patience and love and to our families.

Summary

Using topological degree methods, we give some existence and multiplicity results for nonlinear differential or difference equations. In Chapter 1 some continuation theorems are presented. Chapter 2 deal with nonlinear difference equations. Using Brouwer degree we obtain upper and lower solutions theorems, Ambrosetti and Prodi type results and sharp existence conditions for nonlinearities which are bounded from below or from above. In Chapter 3, using Leray-Schauder degree, we give various existence and multiplicity result for second order differential equations with ϕ -Laplacian. Such equations are in particular motivated by the one-dimensional mean curvature problems and by the acceleration of a relativistic particle of mass one at rest moving on a straight line. In Chapter 4, using Mawhin continuation theorem, sufficient conditions are obtained for the existence of positive periodic solutions for delay Lotka-Volterra systems. In the last chapter of this work we prove some results concerning the multiplicity of solutions for a class of super-linear planar systems. The results of Chapters 2 and 3 are joint work with Prof. Jean Mawhin.

Introduction

The methods we deal with in the present thesis originated long ago. They date back to C.F. GAUSS, L. KRONECKER, H. POINCARÉ and L.E.J. BROUWER who developed the topological theory of continuous mappings in finite dimensional spaces. This theory is called *Brouwer degree* and is one of the main tools used in this work. For a very nice presentation of the Brouwer degree see [59].

The topological theory in infinite dimensional spaces has been initiated in a celebrated paper published in 1922 by G.D. BIRKHOFF and O.D. KELLOGG [18] who extended to some functions spaces the famous *Brouwer fixed point theorem*. They proved the existence of a least one fixed point for continuous mappings from a convex compact subset of $C([a, b])$ into itself. The Birkhoff-Kellogg fixed point theorem has been extended by J. SCHAUDER [62] to the case of a continuous operator T mapping into itself a convex compact subset of a Banach space.

In 1933, SCHAUDER got the opportunity to meet J. LERAY in Paris, and a second important period in infinite dimensional topology started from their collaboration. LERAY and SCHAUDER immediately realized that the topology of completely continuous perturbations of identity in a Banach space was the setting to develop Leray's continuation method introduced in his thesis, and in particular to liberate it from unnecessary uniqueness and regularity assumptions. Hence, LERAY and SCHAUDER, in their fundamental paper [43], extend the Brouwer degree to compact perturbations of the identity in a Banach space. The fundamental ideas of the *Leray-Schauder degree* are explained by LERAY and SCHAUDER themselves:

On peut [...] attacher à l'ensemble des solutions de certaines équations fonctionnelles non linéaires un entier positif, négatif ou nul, l'indice total, qui reste invariant quand l'équation varie continûment et que les solutions restent bornées dans leur ensemble; les équations en question sont du type

$$(0.1) \quad x - \mathcal{F}(x) = 0,$$

où $\mathcal{F}(x)$ est complètement continue, [...] d'où résulte un procédé très général permettant d'obtenir des théorèmes d'existence: soit une équation

du type (0.1). Supposons qu'on la modifie continûment sans qu'elle cesse d'appartenir au type (0.1) et de telle sorte que l'ensemble de ses solutions reste borné [...]; supposons qu'on la transforme ainsi en une équation résoluble $x - \mathcal{F}_0(x) = 0$ et que l'on constate que l'indice total des solutions de cette dernière équation diffère de zéro. Alors l'indice total des solutions de l'équation primitive diffère aussi de zéro; elle admet donc au moins une solution.

The Leray-Schauder degree will be applied by J. LERAY, J. SCHAUDER, E. ROTHE, C.L. DOLPH, M.A. KRASNOSELS'KII and others to various problems for partial differential equations and the second half of the past century will see a tremendous development of the applications of the Leray-Schauder degree to nonlinear problems. For an introduction in the Leray-Schauder degree theory see [25], [53].

An important class of nonlinearities considered in this thesis is the class of so called *Villari type nonlinearities* introduced in 1966 by G. VILLARI [65].

In [65] the existence of T-periodic solutions is studied for the equation

$$(0.2) \quad u''' + au'' + bu' + f(t, u, u', u'') = 0,$$

where $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ is a bounded continuous function, T-periodic with respect to t and satisfying the following condition.

(Vil) There exists $R > 0$ such that

$$(0.3) \quad \begin{aligned} \int_0^T f(t, u(t), u'(t), u''(t)) dt &\geq 0 \quad \text{if } u_L \geq R, \\ \int_0^T f(t, u(t), u'(t), u''(t)) dt &\leq 0 \quad \text{if } u_M \leq -R. \end{aligned}$$

Moreover the constants a, b are such that $b < 0$ or $ab \neq 0$. If the above conditions are satisfied, then (0.2) has a least one T-periodic solution. To prove this result, VILLARI use the Leray-Schauder degree. Let us consider a bounded continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ and $e : \mathbb{R} \rightarrow \mathbb{R}$ a T-periodic, continuous function satisfying the condition

$$(0.4) \quad \limsup_{u \rightarrow -\infty} g(u) < \frac{1}{T} \int_0^T e(s) ds < \liminf_{u \rightarrow +\infty} g(u).$$

It is clear that we can apply the above existence result for $f = g - e$.

In 1968, A.C. LAZER [41] independently considered the existence of T-periodic solutions of second order equations of the form

$$(0.5) \quad u'' + cu' + g(u) = e(t),$$

where $c \in \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded continuous function and $e : \mathbb{R} \rightarrow \mathbb{R}$ a T -periodic, continuous function. If condition (0.4) holds, then (0.5) has at least one T -periodic solution. This is a (non explicit) special case of the existence result in [41]. To prove this result, LAZER applied, in a technically sophisticated way, Schauder theorem to an equivalent fixed point problem in the space of continuous, T -periodic functions.

In 1972 Lazer's result is extended by J. MAWHIN [51] using a continuation theorem introduced in [50] which is a particular case of *Mawhin's continuation theorem* [52]. We state and prove this abstract continuation theorem in Chapter 1. The proof will be based upon Brouwer degree in the finite dimensional case and upon Leray-Schauder degree in the infinite dimensional case.

In Chapter 2 we use Mawhin's continuation theorem in finite dimension to prove existence results for *nonlinear difference equations* with Villari type nonlinearities. Let $n \in \mathbb{N}$ fixed and $(x_1, \dots, x_n) \in \mathbb{R}^n$. Define $(Dx_1, \dots, Dx_{n-1}) \in \mathbb{R}^{n-1}$ and $(D^2x_2, \dots, D^2x_{n-1}) \in \mathbb{R}^{n-2}$ by

$$\begin{aligned} Dx_m &= x_{m+1} - x_m, & (1 \leq m \leq n-1) \\ D^2x_m &= x_{m+1} - 2x_m + x_{m-1}, & (2 \leq m \leq n-1). \end{aligned}$$

Let f_m be continuous functions ($2 \leq m \leq n-1$). Consider the periodic boundary value problem

$$\begin{aligned} D^2x_m + f_m(x_m) &= 0 & (2 \leq m \leq n-1), \\ (0.6) \quad x_1 &= x_n, \quad Dx_1 = Dx_{n-1}. \end{aligned}$$

In Chapter 2 we prove the following result obtained jointly with MAWHIN in [12].

Theorem 0.1. *Suppose that the functions f_m ($2 \leq m \leq n-1$) are all bounded from below or all bounded from above by c , and that for some $R > 0$ and $\varepsilon \in \{-1, 1\}$,*

$$\begin{aligned} \varepsilon \sum_{m=2}^{n-1} f_m(x_m) &\geq 0 & \text{whenever} & \min_{2 \leq j \leq n-1} x_j \geq R \\ (0.7) \quad \varepsilon \sum_{m=2}^{n-1} f_m(x_m) &\leq 0 & \text{whenever} & \max_{2 \leq j \leq n-1} x_j \leq -R. \end{aligned}$$

Then problem (0.6) has at least one solution x .

Analogous results hold also for first order difference equations and for second order difference equations with Dirichlet boundary conditions (see Chapter 2 and our papers [11], [12], [14]).

Existence results for *nonlinear second order differential equations with ϕ -Laplacian* and Villari type nonlinearities are given in Chapter 3. Let us consider equations of the form

$$(0.8) \quad (\phi(u'))' = f(t, u, u'), \quad l(u, u') = 0$$

where $l(u, u') = 0$ denotes the periodic, Neumann or Dirichlet boundary conditions on $[0, T]$, $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$) or $\phi :]-a, a[\rightarrow \mathbb{R}$ ($0 < a < +\infty$) is a homeomorphism such that $\phi(0) = 0$, and $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous. Of course, a *solution* of (0.8) is a function $u \in C^1([0, T])$ such that $\phi \circ u' \in C^1([0, T])$. Interesting models are given by

$$\phi(s) = |s|^{p-2}s \quad (p > 1), \quad \phi(s) = \frac{s}{\sqrt{1+s^2}} \quad \text{and} \quad \phi(s) = \frac{s}{\sqrt{1-s^2}},$$

where, in the first case $(|u'|^{p-2}u')'$ is the p -Laplacian operator, in the second case $(\frac{u'}{\sqrt{1+u'^2}})'$ is the one-dimensional mean-curvature operator and in the last case $(\frac{u'}{\sqrt{1-u'^2}})'$ can be seen as the acceleration of a relativistic particle of mass one at rest moving on a straight line (with the velocity of light normalized to one). Applications of topological degree methods to equations with p -Laplacian can be found in [27].

If $\phi :]-a, a[\rightarrow \mathbb{R}$ ($0 < a \leq +\infty$), following R. MANÁSEVICH and J. MAWHIN [48], the various boundary value problems are reduced to the search of fixed points for some operators defined on the whole space X of function $u \in C^1([0, T])$ such that $l(u, u') = 0$. Those operators are completely continuous, and the Leray-Schauder degree can be used. Using *Manásevich-Mawhin continuation theorem* [47] (this is an adaptation of Mawhin continuation theorem to the ϕ -Laplacian case), we prove the following existence result (see our papers [9], [15]).

Theorem 0.2. *Let $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$) be a homeomorphism such that $\phi(0) = 0$. Assume that f satisfies the following conditions.*

- (1) *There exists a continuous function c on $[0, T]$ such that $\|c^-\|_1 < \frac{a}{2}$ and*

$$(0.9) \quad f(t, u, v) \geq c(t)$$

for all $(t, u, v) \in [0, T] \times \mathbb{R}^2$.

- (2) *There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$(0.10) \quad \begin{aligned} \epsilon \int_0^T f(t, u(t), u'(t)) dt &> 0 \quad \text{if } u_L \geq R, \quad \|u'\|_\infty \leq M, \\ \epsilon \int_0^T f(t, u(t), u'(t)) dt &< 0 \quad \text{if } u_M \leq -R, \quad \|u'\|_\infty \leq M, \end{aligned}$$

where $M = \max\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\}$.

Then (0.8) has at least one solution u if $l(u, u') = 0$ denotes the periodic or Neuman condition.

In a joint work with H. B. THOMPSON [16], we have obtained similar results for difference equations involving the discrete ϕ -Laplacian.

It is interesting to remark that, for $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$), in contrast to the periodic and Neumann cases, the solvability of the Dirichlet problem with bounded right-hand side f does not require any sign condition upon f .

On the other hand, for $\phi :]-a, a[\rightarrow \mathbb{R}$ ($0 < a < +\infty$), a somewhat surprising result is that the Dirichlet problem

$$(\phi(u'))' = f(t, u, u'), \quad u(0) = 0 = u(T)$$

is solvable for *each* right-hand member f (see Chapter 3 and our paper [13]). This is not the case for the other boundary conditions, for which we prove existence results when the right-hand member f only satisfies some sign conditions similar with (0.10).

A powerful tool for solving nonlinear boundary value problems is *the method of lower and upper solutions*. This method originated in a two papers of SCORZA DRAGONI [28], [29]. The method of *lower and upper solution* for the *two-point boundary value problem*

$$(0.11) \quad u'' + f(t, u, u') = 0, \quad u(a) = A, u(b) = B,$$

has been put by M. NAGUMO [61] in 1937 in an almost definitive way by proving, using the shooting method, the existence of at least one solution for (0.11) when there exist two functions α and β of class C^2 such that $\alpha(t) < \beta(t)$ over $[a, b]$ and the following conditions are satisfied:

i) f is continuous as well $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ over the cylinder

$$C = \{(t, u, v) : t \in [a, b], \alpha(t) \leq u \leq \beta(t), v \in \mathbb{R}\};$$

ii) $|f(t, u, v)| \leq \varphi(|v|)$ for all $(t, u, v) \in C$ and some positive continuous function φ such that

$$(0.12) \quad \int_0^\infty \frac{s ds}{\varphi(s)} = +\infty;$$

iii) $\alpha''(t) + f(t, \alpha(t), \alpha'(t)) > 0$, $\beta''(t) + f(t, \beta(t), \beta'(t)) < 0$ for all $t \in [a, b]$;

iv) $\alpha(a) \leq A \leq \beta(a)$, $\alpha(b) \leq B \leq \beta(b)$.

Condition (0.12) is called *Nagumo condition*.

The concept of *lower* and *upper solution* for *second order periodic boundary value problems* has been introduced in 1963 by H.W. KNOBLOCH [39]. A Lipschitz condition was necessary in establishing the existence results in [39] because the method of proof depended on results previously developed by L. CESARI [21] and H.W. KNOBLOCH [38]. Using a result concerning the existence of solutions of two-point boundary value problem (0.11) due to L. JACKSON and K. SCHRADER, K. SCHMITT [63] improved the results of KNOBLOCH, in the sense that a Lipschitz condition is no longer required.

Consider the first order periodic boundary value problem

$$(0.13) \quad x' + f(t, x) = 0, \quad x(0) = x(T),$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Definition 0.3. α (resp. β) $\in C^1([0, T])$ is a lower solution (resp. upper solution) of problem (0.13) if

$$\begin{aligned} \alpha'(t) + f(t, \alpha(t)) &\geq 0 \quad (t \in [0, T]), \quad \alpha(0) \geq \alpha(T) \\ (\text{resp. } \beta'(t) + f(t, \beta(t)) &\leq 0 \quad (t \in [0, T]), \quad \beta(0) \leq \beta(T)). \end{aligned}$$

The basic existence theorem of the method of upper and lower solutions for (0.13) goes as follows.

Theorem 0.4. If problem (0.13) has a lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$ (resp. $\beta(t) \leq \alpha(t)$ for all $t \in [0, T]$), then problem (0.13) has at least one solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [0, T]$ (resp. $\beta(t) \leq x(t) \leq \alpha(t)$ for all $t \in [0, T]$).

The above theorem concerning *lower* and *upper solutions* for *first order periodic boundary value problems* has been proved in 1962 by H.W. KNOBLOCH [37], using a combination of Wazewski's method and Miranda's theorem. A Lipschitz condition upon f was necessary. In 1972 MAWHIN improved the above result, showing that the Lipschitz condition is unnecessary. A simpler proof of this result, based upon Schauder's fixed point theorem is given in [56] (see also Chapter 2).

In a paper with MAWHIN [10], we give a method of upper and lower solutions for the periodic boundary problem for nonlinear difference equations of the first order. If $n \geq 2$ and $f_m : \mathbb{R} \rightarrow \mathbb{R}$ are continuous ($1 \leq m \leq n-1$), one considers the periodic boundary value problem

$$(0.14) \quad Dx_m + f_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n.$$

Definition 0.5. $\alpha = (\alpha_1, \dots, \alpha_n)$ (respectively $\beta = (\beta_1, \dots, \beta_n)$) is called a lower solution (resp. upper solution) for (0.14) if the inequalities

$$\begin{aligned} D\alpha_m + f_m(\alpha_m) &\geq 0, \quad (1 \leq m \leq n-1), \quad \alpha_1 \geq \alpha_n \\ (\text{resp. } D\beta_m + f_m(\beta_m) &\leq 0, \quad (1 \leq m \leq n-1), \quad \beta_1 \leq \beta_n) \end{aligned}$$

hold.

Theorem 0.6. If (0.14) has a lower solution $\alpha = (\alpha_1, \dots, \alpha_n)$ and an upper solution $\beta = (\beta_1, \dots, \beta_n)$ such that $\alpha_m \leq \beta_m$ ($1 \leq m \leq n-1$), then (0.14) has a solution $x = (x_1, \dots, x_n)$ such that $\alpha_m \leq x_m \leq \beta_m$ ($1 \leq m \leq n-1$).

One immediately notices that, in contrast to Theorem 0.4, Theorem 0.6 does not cover the case where $\alpha_m \geq \beta_m$ ($1 \leq m \leq n-1$). In Chapter 2 (see also [10]) one constructs counterexamples to show that, in contrast to ordinary differential case, the lower solution has to be smaller than the upper solution in the case of difference equations, to make the method conclusive. In Chapter 2 (see [11], [12]) we give also an *upper and lower solutions method for second order difference equations with periodic or Dirichlet conditions*.

On the other hand, *upper and lower solutions theorems for nonlinear second order differential equations with ϕ -Laplacian* are given in Chapter 3 (see also our papers [13], [15]). Our results go as follows.

Let $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$) be an increasing homeomorphism such that $\phi(0) = 0$. Consider the periodic boundary value problem

$$(0.15) \quad (\phi(u'))' = f(t, u), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Definition 0.7. A lower solution α (resp. upper solution β) of (0.15) is a function such that $\alpha \in C_{per}^1, \phi(\alpha') \in C^1$ (resp. $\beta \in C_{per}^1, \phi(\beta') \in C^1$) and

$$(0.16) \quad (\phi(\alpha'(t)))' \geq f(t, \alpha(t)) \quad (\text{resp.} \quad (\phi(\beta'(t)))' \leq f(t, \beta(t)))$$

for all $t \in [0, T]$. Such a lower or upper solution will be called *strict* if the inequality (0.16) is strict for all $t \in [0, T]$.

Theorem 0.8. Suppose that (0.15) has a lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$. If there exists a continuous function c on $[0, T]$ such that $\|c^-\|_1 < \frac{a}{2}$ and

$$(0.17) \quad f(t, u) \geq c(t), \quad \text{for all } (t, u) \in [0, T] \times [\alpha_L, \beta_M],$$

then (0.15) has a solution u such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$. Moreover, if α and β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$.

A more general result holds when $\phi :] - a, a[\rightarrow \mathbb{R}$ ($a \neq +\infty$). In this case, in contrast to the classical case, no Nagumo-like growth condition for the dependence of $f(t, u, v)$ with respect to v is required.

A very fruitful result was proved by A. AMBROSETTI and G. PRODI in 1972 [3]. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let us denote by $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ the eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$, and by $\zeta > 0$ the principal eigenfunction. Consider the semilinear Dirichlet problem

$$(0.18) \quad \Delta u + f(u) = v(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $v \in C^{0,\alpha}(\overline{\Omega})$ and $f \in C^2(\mathbb{R})$. The main result in [3] is the following one.

Theorem 0.9. *Assume that f satisfies the following conditions.*

$$(0.19) \quad f''(s) > 0 \quad \text{for all } s \in \mathbb{R},$$

$$(0.20) \quad 0 < \lim_{s \rightarrow -\infty} \frac{f(s)}{s} < \lambda_1 < \lim_{s \rightarrow +\infty} \frac{f(s)}{s} < \lambda_2.$$

Then there exists a closed connected manifold $A_1 \subset C^{0,\alpha}(\overline{\Omega})$ of codimension 1 such that $C^{0,\alpha}(\overline{\Omega}) \setminus A_1 = A_0 \cup A_2$ and (0.18) has exactly zero, one or two solutions according to $v \in A_0, A_1$ or A_2 .

The above result was proved applying a *global inversion theorem* to the operator $Tu = \Delta u + f(u)$ regarded as a differentiable mapping between the Hölder spaces $C^{2,\alpha}(\overline{\Omega})$ and $C^{0,\alpha}(\overline{\Omega})$.

Let

$$v = t\zeta + v_1, \quad \text{with} \quad \int_{\Omega} v_1 \zeta = 0.$$

So that problem (0.18) is equivalent to

$$(0.21) \quad \Delta u + f(u) = t\zeta(x) + v_1(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

A cartesian representation of A_1 was given by M.S. BERGER and E. PODOLAK in 1975 [17].

Theorem 0.10. *Assume that f satisfies (0.19) and (0.20). Then there is a real number $t(v_1)$ depending continuously on v_1 such that (0.21) has exactly zero, one or two solutions according to $t < t(v_1), t = t(v_1)$ or $t > t(v_1)$.*

The proof of Theorem 0.10 is based upon a Hilbert space approach and a Lyapunov-Schmidt procedure. Using upper and lower solutions, J.L. KAZDAN and F.W. WARNER [36] have weakened the assumptions and the conclusions of BERGER-PODOLAK. The multiplicity conclusion of AMBROSETTI-PRODI without exactness has been obtained independently, using a combination of the method of upper and lower solutions and the Leray-Schauder degree, by E.N DANCER in 1978 [24] and H. AMANN-P. HESS in 1979 [4]. An Ambrosetti-Prodi type multiplicity result for a second order ordinary differential equation with periodic conditions was given in 1986 by C. FABRY, J. MAWHIN and M. NKASHAMA [31]. The periodic Ambrosetti-Prodi problem for nonlinear perturbations of the p-Laplacian was discussed in 2006 by MAWHIN [60]. We describe this result below. Consider the periodic boundary value problem

$$(0.22) \quad \begin{aligned} &(|u'|^{p-2}u')' + f(u)u' + g(t, u) = s, \\ &u(0) - u(T) = 0 = u'(0) - u'(T) \end{aligned}$$

where $p > 1$, $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous. The main result in [60] is the following one.

Theorem 0.11. *If g satisfies the condition*

$$(0.23) \quad \lim_{|u| \rightarrow \infty} g(t, u) = +\infty \quad \text{uniformly in } t \in [0, T],$$

then there exists $s_1 \in \mathbb{R}$ such that (0.22) has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$.

When $f = 0$ we generalize the previous result as follows.

Let $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$) be an increasing homeomorphism such that $\phi(0) = 0$. Let $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. For $s \in \mathbb{R}$, consider the periodic boundary-value problem

$$(0.24) \quad (\phi(u'))' + g(t, u) = s, \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

Assume now that the condition (0.23) holds. Consider

$$\sigma_1 = \min_{(t,u) \in [0,T] \times \mathbb{R}} g(t, u), \quad \sigma_2 = \sigma_1 + \frac{a}{2T}.$$

In Chapter 3 (see also [15]) we prove the following Ambrosetti-Prodi type result.

Theorem 0.12. *If the function g satisfies (0.23) and if there is a $u_0 \in \mathbb{R}$ such that*

$$(0.25) \quad \sigma_3 =: \max_{t \in [0,T]} g(t, u_0) < \sigma_2,$$

then there is $s_1 \in [\sigma_1, \sigma_3]$ such that (0.24) has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s_1 < s < \sigma_2$.

A more general Ambrosetti-Prodi type result holds when $\phi :]-a, a[\rightarrow \mathbb{R}$ ($a \neq +\infty$). On the other hand, Ambrosetti-Prodi type results for differences equations are proved in Chapter 2 (see also [11], [12], [16]).

In Chapter 4 (see also [6]), using Mawhin's continuation theorem in infinite dimension we study the existence of positive periodic solutions of some generalizations of competition systems, in particular the May-Leonard model, and of prey-predator systems.

In [68], ZANOLIN has studied the delay-Lotka-Volterra system

$$(0.26) \quad \dot{x}_i(t) = x_i(t) \left[r_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t)x_j(t - \tau_j) \right] \quad (i = 1, 2, \dots, n),$$

where $r_i, a_{ii} > 0, a_{ij} \geq 0$ ($j \neq i$), $(i, j = 1, \dots, n)$ are T-periodic continuous functions, $\tau_j \in \mathbb{R}$ ($j = 1, 2, \dots, n$). If the condition

$$(0.27) \quad \bar{r}_i - \sum_{\substack{j=1 \\ j \neq i}}^n \bar{a}_{ij} \left| \frac{r_j}{a_{jj}} \right|_0 > 0 \quad (i = 1, 2, \dots, n)$$

is satisfied, where $|f|_0 = \sup_{t \in \mathbb{R}} |f(t)|$ denotes the maximum norm and

$\bar{f} = \frac{1}{T} \int_0^T f$ the mean value of the T-periodic continuous function f, then it is proved that system (0.26) has at least one T-periodic, positive solution.

The system (0.26) is generalized by Y. LI in [45] to the delay-Lotka-Volterra system

$$(0.28) \quad \begin{aligned} \dot{x}_i(t) = & x_i(t) [r_i(t) - a_{ii}(t)x_i(t) \\ & - a_{ii}(t)x_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t)x_j(t - \tau_j(t, x_1(t), \dots, x_n(t)))] \\ & (i = 1, 2, \dots, n), \end{aligned}$$

where $\tau_j \in C(\mathbb{R}^{n+1}, \mathbb{R})$ and τ_j ($j = 1, 2, \dots, n$) are T-periodic with respect to their first argument. It is shown that, if condition (0.27) is satisfied and system

$$(0.29) \quad \sum_{j=1}^n \bar{a}_{ij} \exp(y_j) = \bar{r}_i, \quad i = 1, 2, \dots, n$$

has only one solution, then system (0.28) has at least one T-periodic positive solution. We prove that the same conclusion holds if condition (0.27) only is satisfied. A particular case of system (0.28) is the May-Leonard-type system

$$(0.30) \quad \begin{aligned} \dot{x}_1(t) &= x_1(t)[1 - x_1(t) - \alpha_1(t)x_2(t - \tau_2(t, x_1(t), \dots, x_3(t))) \\ &\quad - \beta_1(t)x_3(t - \tau_3(t, x_1(t), \dots, x_3(t)))] \\ \dot{x}_2(t) &= x_2(t)[1 - \beta_2(t)x_1(t - \tau_1(t, x_1(t), \dots, x_3(t))) \\ &\quad - \alpha_2(t)x_3(t - \tau_3(t, x_1(t), \dots, x_3(t)))] \\ \dot{x}_3(t) &= x_3(t)[1 - \alpha_3(t)x_1(t - \tau_1(t, x_1(t), \dots, x_3(t))) \\ &\quad - \beta_3(t)x_2(t - \tau_2(t, x_1(t), \dots, x_3(t))) - x_3(t)] \end{aligned}$$

where $\alpha_i, \beta_i \geq 0$ ($i = 1, 2, 3$) are continuous T-periodic functions, $\tau_j \in C(\mathbb{R}^4, \mathbb{R})$ and τ_j ($j = 1, 2, 3$) are T-periodic with respect to their first argument. In this case condition (0.27) becomes

$$\bar{\alpha}_i + \bar{\beta}_i < 1 \quad (i = 1, 2, 3).$$

In [33] (see also [49]) it is shown that (0.30) has at least one non constant periodic positive solution if

$$0 < \alpha_i < 1 < \beta_i \quad (i = 1, 2, 3),$$

where α_i, β_i ($i = 1, 2, 3$) are constants and $\tau_i \equiv 0$ ($i = 1, 2, 3$). A special case of a result in [5] is that (0.28) (for $n = 3$) has at least one T-periodic positive solution if $\tau_j \equiv 0$ ($j = 1, 2, 3$) and the May-Leonard type condition is satisfied

$$(0.31) \quad \left[\frac{r_i}{r_j} \right]_L > \max \left\{ \left[\frac{a_{ii}}{a_{ji}} \right]_M, \left[\frac{a_{ij}}{a_{jj}} \right]_M \right\} \\ (i, j) \in \{(1, 2), (2, 3), (3, 1)\},$$

where $[f]_L$ denotes the minimum of f and $[f]_M$ denotes the maximum of f . We prove that (0.28) has at least one T-periodic positive solution if condition (0.31) is satisfied.

On the other hand we study the system

$$(0.32) \quad \begin{aligned} \dot{u}(t) &= u(t)[a(t) - b(t)u(t) - c(t)v(t - \beta(t, u(t), v(t)))] \\ \dot{v}(t) &= v(t)[d(t) + f(t)u(t - \alpha(t, u(t), v(t))) - g(t)v(t)] \end{aligned}$$

where a, b, c, d, f, g are continuous T -periodic functions and $\alpha, \beta \in C(\mathbb{R}^3, \mathbb{R})$ are T -periodic with respect to their first variable. It is also assumed that a, b, c, f and g are strictly positive. We prove that if the functions $a, b, \dots, g, \alpha, \beta$ are like above and Gopalsamy's condition

$$(0.33) \quad -\frac{[f]_L}{[b]_M} < \min \left\{ \frac{[d]_L}{[a]_M}, \frac{[d]_L}{[a]_L} \right\} \leq \max \left\{ \frac{[d]_M}{[a]_M}, \frac{[d]_M}{[a]_L} \right\} < \frac{[g]_L}{[c]_M}$$

is satisfied, then system (0.32) has at least one positive T -periodic solution.

Finally, we consider the system

$$(0.34) \quad \begin{aligned} \dot{u}(t) &= u(t)[a(t) - b(t)u(t) - c(t)v(t)] \\ \dot{v}(t) &= \tau(t)v(t)[u(t - \alpha(t, u(t), v(t))) - \sigma(t)] \end{aligned}$$

where a, b, c, τ, σ are continuous T -periodic strictly positive functions and $\alpha \in C(\mathbb{R}^3, \mathbb{R})$ is T -periodic with respect to its first variable. We show that system (0.34) has at least one T -periodic positive solution if condition

$$(0.35) \quad [a]_L - [b]_M[\sigma]_M > 0$$

is satisfied. The autonomous case has been considered in [44].

In the last chapter of this thesis (see also [7]) we prove two multiplicity results for superlinear planar systems. To prove our first main result we use a theorem of J.R. WARD and to prove the second one we use *Capietto-Mawhin-Zanolin continuation theorem*. Consider the following boundary value problem

$$(0.36) \quad u' = -g(t, u, v)v, \quad v' = g(t, u, v)u$$

$$(0.37) \quad u(0) = 0 = u(\pi),$$

where g is a continuous function on $[0, \pi] \times \mathbb{R}^2$. Assume

$$(0.38) \quad g(t, u, v) \rightarrow +\infty \quad \text{as} \quad |u| + |v| \rightarrow \infty$$

uniformly with $t \in [0, \pi]$,

$$(0.39) \quad g(t, 0, 0) = 0 \quad \text{for all} \quad t \in [0, \pi],$$

$$(0.40) \quad g(t, u, v) \geq 0 \quad \text{for all} \quad (t, u, v) \in [0, \pi] \times \mathbb{R}^2.$$

Under these assumptions J. R. WARD [67], among other results, proves, using essentially *Rabinowitz global bifurcation theorem* (see e. g. [32], [53], [58]) and the number of rotations associated to the bifurcations branches furnished by it, that boundary value problem (0.36), (0.37) has infinitely many solutions. Using the same method as in [67], we

prove that (0.38), (0.39) are sufficient for (0.36), (0.37) to have infinitely many solutions. Remark that (0.39) is essential in the method used in [67].

Now, suppose that

$$(0.41) \quad g(0, -u, v) = g(0, u, v) \quad \text{for all } (u, v) \in \mathbb{R}^2.$$

If conditions (0.38), (0.41) hold and if the function $g(0, \cdot)$ is locally Lipschitz on $\mathbb{R}^2$, we prove that boundary value problem (0.36), (0.37) has infinitely many solutions. We use Capietto-Mawhin-Zanolin continuation theorem [20] (see also [19], [34], [57]). We state and prove this continuation theorem in Chapter 1.

CHAPTER I

Continuation theorems

1. MAWHIN'S CONTINUATION THEOREM IN FINITE DIMENSION

Let X, Y be real normed spaces of dimension n . Consider $L : X \rightarrow Y$ a linear mapping which is not injective. Using that $\dim \ker(L) = \text{codim Im}(L)$, it follows that there exists two linear projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that $\ker(Q) = \text{Im}(L)$, $\text{Im}(P) = \ker(L)$ and $\ker(L)$ is isomorphic with $\text{Im}(Q)$. Let $J : \ker(L) \rightarrow \text{Im}(Q)$ be an isomorphism.

Theorem 1.1. *Let $\Omega \subset X$ be an open bounded set and let $F : \bar{\Omega} \rightarrow Y$ be a continuous mapping. Assume*

- (i). $Lx + \lambda F(x) \neq 0$ for each $(\lambda, x) \in]0, 1[\times (\partial\Omega \setminus \ker(L))$;
- (ii). $QF(x) \neq 0$ for each $x \in \partial\Omega \cap \ker(L)$ and $d_B[J^{-1}QF, \Omega \cap \ker(L), 0] \neq 0$.

Then $L + F$ has a least one zero in $\bar{\Omega}$.

Proof. Consider the continuous homotopy $\mathcal{M} : [0, 1] \times \bar{\Omega} \rightarrow Y$ defined by

$$\mathcal{M}(\lambda, x) = Lx + \lambda F(x) + (1 - \lambda)QF(x),$$

which, for $\lambda = 1$, reduces to $L + F$ and for $\lambda = 0$ reduces to $L + QF$. We assume that $L + F$ does not have a zero on $\partial\Omega$ since otherwise we are done with the proof. Consider $(\lambda, x) \in [0, 1] \times \bar{\Omega}$ and notice that, because $\ker(Q) = \text{Im}(L)$, it follows that

$$\mathcal{M}(\lambda, x) = 0$$

if and only if

$$QF(x) = 0, \quad Lx + \lambda F(x) = 0.$$

Using the remark above and hypothesis (i), (ii) it follows that

$$\mathcal{M}(\lambda, x) \neq 0 \quad \text{for each } (\lambda, x) \in [0, 1] \times \partial\Omega.$$

Hence, the invariance of Brouwer degree under a homotopy implies that

$$(1.1) \quad d_B[L + F, \Omega, 0] = d_B[L + QF, \Omega, 0].$$

On the other hand, it is easy to check that $L + JP : X \rightarrow Y$ is an isomorphism and that $(L + JP)^{-1}h = J^{-1}h$ for every $h \in \text{Im}(Q)$ (see also Lemma 2.1 below). It follows that,

$$\begin{aligned} L + QF &= L + JP - JP + QF \\ &= (L + JP)[I + (L + JP)^{-1}(-JP + QF)] \\ &= (L + JP)(I - P + J^{-1}QF). \end{aligned}$$

Consequently, by the product formula of Brouwer degree

$$\begin{aligned} |d_B[L + QF, \Omega, 0]| &= |\text{ind}[L + JP, 0]d_B[I - P + J^{-1}QF, \Omega, 0]| \\ &= |d_B[I - P + J^{-1}QF, \Omega, 0]|. \end{aligned}$$

Now, by the Leray-Schauder reduction theorem for Brouwer degree, we have

$$\begin{aligned} &|d_B[I - P + J^{-1}QF, \Omega, 0]| \\ &= |d_B[(I - P + J^{-1}QF)|_{\ker(L)}, \Omega \cap \ker(L), 0]| \\ &= |d_B[J^{-1}QF, \Omega \cap \ker(L), 0]|, \end{aligned}$$

which, together with (1.1) and hypothesis (ii) implies that

$$d_B[L + F, \Omega, 0] \neq 0.$$

Using the existence property of Brouwer degree the theorem is proved. ■

2. MAWHIN'S CONTINUATION THEOREM IN INFINITE DIMENSION

Let X, Y be two infinite dimensional real normed spaces. A linear mapping $L : D(L) \subset X \rightarrow Y$ is called Fredholm if the following conditions hold.

- (i). $\ker(L)$ has finite dimension;
- (ii). $\text{Im}(L)$ is closed and has finite codimension.

If L is a Fredholm operator, the index of L is the integer

$$i(L) := \dim \ker(L) - \text{codim } \text{Im}(L).$$

In what follows we denote by $L : D(L) \subset X \rightarrow Y$ a Fredholm mapping of index zero which is not injective. Let $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ be continuous projectors such that

$$\begin{aligned} \ker(Q) &= \text{Im}(L), \text{Im}(P) = \ker(L); \\ X &= \ker(L) \oplus \ker(P), Y = \text{Im}(L) \oplus \text{Im}(Q). \end{aligned}$$

Let $J : \ker(L) \rightarrow \operatorname{Im}(Q)$ be an isomorphism. It is clear that the operator $L : D(L) \cap \ker(P) \rightarrow \operatorname{Im}(L)$ is an isomorphism denoted by L_P . Consider the operator $K_{PQ} : Y \rightarrow X$ define by $K_{PQ} = L_P^{-1}(I - Q)$.

Lemma 2.1. *The operator $L + JP : D(L) \rightarrow Y$ is an isomorphism and*

$$(L + JP)^{-1} = K_{PQ} + J^{-1}Q.$$

In particular,

$$(L + JP)^{-1}x = J^{-1}x \quad \text{for all } x \in \operatorname{Im}(Q).$$

Proof. For the injectivity of $L + JP$, let $x \in D(L)$ be such that

$$(2.1) \quad (L + JP)x = 0.$$

From this equality we deduce that

$$Lx \in \operatorname{Im}(L) \cap \operatorname{Im}(J) = \ker(Q) \cap \operatorname{Im}(Q) = \{0\},$$

hence $x \in \ker(L)$. Consequently, $Px = x$ and, taking into account (2.1), $Jx = 0$, therefore $x = 0$. For the surjectivity of $L + JP$, let $y \in Y$. We assert that

$$x = (K_{PQ} + J^{-1}Q)y$$

is a solution of the equation

$$(L + JP)x = y.$$

Indeed, since $J^{-1}Qy \in \ker(L)$, it follows that

$$Lx = LK_{PQ}y = LL_P^{-1}(I - Q)y = (I - Q)y.$$

Since $K_{PQ}y \in D(L) \cap \ker(P)$, it follows that

$$JPx = JJ^{-1}Qy = Qy.$$

Consequently,

$$(L + JP)x = (I - Q)y + Qy = y$$

and

$$(L + JP)^{-1} = K_{PQ} + J^{-1}Q.$$

■

Lemma 2.2. *If $N : \Delta \subset X \rightarrow Y$ is a mapping, the problem*

$$x \in D(L) \cap \Delta, Lx = Nx$$

is equivalent to the fixed point problem

$$x \in \Delta, x = Px + J^{-1}QNx + K_{PQ}Nx.$$

Proof. We have

$$\begin{aligned} & [x \in D(L) \cap \Delta, Lx = Nx] \\ \Leftrightarrow & [x \in D(L) \cap \Delta, (L + JP)x = (N + JP)x] \\ \Leftrightarrow & [x \in \Delta, x = (L + JP)^{-1}(N + JP)x]. \end{aligned}$$

On the other hand, using Lemma 2.1 it follows that

$$\begin{aligned} (L + JP)^{-1}(N + JP) &= (K_{PQ} + J^{-1}Q)(N + JP) \\ &= K_{PQ}N + K_{PQ}JP + J^{-1}QN + J^{-1}QJP. \end{aligned}$$

Since $\text{Im}(J) = \text{Im}(Q) = \ker(I - Q)$, it follows that

$$K_{PQ}JP = L_P^{-1}(I - Q)JP = 0.$$

Using $Q|_{\text{Im}(Q)} = I|_{\text{Im}(Q)}$ and $\text{Im}(J) = \text{Im}(Q)$, we deduce that

$$J^{-1}QJP = J^{-1}JP = P.$$

Consequently

$$(L + JP)^{-1}(N + JP) = P + J^{-1}QN + K_{PQ}N,$$

and the assertion follows. $\blacksquare$

Definition 2.3. Let $\Omega \subset X$ an open bounded set and $N : \overline{\Omega} \rightarrow Y$. We say that N is L -compact if $K_{PQ}N$ is compact, QN is continuous and $QN(\overline{\Omega})$ is a bounded set in Y .

Theorem 2.4. Let $\Omega \subset X$ be an open bounded set and let $N : \overline{\Omega} \rightarrow Y$ be a L -compact operator. Assume

- (i). $Lx \neq \lambda Nx$ for each $(\lambda, x) \in]0, 1[\times (\partial\Omega \cap (D(L) \setminus \ker(L)))$;
- (ii). $QNx \neq 0$ for each $x \in \partial\Omega \cap \ker(L)$ and $d_B[J^{-1}QN, \Omega \cap \ker(L), 0] \neq 0$.

Then $L - N$ has a least one zero in $D(L) \cap \overline{\Omega}$.

Proof. For $\lambda \in [0, 1]$ consider the family of problems

$$(2.2) \quad x \in D(L) \cap \overline{\Omega}, \quad Lx = \lambda Nx + (1 - \lambda)QNx.$$

Let $\mathcal{M} : [0, 1] \times \overline{\Omega} \rightarrow Y$ be the homotopy defined by

$$\mathcal{M}(\lambda, x) = Px + J^{-1}QNx + \lambda K_{PQ}Nx.$$

Using Lemma 2.2 we have that the problem (2.2) is equivalent to the fixed point problem $x \in \overline{\Omega}$ and

$$\begin{aligned} x &= Px + J^{-1}Q(\lambda N + (1 - \lambda)QN)x + K_{PQ}(\lambda N + (1 - \lambda)QN)x \\ &= Px + \lambda J^{-1}QNx + (1 - \lambda)J^{-1}QNx + \lambda K_{PQ}Nx \\ &+ (1 - \lambda)K_{PQ}QNx = \mathcal{M}(\lambda, x). \end{aligned}$$

Hence, the problem (2.2) is equivalent to the fixed point problem

$$(2.3) \quad x \in \overline{\Omega}, \quad x = \mathcal{M}(\lambda, x).$$

If there exists $x \in \partial\Omega$ such that $Lx = Nx$ then we are done. Now assume that

$$(2.4) \quad Lx \neq Nx \quad \text{for each } x \in D(L) \cap \partial\Omega.$$

On the other hand

$$(2.5) \quad Lx \neq \lambda Nx + (1 - \lambda)QNx$$

for each $(\lambda, x) \in]0, 1[\times (D(L) \cap \partial\Omega)$. If

$$Lx = \lambda Nx + (1 - \lambda)QNx$$

for some $(\lambda, x) \in]0, 1[\times (D(L) \cap \partial\Omega)$ we obtain, by applying Q to both members of the preceding equality,

$$Q Nx = 0, \quad Lx = \lambda Nx.$$

The first of those equalities and (ii) imply that $x \notin \ker(L) \cap \partial\Omega$ i. e. $x \in \partial\Omega \cap (D(L) \setminus \ker(L))$ and hence the second contradicts (i). Using again (ii) it follows that

$$(2.6) \quad Lx \neq QNx \quad \text{for each } x \in \partial\Omega \cap D(L).$$

Using (2.4), (2.5) and (2.6) we deduce that

$$(2.7) \quad x \neq \mathcal{M}(\lambda, x) \quad \text{for each } (\lambda, x) \in [0, 1] \times \partial\Omega.$$

Because N is L -compact it follows that $\mathcal{M}$ is compact, hence using (2.7) and the invariance property under compact homotopy of Leray-Schauder degree, we have

$$(2.8) \quad d_{LS}[I - \mathcal{M}(0, \cdot), \Omega, 0] = d_{LS}[I - \mathcal{M}(1, \cdot), \Omega, 0].$$

On the other hand we have that

$$(2.9) \quad d_{LS}[I - \mathcal{M}(0, \cdot), \Omega, 0] = d_{LS}[I - (P + J^{-1}QN), \Omega, 0]$$

But the range of $P + J^{-1}QN$ is contained in $\ker(L)$, so, using the reduction property of the Leray-Schauder degree and the fact that $P|_{\ker(L)} = I|_{\ker(L)}$ it follows that

$$(2.10) \quad \begin{aligned} d_{LS}[I - (P + J^{-1}QN), \Omega, 0] &= d_B[I - (P + J^{-1}QN), \Omega \cap \ker(L), 0] \\ &= d_B[-J^{-1}QN, \Omega \cap \ker(L), 0]. \end{aligned}$$

Using (2.8), (2.9) and (2.10) it follows that $d_{LS}[I - \mathcal{M}(1, \cdot), \Omega, 0] \neq 0$, hence the existence property of Leray-Schauder degree implies that there exists $x \in \Omega$ such that $x = \mathcal{M}(1, x)$ i. e. $x \in D(L) \cap \Omega, Lx = Nx$. ■

3. CAPIETTO-MAWHIN-ZANOLIN CONTINUATION THEOREM

Let X be a normed space endowed with the norm $\|\cdot\|$ and $\mathcal{G} : [0, 1] \times X \rightarrow X$ be a compact homotopy. We denote by Σ the closed set

$$\Sigma = \{(\lambda, x) \in [0, 1] \times X : \mathcal{G}(\lambda, x) = x\}.$$

If $A \subset [0, 1] \times X$ and $\lambda \in [0, 1]$, we denote by A^λ the set

$$A^\lambda = \{x \in X : (\lambda, x) \in A\} \subset X.$$

Now, consider $\varphi : [0, 1] \times X \rightarrow \mathbb{R}_+$ a continuous function satisfying the following conditions.

- (i_1). There exists $R > 0$ such that $\varphi(\lambda, x) \in \mathbb{N}$ for each $(\lambda, x) \in \Sigma$ with $\|x\| \geq R$.
- (i_2). $\varphi^{-1}(n) \cap \Sigma$ is bounded for each $n \in \mathbb{N}$.

Because of our assumptions, it is clear that we can chose $k_0 \in \mathbb{N}$ such that

$$k_0 > \sup\{\varphi(\lambda, x) : (\lambda, x) \in \Sigma, \|x\| \leq R\}.$$

Then, the set $A = \varphi^{-1}[0, k_0]$ is closed in $[0, 1] \times X$ and contains the set $\{(\lambda, x) \in \Sigma : \|x\| \leq R\}$. Let $j \in \mathbb{N}$ such that $j > k_0$. Using the fact that $\mathcal{G}$ is compact, φ is continuous and (i_2), we deduce that $\Sigma_j := \varphi^{-1}(j) \cap \Sigma$ is a compact set. On the other hand, the definition of A implies that $\Sigma_j \cap A = \emptyset$. Hence, there exists a sequence of sets $(\Lambda_j)_{j \geq k_0+1}$ such that Λ_j is open and bounded in X , $\overline{\Lambda_j} \cap \overline{\Lambda_l} = \emptyset$ if $j \neq l$, $\Lambda_j \cap \Sigma^0 = \Sigma_j^0$ and $\partial\Lambda_j \cap \Sigma^0 = \emptyset$. We assume that there exists $k > k_0$ such that

$$(3.1) \quad d_{LS}[\mathcal{G}(0, \cdot), \Lambda_j, 0] \neq 0, \quad (j \geq k).$$

Notice that if $(\Gamma_j)_{j \geq k_0+1}$ is a sequence of sets having the same properties as $(\Lambda_j)_{j \geq k_0+1}$, then by the excision property of Leray-Schauder degree, it follows that

$$(3.2) \quad d_{LS}[\mathcal{G}(0, \cdot), \Lambda_j, 0] = d_{LS}[\mathcal{G}(0, \cdot), \Gamma_j, 0] \quad (j \geq k_0 + 1).$$

Theorem 3.1. *Assume that conditions (i_1), (i_2) and (3.1) hold. Then for each $j \geq k$ there exists $x_j \in X$ such that $\varphi(1, x_j) = j$ and $\mathcal{G}(1, x_j) = x_j$. Moreover, $\lim_{j \rightarrow \infty} \|x_j\| = +\infty$.*

Proof. Using our assumptions, it follows that there exists a sequence of sets $(\mathcal{O}_j)_{j \geq k_0+1}$ such that $\mathcal{O}_j$ is open and bounded in $[0, 1] \times X$, $\overline{\mathcal{O}_j} \cap \overline{\mathcal{O}_l} = \emptyset$, $\mathcal{O}_j \cap \Sigma = \Sigma_j$ and $\partial\mathcal{O}_j \cap \Sigma = \emptyset$. For $j \geq k_0 + 1$ consider $\Gamma_j = \mathcal{O}_j^0$. It is clear that this sequence of sets has the same properties as $(\Lambda_j)_{j \geq k_0+1}$. Hence, (3.1) and (3.2) imply that

$$(3.3) \quad d_{LS}[\mathcal{G}(0, \cdot), \Gamma_j, 0] \neq 0, \quad (j \geq k).$$

On the other hand, using the generalized invariance by homotopy property of Leray-Schauder degree it follows that

$$(3.4) \quad d_{LS}[\mathcal{G}(0, \cdot), \Gamma_j, 0] = d_{LS}[\mathcal{G}(0, \cdot), \mathcal{O}_j^1, 0], \quad (j \geq k).$$

Using (3.3), (3.4) and the existence property of Leray-Schauder degree we deduce that there exists $x_j \in \mathcal{O}_j^1$ such that $\mathcal{G}(1, x_j) = x_j$ for all $j \geq k$. Moreover, $\varphi(1, x_j) = j$ for all $j \geq k$. If $(x_j)_{j \geq k}$ is bounded, using the fact that $\mathcal{G}$ is compact, it follows that we can find a convergent subsequence $(x_{j_l})_{l \geq 1}$. This implies that the sequence $(\varphi(1, x_{j_l}))_{l \geq 1}$ is convergent, contradiction. Consequently, $\lim_{j \rightarrow \infty} \|x_j\| = +\infty$. ■

CHAPTER II

Nonlinear difference equations

1. ONE-SIDE BOUNDED NONLINEARITY

1.1. Periodic solutions of second order difference equations with one-side bounded nonlinearity

Let $n \in \mathbb{N}$ fixed and $(x_1, \dots, x_n) \in \mathbb{R}^n$. Define $(Dx_1, \dots, Dx_{n-1}) \in \mathbb{R}^{n-1}$ and $(D^2x_2, \dots, D^2x_{n-1}) \in \mathbb{R}^{n-2}$ by

$$\begin{aligned} Dx_m &= x_{m+1} - x_m, & (1 \leq m \leq n-1) \\ D^2x_m &= x_{m+1} - 2x_m + x_{m-1}, & (2 \leq m \leq n-1). \end{aligned}$$

Let $f_m : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions ($2 \leq m \leq n-1$). Consider the periodic boundary value problem

$$\begin{aligned} D^2x_m + f_m(x_m) &= 0 & (2 \leq m \leq n-1), \\ (1.1) \quad x_1 &= x_n, \quad Dx_1 = Dx_{n-1}. \end{aligned}$$

Let us introduce the vector space

$$(1.2) \quad V^{n-2} = \{x \in \mathbb{R}^n : x_1 = x_n, Dx_1 = Dx_{n-1}\}$$

endowed with the orientation of $\mathbb{R}^n$. Its elements can be associated to the coordinates $(x_2, \dots, x_{n-1})$ and correspond to the elements of $\mathbb{R}^n$ of the form $(\frac{x_2+x_{n-1}}{2}, x_2, \dots, x_{n-1}, \frac{x_2+x_{n-1}}{2})$, so that the restriction D^2 to V^{n-2} is well defined in terms of $(x_2, \dots, x_{n-1})$. We use the norm $\|x\| := \max_{2 \leq j \leq n-1} |x_j|$ in V^{n-2} and $\max_{1 \leq j \leq n-2} |x_j|$ in $\mathbb{R}^{n-2}$. Now, we define the continuous mapping $G : V^{n-2} \rightarrow \mathbb{R}^{n-2}$ by

$$(1.3) \quad G_m(x_m) = D^2(x_m) + f_m(x_m) \quad (2 \leq m \leq n-1),$$

so that $(\frac{x_2+x_n}{2}, x_2, \dots, x_{n-1}, \frac{x_2+x_n}{2})$ is a solution of (1.1) if and only if $(x_2, \dots, x_{n-1}) \in V^{n-2}$ is a zero of G . We also define $F : V^{n-2} \rightarrow \mathbb{R}^{n-2}$ by

$$F(x) = (f_2(x_2), \dots, f_{n-1}(x_{n-1})),$$

and call $L : V^{n-2} \rightarrow \mathbb{R}^{n-2}$ the restriction of D^2 to V^{n-2} . As the general solution of the difference equation

$$x_{m+1} = 2x_m - x_{m-1} \quad (m \geq 1)$$

is $x_m = c_1 + c_2 m$, we see that

$$\ker(L) = \{(c, \dots, c) : c \in \mathbb{R}\}.$$

Furthermore, if $y \in \text{Im}(L)$, then, for some $(x_2, \dots, x_{m-1}) \in V^{n-2}$,

$$\begin{aligned} \sum_{j=1}^{n-2} y_j &= \frac{x_2 + x_{n-1}}{2} - 2x_2 + x_3 + \sum_{m=3}^{n-2} (x_{m-1} - 2x_m + x_{m+1}) \\ &+ x_{n-2} - 2x_{n-1} + \frac{x_2 + x_{n-1}}{2} = 0. \end{aligned}$$

As $\text{codim Im}(L) = \dim \ker(L) = 1$, it follows that

$$\text{Im}(L) = \{y \in \mathbb{R}^{n-2} : \sum_{j=1}^{n-2} y_j = 0\}.$$

Consider the projectors $P : V^{n-2} \rightarrow V^{n-2}$ and $Q : \mathbb{R}^{n-2} \rightarrow \mathbb{R}^{n-2}$ defined by

$$\begin{aligned} P(x_2, \dots, x_{n-1}) &= (x_2, \dots, x_2), & Q(y_1, \dots, y_{n-2}) &= (\bar{y}, \dots, \bar{y}) \\ \text{with } \bar{y} &:= \frac{1}{n-2} \left(\sum_{j=1}^{n-2} y_j \right), \end{aligned}$$

so that $\ker(Q) = \text{Im}(L)$, $\text{Im}(P) = \ker(L)$.

Lemma 1.1. *Let $x \in V^{n-2}$ such that $\bar{x} = \frac{1}{n-2} \left(\sum_{j=2}^{n-1} x_j \right) = 0$. Then*

$$\|x\| \leq c_n \sum_{j=2}^{n-1} |D^2 x_j|,$$

with c_n is a constant which depends only on n .

Proof. Let

$$V = \{x \in V^{n-2} : \bar{x} = 0\}$$

be a subspace of $\mathbb{R}^n$. The following maps

$$x \rightarrow \|x\|, \quad (x_2, \dots, x_{n-1}) \rightarrow \sum_{j=2}^{n-1} |D^2 x_j|$$

define two norms on V . Because V is finite-dimension, those two norms are equivalent, and the above inequality holds. $\blacksquare$

In order to obtain the existence of zeros of $G = L + F$ we shall apply Mawhin's continuation theorem in finite dimension. We first obtain a priori estimates for the possible zeros of $L + \lambda F$ with $\lambda \in]0, 1]$, or equivalently for the possible solutions of the periodic boundary value problem

$$(1.4) \quad \begin{aligned} D^2 x_m + \lambda f_m(x_m) &= 0 \quad (2 \leq m \leq n-1), \\ x_1 &= x_n, \quad Dx_1 = Dx_{n-1}, \end{aligned}$$

with $\lambda \in]0, 1]$.

Lemma 1.2. *If each function f_m ($2 \leq m \leq n-1$) is bounded from below or from above by c , and if, for some $R > 0$, one has*

$$(1.5) \quad \sum_{m=2}^{n-1} f_m(x_m) \neq 0$$

whenever $\min_{2 \leq j \leq n-1} x_j \geq R$ or $\max_{2 \leq j \leq n-1} x_j \leq -R$, then, for each $\lambda \in]0, 1]$ and each possible solution x of (1.4), one has $\|x\| < 4|c|(n-2)c_n + R$.

Proof. Assume first that each f_m is bounded from below and let c such that $f_m(x) \geq c$ for all $x \in \mathbb{R}$ and all $2 \leq m \leq n-1$. We have, for all $2 \leq m \leq n-1$,

$$(1.6) \quad |f_m(x)| \leq f_m(x) + 2|c|, \quad (x \in \mathbb{R}).$$

Let $\lambda \in]0, 1]$ and x be a solution of (1.4). It follows that

$$(1.7) \quad \sum_{m=2}^{n-1} f_m(x_m) = 0.$$

Denoting by $\tilde{x}_m = x_m - \bar{x}$ ($2 \leq m \leq n-1$) and using (1.4), (1.6) and (1.7), it follows that

$$\sum_{m=2}^{n-1} |D^2 \tilde{x}_m| \leq 2|c|(n-2).$$

The above inequality and Lemma 1.1 imply that

$$(1.8) \quad \max_{2 \leq m \leq n-1} |\tilde{x}_m| \leq 2|c|(n-2)c_n.$$

Using (1.5), (1.7) and (1.8) we deduce that $|\bar{x}| < 2|c|(n-2)c_n + R$, and so, using again (1.8) it follows that $\|x\| < 4|c|(n-2)c_n + R$. In the case where the f_m are bounded from above by c , it suffices to write the problem

$$\tilde{L}x + \tilde{F}(x) = 0,$$

with $\tilde{L} = -L$ and $\tilde{F} = -F$ to reduce it to a problem with $\tilde{F}$ bounded from below, noticing that L and $\tilde{L}$ have the same kernel and the same range. $\blacksquare$

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be the continuous function defined by

$$(1.9) \quad \varphi(x) = \frac{1}{n-2} \left(\sum_{m=2}^{n-1} f_m(x) \right) \quad (x \in \mathbb{R}).$$

Theorem 1.3. *If the functions f_m ($2 \leq m \leq n-1$) satisfy the conditions of Lemma 1.2 and if*

$$(1.10) \quad \varphi(-\rho)\varphi(\rho) < 0,$$

for some $\rho \geq 4|c|(n-2)c_n + R$ then, problem (1.1) has at least one solution x such that $\|x\| < 4|c|(n-2)c_n + R$.

Proof. It follows from Lemma 1.2 that the first assumption (i_1) of Mawhin's continuation theorem in finite dimension is satisfied for $\Omega = B(0, \rho)$. Now, $\ker(L) \simeq \mathbb{R} \simeq \text{Im}(Q)$ and

$$QF(c, \dots, c) = \varphi(c), \quad (c \in \mathbb{R}).$$

Hence, the second assumption (i_2) of Mawhin's continuation theorem in finite dimension is satisfied because of (1.10) and of

$$|d_B[QF, B(0, \rho) \cap \ker(L), 0]| = |d_B[\varphi,] - \rho, \rho[0]| = 1.$$

Consequently, $L + F$ has at least on zero in $B(0, \rho)$, i. e. problem (1.1) has at least one solution. $\blacksquare$

Corollary 1.4. *Suppose that the functions f_m ($2 \leq m \leq n-1$) are all bounded from below or all bounded from above by c , and that for some $R > 0$ and $\varepsilon \in \{-1, 1\}$,*

$$(1.11) \quad \begin{aligned} \varepsilon \sum_{m=2}^{n-1} f_m(x_m) &> 0 \quad \text{whenever} \quad \min_{2 \leq j \leq n-1} x_j \geq R \\ \varepsilon \sum_{m=2}^{n-1} f_m(x_m) &< 0 \quad \text{whenever} \quad \max_{2 \leq j \leq n-1} x_j \leq -R. \end{aligned}$$

Then problem (1.1) has at least one solution x such that $\|x\| < 4|c|(n-2)c_n + R$.

Proposition 1.5. *Suppose that the functions f_m ($2 \leq m \leq n-1$) are all bounded from below or all bounded from above by c , and that for some*

$R > 0$ and $\varepsilon \in \{-1, 1\}$,

$$(1.12) \quad \begin{aligned} \varepsilon \sum_{m=2}^{n-1} f_m(x_m) &\geq 0 \quad \text{whenever} \quad \min_{2 \leq j \leq n-1} x_j \geq R \\ \varepsilon \sum_{m=2}^{n-1} f_m(x_m) &\leq 0 \quad \text{whenever} \quad \max_{2 \leq j \leq n-1} x_j \leq -R. \end{aligned}$$

Then problem (1.1) has at least one solution x such that $\|x\| \leq 4(|c| + 1)(n-2)c_n + R$.

Proof. For definiteness, assume that each f_m is bounded from below by c . For each $k \geq 1$, let us define

$$f_m^{(k)}(x) = f_m(x) + \frac{\varepsilon x}{k(1+|x|)} \quad (2 \leq m \leq n-1),$$

so that $f_m^{(k)}$ is bounded from below by $c-1$ and, using (1.12),

$$(1.13) \quad \begin{aligned} \varepsilon \sum_{m=2}^{n-1} f_m^{(k)}(x_m) &> 0 \quad \text{whenever} \quad \min_{2 \leq j \leq n-1} x_j \geq R \\ \varepsilon \sum_{m=2}^{n-1} f_m^{(k)}(x_m) &< 0 \quad \text{whenever} \quad \max_{2 \leq j \leq n-1} x_j \leq -R. \end{aligned}$$

Let us consider the periodic boundary value problem

$$(1.14) \quad \begin{aligned} D^2 x_m + f_m^{(k)}(x_m) &= 0 \quad (2 \leq m \leq n-1), \\ x_1 &= x_n, \quad Dx_1 = Dx_{n-1}. \end{aligned}$$

Using Corollary 1.4, it follows that problem (1.14) has at least one solution $x^{(k)}$ such that $\|x^{(k)}\| < 4(|c| + 1)(n-2)c_n + R$ for all $k = 1, 2, \dots$. Going if necessary to a subsequence, we can assume that $x^{(k)} \rightarrow x$ which is a solution of (1.1) such that $\|x\| \leq 4(|c| + 1)(n-2)c_n + R$. ■

Corollary 1.6. For $p > 0$, $a_m > 0$, $b_m \in \mathbb{R}$ ($2 \leq m \leq n-1$), the periodic problem

$$(1.15) \quad \begin{aligned} D^2 x_m + a_m(x_m^+)^p - b_m &= 0 \quad (2 \leq m \leq n-1), \\ x_1 &= x_n, \quad Dx_1 = Dx_{n-1}, \end{aligned}$$

has at least one solution if and only if

$$\sum_{m=2}^{n-1} b_m \geq 0.$$

When $a_m < 0$ and $b_m \in \mathbb{R}$ ($2 \leq m \leq n-1$), problem (1.15) has at least one solution if and only if

$$\sum_{m=2}^{n-1} b_m \leq 0.$$

Proof. For the necessity, if problem (1.15) has a solution x , then

$$\sum_{m=2}^{n-1} b_m = \sum_{m=2}^{n-1} a_m (x_m^+)^p \geq 0.$$

For the sufficiency, each function $f_m(x) = a_m(x^+)^p - b_m$ is bounded from below by $-b_m$. Furthermore, if

$$R \geq \left(\frac{\sum_{m=2}^{n-1} b_m}{\sum_{m=2}^{n-1} a_m} \right)^{1/p},$$

then $\sum_{m=2}^{n-1} f_m(x_m) \geq 0$ when $\min_{2 \leq j \leq n-1} x_j \geq R$. On the other hand, $\sum_{m=2}^{n-1} f_m(x_m) = -\sum_{m=2}^{n-1} b_m \leq 0$ when $\max_{2 \leq j \leq n-1} x_j \leq 0$. Hence the result follows from Proposition 1.5. The proof of the other case is similar. $\blacksquare$

Remark 1.7. Conditions (1.5) and (1.10) hold in particular if there exists $R > 0$ such that

$$(1.16) \quad x f_m(x) > 0 \quad \text{or} \quad x f_m(x) < 0 \quad \text{whenever} \quad |x| \geq R.$$

Example 1.8. The problem

$$\begin{aligned} D^2 x_m + \exp x_m - t_m &= 0 \quad (2 \leq m \leq n-1), \\ x_1 &= x_n, \quad D x_1 = D x_{n-1} \end{aligned}$$

has at least one solution if and only if $\sum_{m=2}^{n-1} t_m > 0$.

The necessity follows from summing both members of the equation from 2 to $n-1$, and the sufficiency from Theorem 1.3, if we observe that there exists $R > 0$ such that the function φ defined by $\varphi(x) = \exp x - \frac{1}{n-2} \left(\sum_{m=2}^{n-1} t_m \right)$ is such that $\varphi(x) > 0$ for $x \geq R$ and $\varphi(x) < 0$ for $x \leq -R$.

Example 1.9. If $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and bounded, then, for each $(t_2, \dots, t_{n-1}) \in \mathbb{R}^{n-2}$ such that

$$(1.17) \quad \limsup_{x \rightarrow -\infty} g(x) < \frac{1}{n-2} \left(\sum_{m=2}^{n-1} t_m \right) < \liminf_{x \rightarrow +\infty} g(x),$$

the problem

$$D^2x_m + g(x_m) - t_m = 0 \quad (2 \leq m \leq n-1), \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}$$

has at least one solution.

Condition (1.17) is a Landesman-Lazer-type condition for second order difference equations with periodic boundary conditions.

1.2. Periodic solutions of first order difference equations with one-side bounded nonlinearity

Let $n \in \mathbb{N}$ and f_m continuous functions ($1 \leq m \leq n-1$). Consider the problem

$$(1.18) \quad Dx_m + f_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n.$$

Let us introduce the vector space

$$U^{n-1} = \{x \in \mathbb{R}^n : x_1 = x_n\},$$

where an element of U^{n-1} can be characterized by the coordinates $x_1, \dots, x_{n-1}$ and consider the restriction L of D to U^{n-1} given by

$$(1.19) \quad \begin{aligned} Dx_1 &= x_2 - x_1, \dots, \\ Dx_{n-2} &= x_{n-1} - x_{n-2}, \\ Dx_{n-1} &= x_1 - x_{n-1}. \end{aligned}$$

We use the norm $\|x\| := \max_{1 \leq j \leq n-1} |x_j|$ in U^{n-1} and $\max_{1 \leq j \leq n-1} |x_j|$ in $\mathbb{R}^{n-1}$. It is easy to check that the linear mapping L is such that

$$\begin{aligned} \ker L &= \{(c, \dots, c) \in U^{n-1} : c \in \mathbb{R}\}, \\ \operatorname{Im} L &= \{(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : \sum_{m=1}^{n-1} y_m = 0\}, \end{aligned}$$

The projectors $P : U^{n-1} \rightarrow U^{n-1}$, $Q : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ defined by

$$\begin{aligned} P(x_1, \dots, x_{n-1}) &= (x_1, \dots, x_1), \\ Q(y_1, \dots, y_{n-1}) &= \left(\frac{1}{n-1} \sum_{m=1}^{n-1} y_m, \dots, \frac{1}{n-1} \sum_{m=1}^{n-1} y_m \right) \end{aligned}$$

are such that $\ker Q = \operatorname{Im} L$, $\operatorname{Im} P = \ker L$. Let finally $F : U^{n-1} \rightarrow \mathbb{R}^{n-1}$ be defined by

$$F(x_1, \dots, x_{n-1}) = (f_1(x_1), \dots, f_{n-1}(x_{n-1})),$$

so the zeros of the continuous mapping $L + F$ correspond to the solutions of (1.18). We shall apply Mawhin's continuation theorem in finite dimension. We first obtain a priori estimates for the possible zeros of

$L + \lambda F$ with $\lambda \in]0, 1]$, or equivalently for the possible solutions of the periodic boundary value problem

$$(1.20) \quad Dx_m + \lambda f_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n,$$

with $\lambda \in]0, 1]$.

Lemma 1.10. *If the function f_m ($1 \leq m \leq n-1$), is bounded from below or from above, and if for some $R > 0$*

$$(1.21) \quad \sum_{m=1}^{n-1} f_m(x_m) \neq 0$$

whenever $\min_{1 \leq j \leq n-1} x_j \geq R$ or $\max_{1 \leq j \leq n-1} x_j \leq -R$, then there exists $\rho \geq R$ such that, for each $\lambda \in]0, 1]$ and each possible solution x of (1.20), one has $\|x\| < \rho$.

Proof. Let $\lambda \in]0, 1]$ and x be a possible solution of (1.20). It follows that

$$(1.22) \quad \sum_{m=1}^{n-1} f_m(x_m) = 0.$$

On the other hand, if we assume that each f_m ($1 \leq m \leq n-1$) is bounded from below, say by c , we have, for all $1 \leq m \leq n-1$,

$$|f_m(x)| - |c| \leq |f_m(x) - c| = f_m(x) - c \quad (1 \leq m \leq n-1),$$

so that

$$(1.23) \quad |f_m(x)| \leq f_m(x) + 2c^- \quad (x \in \mathbb{R}).$$

Hence, using (1.20), (1.22) and (1.23), we obtain

$$(1.24) \quad \sum_{m=1}^{n-1} |Dx_m| = \lambda \sum_{m=1}^{n-1} |f_m(x_m)| \leq \sum_{m=1}^{n-1} |f_m(x_m)|$$

$$(1.25) \quad \leq \sum_{m=1}^{n-1} f_m(x_m) + 2(n-1)c^- = 2(n-1)c^-.$$

We deduce

$$\begin{aligned} \max_{1 \leq m \leq n-1} x_m &\leq \min_{1 \leq m \leq n-1} x_m + \sum_{m=1}^{n-1} |Dx_m| \\ &\leq \min_{1 \leq m \leq n-1} x_m + 2(n-1)c^-. \end{aligned}$$

If $1 \leq j \leq n-1$ is such that $x_j = \min_{1 \leq m \leq n-1} x_m$ then, using (1.22) and assumption (1.21), we obtain $x_j < R$. Analogously $-R < \max_{1 \leq m \leq n-1} x_m$. Consequently we have

$$-[R + 2(n-1)c^-] < \min_{1 \leq m \leq n-1} x_m \leq \max_{1 \leq m \leq n-1} x_m < R + 2(n-1)c^-.$$

It follows that we can take any $\rho \geq R + 2(n-1)c^-$. If the f_m are bounded from above, it suffices to consider the equivalent problem $-Lx - F(x) = 0$ with $-F$ bounded from below, as $-L$ has the same kernel and range than L . ■

By using now arguments completely similar to those in the preceding section, we can apply Mawhin's continuation theorem in finite dimension to obtain the following existence result. We define $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(x) = \frac{1}{n-1} \left(\sum_{m=1}^{n-1} f_m(x) \right)$$

Theorem 1.11. *Suppose that the functions f_m ($1 \leq m \leq n-1$) satisfy the conditions of Lemma 1.10 and that*

$$\varphi(-\rho)\varphi(\rho) < 0.$$

Then, problem (1.18) has at least one solution.

Example 1.12. Given positive a_m and b_m ($1 \leq m \leq n-1$), the periodic problem

$$(1.26) \quad Dx_m + a_m \exp x_m - b_m = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n,$$

related to the study of positive periodic solutions of Verhulst equation with variable coefficients has at least one solution.

1.3. Second order difference equations with Dirichlet boundary conditions and one-side bounded nonlinearity

Let $n \in \mathbb{N}$ fixed and $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$. Define $(Dx_0, \dots, Dx_{n-1}) \in \mathbb{R}^n$ and $(D^2x_1, \dots, D^2x_{n-1}) \in \mathbb{R}^{n-1}$ by

$$\begin{aligned} Dx_m &= x_{m+1} - x_m, & (0 \leq m \leq n-1) \\ D^2x_m &= x_{m+1} - 2x_m + x_{m-1}, & (1 \leq m \leq n-1). \end{aligned}$$

Consider the Dirichlet eigenvalue problem

$$(1.27) \quad D^2x_m + \lambda x_m = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n.$$

The following results are classical, but we reproduce them for completion. If we look for a nontrivial solution of the form (for some $\theta \in \mathbb{R}$)

$$(1.28) \quad x_m = A \sin m\theta \quad (0 \leq m \leq n),$$

then one must have

$$\sin(m-1)\theta + (\lambda - 2)\sin m\theta + \sin(m+1)\theta = 0 \quad (1 \leq m \leq n-1)$$

or, equivalently,

$$(\sin m\theta)[2\cos\theta + \lambda - 2] = 0 \quad (1 \leq m \leq n-1).$$

This system of equations is satisfied if we choose

$$(1.29) \quad \lambda = 2 - 2\cos\theta$$

and the Dirichlet boundary conditions $x_0 = 0 = x_n$ hold if and only if $\sin n\theta = 0$, i.e. if and only if

$$\theta = \theta_k := \frac{k\pi}{n} \quad (k = 1, 2, \dots, n-1).$$

Consequently, the eigenvalues of (1.27) (in increasing order) are

$$(1.30) \quad \lambda_k = 2 \left(1 - \cos \frac{k\pi}{n} \right) = 4 \sin^2 \frac{k\pi}{2n} \quad (k = 1, 2, \dots, n-1)$$

and a corresponding eigenvector $\varphi^k = (\varphi_1^k, \dots, \varphi_{n-1}^k)$ is given by

$$(1.31) \quad \varphi^k = \left(\sin \frac{k\pi}{n}, \sin \frac{2k\pi}{n}, \dots, \sin \frac{(n-1)k\pi}{n} \right) \quad (k = 1, 2, \dots, n-1).$$

In particular, the eigenvector φ^1 associated to the first eigenvalue

$$(1.32) \quad \lambda_1 = 2 \left(1 - \cos \frac{\pi}{n} \right) = 4 \sin^2 \frac{\pi}{2n},$$

given by

$$(1.33) \quad \varphi_1 = \left(\sin \frac{\pi}{n}, \sin \frac{2\pi}{n}, \dots, \sin \frac{(n-1)\pi}{n} \right)$$

has all its components positive. Furthermore, as the φ^k constitute a system of eigenvectors of a symmetric matrix, they satisfy the orthogonality conditions $\langle \varphi^j, \varphi^k \rangle = 0$ for $j \neq k$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n-1}$. The norm generated by $\langle \cdot, \cdot \rangle$ will be denoted by $\|\cdot\|_2$.

We need some identities and inequalities for finite sequence satisfying the Dirichlet boundary conditions. The first one is a type of summation by parts.

Lemma 1.13. *If $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ and $(y_0, \dots, y_n) \in \mathbb{R}^{n+1}$ are such that $x_0 = 0 = x_n$ and $y_0 = 0 = y_n$, the identity*

$$(1.34) \quad \sum_{m=1}^{n-1} x_m D^2 y_m = \sum_{m=1}^{n-1} y_m D^2 x_m$$

holds.

Proof. We have

$$\begin{aligned} & \sum_{m=1}^{n-1} x_m (y_{m+1} - 2y_m + y_{m-1}) - \sum_{m=1}^{n-1} y_m (x_{m+1} - 2x_m + x_{m-1}) \\ &= \sum_{m=1}^{n-2} x_m y_{m+1} + \sum_{m=2}^{n-1} x_m y_{m-1} - \sum_{m=1}^{n-2} y_m x_{m+1} - \sum_{m=2}^{n-1} y_m x_{m-1} = 0 \end{aligned}$$

■

Define

$$(1.35) \quad x = (x_1, \dots, x_{n-1}), \quad \bar{x} = \langle x, \varphi^1 \rangle \frac{\varphi^1}{\|\varphi^1\|_2^2}, \quad \tilde{x} = x - \bar{x},$$

so that

$$(\bar{x})_m = \left(\sum_{m=1}^{n-1} x_m \sin \frac{m\pi}{n} \right) \frac{\varphi_m^1}{\|\varphi^1\|_2^2} \quad (1 \leq m \leq n-1), \quad \langle \tilde{x}, \varphi^1 \rangle = 0.$$

Notice that

$$\begin{aligned} D^2 x_m + \lambda_1 x_m &= D^2 (\bar{x})_m + \lambda_1 (\bar{x})_m + D^2 (\tilde{x})_m + \lambda_1 (\tilde{x})_m \\ (1.36) \quad &= D^2 (\tilde{x})_m + \lambda_1 (\tilde{x})_m. \end{aligned}$$

Lemma 1.14. *If $x = (x_1, \dots, x_{n-1})$, then there exists a constant $c_n > 0$ which depends only on n such that*

$$\max_{1 \leq m \leq n-1} |(\tilde{x})_m| \leq c_n \sum_{m=1}^{n-1} |D^2 (\tilde{x})_m + \lambda_1 (\tilde{x})_m| \varphi_m^1.$$

Proof. The applications

$$\begin{aligned} (x_1, \dots, x_{n-1}) &\mapsto \max_{1 \leq m \leq n-1} |x_m|, \\ (x_1, \dots, x_{n-1}) &\mapsto \sum_{m=1}^{n-1} |D^2 x_m + \lambda_1 x_m| \varphi_m^1 \end{aligned}$$

define two norms on the subspace $V = \{x \in \mathbb{R}^{n-1} : \langle x, \varphi^1 \rangle = 0\}$. They are equivalent, and inequality above holds. ■

Let $n \in \mathbb{N}$ and f_m continuous functions ($2 \leq m \leq n-1$). Consider the problem

$$(1.37) \quad D^2 x_m + \lambda_1 x_m + f_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n.$$

Let us introduce the vector space

$$(1.38) \quad V^{n-1} = \{x \in \mathbb{R}^{n+1} : x_0 = 0 = x_n\}$$

endowed with the orientation of $\mathbb{R}^{n+1}$. Its elements can be associated to the coordinates $(x_1, \dots, x_{n-1})$ and correspond to the elements of $\mathbb{R}^{n+1}$ of the form $(0, x_1, \dots, x_{n-1}, 0)$, so that the restriction D^2 to V^{n-1} is well defined in terms of $(x_1, \dots, x_{n-1})$. We use the norm $\|x\| := \max_{1 \leq j \leq n-1} |x_j|$ in V^{n-1} and $\max_{1 \leq j \leq n-1} |x_j|$ in $\mathbb{R}^{n-1}$. We call $L : V^{n-1} \rightarrow \mathbb{R}^{n-1}$ the restriction of $D^2 + \lambda_1 I$ to V^{n-1} . We have

$$\ker L = \{c\varphi^1 : c \in \mathbb{R}\},$$

and, by the properties of symmetric matrices,

$$\operatorname{Im} L = \{y \in \mathbb{R}^{n-1} : \langle y, \varphi^1 \rangle = 0\}.$$

Consider the projectors $P : V^{n-1} \rightarrow V^{n-1}$ and $Q : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ defined by

$$P(x) = \langle x, \varphi^1 \rangle \frac{\varphi^1}{\|\varphi^1\|_2^2}, \quad Q(y) = \langle y, \varphi^1 \rangle \frac{\varphi^1}{\|\varphi^1\|_2^2},$$

so that $\ker Q = \operatorname{Im} L$, $\operatorname{Im} P = \ker L$. We also define $F : V^{n-1} \rightarrow \mathbb{R}^{n-1}$ by

$$F(x) = (f_1(x_1), \dots, f_{n-1}(x_{n-1})),$$

so that $(0, x_1, \dots, x_{n-1}, 0)$ is a solution of (1.37) if and only if $(x_1, \dots, x_{n-1}) \in V^{n-1}$ is a zero of $L + F$. In order to obtain the existence of zeros of $L + F$ we shall apply Mawhin's continuation theorem in finite dimension. We first obtain a priori estimates for the possible zeros of $L + \lambda F$ with $\lambda \in]0, 1]$, or equivalently for the possible solutions of the Dirichlet problem

$$(1.39) \quad D^2 x_m + \lambda_1 x_m + \lambda f_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n,$$

with $\lambda \in]0, 1]$.

Lemma 1.15. *If each function f_m ($1 \leq m \leq n-1$) is bounded from below or from above, and if, for some $R > 0$, one has*

$$(1.40) \quad \sum_{m=1}^{n-1} f_m(x_m) \varphi_m^1 \neq 0$$

whenever $\min_{1 \leq j \leq n-1} x_j \geq R$ or $\max_{1 \leq j \leq n-1} x_j \leq -R$ then there exists $\rho > R$ such that, for each $\lambda \in]0, 1]$ and each possible solution x of (1.39), one has $\|x\| < \rho$.

Proof. Assume first that each f_m is bounded from below and let c such that $f_m(x) \geq c$ for all $x \in \mathbb{R}$ and all $1 \leq m \leq n-1$. We have, for all $1 \leq m \leq n-1$,

$$(1.41) \quad |f_m(x)| \leq f_m(x) + 2|c|, \quad (x \in \mathbb{R}).$$

Let $\lambda \in]0, 1]$ and x be a solution of (1.39). Multiplying (1.39) by φ_m^1 and adding, we obtain

$$\sum_{m=1}^{n-1} [\varphi_m^1 D^2 x_m + \lambda_1 \varphi_m^1 x_m] + \lambda \sum_{m=1}^{n-1} \varphi_m^1 f_m(x_m) = 0.$$

But, using Lemma 1.13,

$$\sum_{m=1}^{n-1} [\varphi_m^1 D^2 x_m + \lambda_1 \varphi_m^1 x_m] = \sum_{m=1}^{n-1} [x_m (D^2 \varphi_m^1 + \lambda_1 \varphi_m^1)] = 0,$$

so that

$$(1.42) \quad \sum_{m=1}^{n-1} f_m(x_m) \varphi_m^1 = 0.$$

On the other hand, using (1.36) we have

$$(1.43) \quad (D^2 \tilde{x})_m + \lambda_1 (\tilde{x})_m + \lambda f_m(x_m) = 0 \quad (1 \leq m \leq n-1).$$

Using (1.41), (1.42) and (1.43) we deduce that

$$\sum_{m=1}^{n-1} |D^2(\tilde{x})_m + \lambda_1(\tilde{x})_m| \varphi_m^1 = \lambda \sum_{m=1}^{n-1} |f_m(x_m)| \varphi_m^1 \leq 2|c| \sum_{m=1}^{n-1} \varphi_m^1,$$

which implies that there exists a constant $R_1 > 0$ such that

$$(1.44) \quad \sum_{m=1}^{n-1} |D^2(\tilde{x})_m + \lambda_1(\tilde{x})_m| \varphi_m^1 \leq R_1.$$

Using (1.44) and Lemma 1.14, it follows that there exists $R_2 > 0$ such that

$$\max_{1 \leq m \leq n-1} |(\tilde{x})_m| \leq R_2.$$

Then, by (1.40) and (1.42), there exists $1 \leq k \leq n-1$ and $1 \leq l \leq n-1$ such that $x_k < R$ and $x_l > -R$. Consequently, $(\bar{x})_k = x_k - (\tilde{x})_k < R + R_2$ and $(\bar{x})_l = x_l - (\tilde{x})_l > -R - R_2$. Therefore, for each $1 \leq m \leq n-1$,

$$\begin{aligned} (\bar{x})_m &= \frac{(\bar{x})_k}{\varphi_k^1} \varphi_m^1 < (R + R_2) \max_{1 \leq m \leq n-1} \frac{\varphi_m^1}{\varphi_k^1} := R_3, \\ (\bar{x})_m &= \frac{(\bar{x})_l}{\varphi_l^1} \varphi_m^1 > -(R + R_2) \max_{1 \leq m \leq n-1} \frac{\varphi_m^1}{\varphi_l^1} := -R_3. \end{aligned}$$

Consequently $\|x\| < \rho$ for some $\rho > 0$.

In the case where the f_m are bounded from above, it suffices to write the problem

$$\tilde{L}x + \tilde{F}(x) = 0,$$

with $\tilde{L} = -L$ and $\tilde{F} = -F$ to reduce it to a problem with $\tilde{F}$ bounded from below, noticing that L and $\tilde{L}$ have the same kernel and the same range. $\blacksquare$

By using now arguments completely similar to those in the preceding sections, we can apply Mawhin's continuation theorem in finite dimension to obtain the following existence result. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be the continuous function defined by

$$(1.45) \quad \varphi(u) = \sum_{m=1}^{n-2} f_m(u\varphi_m^1)\varphi_m^1.$$

Theorem 1.16. *If the functions f_m ($1 \leq m \leq n-1$) satisfy the conditions of Lemma 1.15 and if*

$$(1.46) \quad \varphi(-\rho)\varphi(\rho) < 0,$$

then, problem (1.37) has at least one solution.

Example 1.17. *The problem*

$$D^2x_m + \lambda_1 x_m + \exp x_m - t_m = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n,$$

has at least one solution if and only if $\sum_{m=1}^{n-1} t_m \varphi_m^1 > 0$.

The necessity follows from summing both members of the equation from 1 to $n-1$ after multiplication by φ_m^1 , and the sufficiency from Theorem 1.16, if we observe that there exists $R > 0$ such that the function φ defined by $\varphi(u) = \sum_{m=1}^{n-1} [\exp(u\varphi_m^1) - t_m] \varphi_m^1$ is such that $\varphi(u) > 0$ for $u \geq R$ and $\varphi(u) < 0$ for $u \leq -R$.

Example 1.18. If $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function bounded from below or from above and $(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$(1.47) \quad -\infty < \limsup_{x \rightarrow -\infty} g(x) < \frac{\sum_{m=1}^{n-1} t_m \varphi_m^1}{\sum_{m=1}^{n-1} \varphi_m^1} < \liminf_{x \rightarrow +\infty} g(x) < +\infty,$$

then the problem

$$D^2 x_m + \lambda_1 x_m + g(x_m) - t_m = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n,$$

has at least one solution.

Condition (1.47) is a Landesman-Lazer-type condition for difference equations with Dirichlet boundary conditions. It is easily shown to be necessary if

$$\limsup_{x \rightarrow -\infty} g(x) < g(x) < \liminf_{x \rightarrow +\infty} g(x)$$

for all $x \in \mathbb{R}$.

2. UPPER AND LOWER SOLUTIONS FOR DIFFERENCE EQUATIONS

2.1. First order difference vs first order differential equations

Consider the periodic boundary value problem

$$(2.1) \quad x' + f(t, x) = 0, \quad x(0) = x(T),$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Definition 2.1. α (resp. β) $\in C^1([0, T])$ is a lower solution (resp. upper solution) of problem (2.1) if

$$\begin{aligned} \alpha'(t) + f(t, \alpha(t)) &\geq 0 \quad (t \in [0, T]), \quad \alpha(0) \geq \alpha(T) \\ (\text{resp. } \beta'(t) + f(t, \beta(t)) &\leq 0 \quad (t \in [0, T]), \quad \beta(0) \leq \beta(T)). \end{aligned}$$

The basic existence theorem of the method of upper and lower solutions for (2.1) goes as follows.

Theorem 2.2. If problem (2.1) has a lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$ (resp. $\beta(t) \leq \alpha(t)$ for all $t \in [0, T]$), then problem (2.1) has at least one solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [0, T]$ (resp. $\beta(t) \leq x(t) \leq \alpha(t)$ for all $t \in [0, T]$).

Proof. Let us first consider the case where $\alpha(t) \leq \beta(t)$ ($t \in [0, T]$). Define $\gamma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\gamma(t, x) = \begin{cases} \beta(t), & x > \beta(t) \\ x, & \alpha(t) \leq x \leq \beta(t) \\ \alpha(t), & x < \alpha(t), \end{cases}$$

and consider the modified problem

$$(2.2) \quad x' + f(t, \gamma(t, x)) - x + \gamma(t, x) = 0, \quad x(0) = x(T),$$

which coincides with (2.1) when $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [0, T]$. First it is clear, writing (2.2) in the equivalent form

$$x(t) = \int_0^T G(t, s)[f(s, \gamma(s, x(s))) + \gamma(s, x(s))] ds$$

with $G(t, s)$ the Green function of the linear periodic problem

$$-x' + x = h(t), \quad x(0) = x(T),$$

that the existence of at least one solution to problem (2.2) follows from Schauder's fixed point theorem. It suffices now to show that each solution x of (2.2) is such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [0, T]$, and hence is a solution of (2.1). If $\alpha(t) > x(t)$ for some $t \in [0, T]$, then $\alpha - x$ reaches at some $\tau \in [0, T]$ a positive maximum, so that $\alpha(\tau) > x(\tau)$ and $\gamma(\tau, x(\tau)) = \alpha(\tau)$. The condition $\alpha(0) \geq \alpha(T)$ implies that the maximum cannot be achieved at T without being achieved at 0, so that we can assume that $\tau \in [0, T[$. Then $\alpha'(\tau) - x'(\tau) \leq 0$, and

$$\alpha'(\tau) + f(\tau, \alpha(\tau)) \leq x'(\tau) + f(\tau, \gamma(\tau, x(\tau))) = x(\tau) - \alpha(\tau) < 0,$$

a contradiction with the definition of a lower solution. One shows in a similar way that $x(t) \leq \beta(t)$ for all $t \in [0, T]$.

In the case where $\beta(t) \leq \alpha(t)$ for all $t \in [0, T]$, one defines $\delta : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\delta(t, x) = \begin{cases} \alpha(t), & x > \alpha(t) \\ x, & \beta(t) \leq x \leq \alpha(t) \\ \beta(t), & x < \beta(t), \end{cases}$$

and considers the modified problem

$$(2.3) \quad x' + f(t, \delta(t, x)) + x - \delta(t, x) = 0, \quad x(0) = x(T),$$

which has at least one solution using Schauder's fixed point theorem. An argument like above shows that each solution x of (2.3) is such that $\beta(t) \leq x(t) \leq \alpha(t)$ for all $t \in [0, T]$. $\blacksquare$

If $n \geq 2$ and $f_m : \mathbb{R} \rightarrow \mathbb{R}$ are continuous ($1 \leq m \leq n-1$), one considers the periodic boundary value problem

$$(2.4) \quad Dx_m + f_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n.$$

Definition 2.3. $\alpha = (\alpha_1, \dots, \alpha_n)$ (resp. $\beta = (\beta_1, \dots, \beta_n)$) is called a lower solution (resp. upper solution) for (2.4) if the inequalities

$$\begin{aligned} D\alpha_m + f_m(\alpha_m) &\geq 0, \quad (1 \leq m \leq n-1), \quad \alpha_1 \geq \alpha_n \\ (\text{resp. } D\beta_m + f_m(\beta_m) &\leq 0, \quad (1 \leq m \leq n-1), \quad \beta_1 \leq \beta_n) \end{aligned}$$

hold.

Theorem 2.4. If (2.4) has a lower solution $\alpha = (\alpha_1, \dots, \alpha_n)$ and an upper solution $\beta = (\beta_1, \dots, \beta_n)$ such that $\alpha_m \leq \beta_m$ ($1 \leq m \leq n-1$), then (2.4) has a solution $x = (x_1, \dots, x_n)$ such that $\alpha_m \leq x_m \leq \beta_m$ ($1 \leq m \leq n-1$).

Proof. Let $\gamma_m : \mathbb{R} \rightarrow \mathbb{R}$ ($1 \leq m \leq n-1$) be the continuous functions defined by

$$\gamma_m(x) = \begin{cases} \beta_m, & x > \beta_m \\ x, & \alpha_m \leq x \leq \beta_m \\ \alpha_m, & x < \alpha_m, \end{cases}$$

and consider the modified problem

$$\begin{aligned} Dx_m + f_m \circ \gamma_m(x_m) - x_m + \gamma_m(x_m) &= 0 \quad (1 \leq m \leq n-1), \\ (2.5) \quad x_1 &= x_n. \end{aligned}$$

The existence of at least one solution to (2.5) is an easy consequence of Brouwer fixed point theorem in the space

$$(2.6) \quad U^{n-1} = \{x \in \mathbb{R}^n : x_1 = x_n\},$$

whose elements can be characterized by the coordinates $x_1, \dots, x_{n-1}$. Indeed, the restriction L of $D - I$ to U^{n-1} , given by

$$\begin{aligned} Dx_1 &= x_2 - 2x_1, \dots, \\ Dx_{n-2} &= x_{n-1} - 2x_{n-2}, \\ (2.7) \quad Dx_{n-1} &= x_1 - 2x_{n-1}. \end{aligned}$$

is one-to-one, hence invertible, and (2.5) is equivalent to the fixed point problem

$$x_m = -L^{-1}[f_m \circ \gamma_m(x_m) + \gamma_m(x_m)] \quad (1 \leq m \leq n-1),$$

in U^{n-1} . It remains to show that if $x = (x_1, \dots, x_n)$ is a solution of (2.5), then $\alpha_m \leq x_m \leq \beta_m$ ($1 \leq m \leq n$), so that $(x_1, \dots, x_n)$ is a solution of (2.4). Suppose by contradiction that $\alpha_i - x_i > 0$ for some $1 \leq i \leq n$, so that $\alpha_m - x_m = \max_{1 \leq j \leq n} (\alpha_j - x_j) > 0$. If $1 \leq m \leq n-1$, then

$$\alpha_{m+1} - x_{m+1} \leq \alpha_m - x_m,$$

which gives

$$D\alpha_m + f_m(\alpha_m) \leq Dx_m + f_m \circ \gamma_m(x_m) = x_m - \alpha_m < 0,$$

a contradiction with the definition of lower solution. Now the condition $\alpha_1 \geq \alpha_n$, shows that the maximum is reached at $m = n$ only if it is also reached at $m = 1$, a case already excluded. One shows in the same way that $x \leq \beta$. ■

A simple but useful consequence of Theorem 2.4, goes as follows.

Corollary 2.5. *Assume that there exists numbers $\alpha \leq \beta$ such that*

$$f_m(\alpha) \geq f_m(\beta) \quad (1 \leq m \leq n-1).$$

Then problem (2.4) has at least one solution with $\alpha \leq x_m \leq \beta$ ($1 \leq m \leq n-1$).

Proof. Just observe that $(\alpha, \dots, \alpha)$ is a lower solution and $(\beta, \dots, \beta)$ an upper solution for (2.4). ■

Example 2.6. *For each $p > 0, a_m > 0$ and $b_m \in \mathbb{R}$ ($1 \leq m \leq n-1$) the problem*

$$Dx_m - a_m|x_m|^{p-1}x_m = b_m \quad (1 \leq m \leq n-1), \quad x_1 = x_n$$

has at least one solution, because if

$$R \geq \left(\max_{1 \leq m \leq n-1} \frac{|b_m|}{a_m} \right)^{1/p},$$

then $(-R, \dots, -R)$ is a lower solution and $(R, \dots, R)$ an upper solution

Remark 2.7. When $\beta_m \leq \alpha_m$ ($1 \leq m \leq n-1$), one can try to adapt the argument of Theorem 2.2 for the case where $\beta(t) \leq \alpha(t)$ by defining

$$\delta_m(x) = \begin{cases} \alpha_m, & x > \alpha_m \\ x, & \beta_m \leq x \leq \alpha_m \\ \beta_m, & x < \beta_m, \end{cases}$$

and considering the modified problem

$$(2.8) \quad \begin{aligned} Dx_m + f_m \circ \delta_m(x_m) + x_m - \delta_m(x_m) &= 0 \quad (1 \leq m \leq n-1), \\ x_1 &= x_n. \end{aligned}$$

The Brouwer fixed point theorem still gives the existence of at least one solution for (2.8). If one tries to show that, say, $\beta_m \leq x_m$ ($1 \leq m \leq n-1$), and assume by contradiction that $\beta_i - x_i > 0$ for some $1 \leq i \leq n$, so that $\beta_m - x_m = \max_{1 \leq j \leq n} (\beta_j - x_j) > 0$, one has for $1 \leq m \leq n-1$

$$\beta_{m+1} - x_{m+1} \leq \beta_m - x_m,$$

which gives

$$D\beta_m + f_m(\beta_m) \leq Dx_m + f_m \circ \gamma_m(x_m) = -x_m + \beta_m > 0,$$

implying no contradiction with $D\beta_m + f_m(\beta_m) \leq 0$.

That the argument works in the ordinary differential case and not in the difference case comes from the fact that a local extremum is characterized by an equality (vanishing of the first derivative) in the first case and by two inequalities (with only one usable in the argument) in the second case. This raises the question of the validity of the method of upper and lower solutions in reverse order for difference equations. The question is solved by the negative in what follows.

An *eigenvalue* of the first order difference operator with periodic boundary conditions is a $\lambda \in \mathbb{C}$ such that the problem

$$(2.9) \quad Dx_m = \lambda x_m \quad (1 \leq m \leq n-1), \quad x_1 = x_n$$

has a nontrivial solution. Explicitely, system (2.9) can be written

$$(2.10) \quad \begin{aligned} x_1 - x_n &= 0 \\ x_2 - (1 + \lambda)x_1 &= 0 \\ \dots &\dots \dots \dots \dots \\ x_n - (1 + \lambda)x_{n-1} &= 0 \end{aligned}$$

Looking for solutions of the form $x_m = e^{im\theta}$ ($1 \leq m \leq n-1$) for some $\theta \in \mathbb{R}$, one easily finds that θ must verify the equations

$$\lambda = -1 + e^{i\theta}, \quad e^{i(n-1)\theta} = 1,$$

which gives the $n-1$ distinct eigenvalues

$$(2.11) \quad \lambda_k = -1 + e^{\frac{2k\pi i}{n-1}} \quad (0 \leq k \leq n-2),$$

with the corresponding eigenvectors $x_m^k = e^{\frac{2km\pi i}{n-1}}$ ($1 \leq m \leq n; 0 \leq k \leq n-2$). In particular, $\lambda_0 = 0$ is always a real eigenvalue, and all the other eigenvalues have negative real part. If $n = 2$, 0 is the unique eigenvalue; if $n > 2$ is even, 0 is the unique real eigenvalue; if n is odd, $\lambda_{\frac{n-1}{2}} = -2$ is the unique nonzero real eigenvalue.

For $n \geq 2$ odd and $\lambda = -2$, system (2.10) becomes

$$(2.12) \quad \begin{aligned} x_1 - x_n &= 0 \\ x_2 + x_1 &= 0 \\ \dots &\dots \dots \\ x_n + x_{n-1} &= 0 \end{aligned}$$

and has the eigenvector with components $x_m^{(n-1)/2} = (-1)^{m-1}$ ($1 \leq m \leq n$). The adjoint system

$$\begin{aligned}
 x_1 + x_2 &= 0 \\
 \dots \quad \dots \quad \dots \\
 x_{n-1} + x_n &= 0 \\
 -x_1 + x_n &= 0
 \end{aligned}
 \tag{2.13}$$

has the nontrivial solution with components $x_m = (-1)^{m-1}$ ($1 \leq m \leq n$). As $b_m = \delta_{nm}$ ($1 \leq m \leq n$) (Kronecker symbol) is not orthogonal to the kernel of the adjoint system (2.13), the problem

$$\begin{aligned}
 x_1 - x_n &= 0 \\
 x_2 + x_1 &= 0 \\
 \dots \quad \dots \quad \dots \\
 x_{n-1} + x_{n-2} &= 0 \\
 x_n + x_{n-1} &= 1
 \end{aligned}$$

has no solution, or, equivalently *the problem*

$$\begin{aligned}
 Dx_m + 2x_m &= 0 \quad (1 \leq m \leq n-2), \\
 Dx_{n-1} + 2x_{n-1} &= 1, \\
 x_1 &= x_n
 \end{aligned}
 \tag{2.14}$$

has no solution. However, $\alpha = (1, 1, \dots, 1)$ is a lower solution and $\beta = (0, 0, \dots, 0)$ is an upper solution of (2.14) such that $\beta_m \leq \alpha_m$ ($1 \leq m \leq n$).

If now $n > 2$ is even, the problem

$$\begin{aligned}
 Dx_m + 2x_m &= 0 \quad (1 \leq m \leq n-3), \quad Dx_{n-2} + 2x_{n-2} = 1, \\
 Dx_{n-1} &= 0, \quad x_1 = x_n
 \end{aligned}
 \tag{2.15}$$

is of course equivalent to the problem

$$Dx_m + 2x_m = 0 \quad (1 \leq m \leq n-3), \quad Dx_{n-2} + 2x_{n-2} = 1, \quad x_1 = x_{n-1},$$

As $n-1$ is odd, it follows from the counterexample (2.14) that *problem (2.15) has no solution. However $\alpha = (1, 1, \dots, 1)$ is a lower solution and $\beta = (0, 0, \dots, 0)$ is an upper solution of (2.15) such that $\beta_m \leq \alpha_m$ ($1 \leq m \leq n$).*

For $n = 2$, problem (2.4) is equivalent to the unique scalar equation

$$f_1(x_1) = 0$$

and, in this case, the validity of the method of upper and lower solutions, independently of their order, follows from its equivalence with Bolzano's theorem applied to the real function f_1 .

Remark 2.8. Notice that, in contrast to the periodic problem for first order difference equations, whose eigenvalues are in the left half-plane, all the eigenvalues $\lambda_k = \frac{2k\pi i}{T}$ ($k \in \mathbb{Z}$) of the differential operator $\frac{d}{dt}$ with periodic boundary conditions on $[0, T]$ are on the imaginary axis. Hence the situation for the method of lower and upper solutions for periodic solutions of first order difference equations is more akin to the one of periodic solutions of second order ordinary differential or difference equations, whose eigenvalues are also located in a half-plane, and where the lower solution has to be smaller than the upper solution to make the method of upper and lower solutions fully conclusive.

2.2. Upper and lower solutions for second order difference equations with periodic boundary conditions

Let $f_m : \mathbb{R} \rightarrow \mathbb{R}$, ($2 \leq m \leq n-1$) be continuous functions. We study the existence of solutions for the periodic boundary value problem

$$(2.16) \quad \begin{aligned} D^2x_m + f_m(x_m) &= 0 \quad (2 \leq m \leq n-1), \\ x_1 &= x_n, \quad Dx_1 = Dx_{n-1}. \end{aligned}$$

If $\alpha, \beta \in \mathbb{R}^p$, we write $\alpha \leq \beta$ (resp. $\alpha < \beta$) if $\alpha_i \leq \beta_i$ for all $1 \leq i \leq p$ (resp. $\alpha_i < \beta_i$ for all $1 \leq i \leq p$).

Definition 2.9. $\alpha = (\alpha_1, \dots, \alpha_n)$ (resp. $\beta = (\beta_1, \dots, \beta_n)$) is called a lower solution (resp. upper solution) for (2.16) if

$$\alpha_1 = \alpha_n, \quad D\alpha_1 \geq D\alpha_{n-1} \quad (\text{resp. } \beta_1 = \beta_n, \quad D\beta_1 \leq D\beta_{n-1})$$

and the inequalities

$$(2.17) \quad \begin{aligned} D^2\alpha_m + f_m(\alpha_m) &\geq 0 \quad (\text{resp. } D^2\beta_m + f_m(\beta_m) \leq 0) \\ (2 \leq m \leq n-1) \end{aligned}$$

hold. Such a lower or upper solution will be called strict if the inequalities (2.17) are strict.

Theorem 2.10. If (2.16) has a lower solution $\alpha = (\alpha_1, \dots, \alpha_n)$ and an upper solution $\beta = (\beta_1, \dots, \beta_n)$ such that $\alpha \leq \beta$, then (2.16) has a solution $x = (x_1, \dots, x_n)$ such that $\alpha \leq x \leq \beta$. Moreover, if α and β are strict, then $\alpha < x < \beta$.

Proof. I. A modified problem.

Let $\gamma_m : \mathbb{R} \rightarrow \mathbb{R}$, ($2 \leq m \leq n-1$) be the continuous functions defined

by

$$\gamma_m(x) = \begin{cases} \beta_m, & x > \beta_m \\ x, & \alpha_m \leq x \leq \beta_m \\ \alpha_m, & x < \alpha_m, \end{cases}$$

and define $F_m = f_m \circ \gamma_m$, ($2 \leq m \leq n-1$). We consider the modified problem

$$(2.18) \quad \begin{aligned} D^2 x_m + F_m(x_m) - [x_m - \gamma_m(x_m)] &= 0 \quad (2 \leq m \leq n-1), \\ x_1 = x_n, \quad Dx_1 &= Dx_{n-1}, \end{aligned}$$

and show that if $x = (x_1, \dots, x_n)$ is a solution of (2.18) then $\alpha \leq x \leq \beta$ and hence x is a solution of (2.16). Suppose by contradiction that there is some $1 \leq i \leq n$ such that $\alpha_i - x_i > 0$ so that $\alpha_m - x_m = \max_{1 \leq j \leq n} (\alpha_j - x_j) > 0$. Using the fact that $\alpha_1 - x_1 = \alpha_n - x_n$ and $D(\alpha_1 - x_1) \geq D(\alpha_{n-1} - x_{n-1})$ we obtain that $2 \leq m \leq n-1$, because, if $\alpha_j - x_j < \alpha_1 - x_1 = \alpha_n - x_n$ for all $2 \leq j \leq n-1$, then

$$0 > \alpha_2 - x_2 - (\alpha_1 - x_1) \geq \alpha_n - x_n - (\alpha_{n-1} - x_{n-1}) > 0,$$

a contradiction. Hence

$$D^2(\alpha_m - x_m) = (\alpha_{m+1} - x_{m+1}) - 2(\alpha_m - x_m) + (\alpha_{m-1} - x_{m-1}) \leq 0,$$

and

$$\begin{aligned} D^2 \alpha_m &\leq D^2 x_m = -F_m(x_m) + (x_m - \gamma_m(x_m)) \\ &= -f_m(\alpha_m) + (x_m - \alpha_m) < -f_m(\alpha_m) \leq D^2 \alpha_m, \end{aligned}$$

a contradiction. Analogously we can show that $x \leq \beta$. We remark that if α, β are strict, then $\alpha < x < \beta$.

II. Abstract formulation of problem (2.18).

Consider the vector space

$$(2.19) \quad V^{n-2} = \{x \in \mathbb{R}^n : x_1 = x_n, Dx_1 = Dx_{n-1}\}$$

endowed with the orientation of $\mathbb{R}^n$. Its elements can be associated to the coordinates $(x_2, \dots, x_{n-1})$ and correspond to the elements of $\mathbb{R}^n$ of the form $(\frac{x_2 + x_{n-1}}{2}, x_2, \dots, x_{n-1}, \frac{x_2 + x_{n-1}}{2})$, so that the restriction D^2 to V^{n-2} is well defined in terms of $(x_2, \dots, x_{n-1})$. We use the norm $\|x\| := \max_{2 \leq j \leq n-1} |x_j|$ in V^{n-2} and $\max_{1 \leq j \leq n-2} |x_j|$ in $\mathbb{R}^{n-2}$. We define the continuous mapping $G : V^{n-2} \rightarrow \mathbb{R}^{n-2}$ by

$$(2.20) \quad \begin{aligned} G_m(x_m) &= D^2 x_m + F_m(x_m) - [x_m - \gamma_m(x_m)] \\ &\quad (2 \leq m \leq n-1). \end{aligned}$$

It is clear that the solutions of (2.18) are the zeros of G in V^{n-2} . In order to use Brouwer degree to study those zeros, we introduce the homotopy $\mathcal{G} : [0, 1] \times V^{n-2} \rightarrow \mathbb{R}^{n-2}$ defined by

$$\begin{aligned} \mathcal{G}_m(\lambda, x_m) &= (1 - \lambda)(D^2x_m - x_m) + \lambda G_m(x_m) \\ (2.21) \quad &= D^2x_m - x_m + \lambda[F_m(x_m) + \gamma_m(x_m)], \end{aligned}$$

for all $2 \leq m \leq n - 1$. Notice that $\mathcal{G}(1, \cdot) = G$ and that $\mathcal{G}(0, \cdot)$ is linear.

III. A priori estimates for the possible zeros of $\mathcal{G}$.

Let R be any number such that

$$(2.22) \quad R > \max_{2 \leq m \leq n-1} \max_{x \in \mathbb{R}} |F_m(x) + \gamma_m(x)|$$

and let $(\lambda, x_2, \dots, x_{n-1}) \in [0, 1] \times V^{n-2}$ be a possible zero of $\mathcal{G}$. If $x_m = \max_{2 \leq j \leq n-1} x_j$, then, $D^2x_m \leq 0$. This is clear if $3 \leq m \leq n - 2$ and if, say, $m = 2$, then

$$D^2x_2 = x_3 - 2x_2 + \frac{x_2 + x_{n-1}}{2} = x_3 - x_2 + \frac{x_2 + x_{n-1}}{2} - x_2 \leq 0,$$

and similarly if $m = n - 1$. Hence,

$$0 \geq D^2x_m = x_m - \lambda[F_m(x_m) + \gamma_m(x_m)],$$

which implies

$$x_m \leq \max_{x \in \mathbb{R}} |F_m(x) + \gamma_m(x)| < R.$$

Analogously it can be shown that $-R < \min_{2 \leq j \leq n-1} x_j$, and hence

$$(2.23) \quad \|x\| = \max_{2 \leq j \leq n-1} |x_j| < R$$

for each possible zero (λ, x) of $\mathcal{G}$.

IV. The existence of a zero for G .

Using the results of Parts II, III and the invariance under homotopy of the Brouwer degree, we see that the Brouwer degree $d[\mathcal{G}(\lambda, \cdot), B_R(0), 0]$ is well defined and independent of $\lambda \in [0, 1]$. But $\mathcal{G}(0, \cdot)$ is a linear mapping whose set of solutions is bounded, hence equal to $\{0\}$. Consequently, $|d[\mathcal{G}(0, \cdot), B_R(0), 0]| = 1$, so that $|d[G, B_R(0), 0]| = 1$ and the existence property of Brouwer degree implies the existence of at least one zero of G .

V. End of the proof.

We have proved that there is some $x \in V^{n-2}$ such that $G(x) = 0$, so x is a solution of (2.18), which means that $\alpha \leq x \leq \beta$ and x is a solution of (2.16). Moreover if α, β are strict, then $\alpha < x < \beta$. ■

Remark 2.11. Suppose that α (resp. β) is a strict lower (resp. upper) solutions of (2.16). As we have already seen, (2.16) admits at least one solution x such that $\alpha < x < \beta$. Define the open set

$$\Omega_{\alpha\beta} = \{(x_2, \dots, x_{n-1}) \in V^{n-2} : \alpha_m < x_m < \beta_m \quad (2 \leq m \leq n-1)\}.$$

If ρ is large enough, then, using the additivity-excision property of Brouwer degree, we have

$$|d[G, \Omega_{\alpha\beta}, 0]| = |d[G, B_\rho(0), 0]| = 1.$$

On the other hand, if we define the continuous mapping $\tilde{G} : V^{n-2} \rightarrow \mathbb{R}^{n-2}$ by

$$(2.24) \quad \tilde{G}_m(x_m) = D^2x_m + f_m(x_m) \quad (2 \leq m \leq n-1),$$

$\tilde{G}$ is equal to G on $\Omega_{\alpha\beta}$, and then

$$(2.25) \quad |d[\tilde{G}, \Omega_{\alpha\beta}, 0]| = 1.$$

2.3. Upper and lower solutions for second order difference equations with Dirichlet boundary conditions

Let $f_m : \mathbb{R} \rightarrow \mathbb{R}$, $(1 \leq m \leq n-1)$ be continuous functions. We study the existence of solutions for the Dirichlet boundary value problem

$$(2.26) \quad D^2x_m + f_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n.$$

Definition 2.12. $\alpha = (\alpha_0, \dots, \alpha_n)$ (resp. $\beta = (\beta_0, \dots, \beta_n)$) is called a lower solution (resp. upper solution) for (2.26) if

$$(2.27) \quad \alpha_0 \leq 0, \alpha_n \leq 0 \quad (\text{resp. } \beta_0 \geq 0, \beta_n \geq 0)$$

and the inequalities

$$(2.28) \quad D^2\alpha_m + f_m(\alpha_m) \geq 0 \quad (\text{resp. } D^2\beta_m + f_m(\beta_m) \leq 0) \quad (1 \leq m \leq n-1)$$

hold. Such a lower or upper solution will be called strict if the inequalities (2.28) are strict.

Theorem 2.13. If (2.26) has a lower solution $\alpha = (\alpha_0, \dots, \alpha_n)$ and an upper solution $\beta = (\beta_0, \dots, \beta_n)$ such that $\alpha \leq \beta$, then (2.26) has a solution $x = (x_0, \dots, x_n)$ such that $\alpha \leq x \leq \beta$. Moreover, if α and β are strict, then $\alpha_m < x_m < \beta_m$ $(1 \leq m \leq n-1)$.

Proof. I. A modified problem.

Let $\gamma_m : \mathbb{R} \rightarrow \mathbb{R}$, $(1 \leq m \leq n-1)$ be the continuous functions defined

by

$$\gamma_m(x) = \begin{cases} \beta_m, & x > \beta_m \\ x, & \alpha_m \leq x \leq \beta_m \\ \alpha_m, & x < \alpha_m, \end{cases}$$

and define $F_m = f_m \circ \gamma_m$ ($1 \leq m \leq n-1$). We consider the modified problem

$$(2.29) \quad \begin{aligned} D^2 x_m + F_m(x_m) - [x_m - \gamma_m(x_m)] &= 0 \quad (1 \leq m \leq n-1), \\ x_0 = 0 = x_n, \end{aligned}$$

and show that if $x = (x_0, \dots, x_n)$ is a solution of (2.29) then $\alpha \leq x \leq \beta$ and hence x is a solution of (2.26). Suppose by contradiction that there is some $0 \leq i \leq n$ such that $\alpha_i - x_i > 0$ so that $\alpha_m - x_m = \max_{0 \leq j \leq n} (\alpha_j - x_j) > 0$. Using the inequalities (2.27), we obtain that $1 \leq m \leq n-1$. Hence

$$D^2(\alpha_m - x_m) = (\alpha_{m+1} - x_{m+1}) - 2(\alpha_m - x_m) + (\alpha_{m-1} - x_{m-1}) \leq 0,$$

and

$$\begin{aligned} D^2 \alpha_m &\leq D^2 x_m = -F_m(x_m) + (x_m - \gamma_m(x_m)) \\ &= -f_m(\alpha_m) + (x_m - \alpha_m) < -f_m(\alpha_m) \leq D^2 \alpha_m, \end{aligned}$$

a contradiction. Analogously we can show that $x \leq \beta$. We remark that if α, β are strict, then $\alpha_m < x_m < \beta_m$ ($1 \leq m \leq n-1$).

II. Abstract formulation of problem (2.29).

Let us introduce the vector space

$$(2.30) \quad V^{n-1} = \{x \in \mathbb{R}^{n+1} : x_0 = 0 = x_n\}$$

endowed with the orientation of $\mathbb{R}^{n+1}$. Its elements can be associated to the coordinates $(x_1, \dots, x_{n-1})$ and correspond to the elements of $\mathbb{R}^{n+1}$ of the form $(0, x_1, \dots, x_{n-1}, 0)$, so that the restriction D^2 to V^{n-1} is well defined in terms of $(x_1, \dots, x_{n-1})$. We use the norm $\|x\| := \max_{1 \leq j \leq n-1} |x_j|$ in V^{n-1} and $\max_{1 \leq j \leq n-1} |x_j|$ in $\mathbb{R}^{n-1}$. We define the continuous mapping $G : V^{n-1} \rightarrow \mathbb{R}^{n-1}$ by

$$(2.31) \quad G_m(x) = D^2 x_m + F_m(x_m) - [x_m - \gamma_m(x_m)],$$

for all $1 \leq m \leq n-1$. It is clear that the solutions of (2.29) are the zeros of G in V^{n-1} . In order to use Brouwer degree to study those zeros, we introduce the homotopy $\mathcal{G} : [0, 1] \times V^{n-1} \rightarrow \mathbb{R}^{n-1}$ defined by

$$(2.32) \quad \begin{aligned} \mathcal{G}_m(\lambda, x) &= (1 - \lambda)(D^2 x_m - x_m) + \lambda G_m(x) \\ &= D^2 x_m - x_m + \lambda[F_m(x_m) + \gamma_m(x_m)], \end{aligned}$$

for all $1 \leq m \leq n-1$. Notice that $\mathcal{G}(1, \cdot) = G$ and that $\mathcal{G}(0, \cdot)$ is linear.

III. A priori estimates for the possible zeros of $\mathcal{G}$.

Let R be any number such that

$$(2.33) \quad R > \max_{1 \leq m \leq n-1} \max_{x \in \mathbb{R}} |F_m(x) + \gamma_m(x)|$$

and let $(\lambda, x_1, \dots, x_{n-1}) \in [0, 1] \times V^{n-1}$ be a possible zero of $\mathcal{G}$. If $0 \leq x_m = \max_{1 \leq j \leq n-1} x_j$, then $D^2 x_m \leq 0$. This is clear if $2 \leq m \leq n-2$ and if, say, $m = 1$, then

$$D^2 x_1 = x_2 - 2x_1 = x_2 - x_1 - x_1 \leq 0,$$

and similarly if $m = n-1$. Hence,

$$0 \geq D^2 x_m = x_m - \lambda[F_m(x_m) + \gamma_m(x_m)],$$

which implies

$$x_m \leq \max_{x \in R} |F_m(x) + \gamma_m(x)| < R.$$

Analogously it can be shown that $-R < \min_{1 \leq j \leq n-1} x_j$, and hence

$$(2.34) \quad \|x\| = \max_{1 \leq j \leq n-1} |x_j| < R$$

for each possible zero (λ, x) of $\mathcal{G}$.

IV. The existence of a zero for G .

Using the results of Parts II, III and the invariance under homotopy of the Brouwer degree, we see that the Brouwer degree $d[\mathcal{G}(\lambda, \cdot), B_R(0), 0]$ is well defined and independent of $\lambda \in [0, 1]$. But $\mathcal{G}(0, \cdot)$ is a linear mapping whose set of solutions is bounded, and hence equal to $\{0\}$. Consequently,

$$|d[\mathcal{G}(0, \cdot), B_R(0), 0]| = 1,$$

so that $|d[G, B_R(0), 0]| = 1$ and the existence property of Brouwer degree implies the existence of at least one zero of G .

V. End of the proof.

We have proved that there is some $x \in V^{n-1}$ such that $G(x) = 0$, so x is a solution of (2.29), which means that $\alpha \leq x \leq \beta$ and x is a solution of (2.26). Moreover if α, β are strict, then $\alpha_m < x_m < \beta_m$ ($1 \leq m \leq n-1$). ■

Remark 2.14. Suppose that α (resp. β) is a strict lower (resp. upper) solutions of (2.26). As we have already seen, (2.26) admits at least one solution x such that $\alpha_m < x_m < \beta_m$ ($1 \leq m \leq n-1$). Define the open set

$$\Omega_{\alpha,\beta} = \{(x_1, \dots, x_{n-1}) \in V^{n-1} : \alpha_m < x_m < \beta_m \quad (1 \leq m \leq n-1)\}.$$

If ρ is large enough, then, using the additivity-excision property of Brouwer degree, we have

$$|d[G, \Omega_{\alpha,\beta}, 0]| = |d[G, B_\rho(0), 0]| = 1.$$

On the other hand, if we define the continuous mapping $\tilde{G} : V^{n-1} \rightarrow \mathbb{R}^{n-1}$ by

$$(2.35) \quad \tilde{G}_m(x) = D^2 x_m + f_m(x_m) \quad (1 \leq m \leq n-1),$$

$\tilde{G}$ is equal to G on $\Omega_{\alpha,\beta}$, and then

$$(2.36) \quad |d[\tilde{G}, \Omega_{\alpha,\beta}, 0]| = 1.$$

3. AMBROSETTI-PRODI TYPE MULTIPLICITY RESULTS

3.1. Ambrosetti-Prodi type results for second order difference equations with periodic boundary conditions

In this section we are interested in problems of the type

$$(3.1) \quad \begin{aligned} D^2 x_m + f_m(x_m) &= s \quad (2 \leq m \leq n-1) \\ x_1 &= x_n, \quad Dx_1 = Dx_{n-1}, \end{aligned}$$

where $f_2, \dots, f_{n-1}$ are continuous, $s \in \mathbb{R}$, $n \in \mathbb{N}$ is fixed and

$$(3.2) \quad f_m(x) \rightarrow \infty \text{ as } |x| \rightarrow \infty \quad (2 \leq m \leq n-1).$$

We reformulate (3.1) in order to apply Brouwer degree theory. Consider the space V^{n-2} defined in (2.19) and the continuous mapping $\mathcal{G} : \mathbb{R} \times V^{n-2} \rightarrow \mathbb{R}^{n-2}$ defined by

$$\mathcal{G}_m(s, x_m) = D^2 x_m + f_m(x_m) - s \quad (2 \leq m \leq n-1).$$

Then $(x_1, \dots, x_n)$ is a solution of (3.1) if and only if $(x_2, \dots, x_{n-1}) \in V^{n-2}$ is a zero of $\mathcal{G}(s, \cdot)$.

Lemma 3.1. *If $b \in \mathbb{R}$, then there exists $\rho > 0$ such that any possible solution x of (3.1) with $s \leq b$ belongs to the open ball $B_\rho(0) \subset V^{n-2}$.*

Proof. Let $s \leq b$ and $(x_1, \dots, x_n)$ be a solution of (3.1). We have that

$$(3.3) \quad \sum_{m=2}^{n-1} f_m(x_m) = (n-2)s.$$

From (3.2) we deduce that f_m ($2 \leq m \leq n-1$) is bounded from below. This implies that there exists $c > 0$ such that if $2 \leq m \leq n-1$, then

$$(3.4) \quad |f_m(x)| \leq f_m(x) + c \quad \text{for all } x \in \mathbb{R}.$$

Using (3.3) and (3.4) it follows that

$$\begin{aligned} |f_m(x_m)| &\leq \sum_{m=2}^{n-1} |f_m(x_m)| \leq \sum_{m=2}^{n-1} f_m(x_m) + c(n-2) \\ &= (s+c)(n-2) \leq (b+c)(n-2) \quad (2 \leq m \leq n-1). \end{aligned}$$

Hence, using now (3.2), the conclusion follows. $\blacksquare$

Theorem 3.2. *If the functions f_m satisfy (3.2) ($2 \leq m \leq n-1$), then there is $s_1 \in \mathbb{R}$ such that (3.1) has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$.*

Proof. Let

$$S_j = \{s \in \mathbb{R} : (3.1) \text{ has at least } j \text{ solutions}\} \quad (j \geq 1).$$

(a) $S_1 \neq \emptyset$.

Take $s^* > \max_{2 \leq m \leq n-1} f_m(0)$ and use (3.2) to find $R_-^* < 0$ such that

$$\min_{2 \leq m \leq n-1} f_m(R_-^*) > s^*.$$

Then α with $\alpha_j = R_-^* < 0$ ($1 \leq j \leq n$) is a strict lower solution and β with $\beta_j = 0$ ($1 \leq j \leq n$) is a strict upper solution for (3.1) with $s = s^*$. Hence, using Theorem 2.10, $s^* \in S_1$.

(b) If $\tilde{s} \in S_1$ and $s > \tilde{s}$ then $s \in S_1$.

Let $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a solution of (3.1) with $s = \tilde{s}$, and let $s > \tilde{s}$. Then $\tilde{x}$ is a strict upper solution for (3.1). Take now $R_- < \min_{1 \leq m \leq n} \tilde{x}_m$ such that $\min_{2 \leq m \leq n-1} f_m(R_-) > s$. It follows that α with $\alpha_j = R_-$ ($1 \leq j \leq n$) is a strict lower solution for (3.1), and hence, using Theorem 2.10, $s \in S_1$.

(c) $s_1 = \inf S_1$ is finite and $S_1 \supset]s_1, \infty[$.

Let $s \in \mathbb{R}$ and suppose that (3.1) has a solution $(x_1, \dots, x_n)$. Then (3.3) holds, from where we deduce that $s \geq c$, with $c \in \mathbb{R}$ such that $f_m(x) \geq c$ for all $x \in \mathbb{R}$, and $2 \leq m \leq n-1$. To obtain the second part of claim (c) $S_1 \supset]s_1, \infty[$ we apply (b).

(d) $S_2 \supset]s_1, \infty[$.

Let $s_3 < s_1 < s_2$. Using Lemma 3.1 we find $\rho > 0$ such that each possible zero of $\mathcal{G}(s, \cdot)$ with $s \in [s_3, s_2]$ is such that $\max_{2 \leq m \leq n-1} |x_m| < \rho$. Consequently, the Brouwer degree $d[\mathcal{G}(s, \cdot), B_\rho(0), 0]$ is well defined and does not depend upon $s \in [s_3, s_2]$. However, using (c), we see that $\mathcal{G}(s_3, x) \neq 0$ for all $x \in V^{n-2}$. This implies that $d[\mathcal{G}(s_3, \cdot), B_\rho(0), 0] = 0$, so that $d[\mathcal{G}(s_2, \cdot), B_\rho(0), 0] = 0$ and, by excision property, $d[\mathcal{G}(s_2, \cdot), B_{\rho'}(0), 0] = 0$ if $\rho' > \rho$. Let $s \in]s_1, s_2[$ and $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ be a solution of (3.1) (using (c)). Then $\hat{x}$ is a strict upper solution of (3.1) with $s = s_2$. Let $R < \min_{1 \leq j \leq n} \hat{x}_j$ be such that $\min_{2 \leq m \leq n-1} f_m(R) > s_2$. Then $(R, \dots, R) \in \mathbb{R}^n$ is a strict lower solution of (3.1) with $s = s_2$. Consequently, using Remark 2.11, (3.1) with $s = s_2$ has a solution in $\Omega_{R\hat{x}}$ and

$$|d[\mathcal{G}(s_2, \cdot), \Omega_{R\hat{x}}, 0]| = 1.$$

Taking ρ' sufficiently large, we deduce from the additivity property of Brouwer degree that

$$\begin{aligned} |d[\mathcal{G}(s_2, \cdot), B_{\rho'}(0) \setminus \overline{\Omega}_{R\hat{x}}, 0]| &= |d[\mathcal{G}(s_2, \cdot), B_{\rho'}(0), 0] - d[\mathcal{G}(s_2, \cdot), \Omega_{R\hat{x}}, 0]| \\ &= |d[\mathcal{G}(s_2, \cdot), \Omega_{R\hat{x}}, 0]| = 1, \end{aligned}$$

and (3.1) with $s = s_2$ has a second solution in $B_{\rho'}(0) \setminus \overline{\Omega}_{R\hat{x}}$.

(e) $s_1 \in S_1$.

Taking a decreasing sequence $(\sigma_k)_{k \in \mathbb{N}}$ in $]s_1, \infty[$ converging to s_1 , a corresponding sequence $(x_1^k, \dots, x_n^k)$ of solutions of (3.1) with $s = \sigma_k$ and using Lemma 3.1, we obtain a subsequence $(x_1^{j_k}, \dots, x_n^{j_k})$ which converges to a solution $(x_1, \dots, x_n)$ of (3.1) with $s = s_1$. ■

Similar arguments allow to prove the following result.

Theorem 3.3. *If the functions f_m satisfy condition*

$$(3.5) \quad f_m(x) \rightarrow -\infty \text{ as } |x| \rightarrow \infty, \quad (2 \leq m \leq n-1).$$

then there is $s_1 \in \mathbb{R}$ such that (3.1) has zero, at least one or at least two solutions according to $s > s_1$, $s = s_1$ or $s < s_1$.

Example 3.4. *There exists $s_1 \in \mathbb{R}$ such that the problem*

$$D^2 x_m + |x_m|^{1/2} = s \quad (2 \leq m \leq n-1), \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}$$

has no solution if $s < s_1$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$.

Example 3.5. *There exists $s_1 \in \mathbb{R}$ such that the problem*

$$D^2 x_m - \exp x_m^2 = s \quad (2 \leq m \leq n-1), \quad x_1 = x_n, \quad Dx_1 = Dx_{n-1}$$

has no solution if $s > s_1$, at least one solution if $s = s_1$ and at least two solutions if $s < s_1$.

3.2. Ambrosetti-Prodi type results for second order difference equations with Dirichlet boundary conditions

In this section we are interested in problems of the type

$$(3.6) D^2x_m + \lambda_1 x_m + f_m(x_m) = s\varphi_m^1 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n,$$

where $n \geq 2$ is fixed, $f_1, \dots, f_{n-1} : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, $s \in \mathbb{R}$, λ_1 is defined in (1.32), φ^1 is defined in (1.33) and

$$(3.7) \quad f_m(x) \rightarrow \infty \text{ as } |x| \rightarrow \infty \quad (1 \leq m \leq n-1).$$

We prove an Ambrosetti-Prodi type result for (3.6), which is reminiscent of a multiplicity theorem for second order differential equations with Dirichlet boundary conditions proved in [23].

The next lemma provides a priori bounds for the possible solutions of (3.6).

Lemma 3.6. *Let $a, b \in \mathbb{R}$. Then there is $\rho > 0$ such that any possible solution x of (3.6) with $s \in [a, b]$ belongs to the open ball $B_\rho(0)$.*

Proof. Let $s \in [a, b]$ and $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ be a solution of (3.6). Multiplying each member of the equation by φ_m^1 and adding, we obtain

$$s \left[\sum_{m=1}^{n-1} (\varphi_m^1)^2 \right] = \sum_{m=1}^{n-1} [\varphi_m^1 D^2x_m + \lambda_1 \varphi_m^1 x_m] + \sum_{m=1}^{n-1} \varphi_m^1 f_m(x_m).$$

But, using Lemma 1.13,

$$\sum_{m=1}^{n-1} [\varphi_m^1 D^2x_m + \lambda_1 \varphi_m^1 x_m] = \sum_{m=1}^{n-1} [x_m (D^2\varphi_m^1 + \lambda_1 \varphi_m^1)] = 0,$$

so that

$$(3.8) \quad \sum_{m=1}^{n-1} \varphi_m^1 f_m(x_m) = s \|\varphi^1\|_2^2.$$

Using (3.7) we deduce that there exists a constant $\alpha > 0$ such that

$$(3.9) \quad |f_m(x)| \leq f_m(x) + \alpha, \quad (1 \leq m \leq n-1).$$

Using the equation (3.6), written in the equivalent form

$$(3.10) (D^2\tilde{x})_m + \lambda_1(\tilde{x})_m + f_m(x_m) = s\varphi_m^1 \quad (1 \leq m \leq n-1),$$

and the relations (3.8), (3.9) we have

$$\begin{aligned} \sum_{m=1}^{n-1} |D^2(\tilde{x})_m + \lambda_1(\tilde{x})_m| \varphi_m^1 &= \sum_{m=1}^{n-1} |s\varphi_m^1 - f_m(x_m)| \varphi_m^1 \\ &\leq |s| \|\varphi^1\|_2^2 + \sum_{m=1}^{n-1} |f_m(x_m)| \varphi_m^1 \leq 2|s| \|\varphi^1\|_2^2 + \alpha \sum_{m=1}^{n-1} \varphi_m^1, \end{aligned}$$

which implies that there exists a constant R_1 depending only on a, b, n such that

$$(3.11) \quad \sum_{m=1}^{n-1} |D^2(\tilde{x})_m + \lambda_1(\tilde{x})_m| \varphi_m^1 \leq R_1.$$

Using the relations (3.9), (3.10), (3.11) and Lemma 1.14, we obtain $R_2 > 0$ such that

$$|f_m(x_m)| \leq R_2 \quad (1 \leq m \leq n-1).$$

Hence assumption (3.7) implies the existence of $R_3 > 0$ such that $|x_m| < R_3$ for all $1 \leq m \leq n-1$. $\blacksquare$

Theorem 3.7. *If the functions f_m ($1 \leq m \leq n-1$) satisfy (3.7), then there is $s_1 \in \mathbb{R}$ such that (3.6) has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$.*

Proof. Let

$$S_j = \{s \in \mathbb{R} : (3.6) \text{ has at least } j \text{ solutions}\} \quad (j \geq 1).$$

(a) $S_1 \neq \emptyset$.

Take $s^* > \max_{1 \leq m \leq n-1} \frac{f_m(0)}{\varphi_m^1}$ and use (3.7) to find $R_-^* < 0$ such that

$$f_m(R_-^* \varphi_m^1) > s^* \varphi_m^1 \quad (1 \leq m \leq n-1).$$

Then α with $\alpha_0 = 0 = \alpha_n$ and $\alpha_j = R_-^* \varphi_j^1 < 0$ ($1 \leq j \leq n-1$) is a strict lower solution and β with $\beta_j = 0$ ($1 \leq j \leq n$) is a strict upper solution for (3.6) with $s = s^*$. Hence, using Theorem 2.13, $s^* \in S_1$.

(b) If $\tilde{s} \in S_1$ and $s > \tilde{s}$ then $s \in S_1$.

Let $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ be a solution of (3.6) with $s = \tilde{s}$, and let $s > \tilde{s}$. Then $\tilde{x}$ is a strict upper solution for (3.6). Take now $R_- < \min_{1 \leq m \leq n} \frac{\tilde{x}_m}{\varphi_m^1}$ such that $f_m(R_- \varphi_m^1) > s \varphi_m^1$ ($1 \leq m \leq n-1$). It follows that α with $\alpha_0 = 0 = \alpha_n$ and $\alpha_j = R_- \varphi_j^1$ ($1 \leq j \leq n$) is a strict lower solution for (3.6), and hence, using Theorem 2.13, $s \in S_1$.

(c) $s_1 = \inf S_1$ is finite and $S_1 \supset]s_1, \infty[$.

Let $s \in \mathbb{R}$ and suppose that (3.6) has a solution $(x_1, \dots, x_n)$. Then

(3.8) holds, from where we deduce that $s \geq \|\varphi^1\|_2^{-2} \sum_{m=1}^{n-1} \varphi_m^1 \min_{\mathbb{R}} f_m$. To obtain the second part of claim (c) $S_1 \supset]s_1, \infty[$ we apply (b).

(d) $S_2 \supset]s_1, \infty[$.

We reformulate (3.6) to apply Brouwer degree theory. Consider the space V^{n-1} defined in (1.38) and the continuous mapping $\mathcal{G} : \mathbb{R} \times V^{n-1} \rightarrow \mathbb{R}^{n-1}$ defined by

$$\mathcal{G}_m(s, x) = D^2 x_m + \lambda_1 x_m + f_m(x_m) - s \varphi_m^1 \quad (1 \leq m \leq n-1).$$

Then $(x_0, \dots, x_n)$ is a solution of (3.6) if and only if $(x_1, \dots, x_{n-1}) \in V^{n-1}$ is a zero of $\mathcal{G}(s, \cdot)$. Let $s_3 < s_1 < s_2$. Using Lemma 3.6 we find $\rho > 0$ such that each possible zero of $\mathcal{G}(s, \cdot)$ with $s \in [s_3, s_2]$ is such that $\max_{1 \leq m \leq n-1} |x_m| < \rho$. Consequently, the Brouwer degree $d[\mathcal{G}(s, \cdot), B_\rho(0), 0]$ is well defined and does not depend upon $s \in [s_3, s_2]$. However, using (c), we see that $\mathcal{G}(s_3, x) \neq 0$ for all $x \in V^{n-1}$. This implies that $d[\mathcal{G}(s_3, \cdot), B_\rho(0), 0] = 0$, so that $d[\mathcal{G}(s_2, \cdot), B_\rho(0), 0] = 0$ and, by excision property, $d[\mathcal{G}(s_2, \cdot), B_{\rho'}(0), 0] = 0$ if $\rho' > \rho$. Let $s \in]s_1, s_2[$ and $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ be a solution of (3.6) (using (c)). Then $\hat{x}$ is a strict upper solution of (3.6) with $s = s_2$. Let $R < \min_{1 \leq m \leq n} \frac{\hat{x}_m}{\varphi_m^1}$ be such that $f_m(R \varphi_m^1) > s_2 \varphi_m^1$ ($1 \leq m \leq n-1$). Then $(0, R \varphi_1^1, \dots, R \varphi_{n-1}^1, 0) \in \mathbb{R}^{n+1}$ is a strict lower solution of (3.6) with $s = s_2$. Consequently, using Remark 2.14, (3.6) with $s = s_2$ has a solution in $\Omega_{R \varphi^1, \hat{x}}$ and

$$|d[\mathcal{G}(s_2, \cdot), \Omega_{R \varphi^1, \hat{x}}, 0]| = 1.$$

Taking ρ' sufficiently large, we deduce from the additivity property of Brouwer degree that

$$\begin{aligned} & |d[\mathcal{G}(s_2, \cdot), B_{\rho'}(0) \setminus \overline{\Omega}_{R \varphi^1, \hat{x}}, 0]| \\ &= |d[\mathcal{G}(s_2, \cdot), B_{\rho'}(0), 0] - d[\mathcal{G}(s_2, \cdot), \Omega_{R \varphi^1, \hat{x}}, 0]| \\ &= |d[\mathcal{G}(s_2, \cdot), \Omega_{R \varphi^1, \hat{x}}, 0]| = 1, \end{aligned}$$

and (3.6) with $s = s_2$ has a second solution in $B_{\rho'}(0) \setminus \overline{\Omega}_{R \varphi^1, \hat{x}}$.

(e) $s_1 \in S_1$.

Taking a decreasing sequence $(\sigma_k)_{k \in \mathbb{N}}$ in $]s_1, \infty[$ converging to s_1 , a corresponding sequence $(x_1^k, \dots, x_n^k)$ of solutions of (3.6) with $s = \sigma_k$ and using Lemma 3.6, we obtain a subsequence $(x_1^{j_k}, \dots, x_n^{j_k})$ which converges to a solution $(x_1, \dots, x_n)$ of (3.6) with $s = s_1$. ■

Similar arguments allow to prove the following result.

Theorem 3.8. *If the functions f_m satisfy condition*

$$(3.12) \quad f_m(x) \rightarrow -\infty \text{ as } |x| \rightarrow \infty, \quad (1 \leq m \leq n-1).$$

then there is $s_1 \in \mathbb{R}$ such that (3.6) has zero, at least one or at least two solutions according to $s > s_1$, $s = s_1$ or $s < s_1$.

Example 3.9. *There exists $s_1 \in \mathbb{R}$ such that the problem*

$$D^2x_m + \lambda_1 x_m + |x_m|^{1/2} = s\varphi_m^1 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n$$

has no solution if $s < s_1$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$.

Example 3.10. *There exists $s_1 \in \mathbb{R}$ such that the problem*

$$D^2x_m + \lambda_1 x_m - \exp x_m^2 = s\varphi_m^1 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n$$

has no solution if $s > s_1$, at least one solution if $s = s_1$ and at least two solutions if $s < s_1$.

CHAPTER III

Second order differential equations with ϕ -Laplacian

1. NOTATION AND PRELIMINARIES

We denote the usual norms in $L^1(0, T)$ and $L^\infty(0, T)$ respectively by $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Let C be the Banach space of continuous functions on $[0, T]$ endowed with the norm $\|\cdot\|_\infty$, C^1 denote the Banach space of continuously differentiable functions on $[0, T]$ equipped with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$, C_0^1 denotes the closed subspace of C^1 defined by $C_0^1 = \{u \in C^1 : u(0) = 0 = u(T)\}$, $C_\#^1$ denotes the closed subspace of C^1 defined by $C_\#^1 = \{u \in C^1 : u'(0) = 0 = u'(T)\}$, C_{per}^1 denotes the closed subspace of C^1 defined by $C_{per}^1 = \{u \in C^1 : u(0) = u(T), u'(0) = u'(T)\}$. We denote by P, Q the projectors

$$P, Q : C \rightarrow C, \quad Pu(t) = u(0), \quad Qu(t) = \frac{1}{T} \int_0^T u(s) ds,$$

and we define $H : C \rightarrow C^1$ by

$$Hu(t) = \int_0^t u(s) ds.$$

If $u \in C$, we write

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}, \quad u_L = \min_{[0, T]} u, \quad u_M = \max_{[0, T]} u.$$

Lemma 1.1. *If $w \in L^\infty(0, T)$, then*

$$(1.1) \quad \|H(I - Q)w\|_\infty \leq \frac{T}{2} \|w\|_\infty.$$

Proof. If $w \in L^\infty(0, T)$, we have, for $t \in [0, T]$,

$$\begin{aligned}
 H(I - Q)w(t) &= \int_0^t w(s) ds - \frac{t}{T} \int_0^T w(s) ds \\
 (1.2) \qquad &= \left(1 - \frac{t}{T}\right) \int_0^t w(s) ds - \frac{t}{T} \int_t^T w(s) ds \\
 &= \int_0^T G(t, s) w(s) ds,
 \end{aligned}$$

where

$$(1.3) \qquad G(t, s) = \begin{cases} 1 - \frac{t}{T} & \text{if } 0 \leq s \leq t \\ -\frac{t}{T} & \text{if } t < s \leq T. \end{cases}$$

Hence, for each $t \in [0, T]$, one has

$$\begin{aligned}
 |H(I - Q)w(t)| &\leq \int_0^T |G(t, s)| |w(s)| ds \\
 &\leq \|w\|_\infty \int_0^T |G(t, s)| ds = 2t \left(1 - \frac{t}{T}\right) \|w\|_\infty \\
 &\leq \frac{T}{2} \|w\|_\infty.
 \end{aligned}$$

■

Remark 1.2. *Inequality (1.1) is sharp* as shown by the function $w \in L^\infty(0, T)$ defined by

$$(1.4) \qquad w(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{T}{2} \\ -1 & \text{if } \frac{T}{2} < t \leq T, \end{cases}$$

which is such that $Qw = 0$, $\|w\|_\infty = 1$, and

$$\int_0^t w(s) ds = \begin{cases} t & \text{if } 0 \leq t \leq \frac{T}{2} \\ T - t & \text{if } \frac{T}{2} \leq t \leq T. \end{cases}$$

Consequently,

$$\|Hw\|_\infty = \frac{T}{2} = \frac{T}{2} \|w\|_\infty.$$

It is sharp also in the space C , as shown by the continuous functions w_ε ($0 < \varepsilon < \frac{T}{2}$) defined by

$$(1.5) \qquad w_\varepsilon(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{T}{2} - \varepsilon \\ \frac{1}{\varepsilon} \left(\frac{T}{2} - t\right) & \text{if } \frac{T}{2} - \varepsilon < t < \frac{T}{2} + \varepsilon \\ -1 & \text{if } \frac{T}{2} + \varepsilon < t \leq T, \end{cases}$$

which are such that $Qw_\varepsilon = 0$, $\|w_\varepsilon\|_\infty = 1$,

$$\int_0^t w_\varepsilon(s) ds = \begin{cases} t & \text{if } 0 \leq t \leq \frac{T}{2} - \varepsilon \\ \frac{T-\varepsilon}{2} - \frac{1}{2\varepsilon} \left(\frac{T}{2} - t\right)^2 & \text{if } \frac{T}{2} - \varepsilon < t < \frac{T}{2} + \varepsilon \\ T - t & \text{if } \frac{T}{2} \leq t \leq T, \end{cases}$$

and hence,

$$\|Hw_\varepsilon\|_\infty = \frac{T}{2} - \frac{\varepsilon}{2} = \left(\frac{T-\varepsilon}{2}\right) \|w_\varepsilon\|_\infty.$$

The next Lemma gives an inequality similar to (1.1), except that the norm $\|\cdot\|_\infty$ is replaced by the norm $\|\cdot\|_1$ in the right-hand member. Again, the constant is the best possible.

Lemma 1.3. *If $w \in L^1(0, T)$, then*

$$(1.6) \quad \|H(I - Q)w\|_\infty \leq \|w\|_1.$$

Proof. We use again formula (1.2) with G defined in (1.3). Then, for each $t \in [0, T]$, we have

$$\begin{aligned} |H(I - Q)w(t)| &= \left| \left(1 - \frac{t}{T}\right) \int_0^t w(s) ds - \frac{t}{T} \int_t^T w(s) ds \right| \\ &\leq \left(1 - \frac{t}{T}\right) \int_0^t |w(s)| ds + \frac{t}{T} \int_t^T |w(s)| ds \leq \|w\|_1. \end{aligned}$$

■

Lemma 1.4. *Let $\phi :]-a, a[\rightarrow \mathbb{R}$, $0 < a \leq \infty$ be a homeomorphism such that $\phi(0) = 0$, and let $u \in C_{per}^1$ be such that $\|u'\|_\infty < a$, $\phi(u')$ is of class C^1 , and $(\phi(u'))'(0) = (\phi(u'))'(T)$. Then*

$$\|\phi(u')\|_\infty \leq \|[(\phi(u'))']^\pm\|_1.$$

Proof. Let us extend u to $\mathbb{R}$ as a C^1 T -periodic function, $(\phi(u'))'$ to $\mathbb{R}$ as a continuous T -periodic function, and let $t_0 \in [0, T]$ be such that $u(t_0) = u_L$. Then $u'(t_0) = u'(t_0 + T) = 0$, which implies that if $t \in [t_0, t_0 + T]$, then

$$\begin{aligned} \phi(u'(t)) &= \int_{t_0}^t (\phi(u'(s)))' ds \leq \int_{t_0}^t [(\phi(u'(s)))']^+ ds \\ &\leq \int_{t_0}^{t_0+T} [(\phi(u'(s)))']^+ ds = \|[(\phi(u'))']^+\|_1. \end{aligned}$$

On the other hand, in a similar way, we have

$$\phi(u'(t)) \geq -\|[(\phi(u'))']^+\|_1.$$

The proof in the case of the negative part uses similar arguments. $\blacksquare$

Lemma 1.5. *Let $\phi : \mathbb{R} \rightarrow]-a, a[$, $0 < a < \infty$ be a homeomorphism such that $\phi(0) = 0$, and let $u \in C_{per}^1$ be such that $\phi(u')$ is of class C^1 and $(\phi(u'))'(0) = (\phi(u'))'(T)$. Then*

$$\|\phi(u')\|_\infty \leq \|[(\phi(u'))']^\pm\|_1$$

The next Lemma is an adaptation of a result of [47] to the case of an homeomorphism which is not defined everywhere.

Lemma 1.6. *Let $\phi : \mathbb{R} \rightarrow]-a, a[$, $0 < a < \infty$ be a homeomorphism, $\phi(0) = 0$ and $B = \{h \in C : \|h\|_\infty < \frac{a}{2}\}$. For each $h \in B$, there exists a unique $\alpha \in \mathbb{R}$ such that*

$$\int_0^T \phi^{-1}(h(t) - \alpha) dt = 0.$$

Moreover, $\alpha \in \text{Range}(h)$. The function $Q_\phi : B \rightarrow \mathbb{R}$ defined by $Q_\phi(h) := \alpha$ is continuous. If $\phi :]-a, a[\rightarrow \mathbb{R}$, $0 < a \leq \infty$ is a homeomorphism such that $\phi(0) = 0$, then the operator Q_ϕ is defined on C .

Proof. Let $h \in B$. We first prove uniqueness. Let $\alpha_i \in \mathbb{R}$ be such that $h(t) - \alpha_i \in]-a, a[$ for all $t \in [0, T]$ and $\int_0^T \phi^{-1}(h(t) - \alpha_i) dt = 0$ ($i = 1, 2$). It follows that there exists $t_0 \in [0, T]$ such that $\phi^{-1}(h(t_0) - \alpha_1) = \phi^{-1}(h(t_0) - \alpha_2)$, and using the injectivity of ϕ^{-1} we deduce that $\alpha_1 = \alpha_2$. For existence, it is clear that the function

$$\gamma : [h_L, h_M] \rightarrow \mathbb{R}, \quad s \mapsto \int_0^T \phi^{-1}(h(t) - s) dt$$

is well defined and continuous. On the other hand, because ϕ^{-1} is strictly monotone and $\phi^{-1}(0) = 0$, we see that $\gamma(h_L)\gamma(h_M) \leq 0$, and the existence of $\alpha \in [h_L, h_M]$ such that $\gamma(\alpha) = 0$ follows. Finally, we show that Q_ϕ is continuous on B . Let $(h_n)_n \subset B$ such that $h_n \rightarrow h_0$ in C and $h_0 \in B$. Without loss of generality, we may assume that there is $0 < \varepsilon < \frac{a}{2}$ such that $(\|h_n\|_\infty)_n \subset [-\varepsilon, \varepsilon]$, and, passing if necessary to a subsequence, we may assume that $Q_\phi(h_n) \rightarrow \alpha_0 \in [-\varepsilon, \varepsilon]$. Using the dominated convergence theorem we deduce that $\int_0^T \phi^{-1}(h_0(t) - \alpha_0) dt = 0$, so we have that $\alpha_0 = Q_\phi(h_0)$. Hence, the function Q_ϕ is continuous. $\blacksquare$

Remark 1.7. The above result shows that the function Q_ϕ verifies the identity

$$(1.7) \quad Q \circ \phi^{-1} \circ (I - Q_\phi) \circ u = 0.$$

Finally, to each continuous function $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$, we associate its Nemytskii operator $N_f : C^1 \rightarrow C$ defined by

$$N_f(u)(t) = f(t, u(t), u'(t)).$$

All the above defined operators P, Q, H, N_f are continuous.

2. FORCED ϕ -LAPLACIAN WITH VARIOUS BOUNDARY CONDITIONS

To motivate the assumptions of the theorems proved here and the construction of the associated fixed point operators, we first study the solvability of the forced equation

$$(2.1) \quad (\phi(u'))' = f(t)$$

with $f \in L^1(0, T)$ and $\phi : \mathbb{R} \rightarrow]-a, a[, 0 < a < \infty$ is a homeomorphism, $\phi(0) = 0$, submitted to various boundary conditions. For each $\tau \in [0, T]$, we define $F_\tau : [0, T] \rightarrow \mathbb{R}$ by

$$(2.2) \quad F_\tau(t) := \int_\tau^t f(s) ds,$$

so that

$$F_\tau(t) = F_0(t) - F_0(\tau).$$

We first consider the Neumann boundary conditions

$$(2.3) \quad u'(0) = 0 = u'(T).$$

Proposition 2.1. *Problem (2.1)-(2.3) has a solution if and only if*

$$(2.4) \quad \int_0^T f(s) ds = 0$$

and

$$(2.5) \quad \|F_0\|_\infty < a,$$

in which case problem (2.1)-(2.3) has the family of solutions

$$(2.6) \quad u(t) = u(0) + \int_0^t \phi^{-1}(F_0(s)) ds \quad (t \in [0, T]).$$

Proof. If u is a solution of problem (2.1)-(2.3), then (2.4) follows from integrating both members of (2.1) on $[0, T]$ and using the boundary condition (2.3). If (2.4) holds, we get, by integrating both members of (2.1) on $[0, t]$ and using the boundary condition (2.3)

$$(2.7) \quad \phi(u'(t)) = F_0(t) \quad (t \in [0, T]),$$

which implies (2.5). Now, if conditions (2.4) and (2.5) hold, problem (2.1)-(2.3) is equivalent to (2.7), hence to

$$u'(t) = \phi^{-1}(F_0(t)) \quad (t \in [0, T]),$$

which gives (2.6) by integration from 0 to t . ■

Remark 2.2. If $\phi :]-a, a[\rightarrow \mathbb{R}$ is a homeomorphism such that $\phi(0) = 0$ and $0 < a \leq +\infty$, then problem (2.1)-(2.3) has a solution if and only if (2.4) holds.

Example 2.3. *The problem*

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' = \alpha \cos t, \quad u'(0) = 0 = u'(\pi),$$

is solvable if and only if $|\alpha| < 1$.

Example 2.4. *The problem*

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' = \alpha \left(t - \frac{1}{2} \right), \quad u'(0) = 0 = u'(1),$$

is solvable if and only if $|\alpha| < 8$.

Example 2.5. *If w is defined in (1.4), the problem*

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' = \alpha w(t), \quad u'(0) = 0 = u'(T),$$

is solvable if and only if $|\alpha| < \frac{2}{T}$.

Remark 2.6. Using the inequality (1.1), we see that if $f \in L^\infty(0, T)$, the conditions

$$(2.8) \quad \int_0^T f(s) ds = 0, \quad \|f\|_\infty < \frac{2a}{T},$$

are sufficient for the solvability of problem (2.1)-(2.3). The second inequality gives $|\alpha| < \frac{2}{\pi}$ in Example 2.3, $|\alpha| < 2$ in Example 2.4, and $|\alpha| < \frac{2}{T}$ in Example 2.5.

If we consider now the Dirichlet boundary conditions

$$(2.9) \quad u(0) = 0 = u(T),$$

we obtain the following necessary and sufficient conditions for the solvability of problem (2.1)-(2.9).

Proposition 2.7. *Problem (2.1)-(2.9) has a solution if and only if there exists $\tau \in [0, T]$ such that*

$$(2.10) \quad \|F_\tau\|_\infty < a \quad \text{and} \quad \int_0^T \phi^{-1}(F_\tau(s)) ds = 0,$$

in which case problem (2.1)-(2.9) has the solution

$$(2.11) \quad u(t) = \int_0^t \phi^{-1}(F_\tau(s)) ds \quad (t \in [0, T]).$$

Proof. If u is a solution of problem (2.1)-(2.9), it follows from (2.9) and Rolle's theorem that there exists $\tau \in [0, T]$ such that $u'(\tau) = 0$. Then equation (2.1) gives

$$\phi(u'(t)) = F_\tau(t) \quad (t \in [0, T]),$$

which implies the first condition in (2.10) and the equivalent form

$$u'(t) = \phi^{-1}(F_\tau(t)) \quad (t \in [0, T]),$$

which, using the first boundary condition, gives (2.11). Then the second boundary condition implies the second condition in (2.10). Now, if condition (2.10) holds, it is immediate to check that (2.11) solves (2.1)-(2.9). $\blacksquare$

Example 2.8. For the problem

$$(2.12) \quad \left(\frac{u'}{\sqrt{1+u'^2}} \right)' = \alpha, \quad u(0) = 0 = u(1),$$

$F_\tau(t) = \alpha(t - \tau)$, and it is easily checked that

$$\int_0^1 \frac{\alpha(s - \tau)}{\sqrt{1 - \alpha^2(s - \tau)^2}} ds = \frac{1}{\alpha} \left[\sqrt{1 - \alpha^2\tau^2} - \sqrt{1 - \alpha^2(1 - \tau)^2} \right] = 0$$

if and only if $\tau = \frac{1}{2}$. Consequently, *Problem (2.12) is solvable if and only if $|\alpha| < 2$* . Elementary computations show that the (unique) solution is given by

$$u(t) = \frac{1}{\alpha} \left[\sqrt{1 - \frac{\alpha^2}{4}} - \sqrt{1 - \alpha^2 \left(t - \frac{1}{2} \right)^2} \right] \quad (t \in [0, 1]).$$

Remark 2.9. The computation of τ in conditions (2.10) may be difficult. As $F_\tau(\tau) = 0$, one has the inequalities

$$(2.13) \quad \frac{1}{2} \text{Osc}_{[0, T]} F_0 \leq \|F_\tau\|_\infty \leq \text{Osc}_{[0, T]} F_0 \leq \|f\|_1,$$

which provide the less sharp but more explicit necessary condition for solvability of (2.1)-(2.9)

$$Osc_{[0,T]} F_0 < 2a.$$

and the less sharp versions of the first sufficient condition in (2.10)

$$Osc_{[0,T]} F_0 < a, \quad \text{or} \quad \|f\|_1 < a,$$

or, noticing that, for $f \in L^\infty(0, T)$, $\|f\|_1 < T\|f\|_\infty$,

$$\|f\|_\infty < \frac{a}{T}.$$

In the Example 2.8, all those conditions reduce to $|\alpha| < 1$.

Remark 2.10. The existence of $\tau \in [0, T]$ such that

$$\int_0^T \phi^{-1}(F_\tau(s)) ds = 0$$

i.e. such that

$$\int_0^T \phi^{-1}(F_0(s) - F_0(\tau)) ds = 0$$

is equivalent to the existence of $c \in \text{Range } F_0$ such that

$$\int_0^T \phi^{-1}(F_0(s) - c) ds = 0,$$

and hence is guaranteed by Lemma 1.6 when $\|F_0\|_\infty < \frac{a}{2}$.

Remark 2.11. If $\phi :]-a, a[\rightarrow \mathbb{R}$ is a homeomorphism such that $\phi(0) = 0$ and $0 < a \leq +\infty$, then problem (2.1)-(2.9) has a least one solution for each $f \in L^1(0, T)$. Moreover the solution is given in (2.11), where $\tau \in [0, T]$ is such that $Q_\phi(F_0) = F_0(\tau)$. This follows by Lemma 1.6 and the proof above.

Finally, for the periodic boundary conditions

$$(2.14) \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

we have the following necessary and sufficient condition for the solvability of (2.1)-(2.14).

Proposition 2.12. *Problem (2.1)-(2.14) has a solution if and only if*

$$(2.15) \quad \int_0^T f(s) ds = 0$$

and if there exists $\tau \in [0, T]$ such that

$$(2.16) \quad \|F_\tau\|_\infty < a \quad \text{and} \quad \int_0^T \phi^{-1}(F_\tau(s)) ds = 0,$$

in which case problem (2.1)-(2.14) has the family of solutions

$$(2.17) \quad u(t) = u(0) + \int_0^t \phi^{-1}(F_\tau(s)) ds \quad (t \in [0, T]).$$

Proof. It is a combination of the ideas of the proofs of the Neumann and Dirichlet cases, and the details are left to the reader. ■

Remark 2.13. If $\phi :]-a, a[\rightarrow \mathbb{R}$ is a homeomorphism such that $\phi(0) = 0$ and $0 < a \leq +\infty$, then problem (2.1)-(2.14) has a solution if and only if (2.15) holds.

Example 2.14. Consider the problem

$$(2.18) \quad \left(\frac{u'}{\sqrt{1+u'^2}} \right)' = \alpha w(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where $\alpha \in \mathbb{R}$ and w is defined in (1.4). Then for each $t \in [0, T]$,

$$W_\tau(t) := \int_\tau^t w(s) ds = \left| \tau - \frac{T}{2} \right| - \left| t - \frac{T}{2} \right|.$$

Hence

$$\begin{aligned} & \int_0^T \frac{\alpha W_\tau(s)}{\sqrt{1 - \alpha^2 W_\tau^2(s)}} ds \\ &= \frac{2}{\alpha} \left[\sqrt{1 - \alpha^2 \left(\left| \tau - \frac{T}{2} \right| - \frac{T}{2} \right)^2} - \sqrt{1 - \alpha^2 \left| \tau - \frac{T}{2} \right|^2} \right] = 0, \end{aligned}$$

if and only if $\tau = \frac{T}{4}$ or $\tau = \frac{3T}{4}$. Now

$$\|W_{T/4}\|_\infty = \|W_{3T/4}\|_\infty = \frac{|\alpha|T}{4},$$

so that Problem (2.18) is solvable if and only if $|\alpha| < \frac{4}{T}$.

3. FIXED POINT FORMULATIONS

In this section $\phi :]-a, a[\rightarrow \mathbb{R}$, $0 < a \leq \infty$ is a homeomorphism such that $\phi(0) = 0$ and $N : \Delta \subset C^1 \rightarrow C$ denotes an operator.

Consider the abstract periodic boundary value problem

$$(3.1) \quad (\phi(u'))' = N(u), \quad u(0) - u(T) = u'(0) - u'(T).$$

On the other hand, suppose that $\Delta \cap C_{per}^1 \neq \emptyset$. If $u \in \Delta \cap C_{per}^1$, let us consider the function

$$(3.2) \quad \begin{aligned} \mathcal{P}_N(u) &= Pu + QN(u) \\ &+ H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)N](u). \end{aligned}$$

Proposition 3.1. *The operator $\mathcal{P}_N : \Delta \cap C_{per}^1 \rightarrow C_{per}^1$ is well defined and if $u \in \Delta \cap C_{per}^1$, then u is a solution of (3.1) if and only if $u = \mathcal{P}_N(u)$.*

Proof. Let $u \in \Delta \cap C_{per}^1$. It is clear that $\mathcal{P}_N(u) \in C^1$. We show that in fact $\mathcal{P}_N(u) \in C_{per}^1$. Using Remark 1.7, we deduce that

$$\begin{aligned} \mathcal{P}_N(u)(T) &= Pu + QN(u) + TQ \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)N](u) \\ &= Pu + QN(u) = \mathcal{P}_N(u)(0). \end{aligned}$$

On the other hand we have that

$$(\mathcal{P}_N(u))' = \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)N](u),$$

which implies that

$$(\mathcal{P}_N(u))'(0) = \phi^{-1}(-Q_\phi(H(I - Q)N(u))) = (\mathcal{P}_N(u))'(T).$$

Consequently $\mathcal{P}_N(u) \in C_{per}^1$ and the operator $\mathcal{P}_N : \Delta \cap C_{per}^1 \rightarrow C_{per}^1$ is well defined. Now suppose that $u \in \Delta \cap C_{per}^1$ is such that $u = \mathcal{P}_N(u)$. It follows that

$$u - Pu - H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)N](u) = QN(u),$$

which gives

$$u = Pu + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)N](u), \quad QN(u) = 0,$$

so that $u \in C_{per}^1$ and u is a solution for (3.1) by differentiating the first equation, applying ϕ to both of its members, differentiating again and using the second equation. Now, the proof of the other implication is obvious. $\blacksquare$

Consider the abstract Neumann boundary value problem

$$(3.3) \quad (\phi(u'))' = N(u), \quad u'(0) = 0 = u'(T).$$

On the other hand, suppose that $\Delta \cap C_\#^1 \neq \emptyset$. If $u \in \Delta \cap C_\#^1$, let us consider the function

$$(3.4) \quad \mathcal{N}_N(u) = Pu + QN(u) + H \circ \phi^{-1} \circ [H(I - Q)N](u).$$

Proposition 3.2. *The operator $\mathcal{N}_N : \Delta \cap C_\#^1 \rightarrow C_\#^1$ is well defined and if $u \in \Delta \cap C_\#^1$, then u is a solution of (3.3) if and only if $u = \mathcal{N}_N(u)$.*

Proof. Let $u \in \Delta \cap C_{\#}^1$. It is clear that $\mathcal{N}_N(u) \in C^1$ and

$$(\mathcal{N}_N(u))' = \phi^{-1} \circ [H(I - Q)N](u),$$

which implies that

$$(\mathcal{N}_N(u))'(0) = 0 = (\mathcal{P}_N(u))'(T).$$

This implies that $\mathcal{N}_N(u) \in C_{\#}^1$. Now suppose that $u \in \Delta \cap C_{\#}^1$ is such that $u = \mathcal{N}_N(u)$. It follows that

$$u - Pu - H \circ \phi^{-1} \circ [H(I - Q)N](u) = QN(u),$$

which gives

$$u = Pu + H \circ \phi^{-1} \circ [H(I - Q)N](u), \quad QN(u) = 0,$$

so that $u \in C_{\#}^1$ and u is a solution for (3.3) by differentiating the first equation, applying ϕ to both of its members, differentiating again and using the second equation. Now, the proof of the other implication is obvious. $\blacksquare$

Finally, let us consider the abstract Dirichlet boundary value problem

$$(3.5) \quad (\phi(u'))' = N(u), \quad u(0) = 0 = u(T).$$

On the other hand, suppose that $\Delta \cap C_0^1 \neq \emptyset$. If $u \in \Delta \cap C_0^1$, let us consider the function

$$(3.6) \quad \mathcal{D}_N(u) = H \circ \phi^{-1} \circ (I - Q_{\phi}) \circ H \circ N(u).$$

Proposition 3.3. *The operator $\mathcal{D}_N : \Delta \cap C_0^1 \rightarrow C_0^1$ is well defined and if $u \in \Delta \cap C_0^1$, then u is a solution of (3.5) if and only if $u = \mathcal{D}_N(u)$.*

Proof. Let $u \in \Delta \cap C_0^1$. It is clear that $\mathcal{D}_N(u) \in C^1$. We show that in fact $\mathcal{D}_N(u) \in C_0^1$. Using Remark 1.7, we deduce that

$$\begin{aligned} \mathcal{D}_N(u)(T) &= TQ \circ \phi^{-1} \circ (I - Q_{\phi}) \circ H \circ N(u) \\ &= 0 = \mathcal{D}_N(u)(0). \end{aligned}$$

Consequently $\mathcal{D}_N(u) \in C_0^1$. Now suppose that $u \in C_0^1$ is such that $u = \mathcal{D}_N(u)$. It follows that

$$\begin{aligned} u' &= \phi^{-1} \circ (I - Q_{\phi}) \circ H \circ N(u) \\ \phi(u') &= (I - Q_{\phi}) \circ H \circ N(u) \\ (\phi(u'))' &= N(u). \end{aligned}$$

Now, the proof of the other implication is obvious. $\blacksquare$

4. HOMOTOPY TO THE AVERAGED NONLINEARITY

In this section $\phi :]-a, a[\rightarrow \mathbb{R}, 0 < a \leq \infty$ is a homeomorphism such that $\phi(0) = 0$ and $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function. Our aim is to extend Mawhin's continuation theorem in infinite dimension to quasilinear periodic boundary value problems of the following type

$$(4.1) \quad (\phi(u'))' = f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

The following continuation theorem is due to Mawhin and Manásevich in the case $a = +\infty$ and it is taken from [47].

Theorem 4.1. *Assume that Ω is an open bounded set in C_{per}^1 such that the following conditions hold.*

(1) *For each $\lambda \in]0, 1[$ the problem*

$$(4.2) \quad (\phi(u'))' = \lambda f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has no solution on $\partial\Omega$.

(2) *The equation*

$$(4.3) \quad F(d) = \frac{1}{T} \int_0^T f(t, d, 0) dt = 0,$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(3) *The Brouwer degree*

$$(4.4) \quad d_B[F, \Omega \cap \mathbb{R}, 0] \neq 0.$$

Then problem (4.1) has a least one solution in $\overline{\Omega}$.

Proof. Following the same strategy as in the proof of Mawhin's continuation theorem in infinite dimension, let us embed problem (4.1) into the one parameter family of problems

$$(4.5) \quad \begin{aligned} (\phi(u'))' &= \lambda N_f(u) + (1 - \lambda)QN_f(u), \\ u(0) - u(T) &= 0 = u'(0) - u'(T). \end{aligned}$$

For $\lambda \in]0, 1]$, observe that in both case, u is a solution to problem (4.2) or u is a solution to problem (4.5), we have necessarily

$$(4.6) \quad QN_f(u) = 0.$$

It follows that for $\lambda \in]0, 1]$, problems (4.2) and (4.5) have the same solutions. For $(\lambda, u) \in [0, 1] \times C_{per}^1$ consider

$$N_\lambda(u) = \lambda N_f(u) + (1 - \lambda)QN_f(u).$$

So, if we consider the nonlinear homotopy

$$\mathcal{G}_f : [0, 1] \times \overline{\Omega} \rightarrow C_{per}^1, \quad \mathcal{G}_f(\lambda, u) = \mathcal{P}_{N_\lambda}(u),$$

that is

$$\mathcal{G}_f(\lambda, u) = Pu + QN_f(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H(I - Q)N_f](u),$$

then using Proposition 3.1 it follows that problem (4.5) can be written in the equivalent form

$$(4.7) \quad u = \mathcal{G}_f(\lambda, u).$$

It is not difficult to check that $\mathcal{G}_f$ is a completely continuous homotopy. We assume that for $\lambda = 1$, (4.7) does not have a solution on $\partial\Omega$ since otherwise we are done with the proof. Now by hypothesis (1) it follows that (4.7) has no solutions for $(\lambda, u) \in]0, 1] \times \partial\Omega$. For $\lambda = 0$, (4.5) is equivalent to the problem

$$(\phi(u'))' = QN_f(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

and thus if u is a solution to this problem, we must have (4.6). Hence

$$u'(t) = \phi^{-1}(c),$$

where $c \in \mathbb{R}$ is a constant. Integrating this last equation on $[0, T]$ we obtain that $\phi^{-1}(c) = 0$, and thus $u(t) = d$, where d is a constant. Thus, by (4.6)

$$\int_0^T f(t, d, 0)dt = 0,$$

which together with hypothesis (2) imply that $u = d \notin \partial\Omega$. Thus we have proved that (4.7) has no solution $(\lambda, u) \in [0, 1] \times \partial\Omega$. Then we have that for each $\lambda \in [0, 1]$, the Leray-Schauder degree $d_{LS}[I - \mathcal{G}_f(\lambda, \cdot), \Omega, 0]$ is well defined and using the invariance under a homotopy of this degree we have

$$(4.8) \quad d_{LS}[I - \mathcal{G}_f(1, \cdot), \Omega, 0] = d_{LS}[I - \mathcal{G}_f(0, \cdot), \Omega, 0].$$

On the other hand, we have that

$$I - \mathcal{G}_f(0, \cdot) = I - (P + QN_f),$$

and the range of $P + QN_f$ is contained in the subspace of constant functions, isomorphic to $\mathbb{R}$, so, using the reduction property of the Leray-Schauder degree we have that

$$\begin{aligned} d_{LS}[I - \mathcal{G}_f(0, \cdot), \Omega, 0] &= d_B[I - (P + QN_f)|_{\mathbb{R}}, \Omega \cap \mathbb{R}, 0] \\ &= d_B[-QN_f, \Omega \cap \mathbb{R}, 0] = -d_B[F, \Omega \cap \mathbb{R}, 0]. \end{aligned}$$

Since by hypothesis (3) this last degree is different from zero, using (4.8), it follows that the degree $d_{LS}[I - \mathcal{G}_f(1, \cdot), \Omega, 0]$ is different from zero. Using now the existence property of Leray-Schauder degree we

deduce that there exists $u \in \Omega$ such that $u = \mathcal{G}_f(1, u)$, that is u is a solution of (4.1). $\blacksquare$

Remark 4.2. Using the same strategy as in the proof above and Proposition 3.2, the theorem above holds also if we consider the Neumann condition instead of the periodic condition.

5. VILLARI TYPE NONLINEARITIES

5.1. ϕ -Laplacian with ϕ defined on $\mathbb{R}$

Let $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$) be a homeomorphism such that $\phi(0) = 0$ and $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

Periodic boundary value problems

Consider the periodic boundary value problems of the following type

$$(5.1) \quad (\phi(u'))' = f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

In order to apply the continuation theorem proved in the last section we consider the family of problems

$$(5.2) \quad (\phi(u'))' = \lambda f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

where $\lambda \in]0, 1]$. The following lemma, giving a priori bounds for the possible solutions of (5.2), adapts a technique introduced by Ward [66].

Lemma 5.1. *Assume that f satisfies the following conditions.*

- (1) *There exists a continuous function c on $[0, T]$ such that $\|c^-\|_1 < \frac{a}{2}$ and*

$$(5.3) \quad f(t, u, v) \geq c(t)$$

for all $(t, u, v) \in [0, T] \times \mathbb{R}^2$.

- (2) *There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$(5.4) \quad \begin{aligned} \epsilon \int_0^T f(t, u(t), u'(t)) dt &> 0 \quad \text{if } u_L \geq R, \quad \|u'\|_\infty \leq M, \\ \epsilon \int_0^T f(t, u(t), u'(t)) dt &< 0 \quad \text{if } u_M \leq -R, \quad \|u'\|_\infty \leq M, \end{aligned}$$

where $M = \max\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\}$.

If u is a solution of (5.2), then $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.

Proof. Let u be a solution of (5.2). This implies that

$$(5.5) \quad QN_f(u) = 0.$$

Using the fact that f is bounded from below by c , we deduce the elementary inequality

$$(5.6) \quad |f(t, u, v)| \leq f(t, u, v) + 2c^-(t) \quad \forall (t, u, v) \in [0, T] \times \mathbb{R}^2.$$

From (5.5), (5.2) and (5.6) it follows that

$$(5.7) \quad \begin{aligned} \|(\phi(u'))'\|_1 &= \lambda \|N_f(u)\|_1 \\ &\leq \int_0^T N_f(u)(s) ds + 2\|c^-\|_1 = 2\|c^-\|_1. \end{aligned}$$

Because $u \in C^1$ is such that $u(0) = u(T)$, there exists $\xi \in [0, T]$ such that $u'(\xi) = 0$, which implies $\phi(u'(\xi)) = 0$ and

$$\phi(u'(t)) = \int_{\xi}^t (\phi(u'(s)))' ds \quad (t \in [0, T]).$$

Using the equality above and (5.7) we have that

$$|\phi(u'(t))| \leq 2\|c^-\|_1 \quad (t \in [0, T]),$$

and hence

$$(5.8) \quad \|u'\|_{\infty} \leq M.$$

If $u_M \leq -R$ (respectively $u_L \geq R$) then, from (5.8) and (5.4), it follows that

$$\epsilon \int_0^T f(t, u(t), u'(t)) dt < 0 \quad (\text{respectively} \quad \epsilon \int_0^T f(t, u(t), u'(t)) dt > 0).$$

Using (5.5) we have that

$$(5.9) \quad u_M > -R \text{ and } u_L < R.$$

It is clear that

$$(5.10) \quad u_M \leq u_L + \int_0^T |u'(t)| dt.$$

From relations (5.8), (5.9) and (5.10), we obtain that

$$-(R + MT) < u_L \leq u_M < R + MT.$$

It follows that $\|u\|_{\infty} < R + MT$. ■

Theorem 5.2. *Let f be a continuous function which satisfies the conditions (1) and (2) of Lemma 5.1. Then (5.1) has at least one solution u such that $\|u'\|_{\infty} \leq M$ and $\|u\|_{\infty} < R + MT$.*

Proof. The case $a = +\infty$.

Let $\rho > R + M(T + 1)$ and $\Omega \subset C_{per}^1$ be the open bounded set defined by

$$\Omega = \{u \in C_{per}^1 : \|u\| < \rho\}.$$

Using Lemma 5.1 it follows that (1) in Theorem 4.1 holds, with Ω defined above. Notice that $\Omega \cap \mathbb{R} =]-\rho, \rho[$ and using (5.4) it follows that conditions (2) and (3) in Theorem 4.1 hold. Applying Theorem 4.1 we deduce that (5.1) has at least one solution.

The case $a < +\infty$.

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a homeomorphism such that φ coincides with ϕ on $[-(M + 1), M + 1]$. It follows that

$$(5.11) \quad M = \max\{|\varphi^{-1}(2\|c^-\|_1)|, |\varphi^{-1}(-2\|c^-\|_1)|\}.$$

It is clear that if, for $\lambda \in]0, 1]$, we consider the problem

$$(5.12) \quad (\varphi(u'))' = \lambda f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

then Lemma 5.1 holds also if we change ϕ with φ . Using (5.11) and Lemma 5.1, we deduce that if u is a solution of (5.12), then

$$(5.13) \quad \|u'\|_\infty \leq M \quad \text{and} \quad \|u\|_\infty < R + MT.$$

Hence, the solutions of (5.12) coincide with the solutions of (5.2). But, we have proved that (5.12) has at least one solution for $\lambda = 1$, which is also a solution of (5.1). $\blacksquare$

Remark 5.3. The conclusion of Theorem 5.2 also holds if $f(t, u, v) \leq c(t)$ for some $c \in C$ such that $\|c^+\|_1 < \frac{a}{2}$ and all $(t, u, v) \in [0, T] \times \mathbb{R}^2$, and f satisfies the sign condition (5.4). It suffices to replace ϕ by $-\phi$, f by $-f$ and to apply Theorem 5.2.

Example 5.4. Using Theorem 5.2 and Remark 5.3, we see that the equations

$$\begin{aligned} \left(\frac{u'}{\sqrt{1+u'^2}}\right)' - \exp u - \mu|u'| + h(t) &= 0, \\ u(0) - u(T) &= 0 = u'(0) - u'(T), \\ \left(\frac{u'}{\sqrt{1+u'^2}}\right)' + \exp u + \mu|u'| - h(t) &= 0, \\ u(0) - u(T) &= 0 = u'(0) - u'(T), \end{aligned}$$

have at least one solution if $h \in C$ is such that

$$\|h^-\|_1 < \|h^+\|_1 < \frac{1}{2}$$

and μ is positive and sufficiently small. This is in particular the case if

$$0 < h_L \leq h_M < \frac{1}{2T}.$$

Example 5.5. Let $p > 1$ and $h \in C$. The periodic boundary value problem

$$(|u'|^{p-2}u')' + \exp u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution if and only if $\|h^-\|_1 < \|h^+\|_1$.

An immediate but useful consequence of Theorem 5.2 is the following result.

Corollary 5.6. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $g < 0$, and $h \in C$ such that $\|h^-\|_1 < \frac{a}{2}$. If the following Landesman-Lazer type condition is satisfied

$$(5.14) \quad -\infty < \limsup_{u \rightarrow -\infty} g(u) < \frac{1}{T} \int_0^T h(s) ds < \liminf_{u \rightarrow +\infty} g(u),$$

then the periodic boundary value problem

$$(5.15) \quad (\phi(u'))' + g(u) = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution.

Proposition 5.7. The conclusions of Theorem 5.2 and Remark 5.3 still hold if the sign condition (5.4) is weakened into

(2') There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that

$$(5.16) \quad \begin{aligned} \epsilon \int_0^T f(t, u(t), u'(t)) dt &\geq 0 \quad \text{if } u_L \geq R, \quad \|u'\|_\infty \leq M', \\ \epsilon \int_0^T f(t, u(t), u'(t)) dt &\leq 0 \quad \text{if } u_M \leq -R, \quad \|u'\|_\infty \leq M', \end{aligned}$$

where $M' > \max\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\}$.

Proof. Letting

$$f_n(t, u, v) = f(t, u, v) + \frac{\epsilon}{n} \frac{u}{\sqrt{1+u^2}},$$

it is easy to see that, for $n \geq N$ with N sufficiently large, the problems

$$(5.17) \quad (\phi(u'))' = f_n(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

satisfy the conditions of Theorem 5.2 for c replaced by a suitable c_N with $\|c_N^-\|_1 < \frac{a}{2}$ and M by the corresponding $M_N < M'$. Consequently, each problem (5.17) for $n \geq N$ admits at least one solution u_n satisfying

$$\|u'_n\|_\infty \leq M_N, \quad \|u_n\|_\infty < R + M_N T.$$

which allows us to extract a convergent subsequence whose limit is a solution of (5.1). $\blacksquare$

Corollary 5.8. *Let $p > 0$, $h \in C$ such that $h > 0$ and $k \in C$ such that $\|k^+\|_1 < \frac{a}{2}$. Then the periodic boundary value problem*

$$(5.18) \quad \begin{aligned} &(\phi(u'))' + h(t)(u^+)^p = k(t), \\ &u(0) - u(T) = 0 = u'(0) - u'(T) \end{aligned}$$

has at least one solution if and only if $\int_0^T k(t)dt \geq 0$.

If $h \in C$ such that $h < 0$ and $k \in C$ such that $\|k^-\|_1 < \frac{a}{2}$, problem (5.18) has at least one solution if and only if $\int_0^T k(t)dt \leq 0$.

Proof. For the necessity, if problem (5.18) has a solution u then

$$\int_0^T k(t)dt = \int_0^T h(t)(u^+)^p dt \geq 0.$$

For the sufficiency, the function

$$f(t, u) = k(t) - h(t)(u^+)^p,$$

is bounded from above by k . Furthermore, if

$$R \geq \left(\frac{\int_0^T k(t)dt}{\int_0^T h(t)dt} \right)^{1/p},$$

then $\int_0^T f(t, u(t))dt \geq 0$ when $u_L \geq R$. On the other hand $\int_0^T f(t, u(t))dt = -\int_0^T k(t)dt \leq 0$ when $u_M \leq 0$. Hence the result follows from Proposition 5.7. The proof of the other case is similar. $\blacksquare$

Let $f : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $(t, u, v) \mapsto f(t, u, v)$ be T -periodic with respect to t and continuous, and let us consider the existence of a T -periodic solution u (i.e., a solution such that $u(t+T) = u(t)$ for all $t \in \mathbb{R}$) of the equation

$$(5.19) \quad (\phi(u'))' = f(t, u, u').$$

Of course, those solutions are continuations over $\mathbb{R}$, by T -periodicity, of the solutions over $[0, T]$ such that $u(0) - u(T) = 0 = u'(0) - u'(T)$, so that the previous theory can be applied. Moreover, in this case the results above can be improved.

Lemma 5.9. *Assume that f is T -periodic with respect to the first variable and satisfies the conditions (5.3) and (5.4) of Lemma 5.1 with $\|c^-\|_1 < a$ and*

$$M = \max\{|\phi^{-1}(\|c^-\|_1)|, |\phi^{-1}(-\|c^-\|_1)|\}.$$

If u is a solution of (5.2), then $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.

Proof. Let u be a solution of (5.2). This implies that (5.5) holds. On the other hand the continuation over $\mathbb{R}$, by T -periodicity, of u denoted also by u satisfies the equation

$$(5.20) \quad (\phi(u'))'(t) = \lambda f(t, u(t), u'(t)) \quad \text{for all } t \in \mathbb{R}.$$

From (5.20), (5.5) and Lemma 1.5 it follows that

$$(5.21) \quad \|\phi(u')\|_\infty \leq \|[(\phi(u'))']^-\|_1 \leq \|[N_f(u)]^-\|_1 \leq \|c^-\|_1 < a,$$

and hence (5.8) holds. To finish the proof it suffices to use the same arguments as in the proof of Lemma 5.1. $\blacksquare$

Theorem 5.10. *Assume that f is T -periodic with respect to the first variable and satisfies the conditions of Lemma 5.9. Then (5.19) has at least one T -periodic solution u such that $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.*

Proof. See the proof of Theorem 5.2. $\blacksquare$

Let $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function satisfying the condition

$$(5.22) \quad |f(t, u, v)| \leq c < \frac{a}{T} \quad \text{for all } (t, u, v) \in [0, T] \times \mathbb{R}^2.$$

Lemma 5.11. *Suppose that condition (5.22) holds and that there exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$(5.23) \quad \begin{aligned} \epsilon \int_0^T f(t, u(t), u'(t)) dt &> 0 \quad \text{if } u_L \geq R, \quad \|u'\|_\infty \leq M, \\ \epsilon \int_0^T f(t, u(t), u'(t)) dt &< 0 \quad \text{if } u_M \leq -R, \quad \|u'\|_\infty \leq M, \end{aligned}$$

where $M = \max\{|\phi^{-1}(-cT)|, |\phi^{-1}(cT)|\}$. If u is a solution of (5.2), then $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.

Proof. Let u be a solution of (5.2). From (5.2) and (5.22), it follows that

$$(5.24) \quad \int_0^T |(\phi(u'(t)))'| dt \leq cT.$$

To finish the proof it suffices to use the same arguments as in the proof of Lemma 5.1. $\blacksquare$

Theorem 5.12. *Let f be a continuous function which satisfies the conditions (5.22) and (5.23) of Lemma 5.11. Then (5.1) has at least one solution u such that $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.*

Proof. See the proof of Theorem 5.2. ■

Remark 5.13. For f bounded, condition (5.22) is better than the condition

$$|f(t, u, v)| \leq c < \frac{a}{2T}$$

given by Theorem 5.2 or Remark 5.3.

Remark 5.14. In Theorem 5.12 one can weaken the sign condition (5.23) in a similar way as in Proposition 5.7.

Example 5.15. *Using Theorem 5.12 we obtain that the periodic boundary value problem*

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' = \alpha(\arctan u + \sin t), \quad u(0) - u(1) = 0 = u'(0) - u'(1),$$

has at least one solution if $|\alpha| \leq 0.4145$.

Example 5.16. *Using Theorem 5.12 we obtain that the periodic boundary value problem*

$$\begin{aligned} \left(\frac{u'}{\sqrt{1+u'^2}} \right)' &= \frac{1}{4} \arctan(u+t) + \frac{1}{3} \sin(u'+t^2), \\ u(0) - u(1) &= 0 = u'(0) - u'(1), \end{aligned}$$

has at least one non constant solution.

Neumann boundary value problems

Consider the Neumann boundary value problems of the following type

$$(5.25) \quad (\phi(u'))' = f(t, u, u'), \quad u'(0) = 0 = u'(T).$$

In order to apply the continuation theorem proved in the last section for the Neumann boundary condition, we consider the family of problems

$$(5.26) \quad (\phi(u'))' = \lambda f(t, u, u'), \quad u'(0) = 0 = u'(T).$$

where $\lambda \in]0, 1]$.

Lemma 5.17. *Assume that f satisfies the conditions (1) and (2) of Lemma 5.1. If u is a solution of (5.26), then $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.*

Proof. Let u be a solution of (5.26). This implies that (5.5) holds. Using (5.26) and (5.5) it follows that

$$(5.27) \quad \phi(u') = \lambda H N_f(u) = \lambda H(I - Q)N_f(u).$$

Using (5.5), (5.27), (5.6) and (1.6), it follows that

$$\begin{aligned} \|\phi(u')\|_\infty &= \|\lambda H(I - Q)N_f(u)\|_\infty \leq \|N_f(u)\|_1 \\ &\leq \int_0^T N_f(u)(s) ds + 2\|c^-\|_1 = 2\|c^-\|_1 < a, \end{aligned}$$

which implies $\|u'\|_\infty \leq M$. The end of the proof is then entirely similar to that of Lemma 5.1. $\blacksquare$

Using the previous Lemma and Remark 4.2 we have the following result.

Theorem 5.18. *Let f be a continuous function which satisfies the conditions (1) and (2) of Lemma 5.1. Then (5.25) has at least one solution u such that $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.*

Now, suppose that f satisfies

$$(5.28) \quad |f(t, u, v)| \leq c < \frac{2a}{T} \quad \text{for all } (t, u, v) \in [0, T] \times \mathbb{R}^2.$$

Lemma 5.19. *Assume that f satisfies (5.28) and condition (2) of Lemma 5.1 with $M = \max\{|\phi^{-1}(-\frac{cT}{2})|, |\phi^{-1}(\frac{cT}{2})|\}$. Then (5.25) has at least one solution u such that $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.*

Proof. Let u be a solution of (5.26). As in the proof of Lemma 5.17, we have

$$\|\phi(u')\|_\infty = \|\lambda H(I - Q)N_f(u)\|_\infty,$$

which together with (5.28) and (1.1) imply that $\|u'\|_\infty \leq M$. The end of the proof is then entirely similar to that of Lemma 5.1. $\blacksquare$

Using the previous Lemma and Remark 4.2 we have the following result.

Theorem 5.20. *Let f as in Lemma 5.19. Then (5.25) has at least one solution u such that $\|u'\|_\infty \leq M$ and $\|u\|_\infty < R + MT$.*

Dirichlet boundary value problems

We finally consider Dirichlet problems of the form

$$(5.29) \quad (\phi(u'))' = f(t, u, u'), \quad u(0) = u(T) = 0$$

where $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and such that

$$(5.30) \quad |f(t, u, v)| \leq c < \frac{a}{T} \quad \text{for all } (t, u, v) \in [0, T] \times \mathbb{R}^2.$$

For $\lambda \in [0, 1]$, consider the family of Dirichlet problems

$$(5.31) \quad (\phi(u'))' = \lambda f(t, u, u'), \quad u(0) = u(T) = 0,$$

which reduces to (5.29) for $\lambda = 1$ and to

$$(5.32) \quad (\phi(u'))' = 0, \quad u(0) = u(T) = 0$$

for $\lambda = 0$. This last problem has only the trivial solution.

Lemma 5.21. *Let $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous and satisfying condition (5.30). If u is a solution of (5.31) then*

$$\|u'\|_\infty \leq \max\{|\phi^{-1}(-cT)|, |\phi^{-1}(cT)|\} := M \quad \text{and} \quad \|u\|_\infty \leq TM.$$

Proof. Let u be a solution of (5.31). Then, using (5.31) and (5.30) it follows that

$$\|\phi(u')\|_\infty = \|H \circ (\lambda N_f)(u)\|_\infty \leq cT < a.$$

Consequently,

$$\|u'\|_\infty \leq M$$

and

$$\|u\|_\infty = \left\| \int_0^\cdot u'(s) ds \right\|_\infty \leq TM.$$

■

Theorem 5.22. *Let $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous and satisfying condition (5.30). There exists a solution u of (5.29) such that $\|u'\|_\infty \leq M$ and $\|u\|_\infty \leq TM$ where M is defined above.*

Proof. The case $a = +\infty$.

Consider the completely continuous homotopy $\mathcal{M} : [0, 1] \times C_0^1 \rightarrow C_0^1$ defined by

$$\mathcal{M}(\lambda, u) = \mathcal{D}_{\lambda N_f}(u) = H \circ \phi^{-1} \circ (I - Q_\phi) \circ H \circ (\lambda N_f)(u).$$

Using Proposition 3.3 we deduce that if $u \in C_0^1$ then u is a solution of (5.31) iff $u = \mathcal{M}(\lambda, u)$. Therefore, if

$$\Omega = \{u \in C_0^1 : \|u\|_\infty < TM + 1, \quad \|u'\|_\infty < M + 1\},$$

it follows from Lemma 5.21 that

$$u \neq \mathcal{M}(\lambda, u), \quad \forall (\lambda, u) \in \partial\Omega.$$

It follows from the homotopy invariance of Leray-Schauder degree that $d_{LS}[I - \mathcal{M}(\lambda, \cdot), \Omega, 0]$ is independent of $\lambda \in [0, 1]$ so that, if we notice that $\mathcal{M}(0, \cdot) = 0$,

$$d_{LS}[I - \mathcal{M}(1, \cdot), \Omega, 0] = d_{LS}[I - \mathcal{M}(0, \cdot), \Omega, 0] = 1.$$

Hence, using the existence property of Leray-Schauder degree we deduce that $\mathcal{M}(1, \cdot)$ has a fixed point u , which is a solution of (5.29).

The case $a < +\infty$.

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a homeomorphism such that φ coincides with ϕ on $[-(M+1), M+1]$. It follows that

$$(5.33) \quad M = \max\{|\varphi^{-1}(-cT)|, |\varphi^{-1}(cT)|\}.$$

It is clear that if, for $\lambda \in [0, 1]$, we consider the problem

$$(5.34) \quad (\varphi(u'))' = \lambda f(t, u, u'), \quad u(0) = u(T) = 0,$$

then Lemma 5.21 holds also if we change ϕ with φ . Using (5.33) and Lemma 5.21, we deduce that if u is a solution of (5.34), then

$$(5.35) \quad \|u'\|_\infty \leq M \quad \text{and} \quad \|u\|_\infty \leq MT.$$

Hence, the solutions of (5.34) coincide with the solutions of (5.31). But, we have proved that (5.34) has at least one solution for $\lambda = 1$, which is also a solution of (5.29). $\blacksquare$

Example 5.23. It follows from Theorem 5.22 that *the Dirichlet problem*

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' = \alpha(\sin u + \cos t), \quad u(0) = 0 = u(\pi)$$

has at least one solution if $|\alpha| < \frac{1}{2\pi}$ and the Dirichlet problem

$$(|u'|^{p-2}u')' = \alpha(\sin u + \cos t), \quad u(0) = 0 = u(\pi)$$

has at least one solution for all $\alpha \in \mathbb{R}$.

Remark 5.24. In contrast to the periodic and Neumann cases, the solvability of the Dirichlet problem with bounded right-hand side f does not require any sign condition upon f . This is related to the absence of a necessary condition like (2.4) for the solvability of the simple Dirichlet problem

$$(\phi(u'))' = f(t), \quad u(0) = 0 = u(T).$$

On the other hand, the approach used to study periodic or Neumann problems with one-sided bounded nonlinearities does not work in the Dirichlet case.

5.2. ϕ -Laplacian with ϕ defined on $] -a, a[$, $a \neq +\infty$ **Periodic or Neumann boundary value problems**

Consider the periodic boundary value problems of the type

$$(5.36) \quad (\phi(u'))' = f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where $\phi :] -a, a[\rightarrow \mathbb{R}$ ($0 < a < \infty$) is a homeomorphism, $\phi(0) = 0$ and $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

For $\lambda \in]0, 1]$, consider the family of periodic boundary value problems

$$(5.37) \quad (\phi(u'))' = \lambda f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

Lemma 5.25. *Assume that f satisfies the following conditions. There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$(5.38) \quad \begin{aligned} \epsilon \int_0^T f(t, u(t), u'(t)) dt &> 0 \quad \text{if } u_L \geq R, \quad \|u'\|_\infty < a, \\ \epsilon \int_0^T f(t, u(t), u'(t)) dt &< 0 \quad \text{if } u_M \leq -R, \quad \|u'\|_\infty < a, \end{aligned}$$

If u is a solution of (5.37), then $\|u\| < R + a(T + 1)$.

Proof. Let u be a solution of (5.37). This implies that

$$(5.39) \quad \|u'\|_\infty < a,$$

and

$$(5.40) \quad QN_f(u) = 0.$$

If $u_M \leq -R$ (respectively $u_L \geq R$) then, from (5.39) and (5.38), it follows that

$$\epsilon \int_0^T f(t, u(t), u'(t)) dt < 0 \quad (\text{respectively } \epsilon \int_0^T f(t, u(t), u'(t)) dt > 0).$$

Using (5.40) we have that

$$(5.41) \quad u_M > -R \text{ and } u_L < R.$$

It is clear that

$$(5.42) \quad u_M \leq u_L + \int_0^T |u'(t)| dt.$$

From relations (5.39), (5.41) and (5.42), we obtain that

$$(5.43) \quad -(R + aT) < u_L \leq u_M < R + aT.$$

Using (5.39) and (5.43) it follows that $\|u\| < R + a(T + 1)$. ■

Using Lemma 5.25 and Theorem 4.1 we obtain the following existence result.

Proposition 5.26. *Let f be continuous and satisfying condition (5.38) of Lemma 5.25. Then (5.36) has at least one solution u such that $\|u\| < R + a(T + 1)$.*

Proof. See the proof of Theorem 5.2 in the case $a = +\infty$. ■

Theorem 5.27. *Assume that f satisfies the following conditions. There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$(5.44) \quad \begin{aligned} \epsilon \int_0^T f(t, u(t), u'(t)) dt &\geq 0 \quad \text{if } u_L \geq R, \quad \|u'\|_\infty < a, \\ \epsilon \int_0^T f(t, u(t), u'(t)) dt &\leq 0 \quad \text{if } u_M \leq -R, \quad \|u'\|_\infty < a, \end{aligned}$$

Then (5.36) has at least one solution.

Proof. Consider the continuous functions

$$f_n(t, u, v) = f(t, u, v) + \frac{\epsilon}{n} \frac{u}{\sqrt{1 + u^2}},$$

with associated fixed point operators $\mathcal{P}_{N_{f_n}}$. Then, using Propositions 5.26 and 3.1, we deduce that there exists $u_n \in C_{per}^1$ such that $u_n = \mathcal{P}_{N_{f_n}}(u_n)$ and $\|u_n\| < R + a(T + 1)$. So, applying Arzela-Ascoli theorem, passing if necessarily to a subsequence, we have that $u_n \rightarrow u$ in C^1 , which implies that $\mathcal{P}_{N_{f_n}}(u_n) \rightarrow \mathcal{P}_{N_f}(u)$ in C^1 . Consequently, $u = \mathcal{P}_{N_f}(u)$, which implies that u is a solution for (5.36). ■

Corollary 5.28. *Let $h : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous, with h bounded on $[0, T] \times \mathbb{R} \times]-a, a[$ and g satisfying one of the condition*

$$\begin{aligned} \lim_{u \rightarrow -\infty} g(t, u) &= +\infty, & \lim_{u \rightarrow +\infty} g(t, u) &= -\infty \\ \lim_{u \rightarrow -\infty} g(t, u) &= -\infty, & \lim_{u \rightarrow +\infty} g(t, u) &= +\infty, \end{aligned}$$

uniformly in $t \in [0, T]$. Then the problem

$$(\phi(u'))' + h(t, u, u') + g(t, u) = 0, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution.

Corollary 5.29. *Let $h : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be bounded on $[0, T] \times \mathbb{R} \times]-a, a[$ and continuous. Then, for each $\mu \neq 0$, the problem*

$$(\phi(u'))' + \mu u = h(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution, and

$$d_{LS}[I - \mathcal{P}_{N_g}, B_\rho, 0] = \text{sign}(\mu),$$

where $g(t, u, v) = h(t, u, v) - \mu u$ and ρ sufficiently large.

Example 5.30. If $e \in C$, $c \in \mathbb{R} \setminus \{0\}$, $d \in \mathbb{R}$, $q \geq 0$ and $p > 1$, the problem

$$\begin{aligned} \left(\frac{u'}{\sqrt{1-u'^2}} \right)' + d|u'|^q + c|u|^{p-1}u &= e(t), \\ u(0) - u(T) = 0 &= u'(0) - u'(T), \end{aligned}$$

has at least one solution.

Another easy consequence is a Landesman-Lazer-type theorem.

Corollary 5.31. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then problem

$$(5.45) \quad (\phi(u'))' + g(u) = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one solution for each $e \in C$ such that

$$(5.46) \quad \limsup_{u \rightarrow -\infty} g(u) < \frac{1}{T} \int_0^T e(s) ds < \liminf_{u \rightarrow +\infty} g(u)$$

or

$$(5.47) \quad \limsup_{u \rightarrow +\infty} g(u) < \frac{1}{T} \int_0^T e(s) ds < \liminf_{u \rightarrow -\infty} g(u).$$

Remark 5.32. Similar results hold also for Neumann boundary value problems.

Dirichlet boundary value problems

We finally consider Dirichlet problems of the form

$$(5.48) \quad (\phi(u'))' = f(t, u, u'), \quad u(0) = u(T) = 0$$

where $\phi :]-a, a[\rightarrow \mathbb{R}$ ($0 < a < \infty$) is a homeomorphism such that $\phi(0) = 0$, $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function. For $\lambda \in [0, 1]$, consider the family of Dirichlet problems

$$(5.49) \quad (\phi(u'))' = \lambda f(t, u, u'), \quad u(0) = u(T) = 0,$$

which reduces to (5.48) for $\lambda = 1$ and to

$$(5.50) \quad (\phi(u'))' = 0, \quad u(0) = u(T) = 0$$

for $\lambda = 0$. This last problem has only the trivial solution.

Theorem 5.33. Let $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous. Then (5.48) has at least one solution.

Proof. Let $\lambda \in [0, 1]$ and u be a possible solution of (5.49). Then

$$\|u'\|_\infty < a$$

and

$$\|u\|_\infty = \left\| \int_0^\cdot u'(s) ds \right\|_\infty < Ta.$$

Therefore, if $\Omega = \{u \in C_0^1 : \|u\|_\infty < Ta, \|u'\|_\infty < a\}$, it follows that (5.49) has no solutions on $\partial\Omega$. The rest of the proof follows exactly in the same way as in the proof of Theorem 5.22. ■

6. UPPER AND LOWER SOLUTIONS AND DEGREE

6.1. ϕ -Laplacian with ϕ defined on $\mathbb{R}$

In this section $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$) denotes an increasing homeomorphism such that $\phi(0) = 0$. We prove existence of solutions for the periodic boundary value problem

$$(6.1) \quad (\phi(u'))' = f(t, u), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Definition 6.1. A lower solution α (resp. upper solution β) of (6.1) is a function such that $\alpha \in C_{per}^1, \phi(\alpha') \in C^1$ (resp. $\beta \in C_{per}^1, \phi(\beta') \in C^1$) and

$$(6.2) \quad (\phi(\alpha'(t)))' \geq f(t, \alpha(t)) \quad (\text{resp.} \quad (\phi(\beta'(t)))' \leq f(t, \beta(t))).$$

for all $t \in [0, T]$. Such a lower or upper solution will be called strict if the inequality (6.2) is strict for all $t \in [0, T]$.

Theorem 6.2. Suppose that (6.1) has a lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$. If there exists a continuous function c on $[0, T]$ such that $\|c^-\|_1 < \frac{a}{2}$ and

$$(6.3) \quad f(t, u) \geq c(t), \quad \text{for all } (t, u) \in [0, T] \times [\alpha_L, \beta_M],$$

then (6.1) has a solution u such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$. Moreover, if α and β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$.

Proof. I. A first modified problem.

Let $\gamma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be the continuous function defined by

$$\gamma(t, u) = \begin{cases} \beta(t), & u > \beta(t) \\ u, & \alpha(t) \leq u \leq \beta(t) \\ \alpha(t), & u < \alpha(t), \end{cases}$$

and define $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, F(t, u) = f(t, \gamma(t, u))$. We consider the modified problem

$$(6.4) \quad \begin{aligned} (\phi(u'))' &= F(t, u) + [u - \gamma(t, u)], \\ u(0) - u(T) &= 0 = u'(0) - u'(T), \end{aligned}$$

and show that if u is a solution of (6.4), then $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$ and hence u is a solution of (6.1). Suppose by contradiction that there is some $t_0 \in [0, T]$ such that $[\alpha - u]_M = \alpha(t_0) - u(t_0) > 0$. If $t_0 \in]0, T[$ then $\alpha'(t_0) = u'(t_0)$ and there are sequences (t_k) in $[t_0 - \varepsilon, t_0[$ and (t'_k) in $]t_0, t_0 + \varepsilon]$ converging to t_0 such that $\alpha'(t_k) - u'(t_k) \geq 0$ and $\alpha'(t'_k) - u'(t'_k) \leq 0$. As ϕ is an increasing homeomorphism, this implies that $\phi(\alpha'(t_k)) - \phi(\alpha'(t_0)) \geq \phi(u'(t_k)) - \phi(u'(t_0))$, $\phi(\alpha'(t'_k)) - \phi(\alpha'(t_0)) \leq \phi(u'(t'_k)) - \phi(u'(t_0))$ and $(\phi(\alpha'(t_0)))' \leq (\phi(u'(t_0)))'$. Hence, because α is a lower solution of (6.1) we have

$$\begin{aligned} (\phi(\alpha'(t_0)))' &\leq (\phi(u'(t_0)))' = f(t_0, \alpha(t_0)) + [u(t_0) - \alpha(t_0)] \\ &< f(t_0, \alpha(t_0)) \leq (\phi(\alpha'(t_0)))', \end{aligned}$$

a contradiction. If $[\alpha - u]_M = \alpha(0) - u(0) = \alpha(T) - u(T)$, then $\alpha'(0) - u'(0) \leq 0, \alpha'(T) - u'(T) \geq 0$. Using that $\alpha, u \in C_{per}^1$, we deduce that $\alpha'(0) - u'(0) = 0 = \alpha'(T) - u'(T)$. This implies that

$$(6.5) \quad \phi(\alpha'(0)) = \phi(u'(0)).$$

On the other hand, $[\alpha - u]_M = \alpha(0) - u(0)$ implies, reasoning in a similar way as for $t_0 \in]0, T[$, that

$$(\phi(\alpha'(0)))' \leq (\phi(u'(0)))'.$$

Using the inequality above and $\alpha'(0) = u'(0)$, we can proceed as in the case $t_0 \in]0, T[$ to obtain again a contradiction. In consequence we have that $\alpha(t) \leq u(t)$ for all $t \in [0, T]$. Analogously, using the fact that β is an upper solution of (6.1), we can show that $u(t) \leq \beta(t)$ for all $t \in [0, T]$. We remark that if α, β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$.

II. A second modified problem.

Let u be a solution of (6.4). Using I we have that u is a solution of (6.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$. Hence, using (6.3) and the same method as in the proof of Lemma 5.1, we deduce that

$$(6.6) \quad \|\phi(u')\|_\infty \leq 2\|c^-\|_1 < a,$$

which implies (5.8), where M is defined as in Lemma 5.1. Let $b > 0$ such that

$$\max(\|\alpha'\|_\infty + 1, \|\beta'\|_\infty + 1, M + 1) < b,$$

and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ an increasing homeomorphism such that $\psi = \phi$ on $[-b, b]$. Hence, if we suppose that u is a solution of the periodic boundary-value problem

$$(6.7) \quad \begin{aligned} (\psi(u'))' &= F(t, u) + [u - \gamma(t, u)], \\ u(0) - u(T) &= 0 = u'(0) - u'(T) \end{aligned}$$

then, as in I, we obtain that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$, which implies that u is a solution of the periodic boundary-value problem

$$(\psi(u'))' = f(t, u), \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

This implies, using again (6.3) and the same method as in the proof of Lemma 5.1, that

$$\|u'\|_\infty \leq \max\{|\psi^{-1}(2\|c^-\|_1)|, |\psi^{-1}(-2\|c^-\|_1)|\} = M.$$

By the choice of ψ , this implies that u is a solution of (6.1).

III. Some homotopies and a priori estimates.

For $\lambda \in [0, 1]$, consider the periodic boundary-value problems

$$(6.8) \quad \begin{aligned} (\psi(u'))' &= u + \lambda[F(t, u) - \gamma(t, u)], \\ u(0) - u(T) &= 0 = u'(0) - u'(T), \end{aligned}$$

$$(6.9) \quad \begin{aligned} (\psi(u'))' &= \lambda u + (1 - \lambda)Q(u), \\ u(0) - u(T) &= 0 = u'(0) - u'(T). \end{aligned}$$

Let

$$\sigma := \max_{(t, u) \in [0, T] \times \mathbb{R}} |F(t, u) - \gamma(t, u)|,$$

and consider $u \in C_{per}^1$ a solution of (6.8). Let $t_0 \in [0, T]$ such that $u(t_0) = u_M$. This implies that $(\psi(u'(t_0)))' \leq 0$, which together with (6.8), implies that $u_M \leq \sigma$. Analogously, we have that $u_L \geq -\sigma$. We deduce that $\|u\|_\infty \leq \sigma$. Let $\xi \in [0, T]$ such that $u'(\xi) = 0$. Then, using (6.8) and the fact that $\|u\|_\infty \leq \sigma$, we deduce that

$$|\psi(u'(t))| = \left| \int_\xi^t (\psi(u'(s)))' ds \right| \leq 2T\sigma,$$

which implies that $\|u'\|_\infty \leq \max(|\psi(\pm 2T\sigma)|)$. Now, let $u \in C_{per}^1$ be a solution of (6.9). This implies that $Q(u) = 0$. It is clear that, if $\lambda = 0$, then $u = 0$. If $\lambda \neq 0$, consider $t_0 \in [0, T]$ such that $u(t_0) = u_M$. This implies that $\lambda u_M = (\psi(u'(t_0)))' \leq 0$ and $u_M \leq 0$. Analogously, we have that $u_L \geq 0$. It follows that $u = 0$. In consequence, there is a $\rho > 0$ such that if $u \in C_{per}^1$ is a solution of (6.8) or (6.9), then $\|u\| < \rho$.

IV. Abstract formulations of (6.8) and (6.9).

For $\lambda \in [0, 1]$, consider the nonlinear operators $N_\lambda^i : C_{per}^1 \rightarrow C$ ($i = 1, 2$) defined by

$$N_\lambda^1(u) = u + \lambda[F(\cdot, u(\cdot)) - \gamma(\cdot, u(\cdot))]$$

and

$$N_\lambda^2(u) = \lambda u + (1 - \lambda)Q(u).$$

On the other hand, consider the homotopies $\mathcal{N}_i : [0, 1] \times C_{per}^1 \rightarrow C_{per}^1$, ($i = 1, 2$)

$$\begin{aligned} \mathcal{N}_i(\lambda, u) &= \mathcal{P}_{N_\lambda^i}(u) \\ &= Pu + QN_\lambda^i(u) + H \circ \psi^{-1} \circ (I - Q_\psi) \circ [H(I - Q)N_\lambda^i](u). \end{aligned}$$

Using Arzela-Ascoli's theorem, it is not difficult to see that $\mathcal{N}_i$ ($i = 1, 2$) is completely continuous and, using Proposition 3.1, that if $u \in C_{per}^1$, then u is a solution of (6.8) (resp. (6.9)) if and only if $\mathcal{N}_1(\lambda, u) = u$ (resp. $\mathcal{N}_2(\lambda, u) = u$). Hence, using III, it follows that

$$\mathcal{N}_i(\lambda, u) \neq u \quad \text{for all } (\lambda, u) \in [0, 1] \times \partial B_\rho, \quad (i = 1, 2).$$

The homotopy invariance of Leray-Schauder degree gives

$$(6.10) \quad d_{LS}[I - \mathcal{N}_i(0, \cdot), B_\rho, 0] = d_{LS}[I - \mathcal{N}_i(1, \cdot), B_\rho, 0],$$

for $i = 1, 2$. But the range of $\mathcal{N}_2(0, \cdot) = P + Q$ is contained in the subspace of constant function, isomorphic to $\mathbb{R}$, so, using the reduction property of the Leray-Schauder degree we have that

$$\begin{aligned} d_{LS}[I - \mathcal{N}_2(0, \cdot), B_\rho, 0] &= d_B[I - (P + Q)|_{\mathbb{R}},] - \rho, \rho[0, 0] \\ &= d_B[-I,] - \rho, \rho[0, 0] = -1, \end{aligned}$$

which, together with (6.10) and the fact that $\mathcal{N}_2(1, \cdot) = \mathcal{N}_1(0, \cdot)$ implies that

$$d_{LS}[I - \mathcal{N}_1(1, \cdot), B_\rho, 0] = -1.$$

Then, from the existence property of the Leray-Schauder degree there is $u \in B_\rho$ such that $\mathcal{N}_1(1, u) = u$, which implies that u is a solution of (6.7).

V. End of the proof.

We have proved in IV that there is u a solution of (6.7). The choice of ψ implies that u is a solution of (6.4). Finally, using I it follows that u is a solution of (6.1) and $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$. Moreover, if α and β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$. ■

Remark 6.3. Suppose that (6.1) has a lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$. Then, the conclusion of Theorem 6.2 also holds if $f(t, u) \leq c(t)$ for some continuous function c on $[0, T]$ such that $\|c^+\|_1 < \frac{a}{2}$ and all $(t, u) \in [0, T] \times [\alpha_L, \beta_M]$.

Remark 6.4. The Remark 6.3 also holds if $f : [0, T] \times]0, +\infty[\rightarrow \mathbb{R}$.

Exemple 6.5. Using Theorem 6.2 we deduce that the equations

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' = u^3 \pm \epsilon \sin t, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi)$$

have at least one solution if $0 < \epsilon < \frac{1}{4\pi}$. To apply Theorem 6.2, take $\alpha = -\epsilon^{\frac{1}{3}}, \beta = \epsilon^{\frac{1}{3}}$ and $c(t) = -\epsilon(1 \pm \sin(t))$ for all $t \in [0, 2\pi]$.

Remark 6.6. Suppose that $a = +\infty$. Under the hypothesis of Theorem 6.2, suppose that α (resp. β) is a strict lower (resp. upper) solution of (6.1). As we have already seen, (6.1) admits at least one solution u such that $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$ and $\|u'\|_\infty \leq M$. Moreover, if u is a solution of (6.1), then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$ and $\|u'\|_\infty \leq M$. If $\rho > M$, define the open, bounded set

$$\Omega_{\alpha, \beta}^\rho = \{u \in C_{per}^1 : \alpha(t) < u(t) < \beta(t) \text{ for all } t \in [0, T], \quad \|u'\|_\infty < \rho\}.$$

If ρ, ρ' are large enough, then, using the additivity-excision property of the Leray-Schauder degree, we have

$$d_{LS}[I - \mathcal{N}_1(1, \cdot), \Omega_{\alpha, \beta}^\rho, 0] = d_{LS}[I - \mathcal{N}_1(1, \cdot), B_{\rho'}, 0] = -1.$$

Because $\mathcal{P}_{N_f} = \mathcal{N}_1(1, \cdot)$ on $\Omega_{\alpha, \beta}^\rho$, we deduce that

$$(6.11) \quad d_{LS}[I - \mathcal{P}_{N_f}, \Omega_{\alpha, \beta}^\rho, 0] = d_{LS}[I - \mathcal{P}_{N_f}, B_{\rho'}, 0] = -1.$$

6.2. ϕ -Laplacian with ϕ defined on $] -a, a[$, $a \neq +\infty$

In this section $\phi :] -a, a[\rightarrow \mathbb{R}$ ($a \neq +\infty$) denotes an increasing homeomorphism such that $\phi(0) = 0$. Let $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function. We study the existence of solutions for the periodic boundary value problem

$$(6.12) \quad (\phi(u'))' = f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

Definition 6.7. A lower solution α (resp. upper solution β) of (6.12) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < a$, $\phi(\alpha') \in C^1$, $\alpha(0) = \alpha(T)$, $\alpha'(0) \geq \alpha'(T)$ (resp. $\beta \in C^1$, $\|\beta'\|_\infty < a$, $\phi(\beta') \in C^1$, $\beta(0) = \beta(T)$, $\beta'(0) \leq \beta'(T)$) and

$$(6.13) \quad \begin{aligned} & (\phi(\alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)) \\ & \text{(resp. } (\phi(\beta'(t)))' \leq f(t, \beta(t), \beta'(t)) \end{aligned}$$

for all $t \in [0, T]$. Such a lower or upper solution is called strict if the inequality (6.13) is strict for all $t \in [0, T]$.

Theorem 6.8. *If (6.12) has a lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$, then problem (6.12) has a solution u such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$. Moreover, if α and β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$.*

Proof. Let $\gamma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be the continuous function defined by

$$\gamma(t, u) = \begin{cases} \beta(t), & u > \beta(t) \\ u, & \alpha(t) \leq u \leq \beta(t) \\ \alpha(t), & u < \alpha(t), \end{cases}$$

and define $F : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F(t, u, v) = f(t, \gamma(t, u), v)$. We consider the modified problem

$$(6.14) \quad \begin{aligned} (\phi(u'))' &= F(t, u, u') + u - \gamma(t, u), \\ u(0) - u(T) &= 0 = u'(0) - u'(T) \end{aligned}$$

and first show that if u is a solution of (6.14) then $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$, so that u is a solution of (6.12). Suppose by contradiction that there is some $t_0 \in [0, T]$ such that $[\alpha - u]_M = \alpha(t_0) - u(t_0) > 0$. If $t_0 \in]0, T[$ then $\alpha'(t_0) = u'(t_0)$ and there are sequences (t_k) in $[t_0 - \varepsilon, t_0[$ and (t'_k) in $]t_0, t_0 + \varepsilon]$ converging to t_0 such that $\alpha'(t_k) - u'(t_k) \geq 0$ and $\alpha'(t'_k) - u'(t'_k) \leq 0$. As ϕ is an increasing homeomorphism, this implies $(\phi(\alpha'(t_0)))' \leq (\phi(u'(t_0)))'$. Hence, because α is a lower solution of (6.12) we obtain

$$\begin{aligned} (\phi(\alpha'(t_0)))' \leq (\phi(u'(t_0)))' &= f(t_0, \alpha(t_0), \alpha'(t_0)) + u(t_0) - \alpha(t_0) \\ &< f(t_0, \alpha(t_0), \alpha'(t_0)) \leq (\phi(\alpha'(t_0)))', \end{aligned}$$

a contradiction. If $[\alpha - u]_M = \alpha(0) - u(0) = \alpha(T) - u(T)$, then $\alpha'(0) - u'(0) \leq 0$, $\alpha'(T) - u'(T) \geq 0$. Using that $\alpha'(0) \geq \alpha'(T)$, we deduce that $\alpha'(0) - u'(0) = 0 = \alpha'(T) - u'(T)$. This implies that

$$(6.15) \quad \phi(\alpha'(0)) = \phi(u'(0)).$$

On the other hand, $[\alpha - u]_M = \alpha(0) - u(0)$ implies, reasoning in a similar way as for $t_0 \in]0, T[$, that

$$(\phi(\alpha'(0)))' \leq (\phi(u'(0)))'.$$

Using the inequality above and $\alpha'(0) = u'(0)$, we can proceed as in the case $t_0 \in]0, T[$ to obtain again a contradiction. In consequence we have $\alpha(t) \leq u(t)$ for all $t \in [0, T]$. Analogously, using the fact that β is an upper solution of (6.12), we can show that $u(t) \leq \beta(t)$ for all

$t \in [0, T]$. We remark that if α, β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$.

We now apply Corollary 5.29 to the modified problem (6.14) to obtain the existence of a solution, and the relation

$$(6.16) \quad d_{LS}[I - \widetilde{M}, B_\rho, 0] = -1$$

for the fixed point operator $\widetilde{M}$ associated to (6.14) and all sufficiently large $\rho > 0$. Moreover, if α and β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$. ■

Example 6.9. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that

$$\liminf_{u \rightarrow -\infty} g(u) = +\infty, \quad \limsup_{u \rightarrow +\infty} g(u) = -\infty.$$

If $h \in C$, then the problem

$$\left(\frac{u'}{\sqrt{1-u'^2}} \right)' + f(u)u' + g(u) = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution.

Remark 6.10. Suppose that α (resp. β) is a strict lower (resp. upper) solution of (6.12). As we have already seen, (6.12) admits at least one solution u such that $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$. Moreover, if u is a solution of (6.12), then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$. Define the bounded, open set

$$\Omega_{\alpha, \beta} = \{u \in C_{per}^1 : \alpha(t) < u(t) < \beta(t) \text{ for all } t \in [0, T], \quad \|u'\| < a\}.$$

If ρ is large enough, then, using the additivity-excision property of the Leray-Schauder degree, we have

$$d_{LS}[I - \widetilde{M}, \Omega_{\alpha, \beta}, 0] = d_{LS}[I - \widetilde{M}, B_\rho, 0] = -1.$$

Because $\mathcal{P}_{N_f} = \widetilde{M}$ on $\Omega_{\alpha, \beta}$, we deduce that

$$d_{LS}[I - \mathcal{P}_{N_f}, \Omega_{\alpha, \beta}, 0] = -1.$$

Remark 6.11. A careful analysis of the above proof implies that Theorem 6.8 holds also if $f : [0, T] \times]0, +\infty[\times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous functions.

7. AMBROSETTI-PRODI TYPE MULTIPLICITY RESULTS

7.1. ϕ -Laplacian with ϕ defined on $\mathbb{R}$

In this section $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$) denotes an increasing homeomorphism such that $\phi(0) = 0$. Let $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a

continuous function. For $s \in \mathbb{R}$, consider the periodic boundary-value problem

$$(7.1) \quad (\phi(u'))' + g(t, u) = s, \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

Assume now that the following condition

$$(7.2) \quad g(t, u) \rightarrow +\infty \quad \text{if} \quad |u| \rightarrow \infty, \quad \text{uniformly in} \quad t \in [0, T]$$

holds. Consider

$$\sigma_1 = \min_{(t,u) \in [0,T] \times \mathbb{R}} g(t, u).$$

The case $a = +\infty$

Lemma 7.1. *There exists an $\rho > 0$ such that any possible solution u of (7.1) with $s \leq b \in \mathbb{R}$ belongs to the open ball B_ρ , that is $\|u\| < \rho$.*

Proof. Let u be a solution of (7.1) with $s \leq b \in \mathbb{R}$. This implies that

$$(7.3) \quad QN_g(u) = s.$$

On the other hand, because $s - g$ is bounded from above by $b - \sigma_1$, it follows, as in the proof of Lemma 5.1, that

$$(7.4) \quad \|u'\|_\infty < M,$$

where M is a constant depending only upon b and σ_1 . Using (7.2) we can find $R > 0$ such that

$$(7.5) \quad g(t, u) > b \quad \text{if} \quad |u| \geq R, t \in [0, T].$$

If $u_L \geq R$, then using (7.5), we deduce that $QN_g(u) > b$, which, together with (7.3) gives $s > b$, a contradiction. So we have $u_L < R$. Analogously we can show that $u_M > -R$. Then using the following inequality

$$u_M \leq u_L + \int_0^T |u'(\tau)| d\tau,$$

and (7.4) we have that

$$(7.6) \quad \|u\|_\infty < R + MT.$$

We can take $\rho \geq R + (T + 1)M$. ■

Theorem 7.2. *If the function g satisfies (7.2), then there is $s_1 \in \mathbb{R}$ such that (7.1) has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$.*

Proof. Let

$$S_j = \{s \in \mathbb{R} : (7.1) \text{ has at least } j \text{ solutions}\} \quad (j \geq 1).$$

(a) $S_1 \neq \emptyset$.

Take $s^* > \max_{t \in [0, T]} g(t, 0)$ and use (7.2) to find $R_-^* < 0$ such that

$$\min_{t \in [0, T]} g(t, R_-^*) > s^*.$$

Then α with $\alpha = R_-^* < 0$ is a strict lower solution and β with $\beta = 0$ is a strict upper solution for (7.1) with $s = s^*$. Hence, using Remark 6.3, $s^* \in S_1$.

(b) If $\tilde{s} \in S_1$ and $s > \tilde{s}$ then $s \in S_1$.

Let $\tilde{u}$ be a solution of (7.1) with $s = \tilde{s}$, and let $s > \tilde{s}$. Then $\tilde{u}$ is a strict upper solution for (7.1). Take now $R_- < \tilde{u}_L$ such that $\min_{t \in [0, T]} g(t, R_-) > s$. It follows that $\alpha = R_-$ is a strict lower solution for (7.1), so, using Remark 6.3, $s \in S_1$.

(c) $s_1 = \inf S_1$ is finite and $S_1 \supset]s_1, \infty[$.

Let $s \in \mathbb{R}$ and suppose that (7.1) has a solution u . Then (7.3) holds, from where we deduce that $s \geq \sigma_1$. To obtain the second part of claim (c) $S_1 \supset]s_1, \infty[$ we apply (b).

(d) $S_2 \supset]s_1, \infty[$.

Let $s_3 < s_1 < s_2$. Consider the nonlinear operator $N_s : C_{per}^1 \rightarrow C$ defined by

$$N_s(u) = s - g(\cdot, u),$$

and the completely continuous homotopy $\mathcal{G} : [s_3, s_2] \times C_{per}^1 \rightarrow C_{per}^1$ defined by

$$\mathcal{G}(s, \cdot) = \mathcal{P}_{N_s}.$$

Hence, if $s \in [s_3, s_2]$ and $u \in C_{per}^1$ then u is a solution of (7.1) if and only if $\mathcal{G}(s, u) = u$. Using Lemma 7.1 we find ρ such that each possible zero of $I - \mathcal{G}(s, \cdot)$ with $s \in [s_3, s_2]$ is such that $u \in B_\rho$. Consequently, the Leray-Schauder degree $d_{LS}[I - \mathcal{G}(s, \cdot), B_\rho, 0]$ is well defined and does not depend upon $s \in [s_3, s_2]$. However, using (c), we see that $u - \mathcal{G}(s_3, u) \neq 0$ for all $u \in C_{per}^1$. This implies that $d_{LS}[I - \mathcal{G}(s_3, \cdot), B_\rho, 0] = 0$, so that $d_{LS}[I - \mathcal{G}(s_2, \cdot), B_\rho, 0] = 0$ and, by excision property, $d_{LS}[I - \mathcal{G}(s_2, \cdot), B_{\rho'}, 0] = 0$ if $\rho' > \rho$. Let $s \in]s_1, s_2[$ and $\hat{u}$ be a solution of (7.1) (using (c)). Then $\hat{u}$ is a strict upper solution of (7.1) with $s = s_2$. Let $R < \hat{u}_L$ be such that $\min_{t \in [0, T]} g(t, R) > s_2$. Then R is a strict lower solution of (7.1) with $s = s_2$. Consequently, using Remark 6.6, (7.1) with $s = s_2$ has a solution in $\Omega_{R, \hat{u}}^{\rho'}$ and

$$d_{LS}[I - \mathcal{G}(s_2, \cdot), \Omega_{R, \hat{u}}^{\rho'}, 0] = -1,$$

for ρ' sufficiently large. Taking $\rho'' > \rho'$ sufficiently large, we deduce from the additivity property of Leray-Schauder degree that

$$\begin{aligned} d_{LS}[I - \mathcal{G}(s_2, \cdot), B_{\rho''} \setminus \overline{\Omega}_{R, \widehat{u}}^{\rho'}, 0] &= d_{LS}[I - \mathcal{G}(s_2, \cdot), B_{\rho''}, 0] \\ -d_{LS}[I - \mathcal{G}(s_2, \cdot), \Omega_{R, \widehat{u}}^{\rho'}, 0] &= -d_{LS}[I - \mathcal{G}(s_2, \cdot), \Omega_{R, \widehat{u}}^{\rho'}, 0] = 1, \end{aligned}$$

and (7.1) with $s = s_2$ has a second solution in $B_{\rho''} \setminus \overline{\Omega}_{R, \widehat{u}}^{\rho'}$.

(e) $s_1 \in S_1$.

Let (τ_k) be a sequence in $]s_1, +\infty[$ which converges to s_1 , and let u_k be a solution of (7.1) with $s = \tau_k$ given by (c). We deduce that

$$(7.7) \quad u_k = \mathcal{G}(\tau_k, u_k).$$

From Lemma 7.1, we know that there exists $\rho > 0$ such that $\|u_k\| < \rho$ for all $k \geq 1$. Then, the complete continuity of $\mathcal{G}$ implies that, up to a subsequence, the right-hand member of (7.7) converges in C_{per}^1 , and then (u_k) converges to some $u \in C_{per}^1$ such that $u = \mathcal{G}(s_1, u)$, i.e. to a solution of (7.1) with $s = s_1$. ■

Example 7.3. Let $h \in C$, then there exists $s_1 \in \mathbb{R}$ such that the problem $(|u'|^{p-2}u')' + \log(1 + |u|) + h(t) = s$, $u(0) - u(T) = 0 = u'(0) - u'(T)$ has no solution if $s < s_1$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$.

The case $a < +\infty$

Let us consider

$$\sigma_2 = \sigma_1 + \frac{a}{2T}.$$

Lemma 7.4. If $\sigma_1 < \delta < \sigma_2$ and u is a solution of (7.1) with $s \leq \delta$, then $\sigma_1 \leq s$ and

$$(7.8) \quad \|u'\|_\infty \leq \max(|\phi^{-1}(\pm 2T(\delta - \sigma_1))|) =: \rho.$$

Proof. Suppose that $u \in C_{per}^1$ is a solution of (7.1). It follows that $\sigma_1 \leq s$ because

$$T\sigma_1 \leq \int_0^T g(t, u(t))dt = sT.$$

On the other hand, using the fact that $s - g$ is bounded from above by $s - \sigma_1$ we deduce the following inequalities

$$\|\phi(u')\|_\infty \leq 2\|[s - \sigma_1]^+\|_1 = 2T(s - \sigma_1) \leq 2T(\delta - \sigma_1) < a,$$

which implies (7.8). ■

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing homeomorphism such that $\varphi = \phi$ on $[-(\rho + 1), \rho + 1]$, and consider the periodic boundary-value problem

$$(7.9) \quad (\varphi(u'))' + g(t, u) = s, \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

Using the same arguments as in the proof of Lemma 7.4 we deduce the following result.

Lemma 7.5. *If $\sigma_1 < \delta < \sigma_2$ and u is a solution of (7.9) with $s \leq \delta$, then $\sigma_1 \leq s$ and*

$$(7.10) \quad \|u'\|_\infty \leq \max(|\varphi^{-1}(\pm 2T(\delta - \sigma_1))|) =: \rho_1.$$

Now, using the definition of φ , it follows that $\rho = \rho_1$. Then, we have the following lemma.

Lemma 7.6. *If $\sigma_1 < \delta < \sigma_2$ and $s \leq \delta$, then u is a solution of (7.1) if and only if u is a solution of (7.9).*

Using Theorem 7.2 we deduce the following result.

Lemma 7.7. *If the function g satisfies (7.2) and if there is a $u_0 \in \mathbb{R}$ such that*

$$(7.11) \quad \sigma_3 =: \max_{t \in [0, T]} g(t, u_0) < \sigma_2,$$

then there is $s_1 \in [\sigma_1, \sigma_3]$ such that (7.9) has zero, at least one or at least two solutions according to $s < s_1, s = s_1$ or $s_1 < s \leq \delta$, where $\sigma_3 < \delta < \sigma_2$

Using Lemma 7.6 and Lemma 7.7 we deduce the following Ambrosetti-Prodi type result.

Theorem 7.8. *If the function g satisfies (7.2) and (7.11), then there is $s_1 \in [\sigma_1, \sigma_3]$ such that (7.1) has zero, at least one or at least two solutions according to $s < s_1, s = s_1$ or $s_1 < s < \sigma_2$.*

Example 7.9. *Let $0 < \epsilon < \frac{1}{8\pi}$ and $p > 0$. There exists $s_1 \in [-\epsilon, \epsilon]$ such that the periodic problem*

$$\left(\frac{u'}{\sqrt{1 + u'^2}} \right)' + |u|^p + \epsilon \sin(t) = s, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi)$$

has zero, at least one or at least two non constant solutions according to $s < s_1, s = s_1$ or $s_1 < s < \frac{1}{4\pi} - \epsilon$.

Example 7.10. *Let $h \in C$ such that*

$$h_M - h_L < \frac{1}{2T}.$$

There exists $s_1 \in [h_L, h_M]$ such that the periodic problem

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' + \log(1+|u|) + h(t) = s,$$

$$u(0) - u(T) = 0 = u'(0) - u'(T)$$

has zero, at least one or at least two non constant solutions according to $s < s_1$, $s = s_1$ or $s_1 < s < h_L + \frac{1}{2T}$.

7.2. ϕ -Laplacian with ϕ defined on $] -a, a[$, $a \neq +\infty$

In this section $\phi :] -a, a[\rightarrow \mathbb{R}$ ($0 < a < +\infty$) denotes an increasing homeomorphism such that $\phi(0) = 0$. Let $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function satisfying the following condition

$$(7.12) \quad f(t, u, v) \rightarrow +\infty \quad \text{if} \quad |u| \rightarrow \infty$$

uniformly for $(t, v) \in [0, T] \times] -a, a[$. We are interested in studying the Ambrosetti-Prodi problem for the solutions of the periodic boundary value problem

$$(7.13) \quad \begin{aligned} &(\phi(u'))' + f(t, u, u') = s, \\ &u(0) - u(T) = 0 = u'(0) - u'(T), \end{aligned}$$

in terms of the value of the forcing term s .

Lemma 7.11. *There exists an $\rho > 0$ such that any possible solution u of (7.13) with $s \leq b \in \mathbb{R}$ belongs to the open ball B_ρ .*

Proof. Let $s \leq b$ and u be a solution of (7.13). This implies that u satisfies

$$(7.14) \quad \|u'\|_\infty < a$$

and

$$(7.15) \quad QN_f(u) = s$$

Using (7.12) we can find $R > 0$ such that

$$(7.16) \quad f(t, u, v) > b \quad \text{if} \quad |u| \geq R, (t, v) \in [0, T] \times [-a, a].$$

If $u_L \geq R$, then using (7.14) and (7.16), we deduce that $QN_f(u) > b$, which, together with (7.15) gives $s > b$, a contradiction. So we have $u_L < R$. Analogously we can show that $u_M > -R$. Then using the following inequality

$$u_M \leq u_L + \int_0^T |u'(\tau)| d\tau,$$

we have that

$$\|u\|_\infty < R + Ta.$$

We can take $\rho \geq R + (T + 1)a$. ■

Using Lemma 7.11, Theorem 6.8, Remark 6.10 and the same methodology as in the proof of Theorem 7.2 we deduce the following result.

Theorem 7.12. *If the function f satisfies (7.12), then there is $s_1 \in \mathbb{R}$ such that (7.13) has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$.*

Corollary 7.13. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous functions such that f is bounded and g satisfies*

$$g(t, u) \rightarrow +\infty \quad \text{if} \quad |u| \rightarrow \infty \quad \text{uniformly in} \quad t \in [0, T].$$

Then, there is $s_1 \in \mathbb{R}$ such that the problem

$$(\phi(u'))' + f(u)u' + g(t, u) = s, \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has no solution if $s < s_1$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$.

Example 7.14. *Let $h \in C$, then there exists $s_1 \in \mathbb{R}$ such that the problem*

$$\begin{aligned} \left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \frac{1}{1+u^2}u' + \exp(u^2) + h(t) &= s, \\ u(0) - u(T) = 0 &= u'(0) - u'(T) \end{aligned}$$

has no solution if $s < s_1$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$.

8. SINGULAR NONLINEARITIES

8.1. ϕ -Laplacian with ϕ defined on $\mathbb{R}$

In this section $\phi : \mathbb{R} \rightarrow]-a, a[$ ($0 < a \leq +\infty$) denotes an increasing homeomorphism such that $\phi(0) = 0$. Consider the periodic boundary value problem

$$(8.1) \quad (\phi(u'))' + g(u) = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where $h \in C$ and $g :]0, +\infty[\rightarrow]0, +\infty[$ is a continuous function such that

$$(8.2) \quad g(u) \rightarrow +\infty \quad \text{as} \quad u \rightarrow 0_+,$$

$$(8.3) \quad g(u) \rightarrow 0 \quad \text{as} \quad u \rightarrow +\infty.$$

Theorem 8.1. *Suppose that g satisfies (8.2), (8.3) and $\|h^+\|_1 < \frac{a}{2}$. Then (8.1) has at least one solution if and only if $Q(h) > 0$.*

Proof. If u is a solution of (8.1), then it is clear that $Q(h) = QN_g(u) > 0$ because $g > 0$. Reciprocally, suppose that $Q(h) > 0$.

The case $a = +\infty$.

Using (8.2), it follows that there exists $\epsilon > 0$ such that $g(\epsilon) > h(t)$ for all $t \in [0, T]$. Hence, $\alpha = \epsilon$ is a strict lower solution for (8.1). On the other hand we know that there exists $w \in C_{per}^1$ such that $(\phi(w'))' = h - Q(h)$. Using (8.3), we deduce that there is $\delta > 0$ such that $\beta(t) = \delta + w(t) > \alpha(t)$ and $g(\beta(t)) < Q(h)$ for all $t \in [0, T]$. Then, β is a strict upper solution for (8.1) and using Remark 6.4 with $f = h - g$ we deduce the result in this case.

The case $a < +\infty$.

If u is a solution of (8.1), then using the fact that the function $h - g$ is bounded from above by h and $\|h^+\|_1 < \frac{a}{2}$, it follows that

$$\|u'\|_\infty \leq |\phi^{-1}(\pm 2\|h^+\|_1)| := M''.$$

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing homeomorphism which coincides with ϕ on $[-(M'' + 1), M'' + 1]$. It follows that u is a solution of

$$(\varphi(u'))' + g(u) = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

if and only if u is a solution of (8.1). Now, the conclusion follows using the above case. $\blacksquare$

Example 8.2. *If $\mu > 0$ and $h \in C$ such that $\|h^+\|_1 < \frac{1}{2}$, then there is at least one positive solution for the equation*

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' + \frac{1}{u^\mu} = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

if and only if $Q(h) > 0$.

Example 8.3. *If $\mu > 0$ and $h \in C$, then there is at least one positive solution for the equation*

$$(|u'|^{p-2}u')' + \frac{1}{u^\mu} = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

if and only if $Q(h) > 0$.

Consider the periodic boundary value problem

$$(8.4) \quad (\phi(u'))' - g(u) = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where $h \in C$ such that $h(0) = h(T)$ and $g :]0, +\infty[\rightarrow]0, +\infty[$ is a continuous function. Suppose that

$$(8.5) \quad \int_0^1 g(t) dt = +\infty.$$

The case $a = +\infty$

Lemma 8.4. *Suppose that $h \in C$ such that $h(0) = h(T)$ and g satisfies conditions (8.2), (8.3), and (8.5). There exists $\epsilon > 0$ such that if $\lambda \in [0, 1]$ and u is any positive solution of*

$$(8.6) \quad \begin{aligned} (\phi(u'))' &= (1 - \lambda)[QN_g(u) + Q(h)] + \lambda[g(u) + h(t)], \\ u(0) - u(T) &= 0 = u'(0) - u'(T), \end{aligned}$$

then $u(t) > \epsilon$ for all $t \in [0, T]$.

Proof. Let $\lambda \in [0, 1]$, let u be a possible positive solution of (8.6), and let us extend u to $\mathbb{R}$ as a C^1 T-periodic function, with $(\phi(u'))'$ continuous and T-periodic. Then

$$(8.7) \quad QN_g(u) + Q(h) = 0$$

and hence, if $\lambda \in]0, 1]$, (8.6) is equivalent to

$$(8.8) \quad (\phi(u'))' = \lambda[g(u) + h(t)], \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

This identity can be extended to $\mathbb{R}$ by T-periodicity. Using Lemma 1.4 and the positivity of g , we deduce that

$$(8.9) \quad \|\phi(u')\|_\infty \leq \|[(\phi(u'))']^-\|_1 = \lambda\|(g(u) + h)^-\|_1 \leq \lambda\|h^-\|_1.$$

Using (8.2), there exists $\xi > 0$ such that

$$(8.10) \quad g(u) > -Q(h) \quad \text{for all } 0 < u \leq \xi.$$

and therefore, by (8.10) and (8.7), there exists $t_1 \in \mathbb{R}$ such that $u(t_1) > \xi$. Now, let

$$(8.11) \quad x(t) = \phi(u'(t))$$

which implies

$$(8.12) \quad u'(t) = \phi^{-1}(x(t))$$

for all $t \in [t_1, t_1 + T]$. Introducing (8.11) in (8.8) we obtain

$$(8.13) \quad x'(t) - \lambda g(u(t)) = \lambda h(t)$$

for all $t \in [t_1, t_1 + T]$. Multiplying (8.12) by $x'(t)$ and (8.13) by $u'(t)$ and subtracting we get

$$x'(t)\phi^{-1}(x(t)) - \lambda g(u(t))u'(t) = \lambda h(t)u'(t)$$

i.e.

$$\left(\int_0^{x(t)} \phi^{-1}(s) ds \right)' - \lambda g(u(t))u'(t) = \lambda h(t)u'(t)$$

for all $t \in [t_1, t_1 + T]$. This implies that

$$\int_0^{x(t)} \phi^{-1}(s) ds - \int_0^{x(t_1)} \phi^{-1}(s) ds - \lambda \int_{u(t_1)}^{u(t)} g(s) ds = \lambda \int_{t_1}^t h(s)u'(s) ds$$

for all $t \in [t_1, t_1 + T]$. Using the fact that $\int_0^v \phi^{-1}(s) ds \geq 0$ for all $v \in \mathbb{R}$, we deduce that

$$(8.14) \quad \lambda \int_{u(t)}^{u(t_1)} g(s) ds \leq \int_0^{x(t_1)} \phi^{-1}(s) ds + \lambda \int_t^{t_1} h(s)u'(s) ds$$

for all $t \in [t_1, t_1 + T]$. Using (8.9), (8.11) and (8.14), we obtain

$$\begin{aligned} \int_{u(t)}^{u(t_1)} g(s) ds &\leq \frac{1}{\lambda} \int_0^{\pm \lambda \|h^-\|_1} \phi^{-1}(s) ds + |\phi^{-1}(\pm \|h^-\|_1)| \|h\|_1 \\ &\leq \max_{[-\|h^-\|_1, \|h^-\|_1]} |\phi^{-1}| \|h^-\|_1 + |\phi^{-1}(\pm \|h^-\|_1)| \|h\|_1 \\ (8.15) \quad &:= c \end{aligned}$$

for all $t \in [t_1, t_1 + T]$. Using (8.5) we can find $0 < \epsilon < \xi$ such that

$$(8.16) \quad \int_{\epsilon}^{\xi} g(t) dt > c.$$

Since $u(t_1) > \xi$ and g is positive, from (8.15) and (8.16) one gets $u(t) > \epsilon$ for all $t \in \mathbb{R}$. Now, for $\lambda = 0$, the solutions of (8.6) are the constant functions u solutions of

$$g(u) + Q(h) = 0$$

and they satisfy $u > \xi > \epsilon$. ■

Theorem 8.5. *Suppose that $h \in C$, $h(0) = h(T)$, and g satisfies conditions (8.2), (8.3), and (8.5). Then (8.4) has at least one positive solution if and only if $Q(h) < 0$.*

Proof. If u is a solution of (8.4), then it is clear that $Q(h) = -QN_g(u) < 0$ because $g > 0$. Reciprocally, suppose that $Q(h) < 0$. We use the homotopy (8.6) and the corresponding homotopy for the associated family of fixed point operators $\mathcal{M}(\lambda, \cdot)$. Let $\lambda \in [0, 1]$ and u be a possible positive solution of (8.6). We already know from Lemma 8.4

that $u(t) > \epsilon$ for some $\epsilon > 0$ and all $t \in [0, T]$. On the other hand, (8.9) implies that

$$(8.17) \quad \|u'\|_\infty \leq |\phi^{-1}(\pm \|h^-\|_1)| := d.$$

From assumption (8.3) follows easily the existence of $R > 0$ such that

$$(8.18) \quad g(u) + Q(h) < 0 \quad \text{if } u \geq R.$$

Hence, because of (8.7), there exists $t_2 \in [0, T]$ such that $u(t_2) < R$, which together with (8.17) implies

$$(8.19) \quad u(t) < R + dT \quad (t \in [0, T]).$$

Hence, all the possible positive solutions of problem (8.6) are contained in the open bounded set

$$\Omega := \{u \in C_{per}^1 : \epsilon < u(t) < R + dT \ (0 \leq t \leq T), \ \|u'\|_\infty < d\}.$$

From the homotopy invariance of Leray-Schauder degree, we obtain

$$\begin{aligned} d_{LS}[I - \mathcal{M}(1, \cdot), \Omega, 0] &= d_{LS}[I - \mathcal{M}(0, \cdot), \Omega, 0] \\ &= d_B[g + Q(h), \Omega \cap \mathbb{R}, 0] = d_B[g + Q(h),]\epsilon, R[, 0] = -1, \end{aligned}$$

so the existence property of Leray-Schauder degree implies that $\mathcal{M}(1, \cdot)$ has a fixed point $u \in \Omega$ which is a positive solution of (8.4). ■

Example 8.6. If $\mu \geq 1$, $h \in C$ and $h(0) = h(T)$, problem

$$(|u'|^{p-2}u')' - \frac{1}{u^\mu} = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one positive solution if and only if $Q(h) < 0$.

The case $a < +\infty$

Theorem 8.7. Suppose that $h \in C$ such that $h(0) = h(T)$, $\|h^-\|_1 < a$ and g satisfies conditions (8.2), (8.3), and (8.5). Then (8.4) has at least one positive solution if and only if $Q(h) < 0$.

Proof. If u is a solution of (8.4), then it is clear that $Q(h) = -QN_g(u) < 0$ because $g > 0$. Reciprocally, suppose that $Q(h) < 0$. Let u be a solution of (8.4). It follows that u satisfies (8.9), with $\lambda = 1$. Hence, using the fact that $\|h^-\|_1 < a$, it follows that (8.17) holds. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing homeomorphism which coincides with ϕ on $[-(d+1), d+1]$. It follows that u is a solution of

$$(8.20) \quad (\varphi(u'))' - g(u) = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

if and only if u is a solution of (8.4). Now, the conclusion follows using Theorem 8.5. ■

Exemple 8.8. *If $\mu \geq 1$ and $h \in C$ such that $\|h^-\|_1 < 1$ and $h(0) = h(T)$, then there is at least one positive solution for the equation*

$$\left(\frac{u'}{\sqrt{1+u'^2}} \right)' - \frac{1}{u^\mu} = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

if and only if $Q(h) < 0$.

Remark 8.9. Theorem 8.1 and Theorem 8.5 hold also if $\phi :]-a, a[\rightarrow \mathbb{R}$ ($0 < a < +\infty$) is an increasing homeomorphism such that $\phi(0) = 0$. For the proofs use the same strategy as above.

CHAPTER IV

Delay-Lotka-Volterra systems

1. COMPETITION SYSTEMS WITH ZANOLIN CONDITION

Let $T > 0$ and

$$C_T = \{x : \mathbb{R} \rightarrow \mathbb{R}^n : x \text{ is a continuous } T\text{-periodic function}\}$$

with the norm $|x|_0 = \sup_{t \in \mathbb{R}} |x(t)|$. $(C_T, |\cdot|_0)$ is a Banach space.

We search a positive function $x \in C_T$ which is a solution of the system

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)[r_i(t) - a_{ii}(t)x_i(t) \\ &\quad - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t)x_j(t - \tau_j(t, x_1(t), \dots, x_n(t)))] \\ (1.1) \quad &(i = 1, 2, \dots, n), \end{aligned}$$

where $r_i > 0, a_{ii} > 0, a_{ij} \geq 0$ ($j \neq i$), $(i, j = 1, \dots, n)$ are T -periodic continuous functions, $\tau_j \in C(\mathbb{R}^{n+1}, \mathbb{R})$ and τ_j ($j = 1, 2, \dots, n$) are T -periodic with respect to their first argument. To find such a function, it is sufficient to show that the following system has T -periodic solutions

$$\begin{aligned} \dot{x}_i(t) &= r_i(t) - a_{ii}(t) \exp(x_i(t)) \\ &\quad - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t) \exp \left[x_j(t - \tau_j(t, \exp x_1(t), \dots, \exp x_n(t))) \right] \\ (1.2) \quad &(i = 1, 2, \dots, n). \end{aligned}$$

We reformulate problem (1.2) to use Mawhin's continuation theorem in infinite dimension. Let $(L, D(L))$ be the operator defined by

$$D(L) = C_T \cap C^1(\mathbb{R}, \mathbb{R}^n), Lx = \dot{x}$$

and $N : C_T \rightarrow C_T, Nx = y$ where

$$\begin{aligned} y_i(t) &= r_i(t) - a_{ii}(t) \exp(x_i(t)) \\ &\quad - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t) \exp \left[x_j(t - \tau_j(t, \exp x_1(t), \dots, \exp x_n(t))) \right] \\ &\quad (i = 1, 2, \dots, n). \end{aligned}$$

It is obvious that $x \in C_T$ is a solution of (1.2) if and only if $x \in D(L)$ and $Lx = Nx$. Define the continuous projectors P, Q as

$$\begin{aligned} Q &: C_T \rightarrow C_T, \quad Qx = \frac{1}{T} \int_0^T x(t) dt = \bar{x}, \\ P &: C_T \rightarrow C_T, \quad Px = x(0). \end{aligned}$$

We know that

$$\begin{aligned} \text{Im}(P) &= \ker(L), \quad \ker(Q) = \text{Im}(L), \\ C_T &= \ker(L) \oplus \ker(P) = \text{Im}(L) \oplus \text{Im}(Q), \\ \ker(L) &= \text{Im}(Q) \simeq \mathbb{R}^n. \end{aligned}$$

Consequently, L is a Fredholm operator of index zero. We recall that the operator $L : D(L) \cap \ker(P) \rightarrow \text{Im}(L)$ is an isomorphism denoted by L_P . For $x \in \text{Im}(L)$, we have that

$$L_P^{-1}x(t) = \int_0^t x(s) ds \quad \text{for all } t \in [0, T].$$

Thus, using the fact that N takes bounded sets into bounded sets, we deduce that N is L -compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset C_T$.

If $x \in C_T$, we write

$$[x]_L = \min_{[0, T]} x, \quad [x]_M = \max_{[0, T]} x.$$

Lemma 1.1. *Suppose that condition*

$$(1.3) \quad \bar{r}_i - \sum_{\substack{j=1 \\ j \neq i}}^n \bar{a}_{ij} \left| \frac{r_j}{a_{jj}} \right|_0 > 0 \quad (i = 1, 2, \dots, n).$$

is satisfied. Then there is a bounded, open set $\Omega \subset C_T$ such that

$$\bigcup_{\lambda \in [0, 1]} \{x \in D(L) : Lx = \lambda Nx\} \subset \Omega.$$

Proof. Let $\lambda \in]0, 1]$ and $x \in D(L)$ such that

$$(1.4) \quad Lx = \lambda Nx.$$

Choose $t_i^1 \in [0, T]$ such that

$$x_i(t_i^1) = [x_i]_M \quad (i = 1, 2, \dots, n).$$

It follows that

$$\dot{x}_i(t_i^1) = 0 \quad (i = 1, 2, \dots, n).$$

In view of this and (1.4), we obtain that

$$\begin{aligned} 0 &= r_i(t_i^1) - a_{ii}(t_i^1) \exp(x_i(t_i^1)) \\ &- \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t_i^1) \exp \left[x_j(t_i^1 - \tau_j(t_i^1, \exp x_1(t_i^1), \dots, \exp x_n(t_i^1))) \right] \\ &\quad (i = 1, 2, \dots, n). \end{aligned}$$

Therefore,

$$r_i(t_i^1) - a_{ii}(t_i^1) \exp(x_i(t_i^1)) > 0 \quad (i = 1, 2, \dots, n),$$

which implies that

$$(1.5) \quad \exp(x_i(t_i^1)) < \left| \frac{r_i}{a_{ii}} \right|_0, \quad x_i(t_i^1) < \ln \left| \frac{r_i}{a_{ii}} \right|_0.$$

It follows from (1.4) and (1.5) that

$$(1.6) \quad \int_0^T |\dot{x}_i(t)| dt \leq T \bar{r}_i + T \left| \frac{r_i}{a_{ii}} \right|_0 \bar{a}_{ii} + T \sum_{\substack{j=1 \\ j \neq i}}^n \bar{a}_{ij} \left| \frac{r_j}{a_{jj}} \right|_0 := B_{i1} \quad (i = 1, 2, \dots, n).$$

Integrating (1.4) from 0 to T , we have

$$\begin{aligned} T \bar{r}_i &= \int_0^T a_{ii}(t) \exp(x_i(t)) dt \\ &+ \int_0^T \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t) \exp \left[x_j(t - \tau_j(t, \exp x_1(t), \dots, \exp x_n(t))) \right] dt \\ &\quad (i = 1, 2, \dots, n). \end{aligned}$$

In view of this, (1.5) and (1.3), we find

$$\int_0^T a_{ii}(t) \exp(x_i(t)) dt > T\bar{r}_i - T \sum_{\substack{j=1 \\ j \neq i}}^n \bar{a}_{ij} \left| \frac{r_j}{a_{jj}} \right|_0 > 0 \quad (i = 1, 2, \dots, n),$$

which implies that there exists positive constants B_{i2} and points $t_i^0 \in [0, T]$ such that

$$x_i(t_i^0) > -B_{i2} \quad (i = 1, 2, \dots, n).$$

According to this and (1.6), we obtain

$$\begin{aligned} x_i(t) &= x_i(t_i^0) + \int_{t_i^0}^t \dot{x}_i(t) dt \\ &\geq x_i(t_i^0) - \int_0^T |\dot{x}_i(t)| dt \\ &> -B_{i1} - B_{i2} \quad (i = 1, 2, \dots, n). \end{aligned}$$

Thus, using again (1.5) it follows that

$$|x_i|_0 < \max \left[\left| \ln \left| \frac{r_i}{a_{ii}} \right| \right|, B_{i1} + B_{i2} \right] \quad (i = 1, 2, \dots, n).$$

■

Lemma 1.2. *Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\varphi(x) = y$, where*

$$y_i = \bar{r}_i - \sum_{j=1}^n \bar{a}_{ij} \exp(x_j) \quad (i = 1, 2, \dots, n).$$

If relation (1.3) holds, then there exists an open, bounded set $\Omega_1 \subset \mathbb{R}^n$ such that

$$\{x \in \mathbb{R}^n : \varphi(x) = 0\} \subset \Omega_1 \quad \text{and} \quad d_B[\varphi, \Omega_1, 0] = (-1)^n.$$

Proof. Let $H : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $H(\lambda, x) = y$, where

$$y_i = \bar{r}_i - \bar{a}_{ii} \exp(x_i) - \lambda \sum_{\substack{j=1 \\ j \neq i}}^n \bar{a}_{ij} \exp(x_j) \quad (i = 1, 2, \dots, n).$$

Let $\lambda \in [0, 1]$ and $x \in \mathbb{R}^n$ such that $H(\lambda, x) = 0$. It follows that

$$\bar{r}_i - \bar{a}_{ii} \exp(x_i) - \lambda \sum_{\substack{j=1 \\ j \neq i}}^n \bar{a}_{ij} \exp(x_j) = 0 \quad (i = 1, 2, \dots, n). \quad (1.7)$$

We deduce that $0 \leq \bar{r}_i - \bar{a}_{ii} \exp(x_i)$, so

$$\exp(x_i) \leq \frac{\bar{r}_i}{\bar{a}_{ii}} \leq \left| \frac{r_i}{a_{ii}} \right|_0, \quad x_i \leq \ln \left| \frac{r_i}{a_{ii}} \right|_0 \quad (i = 1, 2, \dots, n). \quad (1.8)$$

On the other hand, from (1.7), (1.8) and (1.3) we get

$$\bar{a}_{ii} \exp(x_i) \geq \bar{r}_i - \sum_{\substack{j=1 \\ j \neq i}}^n \bar{a}_{ij} \exp(x_j) \geq \bar{r}_i - \sum_{\substack{j=1 \\ j \neq i}}^n \bar{a}_{ij} \left| \frac{r_j}{a_{jj}} \right|_0 > 0 \quad (i = 1, 2, \dots, n).$$

which implies that there exists a constant $M \in \mathbb{R}$ such that

$$M \leq x_i \quad (i = 1, 2, \dots, n). \quad (1.9)$$

From (1.8) and (1.9) we obtain the existence of an open, bounded set $\Omega_1 \subseteq \mathbb{R}^n$ such that

$$\bigcup_{\lambda \in [0, 1]} \{x \in \mathbb{R}^n : H(\lambda, x) = 0\} \subset \Omega_1.$$

Furthermore, $H(0, x) = 0$ has the unique solution x^0 with

$$x_i^0 = \ln \frac{\bar{r}_i}{\bar{a}_{ii}} \quad (i = 1, 2, \dots, n)$$

for which

$$J_{H(0, \cdot)}(x^0) = (-1)^n \prod_{i=1}^n (\exp x_i^0) \bar{a}_{ii}.$$

Consequently, we have that

$$d_B[\varphi, \Omega_1, 0] = d_B[H(1, \cdot), \Omega_1, 0] = d_B[H(0, \cdot), \Omega_1, 0] = (-1)^n.$$

■

Theorem 1.3. *Assume that relation (1.3) holds. Then system (1.1) has at least one T -periodic positive solution.*

Proof. We have noticed that it is enough to show the existence of an element $x \in D(L)$ such that $Lx = Nx$. We see that the restriction of QN to $\mathbb{R}^n$ is φ , the function defined in Lemma 1.2. Using Lemma 1.1, Lemma 1.2 and Mawhin's continuation theorem in infinite dimension, we find an element $x \in D(L)$ such that $Lx = Nx$. ■

A particular case of system (1.1) is the May-Leonard-type system

$$\begin{aligned}
 \dot{x}_1(t) &= x_1(t)[1 - x_1(t) - \alpha_1(t)x_2(t - \tau_2(t, x_1(t), \dots, x_3(t))) \\
 &\quad - \beta_1(t)x_3(t - \tau_3(t, x_1(t), \dots, x_3(t)))] \\
 \dot{x}_2(t) &= x_2(t)[1 - \beta_2(t)x_1(t - \tau_1(t, x_1(t), \dots, x_3(t))) \\
 (1.10) \quad &\quad - x_2(t) - \alpha_2(t)x_3(t - \tau_3(t, x_1(t), \dots, x_3(t)))] \\
 \dot{x}_3(t) &= x_3(t)[1 - \alpha_3(t)x_1(t - \tau_1(t, x_1(t), \dots, x_3(t))) \\
 &\quad - \beta_3(t)x_2(t - \tau_2(t, x_1(t), \dots, x_3(t))) - x_3(t)]
 \end{aligned}$$

where $\alpha_i, \beta_i \geq 0$ ($i = 1, 2, 3$) are continuous T -periodic functions, $\tau_j \in C(\mathbb{R}^4, \mathbb{R})$ and τ_j ($j = 1, 2, 3$) are T -periodic with respect to their first argument.

Corollary 1.4. *If*

$$(1.11) \quad \bar{\alpha}_i + \bar{\beta}_i < 1 \quad (i = 1, 2, 3)$$

then system (1.10) has at least one T -periodic positive solution.

Proof. In this particular case, condition (1.3) is exactly condition (1.11). We apply Theorem 1.3 for:

$$\begin{aligned}
 r_i &\equiv 1 \equiv a_{ii} \quad (i = 1, 2, 3) \\
 a_{12} &= \alpha_1; \quad a_{13} = \beta_1; \quad a_{21} = \beta_2; \quad a_{23} = \alpha_2; \quad a_{31} = \alpha_3; \quad a_{32} = \beta_3.
 \end{aligned}$$

■

2. COMPETITION SYSTEMS WITH MAY-LEONARD CONDITION

We keep the notations of Section 1 with $n = 3$. The proof of the following lemma uses some techniques of Ahmad [2].

Lemma 2.1. *Suppose that condition*

$$\begin{aligned}
 \left[\frac{r_i}{r_j} \right]_L &> \max \left\{ \left[\frac{a_{ii}}{a_{ji}} \right]_M, \left[\frac{a_{ij}}{a_{jj}} \right]_M \right\} \\
 (2.1) \quad &\quad (i, j) \in \{(1, 2), (2, 3), (3, 1)\},
 \end{aligned}$$

holds. Then there is a bounded and open set $\Omega \subset C_T$ such that

$$\bigcup_{\lambda \in]0,1]} \{x \in D(L) : Lx = \lambda Nx\} \subset \Omega.$$

Proof. Let $\lambda \in]0,1]$ and $x \in D(L)$ such that

$$(2.2) \quad Lx = \lambda Nx.$$

In what follows C_i denotes a fixed constant independent of λ and x . Integrating (2.2) we obtain

$$\begin{aligned} T\bar{r}_i &= \int_0^T a_{ii}(t) \exp x_i(t) dt \\ &+ \sum_{\substack{j=1 \\ i \neq j}}^3 \int_0^T a_{ij}(t) \exp[x_j(t - \tau_j(t, \exp x_1(t), \dots, \exp x_3(t)))] dt \\ (2.3) \quad &(i = 1, 2, 3). \end{aligned}$$

On the other hand, from (2.2) we have

$$\begin{aligned} |\dot{x}_i(t)| &\leq r_i(t) + a_{ii}(t) \exp x_i(t) \\ &+ \sum_{\substack{j=1 \\ i \neq j}}^3 a_{ij}(t) \exp[x_j(t - \tau_j(t, \exp x_1(t), \dots, \exp x_3(t)))] \\ (2.4) \quad &(i = 1, 2, 3). \end{aligned}$$

Using (2.3) and (2.4) we deduce that

$$(2.5) \quad \|\dot{x}_i\|_{L^1(0,T)} \leq 2T \max_{1 \leq i \leq 3} \bar{r}_i := C_2 \quad (i = 1, 2, 3).$$

Using again (2.3) we obtain $\bar{r}_1 \geq \sum_{j=1}^3 \bar{a}_{1j} \exp[x_j]_L$ which implies the existence of $C_3 > 0$, such that

$$(2.6) \quad [x_i]_L \leq C_3 \quad (i = 1, 2, 3).$$

By (2.5) and (2.6) we have

$$(2.7) \quad [x_i]_M \leq [x_i]_L + \|\dot{x}_i\|_{L^1(0,T)} \leq C_3 + C_2 := C_4 \quad (i = 1, 2, 3).$$

We prove the existence of a constant C_5 such that

$$(2.8) \quad [x_i]_M \geq C_5 \quad (i = 1, 2, 3).$$

Assume, by contradiction, that (2.8) is not true. Then there exists $(\lambda_n)_n \subset]0, 1]$, $(x^n)_n \subset D(L)$ with $Lx^n = \lambda_n Nx^n$ such that one of the following three possible situations holds :

- I. $[x_i^n]_M \rightarrow -\infty$ for all $i \in \{1, 2, 3\}$
- II. $[x_i^n]_M \rightarrow -\infty$ for all $i \in I \subset \{1, 2, 3\}$, $|I| = 2$
- III. $[x_i^n]_M \rightarrow -\infty$ for all $i \in I \subset \{1, 2, 3\}$, $|I| = 1$.

Let us first deal with situation I. Using (2.3) we obtain that

$$\bar{r}_1 \leq \sum_{j=1}^3 \bar{a}_{1j} \exp[x_j^n]_M \xrightarrow[n]{} 0,$$

which implies that $\bar{r}_1 \leq 0$. But $[r_1]_L > 0$, so we have obtained the desired contradiction.

Consider now situation II. Suppose that $I = \{2, 3\}$, the treatment of the other ones being completely similar. Let C_6 be a constant such that

$$(2.9) \quad [x_1^n]_M \geq C_6, \quad (n \in \mathbb{N})$$

and

$$(2.10) \quad [x_i^n]_M \longrightarrow -\infty \quad (i = 2, 3).$$

In view of (2.5) and (2.9) we have that

$$(2.11) \quad [x_1^n]_L \geq [x_1^n]_M - \|\dot{x}_1^n\|_{L^1(0,T)} \geq C_6 - C_2 = C_7 \quad (n \in \mathbb{N}).$$

Integrating the relation $Lx^n = \lambda_n Nx^n$ ($n \in \mathbb{N}$) we obtain

$$\begin{aligned} T\bar{r}_i &= \int_0^T a_{ii}(t) \exp x_i^n(t) dt \\ &+ \sum_{\substack{j=1 \\ i \neq j}}^3 \int_0^T a_{ij}(t) \exp[x_j^n(t - \tau_j(t, \exp x_1^n(t), \dots, \exp x_3^n(t)))] dt \\ (2.12) \quad &(n \in \mathbb{N}, \quad i = 2, 3). \end{aligned}$$

Using (2.10) and (2.12) we deduce the existence of two sequences $(A_i^n)_n$, $A_i^n \xrightarrow[n]{} 0$ such that

$$\begin{aligned} T\bar{r}_i &= \int_0^T a_{i1}(t) \exp[x_1^n(t - \tau_1(t, \exp x_1^n(t), \dots, \exp x_3^n(t)))] dt + A_i^n \\ (2.13) \quad &(n \in \mathbb{N}, \quad i = 2, 3). \end{aligned}$$

By (2.1) for $(i, j) = (1, 2)$ we have

$$(2.14) \quad a_{21}(t) > a_{11}(t) \frac{\bar{r}_2}{\bar{r}_1} \quad (t \in [0, T]).$$

From (2.13) and (2.14) it follows that

$$T\bar{r}_2 > \frac{\bar{r}_2}{\bar{r}_1} \int_0^T a_{11}(t) \exp[x_1^n(t - \tau_1(t, \exp x_1^n(t), \dots, \exp x_3^n(t)))] dt + A_2^n \quad (n \in \mathbb{N})$$

which implies that

$$T\bar{r}_1 > \int_0^T a_{11}(t) \exp[x_1^n(t - \tau_1(t, \exp x_1^n(t), \dots, \exp x_3^n(t)))] dt + \frac{\bar{r}_1}{\bar{r}_2} A_2^n \quad (n \in \mathbb{N})$$

from which we obtain that

$$(2.15) \quad -T\bar{r}_3\bar{r}_1 < - \int_0^T \bar{r}_3 a_{11}(t) \exp[x_1^n(t - \tau_1(t, \exp x_1^n(t), \dots, \exp x_3^n(t)))] dt - \frac{\bar{r}_3\bar{r}_1}{\bar{r}_2} A_2^n \quad (n \in \mathbb{N}).$$

On the other hand, from (2.13) we have

$$(2.16) \quad T\bar{r}_3\bar{r}_1 = \int_0^T \bar{r}_1 a_{31}(t) \exp[x_1^n(t - \tau_1(t, \exp x_1^n(t), \dots, \exp x_3^n(t)))] dt + A_3^n \bar{r}_1 \quad (n \in \mathbb{N}).$$

By (2.15) and (2.16) we have

$$(2.17) \quad \int_0^T [\bar{r}_1 a_{31}(t) - \bar{r}_3 a_{11}(t)] \exp[x_1^n(t - \tau_1(t, \exp x_1^n(t), \dots, \exp x_3^n(t)))] dt + A_3^n \bar{r}_1 - \frac{\bar{r}_3\bar{r}_1}{\bar{r}_2} A_2^n > 0 \quad (n \in \mathbb{N}).$$

On the other hand, from (2.1) (for $(i, j) = (3, 1)$) we obtain that

$$(2.18) \quad \bar{r}_1 a_{31}(t) - \bar{r}_3 a_{11}(t) < 0 \quad (t \in [0, T]).$$

Using the fact that $A_i^n \xrightarrow[n]{} 0$ ($i = 2, 3$) and (2.11), (2.17), (2.18) we deduce that $0 < 0$, a contradiction.

We consider now the third situation. Suppose that $I = \{3\}$, the other cases being treated in the same manner. Let C_8 be a constant such that

$$(2.19) \quad [x_i^n]_M \geq C_8 \quad (n \in \mathbb{N}, i = 1, 2)$$

and

$$[x_3^n]_M \xrightarrow[n]{} -\infty.$$

Using (2.5) and (2.19) we have

$$(2.20) \quad [x_i^n]_L \geq [x_i^n]_M - \|\dot{x}_i^n\|_{L'(0,T)} \geq C_8 - C_2 := C_9 \quad (i = 1, 2).$$

Let $(t_1^n)_n$ be a sequence such that

$$(2.21) \quad x_1^n(t_1^n) = [x_1^n]_L \quad (n \in \mathbb{N}).$$

Using $Lx^n = \lambda_n N x^n$ ($n \in \mathbb{N}$) and (2.21) we deduce that

$$\begin{aligned} r_1(t_1^n) &= a_{11}(t_1^n) \exp[x_1^n]_L \\ &+ \sum_{\substack{j=1 \\ j \neq 1}}^3 a_{1j}(t_1^n) \exp[x_j^n(t_1^n - \tau_j(t_1^n, \exp x_1^n(t_1^n), \dots, \exp x_3^n(t_1^n)))] \\ &\quad (n \in \mathbb{N}), \end{aligned}$$

which implies

$$(2.22) \quad [r_1]_L \leq [a_{11}]_M \exp[x_1^n]_L + \sum_{\substack{j=1 \\ j \neq 1}}^3 [a_{1j}]_M \exp[x_j^n]_M \quad (n \in \mathbb{N}).$$

Let $(t_2^n)_n$ be a sequence such that

$$(2.23) \quad x_2^n(t_2^n) = [x_2^n]_M \quad (n \in \mathbb{N}).$$

Using again the relation $Lx^n = \lambda_n N x^n$ ($n \in \mathbb{N}$) and (2.23) it follows that

$$\begin{aligned} r_2(t_2^n) &= a_{22}(t_2^n) \exp[x_2^n]_M \\ &+ \sum_{\substack{j=1 \\ j \neq 2}}^3 a_{2j}(t_2^n) \exp[x_j^n(t_2^n - \tau_j(t_2^n, \exp x_1^n(t_2^n), \dots, \exp x_3^n(t_2^n)))] \\ &\quad (n \in \mathbb{N}), \end{aligned}$$

which implies

$$(2.24) \quad [a_{22}]_L \exp[x_2^n]_M + \sum_{\substack{j=1 \\ j \neq 2}}^3 [a_{2j}]_L \exp[x_j^n]_L \leq [r_2]_M \quad (n \in \mathbb{N}).$$

By (2.24) we have that

$$(2.25) \quad \begin{aligned} & - [a_{12}]_M [r_2]_M \leq -[a_{12}]_M [a_{22}]_L \exp[x_2^n]_M \\ & - \sum_{\substack{j=1 \\ j \neq 2}}^3 [a_{12}]_M [a_{2j}]_L \exp[x_j^n]_L \quad (n \in \mathbb{N}). \end{aligned}$$

From (2.22) we obtain that

$$(2.26) \quad \begin{aligned} & [r_1]_L [a_{22}]_L \leq [a_{11}]_M [a_{22}]_L \exp[x_1^n]_L \\ & + \sum_{\substack{j=1 \\ j \neq 1}}^3 [a_{1j}]_M [a_{22}]_L \exp[x_j^n]_M \quad (n \in \mathbb{N}). \end{aligned}$$

In view of (2.25) and (2.26) we have that

$$(2.27) \quad \begin{aligned} & [r_1]_L [a_{22}]_L - [a_{12}]_M [r_2]_M \leq \{[a_{11}]_M [a_{22}]_L - [a_{12}]_M [a_{21}]_L\} \exp[x_1^n]_L \\ & + [a_{13}]_M [a_{22}]_L \exp[x_3^n]_M - [a_{12}]_M [a_{23}]_L \exp[x_3^n]_L \quad (n \in \mathbb{N}). \end{aligned}$$

Using the fact that

$$[x_3^n]_M \xrightarrow[n]{} -\infty$$

and the relations (2.1) (for (i,j)=(1,2)), (2.7), (2.20), (2.27) it follows that

$$(2.28) \quad 0 < [a_{11}]_M [a_{22}]_L - [a_{12}]_M [a_{21}]_L.$$

From (2.22) we have that

$$(2.29) \quad \begin{aligned} & - [a_{21}]_L [r_1]_L \geq -[a_{21}]_L [a_{11}]_M \exp[x_1^n]_L \\ & - \sum_{\substack{j=1 \\ j \neq 1}}^3 [a_{21}]_L [a_{1j}]_M \exp[x_j^n]_M \quad (n \in \mathbb{N}). \end{aligned}$$

By (2.24), we have that

$$(2.30) \quad \begin{aligned} [r_2]_M [a_{11}]_M &\geq [a_{11}]_M [a_{22}]_L \exp[x_2^n]_M \\ &+ \sum_{\substack{j=1 \\ j \neq 2}}^3 [a_{2j}]_L [a_{11}]_M \exp[x_j^n]_L \quad (n \in \mathbb{N}). \end{aligned}$$

Using (2.29) and (2.30), we deduce that

$$(2.31) \quad \begin{aligned} [r_2]_M [a_{11}]_M - [a_{21}]_L [r_1]_L &\geq \{[a_{11}]_M [a_{22}]_L - [a_{21}]_L [a_{12}]_M\} \exp[x_2^n]_M \\ &+ [a_{23}]_L [a_{11}]_M \exp[x_3^n]_L - [a_{21}]_L [a_{13}]_M \exp[x_3^n]_M \\ &\quad (n \in \mathbb{N}). \end{aligned}$$

From

$$[x_3^n]_M \xrightarrow{n} -\infty$$

and (2.1) (for (i,j)=(1,2)), (2.7), (2.19) and (2.31) we have that

$$(2.32) \quad 0 > [a_{11}]_M [a_{22}]_L - [a_{21}]_L [a_{12}]_M.$$

In view of (2.28) and (2.32) we have obtained a contradiction. Consequently relation (2.8) is true.

From (2.5) and (2.8) we have that

$$(2.33) \quad [x_i]_L \geq [x_i]_M - \|\dot{x}_i\|_{L^1(0,T)} \geq C_5 - C_1 \quad (i = 1, 2, 3).$$

By (2.7) and (2.33) we deduce the existence of a constant $C > 0$, independent of $\lambda \in]0, 1]$ such that the relation $Lx = \lambda Nx$, $x \in D(L)$ implies that $|x|_0 \leq C$. $\blacksquare$

Lemma 2.2. *Let $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\varphi(x) = y$ where*

$$y_i = \bar{r}_i - \sum_{j=1}^3 \bar{a}_{ij} \exp(x_j) \quad (i = 1, 2, 3).$$

Suppose that condition (2.1) holds. Then there exist an open, bounded set $\Omega_1 \subset \mathbb{R}^3$ such that

$$\{x \in \mathbb{R}^3 : \varphi(x) = 0\} \subset \Omega_1 \quad \text{and} \quad d_B[\varphi, \Omega_1, 0] \neq 0.$$

Proof. Consider the continuous homotopy $H : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $H(\lambda, x) = y$, where

$$\begin{aligned} y_1 &= \bar{r}_1 - \bar{a}_{11} \exp(x_1) - \lambda[\bar{a}_{12} \exp(x_2)] - \bar{a}_{13} \exp(x_3), \\ y_2 &= \bar{r}_2 - \bar{a}_{21} \exp(x_1) - \bar{a}_{22} \exp(x_2) - \lambda[\bar{a}_{23} \exp(x_3)], \\ y_3 &= \bar{r}_3 - \lambda[\bar{a}_{31} \exp(x_1)] - \bar{a}_{32} \exp(x_2) - \bar{a}_{33} \exp(x_3). \end{aligned}$$

Clearly, $H(1, \cdot) = \varphi$ and the function $H(0, \cdot)$ is denoted by $\psi = (\psi_1, \psi_2, \psi_3)$. Using the same strategy as in the proof of Lemma 2.1, it is not difficult to prove that there exists a positive constant R such that

$$\bigcup_{\lambda \in [0,1]} \{x \in \mathbb{R}^3 : H(\lambda, x) = 0\} \subset]-R, R[^3.$$

Hence, using the invariance and excision properties of Brouwer degree, it follows that

$$(2.34) \quad d_B[\varphi,] - \rho, \rho[^3, 0] = d_B[\psi,] - \rho, \rho[^3, 0],$$

for any $\rho \geq R$. Now, consider the continuous homotopy

$$G : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad G(\lambda, x) = y,$$

where

$$\begin{aligned} y_1 &= (1 - \lambda)\psi_1(x) + \lambda(R - \exp(x_1) - \eta \exp(x_3)), \\ y_2 &= (1 - \lambda)\psi_2(x) + \lambda(R - \eta \exp(x_1) - \exp(x_2)), \\ y_3 &= (1 - \lambda)\psi_3(x) + \lambda(R - \eta \exp(x_2) - \exp(x_3)), \end{aligned}$$

and $\eta \geq 1$ is chosen such that there exists a positive constant R_1 satisfying

$$\bigcup_{\lambda \in [0,1]} \{x \in \mathbb{R}^3 : G(\lambda, x) = 0\} \subset]-R_1, R_1[^3.$$

Hence, using the invariance and excision properties of Brouwer degree, it follows that

$$(2.35) \quad \begin{aligned} d_B[\psi,] - \rho, \rho[^3, 0] &= d_B[G(0, \cdot),] - \rho, \rho[^3, 0] \\ &= d_B[G(1, \cdot),] - \rho, \rho[^3, 0] = -1. \end{aligned}$$

for any $\rho \geq R_1$. Now, the conclusion follows using (2.34) and (2.35). ■

Theorem 2.3. *Assume that relation (2.1) holds. Then system (1.1) has at least one T -periodic positive solution.*

Proof. See the proof of Theorem 1.3. ■

Corollary 2.4. *If*

$$(2.36) \quad 0 < \alpha_i(t) < 1 < \beta_i(t) \quad (t \in [0, T], i = 1, 2, 3),$$

then system (1.10) has at least one T -periodic, positive solution.

Proof. We apply Theorem 2.3 for

$$\begin{aligned} r_i &\equiv 1 \equiv a_{ii} \quad (i = 1, 2, 3) \\ a_{12} &= \alpha_1, \quad a_{13} = \beta_1, \quad a_{21} = \beta_2, \quad a_{23} = \alpha_2, \quad a_{31} = \alpha_3, \quad a_{32} = \beta_3. \end{aligned}$$

■

3. DELAY-PREY-PREDATOR SYSTEMS

Let

$$C_T := \{x : \mathbb{R} \rightarrow \mathbb{R}^2 : x \text{ is a continuous } T\text{-periodic function}\}.$$

It is known that $(C_T, \|\cdot\|)$ is a Banach space with the norm $\|x\| = \sup_{t \in \mathbb{R}} |x(t)|$. We search for a positive function $x \in C_T$ such that x is a solution of the system

$$\begin{aligned} \dot{u}(t) &= u(t)[a(t) - b(t)u(t) - c(t)v(t - \beta(t, u(t), v(t)))], \\ (3.1) \quad \dot{v}(t) &= v(t)[d(t) + f(t)u(t - \alpha(t, u(t), v(t))) - g(t)v(t)], \end{aligned}$$

where a, b, c, d, f, g are continuous T -periodic functions and $\alpha, \beta \in C(\mathbb{R}^3, \mathbb{R})$ are T -periodic with respect to their first variable. It is also assumed that a, b, c, f and g are strictly positive.

Consider the following system

$$\begin{aligned} \dot{u}(t) &= a(t) - b(t) \exp u(t) \\ &\quad - c(t) \exp[v(t - \beta(t, \exp u(t), \exp v(t)))], \\ (3.2) \quad \dot{v}(t) &= d(t) + f(t) \exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] \\ &\quad - g(t) \exp v(t). \end{aligned}$$

It is obvious that if system (3.2) has a T -periodic solution, then system (3.1) has a positive T -periodic solution.

We reformulate problem (3.2) so we can use Mawhin's continuation theorem in infinite dimension. Let $(L, D(L))$ be the operator defined by

$$D(L) = C_T \cap C^1(\mathbb{R}, \mathbb{R}^2), \quad Lx = \dot{x}$$

and $N : C_T \rightarrow C_T, Nx = y$ where

$$\begin{aligned} y_1(t) &= a(t) - b(t) \exp u(t) - c(t) \exp[v(t - \beta(t, \exp u(t), \exp v(t)))] \\ y_2(t) &= d(t) + f(t) \exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] - g(t) \exp v(t). \end{aligned}$$

It is obvious that $x \in C_T$ is a solution of (3.2) iff $x \in D(L)$ and $Lx = Nx$.

Define the continuous projectors P, Q

$$\begin{aligned} P &: C_T \rightarrow C_T, \quad Px = x(0), \\ Q &: C_T \rightarrow C_T, \quad Qx = \bar{x}. \end{aligned}$$

We know that

$$\begin{aligned} \operatorname{Im}(P) &= \ker(L), \quad \ker(Q) = \operatorname{Im}(L), \\ C_T &= \ker(L) \oplus \ker(P) = \operatorname{Im}(L) \oplus \operatorname{Im}(Q), \\ \ker(L) &= \operatorname{Im}(Q) \simeq \mathbb{R}^2. \end{aligned}$$

Consequently, L is a Fredholm operator of index zero. It is easy to prove that N is an L -compact operator on $\bar{\Omega}$ for any open bounded set $\Omega \subset C_T$.

Lemma 3.1. *Suppose that condition*

$$(3.3) \quad -\frac{[f]_L}{[b]_M} < \min \left\{ \frac{[d]_L}{[a]_M}, \frac{[d]_L}{[a]_L} \right\} \leq \max \left\{ \frac{[d]_M}{[a]_M}, \frac{[d]_M}{[a]_L} \right\} < \frac{[g]_L}{[c]_M}$$

holds. Then there is a bounded, open set $\Omega \subset C_T$ such that

$$\bigcup_{\lambda \in]0,1]} \{x \in D(L) : Lx = \lambda Nx\} \subset \Omega.$$

Proof. Let $\lambda \in]0,1]$ and $x \in D(L)$ such that

$$(3.4) \quad Lx = \lambda Nx.$$

In what follows C_i denotes a fixed constant independent of λ and x . Integrating (3.4), we obtain

$$\begin{aligned} \bar{a}T &= \int_0^T b(t) \exp u(t) dt \\ (3.5) \quad &+ \int_0^T c(t) \exp[v(t - \beta(t, \exp u(t), \exp v(t)))] dt, \end{aligned}$$

$$\begin{aligned} \bar{d}T &= - \int_0^T f(t) \exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] dt \\ (3.6) \quad &+ \int_0^T g(t) \exp v(t) dt. \end{aligned}$$

On the other hand, from (3.4) we have that

$$\begin{aligned} |\dot{u}(t)| &\leq a(t) + b(t) \exp u(t) \\ (3.7) \quad &+ c(t) \exp[v(t - \beta(t, \exp u(t), \exp v(t)))]]. \end{aligned}$$

Using (3.5) and (3.7) we deduce that

$$(3.8) \quad \|\dot{u}\|_{L^1(0,T)} \leq 2T\bar{a} := C_2.$$

Using again (3.5) we obtain that

$$\bar{a} \geq \bar{b} \exp[u]_L + \bar{c} \exp[v]_L$$

which implies the existence of a constant $C_3 > 0$, such that

$$(3.9) \quad [u]_L, [v]_L \leq C_3.$$

By (3.8) and (3.9) we have that

$$(3.10) \quad [u]_M \leq [u]_L + \|\dot{u}\|_{L^1(0,T)} \leq C_3 + C_2 := C_4.$$

Using (3.6), we have that

$$(3.11) \quad \begin{aligned} 0 &< [g]_L \int_0^T \exp v(t) dt \leq |\bar{d}|T \\ &+ [f]_M \int_0^T \exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] dt \\ &\leq |\bar{d}|T + [f]_M \exp[u]_M T. \end{aligned}$$

From (3.10) and (3.11) we deduce the existence of a constant $C_5 > 0$ such that

$$(3.12) \quad \int_0^T \exp v(t) dt \leq C_5.$$

Using (3.4), (3.10) and (3.12) we obtain, as for (3.8), the existence of a constant $C_6 > 0$ such that

$$(3.13) \quad \|\dot{v}\|_{L^1(0,T)} \leq C_6.$$

From (3.9) and (3.13) we have that

$$(3.14) \quad [v]_M \leq [v]_L + \|\dot{v}\|_{L^1(0,T)} \leq C_3 + C_6 := C_7.$$

Now we show the existence of a constant $\tilde{c}$ such that:

$$(3.15) \quad [u]_M, [v]_M \geq \tilde{c}.$$

Assume, by contradiction, that the relation (3.15) is not true. Then there exist $(\lambda_n)_n \subset]0, 1]$, $(x_n = (u_n, v_n))_n \subset D(L)$, $Lx_n = \lambda_n Nx_n$ such that one of the three following situations holds:

- I. $[u_n]_M \xrightarrow{n} -\infty, [v_n]_M \xrightarrow{n} -\infty$
- II. $[u_n]_M \rightarrow -\infty, \exists C_8, [v_n]_M \geq C_8 \quad (n \in \mathbb{N})$
- III. $\exists C_9, [u_n]_M \geq C_9, \quad (n \in \mathbb{N}), [v_n]_M \xrightarrow{n} -\infty.$

Let us first deal with situation I. Using (3.5), we obtain that

$$\bar{a} \leq [b]_M \exp[u_n]_M + [c]_M \exp[v_n]_M \xrightarrow{n} 0,$$

but $[a]_L > 0$, so we have obtained the desired contradiction.

Consider now the situation II. Using the relation (3.13) and II we have that

$$(3.16) \quad [v_n]_L \geq [v_n]_M - \|\dot{v}_n\|_{L^1(0,T)} \geq C_8 - C_5 := C_{10} \quad (n \in \mathbb{N}).$$

From (3.14) and (3.16) we obtain that the sequence $(v_n)_n$ is bounded in $C(0, T)$ and equicontinuous (clear from the relations (3.4), (3.10) and (3.14)), so, using Arzela-Ascoli's theorem, we can admit that there is a function $v \in C(0, T)$ such that $\|v - v_n\| \xrightarrow[n]{} 0$. We deal with two situations: **II.1** $[d]_M > 0$.

Consider the sequences $(t_n^1)_n, (t_n^2)_n \subset [0, T]$ such that

$$(3.17) \quad u_n(t_n^1) = [u_n]_M, v_n(t_n^2) = [v_n]_M \quad (n \in \mathbb{N}).$$

Because $[0, T]$ is compact we can assume that there are $t^1, t^2 \in [0, T]$ such that

$$t_n^i \xrightarrow[n]{} t^i \quad (i = 1, 2).$$

Using (3.17) and the fact that $\|v - v_n\| \xrightarrow[n]{} 0$ we deduce that

$$v_n(t_n^2) \xrightarrow[n]{} v(t^2) = [v]_M$$

and

$$v_n(t_n^1 - \beta(t_n^1, \exp u_n(t_n^1), \exp v_n(t_n^1))) = v_n(\tilde{t}_n) \xrightarrow[n]{} v(\tilde{t}).$$

(where $\tilde{t} \in [0, T]$ such that $\tilde{t}_n \xrightarrow[n]{} \tilde{t}$). So, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} a(t_n^1) &= b(t_n^1) \exp[u_n]_M + c(t_n^1) \exp v(\tilde{t}_n), \\ d(t_n^2) &= -f(t_n^2) \exp u_n(t_n^2 - \alpha(t_n^2, \exp u_n(t_n^2), \exp v_n(t_n^2))) \\ &\quad + g(t_n^2) \exp[v_n]_M. \end{aligned}$$

Taking the limit we obtain that

$$a(t^1) = c(t^1) \exp v(\tilde{t}), \quad d(t^2) = g(t^2) \exp[v]_M,$$

so we have that

$$\frac{a}{c}(t^1) \leq \frac{d}{g}(t^2)$$

and from (3.3) and II.1 we have that

$$\left[\frac{a}{c} \right]_L > \left[\frac{d}{g} \right]_M,$$

a contradiction.

II.2 $[d]_M \leq 0$: Using the notation in II.1, we obtain that

$$d(t^2) = g(t^2) \exp[v]_M,$$

which is impossible.

Consider now the last possible situation. Using the same method (see II) we can show that this situation proves to be also impossible (for example we can use the fact that

$$-\frac{[f]_L}{[b]_M} < \min \left\{ \frac{[d]_L}{[a]_M}, \frac{[d]_L}{[a]_L} \right\}.$$

Consequently (3.15) is true. Using (3.8), (3.13) and (3.15) we obtain a constant C_{11} such that

$$(3.18) \quad [u]_L, [v]_L \geq C_{11}.$$

From (3.10), (3.14) and (3.18) we have that there is a constant C_{12} such that

$$\|u\|, \|v\| \leq C_{12},$$

which completes the proof. ■

Lemma 3.2. Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\varphi(x) = y$ where

$$\begin{aligned} y_1 &= \bar{a} - \bar{b} \exp(x_1) - \bar{c} \exp(x_2), \\ y_2 &= \bar{d} + \bar{f} \exp(x_1) - \bar{g} \exp(x_2). \end{aligned}$$

If the relation (3.3) holds, then there is an open, bounded set $\Omega_1 \subset \mathbb{R}^2$ such that

$$\{x \in \mathbb{R}^2 : \varphi(x) = 0\} \subset \Omega_1 \text{ and } d_B[\varphi, \Omega_1, 0] = 1.$$

Proof. It is obvious that from relation (3.3) we can deduce that

$$(3.19) \quad \bar{a} > 0, \quad -\left[\frac{f}{b}\right]_L < \frac{\bar{d}}{\bar{a}} < \left[\frac{g}{c}\right]_L.$$

We have that

$$(3.20) \quad \left[\frac{f}{b}\right]_L \leq \frac{\bar{f}}{\bar{b}}, \quad \left[\frac{g}{c}\right]_L \leq \frac{\bar{g}}{\bar{c}}.$$

From (3.19) and (3.20) we have that

$$(3.21) \quad \bar{a}\bar{g} - \bar{d}\bar{c} > 0, \quad \bar{b}\bar{d} + \bar{f}\bar{a} > 0.$$

Because $b_L, c_L, f_L, g_L > 0$, it follows that

$$(3.22) \quad \bar{b}\bar{g} + \bar{c}\bar{f} > 0.$$

From (3.21) and (3.22) we deduce that there is only one point $(x_1^0, x_2^0) \in \mathbb{R}^2$ such that

$$\varphi(x_1^0, x_2^0) = 0.$$

Furthermore,

$$J_\varphi(x_1^0, x_2^0) = \begin{vmatrix} -\bar{b} \exp(x_1^0) & -\bar{c} \exp(x_2^0) \\ \bar{f} \exp(x_1^0) & -\bar{g} \exp(x_2^0) \end{vmatrix} = \exp(x_1^0) \exp(x_2^0) [\bar{b}\bar{g} + \bar{f}\bar{c}] > 0$$

and $d_B[\varphi, B_R(0), 0] = 1$. We can choose $\Omega_1 = B_R(0)$, $R > 0$ such that $(x_1^0, x_1^0) \in B_R(0)$. ■

Theorem 3.3. *Assume that relation (3.3) holds. Then system (3.1) has at least one T -periodic positive solution.*

Proof. See the proof of Theorem 1.3. ■

Next we state and prove our second result of this section. Consider the system

$$(3.23) \quad \begin{aligned} \dot{u}(t) &= u(t)[a(t) - b(t)u(t) - c(t)v(t)] \\ \dot{v}(t) &= \tau(t)v(t)[u(t - \alpha(t, u(t), v(t))) - \sigma(t)] \end{aligned}$$

where a, b, c, τ, σ are continuous T -periodic strictly positive functions and $\alpha \in C(\mathbb{R}^3, \mathbb{R})$ is T -periodic with respect to its first variable. As in the case of system (3.1), we consider the following system

$$(3.24) \quad \begin{aligned} \dot{u}(t) &= a(t) - b(t) \exp u(t) - c(t) \exp v(t) \\ \dot{v}(t) &= \tau(t)[\exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] - \sigma(t)]. \end{aligned}$$

If system (3.24) has a T -periodic solution, then system (3.23) has a positive T -periodic solution. Let $N_1 : C_T \rightarrow C_T$, $N_1 x = y$ where

$$\begin{aligned} y_1(t) &= a(t) - b(t) \exp u(t) - c(t) \exp v(t), \\ y_2(t) &= \tau(t)[\exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] - \sigma(t)]. \end{aligned}$$

It is obvious that $x \in C_T$ is a solution of (3.24) if and only if $x \in D(L)$ and $Lx = N_1 x$. We notice that N_1 is an L -compact operator.

Lemma 3.4. *Suppose that condition*

$$(3.25) \quad [a]_L - [b]_M[\sigma]_M > 0$$

holds. Then there is a bounded, open set $\Omega \subset C_T$ such that

$$\bigcup_{\lambda \in [0,1]} \{x \in D(L) : Lx = \lambda N_1 x\} \subset \Omega.$$

Proof. The proof is similar to the proof of Lemma 3.1 and will be omitted. ■

Lemma 3.5. *Let $\varphi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\varphi_1(x) = y$ where*

$$y_1 = \bar{a} - \bar{b} \exp(x_1) - \bar{c} \exp(x_2), \quad y_2 = \bar{\tau}[\exp(x_1) - \bar{\sigma}].$$

If relation (3.25) is true, then there is an open, bounded set $\Omega_1 \subset \mathbb{R}^2$ such that

$$\{x \in \mathbb{R}^2 : \varphi_1(x) = 0\} \subset \Omega_1 \text{ and } d_B[\varphi, \Omega_1, 0] = 1.$$

Proof. From relation (3.25) we deduce that

$$(3.26) \quad \bar{a} - \bar{b}\bar{\sigma} > 0$$

From the relation (3.26) we obtain that there is only one point (x_1^0, x_1^0) such that $\varphi_1(x_1^0, x_1^0) = 0$. Furthermore

$$J_\varphi(x_1^0, x_2^0) = \begin{vmatrix} -\bar{b}\exp(x_1^0) & -\bar{c}\exp(x_2^0) \\ \bar{\tau}\exp(x_1^0) & 0 \end{vmatrix} = \bar{\tau}\bar{c}\exp(x_1^0)\exp(x_2^0) > 0.$$

We can choose $\Omega_1 = B_R(0)$, $R > 0$, such that $(x_1^0, x_1^0) \in B_R(0)$. ■

Theorem 3.6. *Assume that relation (3.25) holds. Then system (3.23) has at least one T -periodic positive solution.*

Proof. See the proof of Theorem 1.3. ■

CHAPTER V

Multiplicity results for superlinear planar systems

Consider the following boundary value problem

$$(0.27) \quad u' = -g(t, u, v)v, \quad v' = g(t, u, v)u$$

$$(0.28) \quad u(0) = 0 = u(\pi),$$

where $g : [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function satisfying

$$(0.29) \quad g(t, u, v) \rightarrow +\infty \quad \text{as} \quad |u| + |v| \rightarrow \infty$$

uniformly with $t \in [0, \pi]$. In this chapter we prove two multiplicity results concerning the boundary value problem (0.27), (0.28) for nonlinearities g satisfying (0.29) and some additional conditions.

1. A FIRST RESULT

To prove the main result of this section, we use a theorem of Ward. For the convenience of the reader, we state this theorem below.

Let $G = (g_1, g_2)^T \in C(\mathbb{R} \times [0, \pi] \times \mathbb{R}^2, \mathbb{R}^2)$, $w = (u, v)^T$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and consider the boundary value problem

$$(1.1) \quad w' = \mu Jw + G(\mu, t, w), \quad t \in [0, \pi],$$

$$(1.2) \quad u(0) = 0 = u(\pi).$$

Let

$$X = \{w = (u, v) \in C([0, \pi], \mathbb{R}^2) : u(0) = 0 = u(\pi)\}$$

be a linear space equipped with the norm $\|w\| = \max_{t \in [0, \pi]} |w(t)|$ where, if $w = (u, v) \in \mathbb{R}^2$, then $|w|^2 = u^2 + v^2$. Let $\mathcal{S}$ be the closure in $\mathbb{R} \times X$ of the set of all nontrivial solutions (μ, w) (i. e. $w \neq 0$) of (1.1), (1.2). The following theorem is due to Ward [67].

Theorem 1.1. *Let $G \in C(\mathbb{R} \times [0, \pi] \times \mathbb{R}^2, \mathbb{R}^2)$, $G(\mu, t, 0) = 0$ for all $(\mu, t) \in \mathbb{R} \times [0, \pi]$ and $G(\mu, t, w) = o(|w|)$ as $|w| \rightarrow 0$ uniformly with respect to $t \in [0, \pi]$ and μ in compact sets. Then*

- (W₁) For each $k \in \mathbb{Z}$, $(k, 0)$ is a bifurcation point of (1.1), (1.2). That is, in every neighborhood of $(k, 0) \in \mathbb{R} \times X$ there is a nontrivial solution (μ, w) of (1.1), (1.2).
- (W₂) For each $k \in \mathbb{Z}$ let $C_k \subset \mathbb{R} \times X$ denote the component of $\mathcal{S}$ which meets $(k, 0)$. Each C_k is unbounded in $\mathbb{R} \times X$ and if $(\mu, w) \in C_k, w \neq 0$, then w may be extended to $\mathbb{R}$ as a 2π -periodic function in such a way that if $t \in [-\pi, \pi] \setminus \{-\pi, \pi\}$, then $t \rightarrow w(t)/|w(t)|$ defines a mapping $\varphi(\mu, w)$ of the circle S^1 into itself, and $\text{rot}(\varphi(\mu, w)) = k$. It follows that if $j \neq k$ then $C_j \cap C_k = \emptyset$.

Now, we are in a position to state and prove the first main result of this chapter.

Theorem 1.2. *If $g : [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function satisfying (0.29) and*

$$(1.3) \quad g(t, 0, 0) = 0 \quad \text{for all } t \in [0, \pi],$$

then (0.27), (0.28) has infinitely many topologically distinct solutions. Indeed, for each $k \in \mathbb{N}$ there is a solution $w_k = (u_k, v_k)$ such that the odd/even 2π -periodic extension $\tilde{w}_k$ of w_k has rotation number k .

Proof. Let X be the Banach space defined above. We associate to (0.27), (0.28) the following family of boundary value problems

$$(1.4) \quad u' = -\mu v - g(t, u, v)v, \quad v' = \mu u + g(t, u, v)u$$

$$(1.5) \quad u(0) = 0 = u(\pi).$$

Let $\mathcal{S}$ be the closure in $\mathbb{R} \times X$ of the set of all nontrivial solutions (μ, w) of (1.4), (1.5). For each $k \in \mathbb{N}$ let $C_k \subset \mathbb{R} \times X$ denote the component of $\mathcal{S}$ which meets $(k, 0)$. Using Theorem 1.1 (we can apply this theorem because g satisfies (1.3)) we have that C_k is unbounded in $\mathbb{R} \times X$ for each $k \in \mathbb{N}$. Consider $(\mu, w) \in C_k, w \neq 0$. Let $\tilde{w}$ be the odd/even 2π -periodic extension of w , and let $\tilde{g}$ be the extension of g on $[-\pi, \pi] \times \mathbb{R}^2$ defined by $\tilde{g}(t, u, v) = g(-t, -u, v)$ for all $(t, u, v) \in [-\pi, 0] \times \mathbb{R}^2$. Then, if $t \in [-\pi, \pi] \setminus \{0\}$ we have that

$$\tilde{u}'(t) = -\mu \tilde{v}(t) - \tilde{g}(t, \tilde{u}(t), \tilde{v}(t))\tilde{v}(t), \quad \tilde{v}'(t) = \mu \tilde{u}(t) + \tilde{g}(t, \tilde{u}(t), \tilde{v}(t))\tilde{u}(t).$$

This implies that

$$\frac{d}{dt} |\tilde{w}(t)|^2 = 0 \quad \text{for all } t \in [-\pi, \pi] \setminus \{0\},$$

from where we deduce that $\tilde{u}^2 + \tilde{v}^2$ is constant on $[-\pi, \pi]$. Then $t \mapsto \tilde{w}(t)/|\tilde{w}(t)|$ may be considered as a map of the circle S^1 into itself, denoted by $\varphi(\mu, w)$. Let $\text{rot}(\varphi(\mu, w))$ be the rotation number of $\varphi(\mu, w)$. Using Kronecker formula, we have that

$$\begin{aligned} \text{rot}(\varphi(\mu, w)) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\tilde{v}'\tilde{u} - \tilde{u}'\tilde{v}}{\tilde{u}^2 + \tilde{v}^2} dt \\ (1.6) \quad &= \mu + \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{g}(t, \tilde{u}, \tilde{v}) dt. \end{aligned}$$

Let $k \in \mathbb{N}$, then from Theorem 1.1 we know that $\text{rot}(\varphi(\mu, w)) = k$ for all $(\mu, w) \in C_k, w \neq 0$. Because C_k is unbounded in $\mathbb{R} \times X$, we have three possible situations.

- (I) The projection of C_k onto $\mathbb{R}$ is unbounded from below.
- (II) The projection of C_k onto $\mathbb{R}$ is bounded.
- (III) The projection of C_k onto $\mathbb{R}$ is unbounded from above.

We show that situations (II) and (III) don't hold.

Using (0.29), we deduce that there exists $c \in \mathbb{R}$ such that

$$(1.7) \quad g(t, u, v) \geq c \quad \text{for all } (t, u, v) \in [0, \pi] \times \mathbb{R}^2.$$

Suppose that (II) holds. Then, because C_k is unbounded in $\mathbb{R} \times X$, there is a sequence $(\mu_n, w_n)_n$ in C_k such that $(\mu_n)_n$ is bounded in $\mathbb{R}$ and $\|w_n\| \rightarrow \infty$. As we have already seen, we have for all $n \in \mathbb{N}$ that

$$\|w_n\|^2 = \tilde{u}_n^2 + \tilde{v}_n^2 \quad \text{for all } t \in [-\pi, \pi]$$

so that $|\tilde{u}_n| + |\tilde{v}_n| \rightarrow \infty$ uniformly in $t \in [-\pi, \pi]$. Using (0.29) we deduce that $\tilde{g}(t, \tilde{u}_n(t), \tilde{v}_n(t)) \rightarrow +\infty$ uniformly in $t \in [-\pi, \pi]$. From this, the fact that the sequence $(\mu_n)_n$ is bounded and (1.6) we have that $\text{rot}(\varphi(\mu_n, w_n)) \rightarrow +\infty$, a contradiction with $\text{rot}(\varphi(\mu_n, w_n)) = k$ for all $n \in \mathbb{N}$.

Suppose that (III) holds. Then, there is a sequence $(\mu_n, w_n)_n$ in C_k such that $\mu_n \rightarrow +\infty$. Using (1.6) and (1.7) it follows that $\text{rot}(\varphi(\mu_n, w_n)) \rightarrow +\infty$, which is again a contradiction with $\text{rot}(\varphi(\mu_n, w_n)) = k$ for all $n \in \mathbb{N}$.

Consequently we can only have situation (I), so, from the connectedness of C_k and $(k, 0) \in C_k$ it follows that there exists $w_k \in X$ such that $(0, w_k) \in C_k$, so $w_k \neq 0$ is a solution of (0.27), (0.28). On the other hand, because $\text{rot}(\varphi(0, w_k)) = k$ for all $k \in \mathbb{N}$, we deduce that $w_k \neq w_j$ if $k \neq j$. ■

Example 1.3. Using Theorem 1.2, we deduce that the boundary value problem

$$\begin{aligned} u' &= -[(\sin t + 1)(u^2 - u) + v^4]v, & v' &= [(\sin t + 1)(u^2 - u) + v^4]u, \\ u(0) &= 0 = u(\pi), \end{aligned}$$

has infinitely many topologically distinct solutions.

2. A SECOND RESULT

In this section $g : [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function satisfying (0.29) and

$$(2.1) \quad g(0, -u, v) = g(0, u, v) \quad \text{for all } (u, v) \in \mathbb{R}^2.$$

Moreover, we suppose that function $g(0, \cdot)$ is locally Lipschitz on $\mathbb{R}^2$. The second main result of this chapter is the following one.

Theorem 2.1. *If g is as above, then (0.27), (0.28) has infinitely many topologically distinct solutions.*

To prove the theorem above, we use Capietto-Mawhin-Zanolin continuation theorem. So, we need to make some preparations.

Let X be the linear space of continuous functions $w = (u, v)$ on $[0, \pi]$ with values in $\mathbb{R}^2$ equipped with the usual norm $\|w\| = \max_{t \in [0, \pi]} |w(t)|$. Consider the homotopy $\mathcal{G} : [0, 1] \times X \rightarrow X$ defined by $\mathcal{G}(\lambda, (u, v)) = (x, y)$, where

$$x(t) = - \int_0^t g(\lambda s, u, v) v \, ds, \quad y(t) = v(0) - u(\pi) + \int_0^t g(\lambda s, u, v) u \, ds,$$

for all $t \in [0, \pi]$.

Lemma 2.2. *The homotopy $\mathcal{G}$ is completely continuous on $[0, 1] \times X$.*

Proof. Let $(\lambda_n, w_n)_n \subset [0, 1] \times X$ such that $\lambda_n \rightarrow \lambda_0, w_n \rightarrow w_0$. Then, if $t \in [0, \pi]$, we have

$$\begin{aligned} & \left| \int_0^t g(\lambda_n s, u_n, v_n) v_n \, ds - \int_0^t g(\lambda_0 s, u_0, v_0) v_0 \, ds \right| \\ & \leq \int_0^\pi |g(\lambda_n s, u_n, v_n) v_n - g(\lambda_0 s, u_0, v_0) v_0| \, ds =: \gamma_n, \quad (n \in \mathbb{N}). \end{aligned}$$

Using Lebesgue's dominated convergence theorem, we deduce that $\gamma_n \rightarrow 0$. Now, the continuity of $\mathcal{G}$ follows obviously. Let $(\lambda_n, w_n)_n$ be a bounded sequence in $[0, 1] \times X$. Passing if necessarily to a subsequence, we can assume that $\lambda_n \rightarrow \lambda_0$. For $n \in \mathbb{N}$, define the continuous function x_n by

$$x_n(t) = \int_0^t g(\lambda_n s, u_n, v_n) v_n \, ds, \quad (t \in [0, \pi]).$$

Let $M > 0$ such that $\|w_n\| \leq M$ for all $n \in \mathbb{N}$ and $M' = \sup\{|g(t, u, v)v| : (t, u, v) \in [0, \pi] \times [-M, M]^2\}$. Because

$$|x_n(t)| \leq \int_0^\pi |g(\lambda_n s, u_n, v_n)v_n| ds, \quad (t \in [0, \pi]),$$

we deduce that $\max_{t \in [0, \pi]} |x_n(t)| \leq \pi M'$ for all $n \in \mathbb{N}$. Now, consider $t, t' \in [0, \pi]$ and $n \in \mathbb{N}$. We have

$$|x_n(t) - x_n(t')| \leq \left| \int_t^{t'} |g(\lambda_n s, u_n, v_n)v_n| ds \right| \leq M'|t - t'|.$$

It follows that the sequence $(x_n)_n$ is equicontinuous. So, we can apply Arzela-Ascoli theorem to deduce that $(x_n)_n$ has a convergence subsequence in $C([0, \pi])$. Now, the compactness of $\mathcal{G}$ follows obviously. ■

Consider the family of boundary value problems

$$(2.2) \quad u' = -g(\lambda t, u, v)v, \quad v' = g(\lambda t, u, v)u$$

$$(2.3) \quad u(0) = 0 = u(\pi).$$

Lemma 2.3. *If $(\lambda, w) \in [0, 1] \times X$, then $\mathcal{G}(\lambda, w) = w$ if and only if w is a solution of (2.2), (2.3).*

Proof. Suppose that $\mathcal{G}(\lambda, w) = w$. Then, it is clear that we have (2.2). On the other hand, it follows that $u(0) = 0$ and $v(0) = v(0) - u(\pi)$, so $u(\pi) = 0$. Conversely, suppose that w is a solution of (2.2), (2.3). Integrating on $[0, t]$ the equations in (2.2) and using the boundary condition (2.3), it follows that $\mathcal{G}(\lambda, w) = w$. ■

Let $\tilde{g} : [-\pi, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be an extension of g defined by $\tilde{g}(t, u, v) = g(-t, -u, v)$ for all $(t, u, v) \in [-\pi, 0[\times \mathbb{R}^2$. Using (2.1) it follows that $\tilde{g}$ is continuous. On the other hand, if $(u, v) \neq (0, 0)$ is a solution of (2.2), (2.3), we define the 2π -periodic odd/even continuous extension of (u, v) by $\tilde{u}(t) = -u(-t)$, $\tilde{v}(t) = v(-t)$ for all $t \in [-\pi, 0[$.

Lemma 2.4. *If $(u, v) \neq (0, 0)$ is a solution of (2.2), (2.3), then $(\tilde{u}, \tilde{v}) \in C^1([-\pi, \pi], \mathbb{R}^2)$ and $\tilde{u}' = -\tilde{g}(\lambda t, \tilde{u}, \tilde{v})\tilde{v}$, $\tilde{v}' = \tilde{g}(\lambda t, \tilde{u}, \tilde{v})\tilde{u}$.*

Proof. Let $t \in]0, \pi]$, then $\tilde{u}$ is differentiable in t and

$$\tilde{u}'(t) = u'(t) = -g(\lambda t, u(t), v(t))v(t) = -\tilde{g}(\lambda t, \tilde{u}(t), \tilde{v}(t))\tilde{v}(t).$$

Analogously, $\tilde{v}$ is differentiable in t and

$$\tilde{v}'(t) = \tilde{g}(\lambda t, \tilde{u}(t), \tilde{v}(t))\tilde{u}(t).$$

Now, consider $t \in [-\pi, 0[$. Then, $\tilde{u}$ is differentiable in t and

$$\begin{aligned}\tilde{u}'(t) = u'(-t) &= -g(-\lambda t, u(-t), v(-t))v(-t) \\ &= -g(-\lambda t, -\tilde{u}(t), \tilde{v}(t))\tilde{v}(t) = -\tilde{g}(\lambda t, \tilde{u}(t), \tilde{v}(t))\tilde{v}(t).\end{aligned}$$

Note that the last equality follows by (2.1). On the other hand, $\tilde{v}$ is differentiable in t and

$$\begin{aligned}\tilde{v}'(t) &= -v'(-t) = -g(-\lambda t, u(-t), v(-t))u(-t) \\ &= -g(-\lambda t, -\tilde{u}(t), \tilde{v}(t))(-\tilde{u}(t)) = \tilde{g}(\lambda t, \tilde{u}(t), \tilde{v}(t))\tilde{u}(t).\end{aligned}$$

We have

$$\begin{aligned}\lim_{t \searrow 0} \frac{\tilde{u}(t) - \tilde{u}(0)}{t} &= \lim_{t \searrow 0} \frac{u(t)}{t} = u'(0) = -g(0, 0, v(0))v(0), \\ \lim_{t \nearrow 0} \frac{\tilde{u}(t) - \tilde{u}(0)}{t} &= \lim_{t \nearrow 0} \frac{-u(-t)}{t} = -g(0, 0, v(0))v(0).\end{aligned}$$

It follows that $\tilde{u}$ is differentiable in 0 and $\tilde{u}'(0) = -g(0, 0, v(0))v(0)$. On the other hand we have

$$\begin{aligned}\lim_{t \searrow 0} \frac{\tilde{v}(t) - \tilde{v}(0)}{t} &= v'(0) = g(0, 0, v(0))u(0) = 0, \\ \lim_{t \nearrow 0} \frac{\tilde{v}(t) - \tilde{v}(0)}{t} &= -v'(0) = 0.\end{aligned}$$

So, $\tilde{v}$ is differentiable in 0 and $\tilde{v}'(0) = 0$. Finally, $\tilde{u}, \tilde{v}$ are C^1 because the continuity of $\tilde{g}$. $\blacksquare$

Let $w = (u, v)$ be a non-trivial solution of (2.2), (2.3). Then, using Lemma 2.4 we deduce that $(\tilde{u}^2(t) + \tilde{v}^2(t))' = 0$ for all $t \in [-\pi, \pi]$. It follows that $|\tilde{w}(t)|^2 = c$ for all $t \in [-\pi, \pi]$, where $c > 0$ is a constant. Now, $\tilde{w}(-\pi) = \tilde{w}(\pi)$, and $\tilde{w}(t)/|\tilde{w}(t)| \in S^1$ for all $t \in [-\pi, \pi]$. Identifying S^1 with $[-\pi, \pi]/\{-\pi, \pi\}$ we obtain a mapping $t \mapsto \tilde{w}(t)/|\tilde{w}(t)|$ of S^1 into itself, which we denote by $\psi(\lambda, w)$. The rotation is defined. Using again Kronecker formula and Lemma 2.4 we have that

$$(2.4) \quad \text{rot}(\psi(\lambda, w)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\tilde{v}'\tilde{u} - \tilde{u}'\tilde{v}}{\tilde{u}^2 + \tilde{v}^2} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{g}(\lambda t, \tilde{u}, \tilde{v}) dt.$$

Let $\delta : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\delta(u, v) = \min\{1, (u^2 + v^2)^{-1}\}$. If $(u, v) \in X$, we define as before $(\tilde{u}, \tilde{v})$ to be the odd/even extension (not necessarily continuous) of (u, v) . Consider $\varphi : [0, 1] \times X \rightarrow \mathbb{R}_+$ defined by

$$\varphi(\lambda, (u, v)) = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \tilde{g}(\lambda t, \tilde{u}, \tilde{v})(\tilde{u}^2 + \tilde{v}^2)\delta(\tilde{u}, \tilde{v}) dt \right|.$$

Lemma 2.5. *The function φ defined above is continuous.*

Proof. The proof follows easily using the continuity of $\tilde{g}, \delta$ and the Lebesgue's dominated convergence theorem. ■

Lemma 2.6. *There exists $R > 1$ such that $\varphi(\lambda, w) \in \mathbb{N}$ for all $(\lambda, w) \in \Sigma$ with $\|w\| \geq R$, where $\Sigma = \{(\lambda, w) \in X : \mathcal{G}(\lambda, w) = w\}$.*

Proof. From (0.29) we have that there exists $R > 1$ such that

$$(2.5) \quad \tilde{g}(t, u, v) > 0 \quad \text{if} \quad |(u, v)| \geq R, t \in [-\pi, \pi].$$

Let $(\lambda, w) \in \Sigma$ such that $\|w\| \geq R$. Using Lemma 2.3 and Lemma 2.4 we deduce that

$$(2.6) \quad \tilde{u}^2(t) + \tilde{v}^2(t) = \|w\|^2 \geq R^2 > 1 \quad \text{for all} \quad t \in [-\pi, \pi].$$

The conclusion follows from relation (2.4), (2.5), (2.6) and the definition of φ . ■

Lemma 2.7. *The set $\varphi^{-1}(n) \cap \Sigma$ is bounded for each $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and suppose that the set $\varphi^{-1}(n) \cap \Sigma$ is unbounded. There exists a sequence $(\lambda_k, w_k) \in \Sigma$ such that $\varphi(\lambda_k, w_k) = n$ for all $k \in \mathbb{N}$ and $\|w_k\| \rightarrow \infty$. Using Lemma 2.3 and Lemma 2.4 we deduce that $\tilde{u}_k^2 + \tilde{v}_k^2 = \|w_k\|^2$ on $[-\pi, \pi]$, which implies that $|\tilde{u}_k(t)| + |\tilde{v}_k(t)| \rightarrow \infty$ uniformly with $t \in [-\pi, \pi]$. So, using (0.29), (2.4) and the definition of φ we obtain that $\varphi(\lambda_k, w_k) \rightarrow \infty$. Contradiction. ■

Lemma 2.8. *If $u_0, v_0 \in \mathbb{R}$, then the initial boundary value problem*

$$(2.7) \quad u' = -g(0, u, v)v, \quad v' = g(0, u, v)u, \quad u(0) = u_0, \quad v(0) = v_0$$

has a unique solution $(u(\cdot, (u_0, v_0)), v(\cdot, (u_0, v_0)))$ which is defined on $\mathbb{R}$.

Proof. Because the function $g(0, \cdot)$ is locally Lipschitz on $\mathbb{R}^2$, it follows that (2.7) has a unique maximal solution

$$(u(\cdot, (u_0, v_0)), v(\cdot, (u_0, v_0))) :]a, b[\rightarrow \mathbb{R}^2.$$

We shall prove that $]a, b[= \mathbb{R}$. Remark that (2.7) implies

$$|(u(t, (u_0, v_0)), v(t, (u_0, v_0)))| = |(u_0, v_0)|$$

for all $t \in]a, b[$. Using again (2.7) and the continuity of g , it follows that the function $(u'(\cdot, (u_0, v_0)), v'(\cdot, (u_0, v_0)))$ is bounded on $]a, b[$, which implies that $(u(\cdot, (u_0, v_0)), v(\cdot, (u_0, v_0)))$ has a continuous extension on $]a, b[$, if b is finite. Consider (u_b, v_b) the solution of (2.7) with the initial data $(u(b, (u_0, v_0)), v(b, (u_0, v_0)))$. Let $\varepsilon > 0$ sufficiently small and define $(u, v) :]a, b + \varepsilon[\rightarrow \mathbb{R}^2$ by

$$(u, v) = (u(\cdot, (u_0, v_0)), v(\cdot, (u_0, v_0)))$$

on $]a, b[$, and $(u, v) = (u_b, v_b)$ on $[b, b + \varepsilon]$. It is clear that (u, v) verifies (2.7), contradiction with maximality of $(u(\cdot, (u_0, v_0)), v(\cdot, (u_0, v_0)))$, so $b = +\infty$. Analogously, it follows that $a = -\infty$. ■

Using Lemma 2.8 we can consider the continuous function $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\mathcal{U}(z_1, z_2) = (2z_1, z_2 + u(\pi, (z_1, z_2)))$. It is obvious that if (u, v) is a solution of (2.2), (2.3) with $\lambda = 0$, then $(0, v(0))$ is a fixed point of $\mathcal{U}$, and if (z_1, z_2) is a fixed point of $\mathcal{U}$, then $z_1 = 0$ and $(u(\cdot, (0, z_2)), v(\cdot, (0, z_2)))$ is a solution of (2.2), (2.3) with $\lambda = 0$. If $\alpha > 0$, define

$$\Omega_\alpha = \{w \in X : \|w\| < \alpha\}, \quad G_\alpha = \{\xi \in \mathbb{R}^2 : |\xi| < \alpha\}.$$

Suppose that α is chosen so that there is no solution (u, v) of (2.2), (2.3) with $\lambda = 0$ such that $|v(0)| = \alpha$. The open sets Ω_α, G_α have the following properties: there are no initial values of solutions to (2.2), (2.3) with $\lambda = 0$ on ∂G_α and no solution on $\partial \Omega_\alpha$; the set of initial values in G_α of solutions to (2.2), (2.3) with $\lambda = 0$ equals the set of values at $t = 0$ of solutions in Ω_α to (2.2), (2.3) with $\lambda = 0$. If $G \subset \mathbb{R}^2, \Omega \subset X$ are two bounded open sets having the proprieties above, following Krasnosel'skii and Zabreiko, we say that G, Ω have a common core. Following the same lines as in the proof of [40], Theorem 28.5] we have the following result.

Lemma 2.9. *If $G \subset \mathbb{R}^2, \Omega \subset X$ are two open bounded sets having a common core, then the degrees $d_B[I - \mathcal{U}, G, 0], d_{LS}[I - \mathcal{G}(0, \cdot), \Omega, 0]$ are well defined and equal.*

In what follows we use the notation

$$(u(\cdot, \alpha), v(\cdot, \alpha)) \quad \text{for} \quad (u(\cdot, (0, \alpha)), v(\cdot, (0, \alpha))).$$

If $R > 1$ is the constant from Lemma 2.6 and $\alpha > R$, then, using (2.5) it follows that the range of $(u(\cdot, \alpha), v(\cdot, \alpha))$ is the circle of radius α . Let $\tau(\alpha) > 0$ be such that $u(\tau(\alpha), \alpha) = 0$ and $u(t, \alpha) \neq 0$ for all $t \in]0, \tau(\alpha)[$. On the other hand, from Lemma 2.8 and (2.1), we obtain that $u(\cdot, \alpha)$ is odd and $v(\cdot, \alpha)$ is even. So, we have that $(u(\cdot, \alpha), v(\cdot, \alpha))$ is a bijection from $[-\tau(\alpha), \tau(\alpha)]$ to the circle of radius α .

Lemma 2.10. $\tau(\alpha) \rightarrow 0$ as $\alpha \rightarrow +\infty$.

Proof. Because $(u(\cdot, \alpha), v(\cdot, \alpha))$ is a bijection from $[-\tau(\alpha), \tau(\alpha)]$ to the circle of radius α , we have that

$$\int_{-\tau(\alpha)}^{\tau(\alpha)} (u'^2(t, \alpha) + v'^2(t, \alpha))^{\frac{1}{2}} dt = 2\pi\alpha$$

which implies that

$$(2.8) \quad \tau(\alpha) \inf\{(u'^2(t, \alpha) + v'^2(t, \alpha))^{\frac{1}{2}} : t \in [-\tau(\alpha), \tau(\alpha)]\} \leq \pi\alpha.$$

On the other hand

$$(2.9) \quad u'^2(\cdot, \alpha) + v'^2(\cdot, \alpha) = \alpha^2 [g(0, u(\cdot, \alpha), v(\cdot, \alpha))]^2.$$

Using (0.29) and (2.9) it follows that (2.8) holds only if $\tau(\alpha) \rightarrow 0$ as $\alpha \rightarrow +\infty$. ■

Consider the set $\mathcal{S} = \{\frac{\pi}{n}\}_n$. If $\alpha > R$, then it follows that

$$(u(\cdot + \tau(\alpha), \alpha), v(\cdot + \tau(\alpha), \alpha)) = (u(\cdot, -\alpha), v(\cdot, -\alpha)),$$

which implies that if $|\alpha| > R$ then $(u(\cdot, \alpha), v(\cdot, \alpha))$ is a solution of (2.2), (2.3) with $\lambda = 0$ iff $\tau(|\alpha|) \in \mathcal{S}$. So, if we consider the continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}, \phi(t) = u(\pi, t)$ then, if $\alpha > R$ such that $\tau(\alpha) \notin \mathcal{S}$, it follows that the degrees $d_B[I - \mathcal{U}, G_\alpha, 0], d_B[\phi,] - \alpha, \alpha[0]$ are well defined. Moreover, we have the following result.

Lemma 2.11. *Let $\alpha > R$ such that $\tau(\alpha) \in]\frac{\pi}{n+1}, \frac{\pi}{n}[$ for some $n \in \mathbb{N}$. Then*

$$d_B[I - \mathcal{U}, G_\alpha, 0] = d_B[\phi,] - \alpha, \alpha[0] = (-1)^{n+1}.$$

Proof. Because $\mathcal{U}$ acts in $\mathbb{R}^2$, we have

$$(2.10) \quad d_B[I - \mathcal{U}, G_\alpha, 0] = d_B[\mathcal{U} - I, G_\alpha, 0].$$

If we denote the rectangle $[-\alpha, \alpha] \times [-\alpha, \alpha]$ by $\mathcal{R}$ then, using the excision property of Brouwer degree it follows that

$$(2.11) \quad d_B[\mathcal{U} - I, G_\alpha, 0] = d_B[\mathcal{U} - I, \mathcal{R}, 0].$$

Now, consider the homotopy

$$h : [0, 1] \times \overline{\mathcal{R}} \rightarrow \mathbb{R}^2, \quad h(\lambda, (z_1, z_2)) = (z_1, u(\pi, (\lambda z_1, z_2))).$$

Remark that $h(1, \cdot) = \mathcal{U} - I$ and $h(0, \cdot) = I_{\mathbb{R}} \times \phi$. Moreover, $h(\lambda, (z_1, z_2)) = 0$ if and only if $h(1, (z_1, z_2)) = 0$. It follows that $h(\lambda, (z_1, z_2)) \neq 0$ for all $\lambda \in [0, 1]$ and $(z_1, z_2) \in \partial \mathcal{R}$. So, we can apply the invariance by homotopy property, hence

$$(2.12) \quad \begin{aligned} d_B[\mathcal{U} - I, \mathcal{R}, 0] &= d_B[I_{\mathbb{R}} \times \phi, \mathcal{R}, 0] \\ &= d_B[I_{\mathbb{R}},] - \alpha, \alpha[0] \quad d_B[\phi,] - \alpha, \alpha[0]. \end{aligned}$$

Finally, because ϕ is odd it follows that $d_B(\phi,] - \alpha, \alpha[0) = \text{sgn}(\phi(\alpha))$, and so, using (2.10), (2.11), (2.12) and the definition of $\tau(\alpha)$ the conclusion of lemma follows. ■

Denote, for any subset $A \subset [0, 1] \times X$, the section of A at $\lambda \in [0, 1]$, by $A_\lambda = \{x \in X : (\lambda, x) \in A\}$. Let $R > 1$ be the constant from Lemma 2.6 and let k_0 be an integer such that

$$k_0 > \sup\{\varphi(\lambda, w) : (\lambda, w) \in \Sigma, \|w\| \leq R\}$$

and, using Lemma 2.7, consider, for any integer $j > k_0$, the topological degree $d_{LS}[I - \mathcal{G}(0, \cdot), \Gamma_j, 0]$, where $\Gamma_j \supset (\varphi^{-1}(j) \cap \Sigma)_0$ is an open bounded subset of X for which the Leray-Schauder degree d_{LS} is defined and such that $\Gamma_j \cap \Sigma_0 = (\varphi^{-1}(j) \cap \Sigma)_0$.

Lemma 2.12. *There exists some integer $k > k_0$ such that*

$$d_{LS}[I - \mathcal{G}(0, \cdot), \Gamma_j, 0] \neq 0$$

for all integers $j \geq k$.

Proof. Using the continuity of $\tau(\cdot)$ and Lemma 2.10 it follows that there exists some integer $k > k_0$ such that $\tau^{-1}(\frac{\pi}{j}) \neq \emptyset$ for all integers $j \geq k$. If $j \geq k$, let $\varepsilon > 0$ such that $]\frac{\pi}{j} - \varepsilon, \frac{\pi}{j} + \varepsilon[\subset]\frac{\pi}{j+1}, \frac{\pi}{j-1}[$ and

$$\Delta_j = \{(u(\cdot, \alpha), v(\cdot, \alpha)) : |\alpha| > R, \tau(|\alpha|) \in]\frac{\pi}{j} - \varepsilon, \frac{\pi}{j} + \varepsilon[\}.$$

The sets Δ_j have the same proprieties as the sets Γ_j above. Consider

$$\alpha_j = \max\{\alpha > R : \tau(\alpha) = \frac{\pi}{j} + \varepsilon\}, \quad \beta_j = \min\{\alpha > R : \tau(\alpha) = \frac{\pi}{j} - \varepsilon\}.$$

Using a continuity argument given in [[34], Theorem 5.1], we have that

$$d_{LS}[I - \mathcal{G}(0, \cdot), \Delta_j, 0] = d_{LS}[I - \mathcal{G}(0, \cdot), \Omega_{\beta_j} \setminus \overline{\Omega}_{\alpha_j}, 0].$$

But, using Lemma 2.9, Lemma 2.11 and the additivity property of the Leray-Schauder degree, we deduce that

$$d_{LS}[I - \mathcal{G}(0, \cdot), \Omega_{\beta_j} \setminus \overline{\Omega}_{\alpha_j}, 0] = (-1)^{j+1} - (-1)^j \neq 0.$$

Now, the conclusion follows using the excision property of Leray-Schauder degree. ■

Proof of Theorem 2. Using Lemma 2.2, 2.5, 2.6, 2.7, 2.12 we can apply Capietto-Mawhin-Zanolin continuation theorem to deduce that for all $j \geq k$ there exists $w_j \in X$ such that $\varphi(1, w_j) = j$ and $\mathcal{G}(1, w_j) = w_j$. The conclusion follows using Lemma 2.3 ■

Exemple 2.13. *Using Theorem 2.1, we deduce that the boundary value problem*

$$\begin{aligned} u' &= -[(u - t)^2 + (v - \sin t)^2]v, & v' &= [(u - t)^2 + (v - \sin t)^2]u, \\ u(0) &= 0 = u(\pi), \end{aligned}$$

has infinitely many topologically distinct solutions.

Bibliography

- [1] R.P. Agarwal, P.J.Y. Wong, Advanced topics in difference equations. Mathematics and its Applications, 404. Kluwer Academic Publishers Group, Dordrecht, 1997.
- [2] S. Ahmad, On the nonautonomous Volterra-Lotka competition equations, *Proc. Amer. Math. Soc.* **117** (1993), 199–204.
- [3] A. Ambrosetti, G. Prodi, On the inversion of some differential mappings with singularities between Banach spaces, *Ann. Mat. Pura Appl.* **93** (1972), 231–246.
- [4] H. Amann, P. Hess, A multiplicity result for a class of elliptic boundary value problems, *Proc. Roy. Soc. Edinburgh* **84A** (1979), 145–151.
- [5] A. Batauz, F. Zanolin, Coexistence States for Periodic Competitive Kolmogorov Systems, *J. Math. Anal. Appl.* **219** (1998), 179–199.
- [6] C. Bereanu, Periodic solutions for delay competition systems and delay prey-predator systems, *Adv. Nonlin. Stud.* **5** (2005), 393–410.
- [7] C. Bereanu, On a multiplicity result of J.R. Ward for superlinear planar systems, *Topol. Methods Nonlin. Anal.*, **27** (2006), 289–298.
- [8] C. Bereanu, J. Mawhin, Nonlinear Neumann boundary value problems with ϕ -Laplacian operators, *An. St. Univ. Ovidius Constanta*, (2)**12** (2004), 73–92.
- [9] C. Bereanu, J. Mawhin, Boundary -value problems with non-surjective ϕ -Laplacian and one-side bounded nonlinearity, *Adv. Differential Equations* **11** (2006), 35–60.
- [10] C. Bereanu, J. Mawhin, Upper and lower solutions for periodic problems : first order difference vs first order differential equations, *AIP Conference Proceedings* **835** (2006), 30–36.
- [11] C. Bereanu, J. Mawhin, Existence and multiplicity results for nonlinear difference equations with Dirichlet boundary conditions, *Math. Bohemica* **131** (2006), 145–160.
- [12] C. Bereanu, J. Mawhin, Existence and multiplicity results for periodic solutions of nonlinear difference equations, *J. Difference Equations. Appl.*, **12** (2006), 677–695.
- [13] C. Bereanu, J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ -Laplacian, *J. Differential Equations*, to appear.
- [14] C. Bereanu, J. Mawhin, Periodic solutions of first order nonlinear difference equations, *Rend. Sem. Mat. Univ. Pol. Torino*, to appear.
- [15] C. Bereanu, J. Mawhin, Periodic solutions of some nonlinear perturbations of the ϕ -Laplacians with possibly bounded ϕ , preprint.
- [16] C. Bereanu, H.B. Thompson, Periodic solutions of second order nonlinear difference equations with discrete ϕ -Laplacian, *J. Math. Anal. Appl.*, to appear.

- [17] M.S. Berger, E. Podolak, On the solutions of a nonlinear Dirichlet problem, *Indiana Univ. Math. J.* **24** (1975), 837-846.
- [18] G.D. Birkhoff, O.D. Kellogg, Invariant points in function space, *Trans. Amer. Math. Soc.* **23** (1922), 96-115.
- [19] A. Capietto, M. Henrard, J. Mawhin, F. Zanolin, A continuation approach to some forced superlinear Sturm-Liouville boundary value problems, *Topol. Methods Nonlin. Anal.* **3** (1994), 81-100.
- [20] A. Capietto, J. Mawhin, F. Zanolin, Boundary value problems for forced superlinear second order ordinary differential equations, Collège de France Seminar, vol. XII, Longman, New York, 1994, pp. 55-64.
- [21] L. Cesari, Functional analysis and periodic solutions of nonlinear differential equations, in *Contributions in Differential Equations, Vol. 1*, 149-187, Wiley, New York, 1963.
- [22] C. De Coster, P. Habets, Two-Point Boundary Value Problems. Lower and Upper Solutions, Elsevier, Amsterdam, 2006.
- [23] R. Chiappinelli, J. Mawhin, R. Nugari, Generalized Ambrosetti-Prodi conditions for nonlinear two-point boundary value problems, *J. Differential Equations* **69** (1987), 422-434.
- [24] E.B. Dancer, On the ranges of certain weakly non linear elliptic partial differential equations, *J. Math. Pures Appl.* **57** (1978), 351-366.
- [25] K. Deimling, Nonlinear Functional Analysis, Springer, Berlin, 1985.
- [26] G. Dincă, P. Jebelean, Some existence results for a class of non-linear equations involving a duality mapping, *Nonlinear Anal.* **46** (2001), 347-363.
- [27] G. Dincă, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, *Portugal. Math.* **58** (2001), 339-378.
- [28] G.S. Dragoni, Il problema dei valori ai limiti studiato in grande per gli integrali di una equazione differenziale del secondo ordine, *Giornale di Mat. (Battaglini)* (3) **69** (1931), 77-112.
- [29] G.S. Dragoni, Il problema dei valori ai limiti studiato in grande per le equazioni differenziali del secondo ordine, *Math. Ann.* **105** (1931), 133-143.
- [30] S. Elaydi, An introduction to difference equations. Third edition. Undergraduate Texts in Mathematics. Springer, New York, 2005.
- [31] C. Fabry, J. Mawhin, M. Nkashama, A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations, *Bull. London Math. Soc.* **18** (1986), 173-180.
- [32] K. Gęba, P. Rabinowitz, Topological Methods in Bifurcation Theory, No. 91, Sémin. Math. Sup., Presses Univ. Montréal, Montréal, 1985.
- [33] J. Hofbauer, On the accurence of limit cycle in the Volterra-Lotka equation, *Nonlinear Anal. TMA* **5** (1981), 1003-1007.
- [34] M. Garcia-Huidobro, R. Manásevich, F. Zanolin, Strong nonlinear second-order ODEs with rapidly growing terms, *J. Math. Anal. Appl.* **202** (1996), 1-26.
- [35] P. Jebelean, J. Mawhin, Periodic solutions of singular nonlinear perturbations of the ordinary p-Laplacian, *Adv. Nonlin. Stud.* **2** (2002), 299-312.
- [36] J.L. Kazdan, F.W. Warner, Remarks on some quasilinear elliptic equations, *Comm. Pure Appl. Math.* **28** (1975), 567-597.
- [37] H.W. Knobloch, An existence theorem for periodic solutions of nonlinear ordinary differential equations, *Michigan Math. J.* **9** (1962), 303-309.

- [38] H.W. Knobloch, Remarks on a paper of L. Cesari on functional analysis and nonlinear differential equations, *Michigan Math. J.* **10** (1963), 417-430.
- [39] H.W. Knobloch, Eine neue Methode zur Approximation periodischer Lösungen nicht-linearer Differentialgleichungen zweiter Ordnung, *Math. Z.* **82** (1963), 177-197.
- [40] M.A. Krasnosel'skii, P.P. Zabreiko, Geometrical Methods of Nonlinear Analysis, Springer, Berlin, 1984.
- [41] A.C. Lazer, On Schauder's fixed point theorem and forced second-order nonlinear oscillations, *J. Math. Anal. Appl.* **21** (1968), 421-425.
- [42] A.C. Lazer, S. Solimini, On periodic solutions of nonlinear differential equations with singularities, *Proc. Amer. Math. Soc.* **99** (1987), 109-114.
- [43] J. Leray, J. Schauder, Topologie et équations fonctionnelles, *Ann. Ec. Norm. Sup.* **51** (1934), 45-78.
- [44] A. Leung, Periodic solutions for a prey-predator differential delay equation, *J. Differential Equations* **26** (1977), 391-403.
- [45] Y. Li, Periodic Solutions for Delay Lotka-Volterra Competition Systems, *J. Math. Anal. Appl.* **246** (2000), 230-244.
- [46] Y. Li, Y. Kuang, Periodic Solution of Periodic Delay Lotka-Volterra Equations and Systems, *J. Math. Anal. Appl.* **255** (2001), 260-280.
- [47] R. Manásevich, J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators, *J. Differential Equations* **145** (1998), 367-393.
- [48] R. Manásevich, J. Mawhin, Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators, *J. Korean Math. Soc.* **37** (2000), 665-685.
- [49] R.M. May, W.J. Leonard, Nonlinear aspects of competition between three species, *SIAM, J. Appl. Math.* **29** (1975), 243-253.
- [50] J. Mawhin, Équations intégrales et solutions périodiques des systèmes différentiels non linéaires, *Acad. Roy. Belg. Bull. Cl. Sci.* **55** (1969), 934-947.
- [51] J. Mawhin, An extension of a theorem of A.C. Lazer on forced nonlinear oscillations, *J. Math. Anal. Appl.* **40** (1972) 20-29.
- [52] J. Mawhin, Equivalence theorems for nonlinear operator equations and coincidence degree theory, *J. Differential Equations* **12** (1972), 610-636.
- [53] J. Mawhin, Topological Degree Methods in Nonlinear Boundary Value Problems, CBMS Series No. 40, American Math. Soc., Providence, 1979.
- [54] J. Mawhin, Points fixes, points critiques et problèmes aux limites, Sémin. Math. Sup. No. 92, Presses Univ. Montréal, Montréal, 1985.
- [55] J. Mawhin, Ambrosetti-Prodi type results in nonlinear boundary value problems, Lecture Notes in Mathematics No. 1285, Springer, Berlin, 1986, 390-413.
- [56] J. Mawhin, First order ordinary differential equations with several periodic solutions, *Z. Angew. Math. Phys.* **38** (1987), 257-265.
- [57] J. Mawhin, Leray-Schauder continuation theorems in the absence of a priori bounds, *Topol. Methods Nonlin. Anal.* **9** (1997), 179-200.
- [58] J. Mawhin, Leray-Schauder degree: A half century of extensions and applications, *Topol. Methods Nonlin. Anal.* **14** (1999), 195-228.
- [59] J. Mawhin, A simple approach to Brouwer degree based on differential forms, *Adv. Nonlin. Stud.* **4** (2004), 535-548.
- [60] J. Mawhin, The periodic Ambrosetti-Prodi problem for nonlinear perturbations of the p-Laplacian, *J. European Math. Soc.* **8** (2006), 375-388.

- [61] M. Nagumo, Ueber die Differentialgleichung $y'' = f(x, y, y')$, *Proc. Phys. Math. Soc. Japan* (3) **19** (1937), 861-866.
- [62] J. Schauder, Der Fixpunktsatz in Funktionalräumen, *Studia Math.* **2** (1930), 171-189.
- [63] K. Schmitt, Periodic solutions of nonlinear second order differential equations, *Math. Z.* **98** (1967), 200-207.
- [64] H.B. Thompson, Existence of multiple solutions for finite difference approximations to second-order boundary value problems, *Nonlinear Anal.* **53** (2003), 97-110.
- [65] G. Villari, Soluzioni periodiche di una classe di equazioni differenziali del terz'ordine quasi lineari, *Ann. Mat. Pura Appl.* (4) **73** (1966), 103-110.
- [66] J.R. Ward, Asymptotic conditions for periodic solutions of ordinary differential equations, *Proc. Amer. Math. Soc.* **81** (1981), 415-420.
- [67] J.R. Ward, Rotation numbers and global bifurcation in systems of ordinary differential equations, *Adv. Nonlin. Stud.* **5** (2005), 375-392.
- [68] F. Zanolin, Permanence and positive periodic solutions for Kolmagorov Competing species systems, *Results Math.* **21** (1992), 224-250.
- [69] A. Zitan, R. Ortega, A periodic prey-predator system, *J. Math. Anal. Appl.* **185** (1994), 477-489.