

A point process on the unit circle with antipodal interactions

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June 30, 2025

Abstract

We introduce the point process

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} + e^{i\theta_k}|^\beta \prod_{j=1}^n d\theta_j, \quad \theta_1, \dots, \theta_n \in (-\pi, \pi], \quad \beta > 0,$$

where Z_n is the normalization constant. This point process is *attractive*: it involves n dependent, uniformly distributed random variables on the unit circle that attract each other. (For comparison, the well-studied C β E involves n uniformly distributed random variables on the unit circle that repel each other.)

We consider linear statistics of the form $\sum_{j=1}^n g(\theta_j)$ as $n \rightarrow \infty$, where $g \in C^{1,q}$ and 2π -periodic. We prove that the leading order fluctuations around the mean are of order n and given by $(g(U) - \int_{-\pi}^{\pi} g(\theta) \frac{d\theta}{2\pi})n$, where $U \sim \text{Uniform}(-\pi, \pi]$. We also prove that the subleading fluctuations around the mean are of order $\sqrt{n}$ and of the form $\mathcal{N}_{\mathbb{R}}(0, 4g'(U)^2/\beta)\sqrt{n}$, i.e. that the subleading fluctuations are given by a Gaussian random variable that itself has a random variance.

Our proof uses techniques developed by McKay and Isaev [8, 6] to obtain asymptotics of related n -fold integrals.

AMS SUBJECT CLASSIFICATION (2020): 41A60, 60G55.

KEYWORDS: Smooth statistics, asymptotics, point processes, attractive interactions.

1 Introduction

Gibbs measures are models for collections of locally dependent random points (or particles) and are important in various problems in probability and statistical physics [4]. Interactions between particles can be either attractive or repulsive. A well-known repulsive Gibbs measure is the circular β -ensemble (C β E), given by

$$\frac{1}{\tilde{Z}_n} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{j=1}^n d\theta_j, \quad \theta_1, \dots, \theta_n \in (-\pi, \pi], \quad (1.1)$$

where $\tilde{Z}_n$ is the normalization constant. Here the n points are confined to the unit circle and repel each other according to the two-dimensional Coulomb law at inverse temperature $\beta > 0$. In order to maximize the density of (1.1), the n points must be evenly spaced on the unit circle. This point process has been widely studied, see e.g. [5, Chapter 2].

In comparison, little is known about attractive point processes on the unit circle, and the purpose of this paper is to initiate the study of such a process. More precisely, we are interested in the joint

probability measure

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} + e^{i\theta_k}|^\beta \prod_{j=1}^n d\theta_j, \quad \theta_1, \dots, \theta_n \in (-\pi, \pi], \quad (1.2)$$

where Z_n is the normalization constant. This point process is indeed an attractive Gibbs measure, because the density of (1.2) is maximized for point configurations of the form $(e^{i\theta_1}, \dots, e^{i\theta_n}) = (e^{i\theta}, \dots, e^{i\theta})$ with $\theta \in (-\pi, \pi]$.

In view of (1.1) and (1.2), it is also natural to consider

$$\frac{1}{\widehat{Z}_n} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{-i\theta_k}|^\beta \prod_{j=1}^n d\theta_j, \quad \theta_1, \dots, \theta_n \in (-\pi, \pi], \quad (1.3)$$

where $\widehat{Z}_n$ is the normalization constant. The point process (1.3) is attractive, but also features a repulsion with the real line: for $n \geq 3$, only the point configurations $(e^{i\theta_1}, \dots, e^{i\theta_n}) = (i, \dots, i)$ and $(e^{i\theta_1}, \dots, e^{i\theta_n}) = (-i, \dots, -i)$ maximize the density of (1.3).

By rotational symmetry, the random variables $e^{i\theta_1}, \dots, e^{i\theta_n}$ of both (1.1) and (1.2) are uniformly distributed on the unit circle (but not independently distributed). In the case of (1.1), these random variables repel each other, while in the case of (1.2) they attract each other. On the other hand, for the point process (1.3), the individual distributions of $e^{i\theta_1}, \dots, e^{i\theta_n}$ are not uniform if $n \geq 3$.

In addition to providing concrete examples of attractive point processes on the unit circle, the point processes (1.2) and (1.3) are also valuable from a mathematical point of view, because they can be studied rigorously as $n \rightarrow \infty$ using results from [8, 6] and [9], respectively (we comment more on this below).

Both (1.2) and (1.3) can also be seen as repulsive point processes of a new kind, where the points $e^{i\theta_1}, \dots, e^{i\theta_n}$ do not repel each other, but are repelled by some “image points”. Indeed, the points $e^{i\theta_1}, \dots, e^{i\theta_n}$ of (1.2) are repelled by the image points $-e^{i\theta_1}, \dots, -e^{i\theta_n}$ obtained by reflection across the origin, and the points $e^{i\theta_1}, \dots, e^{i\theta_n}$ of (1.3) are repelled by the image points $e^{-i\theta_1}, \dots, e^{-i\theta_n}$ obtained by reflection across the real line. For these reasons, we say that (1.2) is a point process “with antipodal interactions”, and that (1.3) is a point process “with mirror-type interactions” (where the real line plays the role of the mirror).

In this paper we focus on the point process (1.2) with antipodal interactions. The other point process (1.3) is studied in the companion paper [3]. Further comparisons between (1.1), (1.2) and (1.3) are provided at the end of this section.

Our first result shows that for large n , all points of (1.2) cluster together in an arc of length $\mathcal{O}(n^{-\frac{1}{2}+\epsilon})$ with overwhelming probability, see also Figure 1. More precisely, we have the following.

Theorem 1.1. *Fix $\beta > 0$. For any $\epsilon \in (0, \frac{1}{8})$, there exists $c > 0$ such that, for all large enough n ,*

$$\mathbb{P}\left(|e^{i\theta_j} - e^{i\theta_n}| \leq n^{-\frac{1}{2}+\epsilon} \text{ for all } j \in \{1, \dots, n-1\}\right) \geq 1 - e^{-cn^{2\epsilon}}.$$

In this paper, we study the fluctuations as $n \rightarrow \infty$ of linear statistics of the form $\sum_{j=1}^n g(\theta_j)$, for fixed β and where $g : \mathbb{R} \rightarrow \mathbb{R}$ is 2π -periodic and sufficiently regular. More precisely, for Theorem (1.2), g is assumed to be continuous, but our other results (Theorems 1.3 and 1.4 below) require g to be differentiable and such that g' is Hölder continuous.

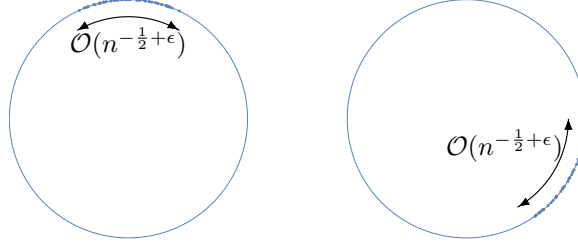


Figure 1: Illustration of the point process (1.2) with $n = 50$. With high probability all points are close to each other.

Let $\mu_n := \frac{1}{n} \sum_{j=1}^n \delta_{\theta_j}$ be the empirical measure of (1.2). From the rotational symmetry of (1.2) together with Theorem 1.1, one expects the average empirical measure $\mathbb{E}[\mu_n]$ to converge as $n \rightarrow \infty$ to the uniform measure on $(-\pi, \pi]$ given by $\frac{d\theta}{2\pi}$. On the other hand, Theorem 1.1 also implies that the support of μ_n will be contained inside an arc of length $\mathcal{O}(n^{-\frac{1}{2}+\epsilon})$ with overwhelming probability for large n . In other words, for large n the measure μ_n “deviates substantially” from $\mathbb{E}[\mu_n]$ with high probability, suggesting that μ_n has no deterministic limit as $n \rightarrow \infty$. The following result makes these ideas more precise.

Theorem 1.2. *Fix $\beta > 0$, and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic and continuous. We have*

$$\int_{(-\pi, \pi]} g(x) d\mu_n(x) - g(\theta_n) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \quad (1.4)$$

Equivalently, $\frac{1}{n} \sum_{j=1}^n g(\theta_j) - g(\theta_n) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$. Since $\theta_n \sim \text{Uniform}(-\pi, \pi]$, (1.4) implies that

$$\int_{(-\pi, \pi]} g(x) d\mu_n(x) \xrightarrow[n \rightarrow \infty]{\text{law}} \int_{(-\pi, \pi]} g(x) d\mu(x), \quad (1.5)$$

where $\mu := \delta_U$ and $U \sim \text{Uniform}(-\pi, \pi]$. Equivalently, $\frac{1}{n} \sum_{j=1}^n g(\theta_j) \xrightarrow[n \rightarrow \infty]{\text{law}} g(U)$.

Proof. The claim (1.4) is a direct consequence of Theorem 1.1 and the Borel–Cantelli Lemma. $\square$

Theorem 1.2 deals only with the leading order fluctuations of $\sum_{j=1}^n g(\theta_j)$. The subleading fluctuations are more intricate and will be given in Theorem 1.4 below. We will proceed by first establishing the large n asymptotics of

$$I\left(\frac{t}{\sqrt{n}}g\right) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} + e^{i\theta_k}|^{\beta} \prod_{j=1}^n e^{\frac{t}{\sqrt{n}}g(\theta_j)} d\theta_j, \quad (1.6)$$

where t lies in a compact subset of $\mathbb{R}$, see Theorem 1.3. We will then derive the large n asymptotics for the generating function of $\frac{1}{\sqrt{n}} \sum_{j=1}^n g(\theta_j)$ simply from the formula

$$\mathbb{E} \left[\exp \left(\frac{t}{\sqrt{n}} \sum_{j=1}^n g(\theta_j) \right) \right] = \frac{I\left(\frac{t}{\sqrt{n}}g\right)}{I(0)}.$$

This paper is inspired by the works [8, 6]. In the study of counting problems on graphs, McKay in [8] introduced techniques to derive large n asymptotics of several types of n -fold integrals, among which is

$$\frac{1}{(2\pi i)^n} \oint \dots \oint \frac{\prod_{1 \leq j < k \leq n} (z_j^{-1} z_k + z_j z_k^{-1})}{z_1 z_2 \dots z_n} dz_1 \dots dz_n, \quad (1.7)$$

where each contour encloses the origin once anticlockwise. In recent years, a more systematic approach to such integrals was developed in [6]. The methods of [8, 6] are remarkably robust and can be adjusted to analyze the integral (1.6) (despite the fact that the integrand in (1.6) is not analytic).

The following theorem gives a precise asymptotic formula, up to and including the constant term, for $I(\frac{t}{\sqrt{n}}g)$ as $n \rightarrow \infty$.

Theorem 1.3. Fix $\beta > 0$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic and $C^{1,q}$, with $0 < q \leq 1$, and let $t \in \mathbb{R}$. Then, for any fixed $0 < \zeta < \frac{q}{2}$, as $n \rightarrow \infty$ we have

$$I(\frac{t}{\sqrt{n}}g) = 2^{\beta \frac{n(n-1)}{2}} \left(\frac{8\pi}{\beta n} \right)^{\frac{n-1}{2}} \sqrt{n} e^{-\frac{1}{2\beta}} (1 + \mathcal{O}(n^{-\zeta})) \int_{-\pi}^{\pi} \exp \left(t\sqrt{n}g(\theta) + \frac{2g'(\theta)^2}{\beta} t^2 \right) d\theta, \quad (1.8)$$

uniformly for t in compact subsets of $\mathbb{R}$. If $g \equiv 0$, then the error term can be improved: for any fixed $0 < \zeta < 1$, we have

$$Z_n = I(0) = 2^{\beta \frac{n(n-1)}{2}} \left(\frac{8\pi}{\beta n} \right)^{\frac{n-1}{2}} \sqrt{n} e^{-\frac{1}{2\beta}} 2\pi (1 + \mathcal{O}(n^{-\zeta})), \quad \text{as } n \rightarrow \infty. \quad (1.9)$$

Our next result on the generating function of $\frac{1}{\sqrt{n}} \sum_{j=1}^n g(\theta_j)$ follows directly from Theorem 1.3.

Theorem 1.4. Fix $\beta > 0$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic and $C^{1,q}$, with $0 < q \leq 1$, and let $t \in \mathbb{R}$. For any fixed $0 < \zeta < \frac{q}{2}$, as $n \rightarrow \infty$ we have

$$\mathbb{E} \left[e^{\frac{t}{\sqrt{n}} \sum_{j=1}^n g(\theta_j)} \right] = \frac{I(\frac{t}{\sqrt{n}}g)}{I(0)} = \frac{1 + \mathcal{O}(n^{-\zeta})}{2\pi} \int_{-\pi}^{\pi} \exp \left(t\sqrt{n}g(\theta) + \frac{2g'(\theta)^2}{\beta} t^2 \right) d\theta, \quad (1.10)$$

uniformly for t in compact subsets of $\mathbb{R}$.

The asymptotic formula (1.10) can be rewritten as

$$\mathbb{E} \left[e^{\frac{t}{\sqrt{n}} \sum_{j=1}^n g(\theta_j)} \right] = (1 + o(1)) \mathbb{E} [e^{t[\sqrt{n}g(U) + \mathcal{N}_{\mathbb{R}}(0, \frac{4g'(U)^2}{\beta})}]}, \quad (1.11)$$

where $U \sim \text{Uniform}(-\pi, \pi]$, and where $\mathcal{N}_{\mathbb{R}}(0, \frac{4g'(u)^2}{\beta}) := 0$ if $g'(u) = 0$. Informally, one can interpret (1.11) as

$$\sum_{j=1}^n g(\theta_j) = n g(U) + \sqrt{n} N_U + o(\sqrt{n}), \quad \text{where } N_U \sim \mathcal{N}_{\mathbb{R}}(0, \frac{4g'(U)^2}{\beta}). \quad (1.12)$$

Therefore, for a non-constant g , Theorem 1.4 means that the leading order fluctuations of $\sum_{j=1}^n g(\theta_j)$ around the mean $n \int_{-\pi}^{\pi} g(\theta) \frac{d\theta}{2\pi}$ are of order n and given by $n(g(U) - \int_{-\pi}^{\pi} g(\theta) \frac{d\theta}{2\pi})$. Moreover, the subleading fluctuations are of order $\sqrt{n}$ and given by $\mathcal{N}_{\mathbb{R}}(0, \frac{4g'(U)^2}{\beta})\sqrt{n}$, i.e. by a Gaussian random variable whose variance is itself random.

Other point processes on the unit circle. It is interesting to compare (1.2) with other point processes on the unit circle, such as

- (a) the C β E (1.1), i.e. $\tilde{Z}_n^{-1} \prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{j=1}^n d\theta_j$,
- (b) the point process (1.3), i.e. $\hat{Z}_n^{-1} \prod_{j < k} |e^{i\theta_j} - e^{-i\theta_k}|^\beta \prod_{j=1}^n d\theta_j$.

In sharp contrast with (1.2), the empirical measure $\frac{1}{n} \sum_{j=1}^n \delta_{\theta_j}$ of the C β E converges almost surely to the uniform measure on $(-\pi, \pi]$, the associated smooth linear statistics have Gaussian fluctuations of order 1, and the test function only affects the variance of this Gaussian. More informally, for the C β E we have

$$\sum_{j=1}^n g(\theta_j) = n \int_{-\pi}^{\pi} g(\theta) \frac{d\theta}{2\pi} + \mathcal{N}_{\mathbb{R}}(0, \sigma^2) + o(1), \quad \text{as } n \rightarrow \infty \quad (1.13)$$

where $\sigma^2 = \frac{4}{\beta} \sum_{k=1}^{\infty} k |g_k|^2$ and $g_k := \int_{-\pi}^{\pi} g(\theta) e^{-ik\theta} \frac{d\theta}{2\pi}$, see [7]. Many other point processes with different types of repulsive interactions have been considered in the literature, see e.g. [5, 2]. It is typically the case for such processes that (i) the associated empirical measures have deterministic limits, and (ii) the smooth statistics have Gaussian leading order fluctuations. The point process (b) listed above is very different: in fact, just like (1.2), its empirical measure μ_n^b has no deterministic limit. Indeed, it is shown in [3] that if $g : \mathbb{R} \rightarrow \mathbb{R}$ is 2π -periodic, bounded, and $C^{2,q}$ in neighborhoods of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ with $0 < q \leq 1$, then

$$\int_{(-\pi, \pi]} g(x) d\mu_n^b(x) \xrightarrow[n \rightarrow \infty]{\text{law}} \int_{(-\pi, \pi]} g(x) d\mu^b(x),$$

where $\mu^b = B\delta_{\frac{\pi}{2}} + (1-B)\delta_{-\frac{\pi}{2}}$ and $B \sim \text{Bernoulli}(\frac{1}{2})$ (i.e. $\mathbb{P}(B=1) = \mathbb{P}(B=0) = 1/2$). In the generic case where $g(\frac{\pi}{2}) \neq g(-\frac{\pi}{2})$ and $g'(\frac{\pi}{2}) \neq g'(-\frac{\pi}{2})$, it is also proved in [3] that the leading order fluctuations of the smooth linear statistics of (b) are of order n and purely Bernoulli, and that the subleading fluctuations are of order 1 and of the form $BN_1 + (1-B)N_2$, where N_1, N_2 are two Gaussian random variables, and N_1, N_2, B are independent from each other. Informally, the results from [3] can be rewritten as

$$\sum_{j=1}^n g(\theta_j) = n \left(g\left(\frac{\pi}{2}\right)B + g\left(-\frac{\pi}{2}\right)(1-B) \right) + BN_1 + (1-B)N_2 + o(1), \quad \text{as } n \rightarrow \infty. \quad (1.14)$$

It is interesting to compare (1.12), (1.13) and (1.14). In particular, for the point processes (a) and (b), there are no fluctuations of order $\sqrt{n}$.

Another difference between (b) and (1.2) is the following: for (b), there are some non-generic test functions for which the leading order fluctuations around the mean are not of order n (if $g(\frac{\pi}{2}) = g(-\frac{\pi}{2})$), but of order 1 or even of order $o(1)$. In comparison, for (1.2), the only scenario where the leading order fluctuations are not of order n corresponds to the trivial situation where g is a constant, in which case there are no fluctuations at all.

Conclusion. In this paper, we studied the smooth linear statistics of (1.2). We proved formula (1.10) concerning the leading and subleading order fluctuations of $\sum_{j=1}^n g(\theta_j)$. There are other problems concerning (1.2) that are also of interest for future research, for example:

- In this paper $\beta > 0$ is fixed. The asymptotic formula (1.10) suggests that a critical transition occurs when $\beta \asymp n^{-\frac{1}{2}}$. It would be interesting to figure that out.

2 Preliminary lemma

Define

$$U_n(t) = \{\mathbf{x} \in \mathbb{R}^n : |x_i| \leq t, i = 1, \dots, n\} \quad \text{for } t \geq 0.$$

The following lemma is proved using techniques from [8, 6] and will be used in Section 3 to obtain large n asymptotics for $I(\frac{t}{\sqrt{n}}g)$.

Lemma 2.1. *Let $0 < \epsilon < \frac{1}{8}$, $a > 0$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, and $n \geq 2$ be an integer. Define*

$$J = \int_{U_{n-1}(n^{-\frac{1}{2}+\epsilon})} \exp \left(-a \sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^2 + b \sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^4 + \frac{c}{\sqrt{n}} \sum_{j=1}^{n-1} \theta_j \right) \prod_{j=1}^{n-1} d\theta_j,$$

where the integral is over $\boldsymbol{\theta}' := (\theta_1, \dots, \theta_{n-1}) \in U_{n-1}(n^{-\frac{1}{2}+\epsilon})$ with $\theta_n = 0$. Then, as $n \rightarrow \infty$,

$$J = n^{1/2} \left(\frac{\pi}{an} \right)^{\frac{n-1}{2}} \exp \left(\frac{c^2}{4a} + \frac{3b}{2a^2} + \mathcal{O}(n^{-1+8\epsilon}) \right). \quad (2.1)$$

Proof. If $c = 0$, and if the error term $\mathcal{O}(n^{-1+8\epsilon})$ in (2.1) is replaced by the weaker estimate $\mathcal{O}(n^{-\frac{1}{2}+4\epsilon})$, then the statement directly follows from [8, Theorem 2.1].

Changing variables $\theta_j = \alpha_j + \frac{c}{2a\sqrt{n}}$, $j = 1, \dots, n-1$, we get

$$\begin{aligned} J = \exp \left(\frac{n-1}{n} \frac{c^2}{4a} + \frac{bc^4}{16a^4} \frac{n-1}{n^2} \right) \int_{-\frac{c}{2a\sqrt{n}} \mathbf{1} + U_{n-1}(n^{-\frac{1}{2}+\epsilon})} \exp \left(-a \sum_{1 \leq j < k \leq n} (\alpha_j - \alpha_k)^2 \right. \\ \left. + b \sum_{1 \leq j < k \leq n} (\alpha_j - \alpha_k)^4 + \sum_{j=1}^{n-1} \left[\frac{2bc}{a} \frac{\alpha_j^3}{\sqrt{n}} + \frac{3bc^2}{2a^2} \frac{\alpha_j^2}{n} + \frac{bc^3}{2a^3} \frac{\alpha_j}{n^{3/2}} \right] \right) \prod_{j=1}^{n-1} d\alpha_j, \end{aligned}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{n-1}$ and $\alpha_n := 0$. For $(\alpha_1, \dots, \alpha_{n-1}) \in -\frac{c}{2a\sqrt{n}} \mathbf{1} + U_{n-1}(n^{-\frac{1}{2}+\epsilon})$, we have

$$\sum_{j=1}^{n-1} \left[\frac{2bc}{a} \frac{\alpha_j^3}{\sqrt{n}} + \frac{3bc^2}{2a^2} \frac{\alpha_j^2}{n} + \frac{bc^3}{2a^3} \frac{\alpha_j}{n^{3/2}} \right] = \mathcal{O}(n^{-1+3\epsilon}), \quad \text{as } n \rightarrow \infty,$$

and thus (resetting $\alpha_j = \theta_j$)

$$\begin{aligned} J = \exp \left(\frac{c^2}{4a} + \mathcal{O}(n^{-1+3\epsilon}) \right) \\ \times \int_{-\frac{c}{2a\sqrt{n}} \mathbf{1} + U_{n-1}(n^{-\frac{1}{2}+\epsilon})} \exp \left(-a \sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^2 + b \sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^4 \right) \prod_{j=1}^{n-1} d\theta_j. \quad (2.2) \end{aligned}$$

Since the integrand in (2.2) is positive, and since

$$U_{n-1}(\frac{1}{2}n^{-\frac{1}{2}+\epsilon}) \subset -\frac{c}{2a\sqrt{n}} \mathbf{1} + U_{n-1}(n^{-\frac{1}{2}+\epsilon}) \subset U_{n-1}(2n^{-\frac{1}{2}+\epsilon})$$

holds for all sufficiently large n , to conclude the proof it remains to prove the claim for $c = 0$.

Assume from now on that $c = 0$. As mentioned, if $\mathcal{O}(n^{-1+8\epsilon})$ in (2.1) is replaced by $\mathcal{O}(n^{-\frac{1}{2}+4\epsilon})$, then the statement follows from [8, Theorem 2.1]. The improved error term can be proved using the general method from [6]. Let $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^n$ and define

$$\Omega = U_n(n^{-\frac{1}{2}+\epsilon}), \quad F(\mathbf{x}) = \exp\left(-a \sum_{1 \leq j < k \leq n} (x_j - x_k)^2 + b \sum_{1 \leq j < k \leq n} (x_j - x_k)^4\right),$$

$$Q\mathbf{x} = \mathbf{x} - x_n \mathbf{1}, \quad W\mathbf{x} = \frac{\sqrt{a}}{\sqrt{n}}(x_1 + \dots + x_n) \mathbf{1}, \quad P\mathbf{x} = \mathbf{x} - \frac{1}{n}(x_1 + \dots + x_n) \mathbf{1}, \quad R\mathbf{x} = \frac{1}{\sqrt{an}} \mathbf{x}.$$

Clearly, $F(Q\mathbf{x}) = F(\mathbf{x})$, $\dim \ker Q = 1$, $\dim \ker W = n-1$, $\ker Q \cap \ker W = \{\mathbf{0}\}$ and $\text{span}(\ker Q, \ker W) = \mathbb{R}^n$. Applying [6, Lemma 4.6] with $\rho = \sqrt{an}^\epsilon$, $\rho_1 = \rho_2 = n^{-\frac{1}{2}+\epsilon}$, we obtain

$$J = \int_{U_{n-1}(n^{-\frac{1}{2}+\epsilon})} F(\boldsymbol{\theta}) d\boldsymbol{\theta}' = \int_{\Omega \cap Q(\mathbb{R}^n)} F(\mathbf{y}) d\mathbf{y}$$

$$= (1 - K)^{-1} \pi^{-\frac{1}{2}} \det(Q^T Q + W^T W)^{1/2} \int_{\Omega_\rho} F(\mathbf{x}) e^{-\mathbf{x}^T W^T W \mathbf{x}} d\mathbf{x},$$

where $\det(Q^T Q + W^T W)^{1/2} = \sqrt{an}$,

$$\Omega_\rho = \{\mathbf{x} \in \mathbb{R}^n : Q\mathbf{x} \in \Omega \text{ and } W\mathbf{x} \in U_n(\rho)\},$$

$$0 \leq K < \min(1, ne^{-\frac{\rho^2}{\kappa^2}}) = ne^{-an^{1+2\epsilon}}, \quad \kappa = \sup_{W\mathbf{x} \neq 0} \frac{\|W\mathbf{x}\|_\infty}{\|W\mathbf{x}\|_2} = \frac{1}{\sqrt{n}}.$$

We thus have $J = (1 + \mathcal{O}(e^{-cn^{1+2\epsilon}})) \frac{\sqrt{an}}{\sqrt{\pi}} \int_{\Omega_\rho} F(\mathbf{x}) e^{-a(x_1 + \dots + x_n)^2} d\mathbf{x}$. Since

$$\frac{\rho_1}{\|Q\|_\infty} = \frac{n^{-\frac{1}{2}+\epsilon}}{2}, \quad \frac{\rho}{\|W\|_\infty} = \frac{\sqrt{an}^\epsilon}{\sqrt{an}} = n^{-\frac{1}{2}+\epsilon},$$

$$\|P\|_\infty \rho_2 + \|R\|_\infty \rho = \frac{2(n-1)}{n} n^{-\frac{1}{2}+\epsilon} + \frac{1}{\sqrt{an}} \sqrt{an}^\epsilon \leq 3n^{-\frac{1}{2}+\epsilon},$$

we also obtain from [6, Lemma 4.6] that

$$U_n(\frac{1}{2}n^{-\frac{1}{2}+\epsilon}) \subseteq \Omega_\rho \subseteq U_n(3n^{-\frac{1}{2}+\epsilon}).$$

Furthermore, a direct computation gives

$$F(\mathbf{x}) e^{-a(x_1 + \dots + x_n)^2} = \exp\left(-an \sum_{j=1}^n x_j^2 + b \sum_{1 \leq j < k \leq n} (x_j - x_k)^4\right).$$

Let $\mathbf{f}(\mathbf{x}) = b \sum_{j < k} (x_j - x_k)^4$, and let $\mathbf{X}$ be a Gaussian random variable with density $(\frac{an}{\pi})^{\frac{n}{2}} e^{-an\mathbf{x}^T \mathbf{x}}$.

Note that $|\partial \mathbf{f} / \partial x_j| = \mathcal{O}(n^{-\frac{1}{2}+3\epsilon})$ for $j = 1, \dots, n$ and $\mathbf{x} \in \Omega_\rho$, and that

$$\mathbb{E}[\mathbf{f}(\mathbf{X})] = b \mathbb{E}\left[(n^2 - 4n)x_1^4 + 3\left(\sum_{j=1}^n x_j^2\right)^2\right] = b\left((n^2 - n)\mathbb{E}[x_1^4] + 3n(n-1)\mathbb{E}[x_1^2]^2\right) = \frac{3b(n-1)}{2a^2n}.$$

Applying [6, Theorem 4.3] with $A = anI$, $T = \frac{1}{\sqrt{an}}I$, $\rho_1 = \frac{\sqrt{a}}{2}n^\epsilon$, $\rho_2 = 3\sqrt{a}n^\epsilon$, $\phi_1, \phi_2 \asymp n^{-\frac{1}{2}+4\epsilon}$ (with the functions f, g and h in [6, Theorem 4.3] equal to $\mathbf{f}$, $\mathbf{f}$ and 0 here, respectively, and with Ω in [6, Theorem 4.3] equal to Ω_ρ here), we then find

$$J = (1 + \mathcal{O}(e^{-cn^{1+2\epsilon}})) \frac{\sqrt{an}}{\sqrt{\pi}} (1 + \mathcal{O}(n^{-1+8\epsilon})) \left(\frac{\pi}{an}\right)^{\frac{n}{2}} e^{\frac{3b(n-1)}{2a^2n}}, \quad \text{as } n \rightarrow \infty,$$

and (2.1) follows. $\square$

3 Proof of Theorems 1.1 and 1.3

We start with Theorem 1.3. Our proof is inspired by [8, Proof of Theorem 3.1].

Using $|e^{i\theta_j} + e^{i\theta_k}|^\beta = 2^\beta |\cos \frac{\theta_j - \theta_k}{2}|^\beta$, we first rewrite $I(\frac{t}{\sqrt{n}}g)$ as

$$I(\frac{t}{\sqrt{n}}g) = 2^{\beta \frac{n(n-1)}{2}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{1 \leq j < k \leq n} |\cos \frac{\theta_j - \theta_k}{2}|^\beta \prod_{j=1}^n e^{\frac{t}{\sqrt{n}}g(\theta_j)} d\theta_j. \quad (3.1)$$

Given $x \in \mathbb{R}$, let $x \bmod 2\pi$ be such that $x \bmod 2\pi = x + 2\pi k$, $k \in \mathbb{Z}$ and $x \bmod 2\pi \in (-\pi, \pi]$. Fix $\theta_n \in (-\pi, \pi]$ and $\epsilon \in (0, \frac{1}{8})$, and let I_1 be the contribution to $I(\frac{t}{\sqrt{n}}g)$ of those θ such that $|(\theta_j - \theta_n) \bmod 2\pi| \leq n^{-\frac{1}{2}+\epsilon}$ for $1 \leq j \leq n$, i.e.

$$I_1 := 2^{\beta \frac{n(n-1)}{2}} \int_{-\pi}^{\pi} e^{\frac{t}{\sqrt{n}}g(\theta_n)} \left[\int_{\theta_n + U_{n-1}(n^{-\frac{1}{2}+\epsilon})} \prod_{1 \leq j < k \leq n} |\cos \frac{\theta_j - \theta_k}{2}|^\beta \prod_{j=1}^{n-1} e^{\frac{t}{\sqrt{n}}g(\theta_j)} d\theta_j \right] d\theta_n,$$

where $\theta_n + U_{n-1}(n^{-\frac{1}{2}+\epsilon})$ is equal to

$$\{\theta' = (\theta_1, \dots, \theta_{n-1}) \in (-\pi, \pi]^{n-1} : |(\theta_j - \theta_n) \bmod 2\pi| \leq n^{-\frac{1}{2}+\epsilon} \text{ for } 1 \leq j \leq n-1\}.$$

Since $g \in C^{1,q}$, as $\theta \rightarrow \theta_n$ we have

$$\begin{aligned} \log [2^\beta |\cos \frac{\theta - \theta_n}{2}|^\beta] &= \beta \log 2 - \frac{\beta}{8}(\theta - \theta_n)^2 - \frac{\beta}{192}(\theta - \theta_n)^4 + \mathcal{O}((\theta - \theta_n)^6), \\ g(\theta) &= g(\theta_n) + g'(\theta_n)(\theta - \theta_n) + \mathcal{O}(|\theta - \theta_n|^{1+q}), \end{aligned} \quad (3.2)$$

where the implied constants are independent of $\theta_n \in (-\pi, \pi]$. For $\theta' = (\theta_1, \dots, \theta_{n-1}) \in \theta_n + U_{n-1}(n^{-\frac{1}{2}+\epsilon})$,

$$\mathcal{O}\left(\sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^6\right) = \mathcal{O}(n^{-1+6\epsilon}), \quad \mathcal{O}\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} (\theta_j - \theta_n)^{1+q}\right) = \mathcal{O}(n^{-\frac{q}{2}+(1+q)\epsilon}), \quad n \rightarrow \infty, \quad (3.3)$$

and thus

$$\begin{aligned} \prod_{1 \leq j < k \leq n} |\cos \frac{\theta_j - \theta_k}{2}|^\beta \prod_{j=1}^n e^{\frac{t}{\sqrt{n}}g(\theta_j)} &= e^{t\sqrt{n}g(\theta_n)} \exp\left(-\frac{\beta}{8} \sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^2 \right. \\ &\quad \left. - \frac{\beta}{192} \sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^4 + \frac{tg'(\theta_n)}{\sqrt{n}} \sum_{j=1}^{n-1} (\theta_j - \theta_n) + \mathcal{O}(n^{-\frac{q}{2}+(1+q)\epsilon} + n^{-1+6\epsilon})\right), \end{aligned}$$

as $n \rightarrow \infty$, uniformly for $\theta_n \in (-\pi, \pi]$, for $\theta' \in \theta_n + U_{n-1}(n^{-\frac{1}{2}+\epsilon})$, and for t in compact subsets of $\mathbb{R}$. Because the integrand is positive, we then find

$$\begin{aligned} I_1 &= 2^{\beta \frac{n(n-1)}{2}} \exp\left(\mathcal{O}(n^{-\frac{q}{2}+(1+q)\epsilon} + n^{-1+6\epsilon})\right) \int_{-\pi}^{\pi} e^{t\sqrt{n}g(\theta_n)} I_1'(\theta_n) d\theta_n, \\ I_1'(\theta_n) &:= \int_{\theta_n + U_{n-1}(n^{-\frac{1}{2}+\epsilon})} \exp\left(-\frac{\beta}{8} \sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^2 - \frac{\beta}{192} \sum_{1 \leq j < k \leq n} (\theta_j - \theta_k)^4 \right. \end{aligned} \quad (3.4)$$

$$+ \frac{tg'(\theta_n)}{\sqrt{n}} \sum_{j=1}^{n-1} (\theta_j - \theta_n) \prod_{j=1}^{n-1} d\theta_j,$$

as $n \rightarrow \infty$ uniformly for t in compact subsets of $\mathbb{R}$. Using Lemma 2.1 with $a = \frac{\beta}{8}$, $b = -\frac{\beta}{192}$ and $c = tg'(\theta_n)$, we obtain

$$I'_1(\theta_n) = n^{\frac{1}{2}} \left(\frac{8\pi}{\beta n} \right)^{\frac{n-1}{2}} \exp \left(-\frac{1}{2\beta} + \frac{2t^2 g'(\theta_n)^2}{\beta} + \mathcal{O}(n^{-1+8\epsilon}) \right). \quad (3.5)$$

Substituting the above in (3.4) yields

$$\begin{aligned} I_1 &= 2^{\beta \frac{n(n-1)}{2}} \left(\frac{8\pi}{\beta n} \right)^{\frac{n-1}{2}} n^{\frac{1}{2}} e^{-\frac{1}{2\beta}} \exp \left(\mathcal{O}(n^{-\frac{q}{2}+(1+q)\epsilon} + n^{-1+8\epsilon}) \right) \\ &\times \int_{-\pi}^{\pi} e^{t\sqrt{n}g(\theta_n) + \frac{2g'(\theta_n)^2}{\beta} t^2} d\theta_n, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.6)$$

The rest of the proof consists in showing that $I(\frac{t}{\sqrt{n}}g) - I_1$ is negligible in comparison to I_1 . Let $\tau = \pi/8$, and for $j \in \{-15, -14, \dots, 16\}$, define $A_j = ((j-1)\frac{\tau}{2}, j\frac{\tau}{2}]$. For any $\theta \in (-\pi, \pi]^n$, at least one of the 16 intervals $A_{16} \cup A_{-15}$, $A_{-14} \cup A_{-13}$, $\dots$, $A_0 \cup A_1, \dots, A_{14} \cup A_{15}$ contains $\geq n/16$ of the θ_j . Thus

$$I(\frac{t}{\sqrt{n}}g) - I_1 \leq 16\tilde{I}, \quad \tilde{I} := 2^{\beta \frac{n(n-1)}{2}} e^{\sqrt{n}M(tg)} \int_{\mathcal{J}} \prod_{1 \leq j < k \leq n} |\cos \frac{\theta_j - \theta_k}{2}|^{\beta} \prod_{j=1}^n d\theta_j, \quad (3.7)$$

where $M(tg) = \max_{\theta \in (-\pi, \pi]} tg(\theta)$ and

$$\mathcal{J} = \{\theta \in (-\pi, \pi]^n : \theta' \notin \theta_n + U_{n-1}(n^{-\frac{1}{2}+\epsilon}) \text{ and } \#\{\theta_j \in A_0 \cup A_1\} \geq \frac{n}{16}\}. \quad (3.8)$$

Define $S'_0 = S'_0(\theta)$, $S'_1 = S'_1(\theta)$ and $S'_2 = S'_2(\theta)$ by

$$S'_0 = \{j : |\theta_j| \leq \frac{\tau}{2}\}, \quad S'_1 = \{j : \frac{\tau}{2} < |\theta_j| \leq \tau\}, \quad S'_2 = \{j : \tau < |\theta_j| \leq \pi\},$$

and let $s'_2 = \#S'_2$. If $j \in S'_2$ and $k \in S'_0$, then $|\cos \frac{\theta_j - \theta_k}{2}| \leq \cos(\tau/4)$. Thus the contribution to $\tilde{I}$ of all the cases where $s'_2 \geq n^\epsilon$ is at most

$$2^{\beta \frac{n(n-1)}{2}} e^{\sqrt{n}M(tg)} (\cos \frac{\tau}{4})^{\frac{\beta}{16} n^{1+\epsilon}} (2\pi)^n \leq \exp(-c_1 n^{1+\epsilon}) I_1 \quad (3.9)$$

for some $c_1 > 0$ and for all sufficiently large n . Let us now define $S_0 = S_0(\theta)$, $S_1 = S_1(\theta)$ and $S_2 = S_2(\theta)$ by

$$S_0 = \{j : |\theta_j| \leq \tau\}, \quad S_1 = \{j : \tau < |\theta_j| \leq 2\tau\}, \quad S_2 = \{j : 2\tau < |\theta_j| \leq \pi\},$$

and let $s_0 = \#S_0$, $s_1 = \#S_1$ and $s_2 = \#S_2$. The case $s'_2 = s_1 + s_2 \geq n^\epsilon$ is already handled by (3.9), and we now focus on the case $s'_2 = s_1 + s_2 < n^\epsilon$. Let $I_2(m_2)$ be the contribution to $\tilde{I}$ of those θ such that $s_2(\theta) = m_2$ and $s_1(\theta) + s_2(\theta) < n^\epsilon$. For $\theta \in (-\pi, \pi]^n$, we have

$$|\cos \frac{\theta_j - \theta_k}{2}|^{\beta} \leq \begin{cases} \exp(-\frac{\beta}{8}(\theta_j - \theta_k)^2), & \text{if } j, k \in S_0 \cup S_1, \\ (\cos \frac{\tau}{2})^{\beta}, & \text{if } j \in S_0, k \in S_2, \\ 1, & \text{if } j, k \in S_2, \end{cases} \quad (3.10)$$

where we have used the fact that $|\cos \frac{x}{2}| \leq \exp(-\frac{x^2}{8})$ holds for all $|x| \leq \pi$. Thus

$$\prod_{1 \leq j < k \leq n} |\cos \frac{\theta_j - \theta_k}{2}|^\beta \leq \exp \left(-\frac{\beta}{8} \sum_{\substack{1 \leq j < k \leq n \\ j, k \in S_0 \cup S_1}} (\theta_j - \theta_k)^2 - \alpha s_0 s_2 \right)$$

with $\alpha := -\beta \log \cos \frac{\tau}{2}$, and we find

$$\begin{aligned} I_2(m_2) &\leq 2^{\beta \frac{n(n-1)}{2}} e^{\sqrt{n} M(tg)} \binom{n}{m_2} \int_{|\theta_1|, \dots, |\theta_{m_2}| \in (2\tau, \pi]} \int_{\substack{|\theta_{m_2+1}|, \dots, |\theta_n| \leq 2\tau \\ s_0(\theta) \geq n-n^\epsilon}} \prod_{1 \leq j < k \leq n} |\cos \frac{\theta_j - \theta_k}{2}|^\beta \prod_{j=1}^n d\theta_j \\ &\leq 2^{\beta \frac{n(n-1)}{2}} e^{\sqrt{n} M(tg)} e^{-\alpha m_2(n-n^\epsilon)} (2\pi - 4\tau)^{m_2} \binom{n}{m_2} I'_2(n - m_2), \end{aligned} \quad (3.11)$$

with

$$\begin{aligned} I'_2(m) &= \int_{U_m(2\tau)} \exp \left(-\frac{\beta}{8} \sum_{1 \leq j < k \leq m} (\theta_j - \theta_k)^2 \right) \prod_{j=1}^m d\theta_j \leq 4\tau I''_2(m), \\ I''_2(m) &= \int_{U_{m-1}(4\tau)} \exp \left(-\frac{\beta}{8} \sum_{1 \leq j < k \leq m} (\theta_j - \theta_k)^2 \right) \prod_{j=1}^{m-1} d\theta_j, \end{aligned} \quad (3.12)$$

and where in the definition of $I''_2(m)$ we set $\theta_m := 0$. We can estimate $I''_2(m)$ using the linear transformation T of [8, Proof of Theorem 2.1]. This transformation $T : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{m-1}$ is defined by $T : (\theta_1, \dots, \theta_{m-1}) \mapsto \mathbf{y} = (y_1, \dots, y_{m-1})$, where

$$y_j = \theta_j - \sum_{k=1}^{m-1} \frac{\theta_k}{m + m^{1/2}}, \quad 1 \leq j \leq m-1.$$

Let $\mu_k := \sum_{j=1}^{m-1} y_j^k$ for $k \geq 1$ and $V_1 := T(U_{m-1}(4\tau))$. The following identities hold:

$$V_1 = \{\mathbf{y} \in \mathbb{R}^{m-1} : |y_j + \mu_1/(m^{1/2} + 1)| \leq 4\tau \text{ for } 1 \leq j \leq m-1\}, \quad (3.13)$$

$$\mu_1 = m^{-1/2} \sum_{j=1}^{m-1} \theta_j, \quad (3.14)$$

$$\sum_{1 \leq j < k \leq m} (\theta_j - \theta_k)^2 = m\mu_2, \quad (3.15)$$

$$(\det T)^{-1} = m^{1/2}. \quad (3.16)$$

Hence, by (3.15) and (3.16), and since the integrand of $I''_2(m)$ is positive,

$$I''_2(m) = \sqrt{m} \int_{V_1} \exp \left(-\frac{\beta}{8} m\mu_2 \right) \prod_{j=1}^{m-1} dy_j \leq \sqrt{m} \int_{\mathbb{R}^{m-1}} \exp \left(-\frac{\beta}{8} m\mu_2 \right) \prod_{j=1}^{m-1} dy_j \leq \sqrt{m} \left(\frac{8\pi}{\beta m} \right)^{\frac{m-1}{2}}.$$

Substituting the above in (3.12) and (3.11) yields $I'_2(m) \leq 4\tau \sqrt{m} \left(\frac{8\pi}{\beta m} \right)^{\frac{m-1}{2}}$ and

$$I'_2(n - m_2) \leq e^{\mathcal{O}(m_2 \log n)} \left(\frac{8\pi}{\beta n} \right)^{\frac{n-1}{2}},$$

$$I_2(m_2) \leq 2^{\beta \frac{n(n-1)}{2}} e^{\sqrt{n}M(tg)} e^{-\alpha m_2(n-n^\epsilon)} (2\pi - 4\tau)^{m_2} n^{m_2} e^{\mathcal{O}(m_2 \log n)} \left(\frac{8\pi}{\beta n}\right)^{\frac{n-1}{2}} \leq \exp\left(-\frac{\alpha}{2} m_2 n\right) I_1,$$

for all sufficiently large n . Hence

$$\sum_{m_2=1}^{n^\epsilon} I_2(m_2) \leq \exp\left(-c_2 n\right) I_1 \quad (3.17)$$

for some $c_2 > 0$ and all large enough n . It only remains to analyze $I_2(0)$. Let $I_3(\mathbf{h})$ be the contribution to $\tilde{I}$ of those $\boldsymbol{\theta}$ such that

- (i) $\#\{j : n^{-\frac{1}{2}+\epsilon} < |\theta_j - \theta_n| \leq 4\tau\} = \mathbf{h}$,
- (ii) $\#\{j : |\theta_j - \theta_n| \leq n^{-\frac{1}{2}+\epsilon}\} = n - \mathbf{h}$, and
- (iii) $|\theta_n| \leq 2\tau$.

By (3.7) and (3.8), and because the integrand is positive, we have $I_2(0) \leq \sum_{\mathbf{h}=1}^{n-1} I_3(\mathbf{h})$ and

$$I_3(\mathbf{h}) \leq 4\tau 2^{\beta \frac{n(n-1)}{2}} e^{\sqrt{n}M(tg)} I'_3(\mathbf{h}), \quad \text{with} \quad I'_3(\mathbf{h}) = \int_{\mathcal{J}_{\mathbf{h}}} \prod_{1 \leq j < k \leq n} |\cos \frac{\theta_j - \theta_k}{2}|^\beta \prod_{j=1}^{n-1} d\theta_j,$$

where in the definition of $I'_3(\mathbf{h})$ we set $\theta_n = 0$ and

$$\mathcal{J}_{\mathbf{h}} = \{\boldsymbol{\theta}' \in U_{n-1}(4\tau) : \#\{\theta_j : n^{-\frac{1}{2}+\epsilon} < |\theta_j| \leq 4\tau\} = \mathbf{h}\}.$$

For $\boldsymbol{\theta}' \in U_{n-1}(4\tau)$, we have $|\cos \frac{\theta_j - \theta_k}{2}|^\beta \leq \exp(-\frac{\beta}{8}(\theta_j - \theta_k)^2)$ and thus

$$I'_3(\mathbf{h}) \leq I''_3(\mathbf{h}), \quad \text{where} \quad I''_3(\mathbf{h}) = \int_{\mathcal{J}_{\mathbf{h}}} \prod_{1 \leq j < k \leq n} \exp\left(-\frac{\beta}{8}(\theta_j - \theta_k)^2\right) \prod_{j=1}^{n-1} d\theta_j.$$

Applying the transformation T defined above (but with m replaced by n), we obtain

$$I''_3(\mathbf{h}) = \sqrt{n} \int_{T(\mathcal{J}_{\mathbf{h}})} \exp\left(-\frac{\beta}{8} n \mu_2\right) \prod_{j=1}^{n-1} dy_j,$$

where now $\mu_k := \sum_{j=1}^{n-1} y_j^k$, $k \geq 1$. The set $T(\mathcal{J}_{\mathbf{h}})$ consists of all $\mathbf{y} \in \mathbb{R}^{n-1}$ such that

- (i) $n^{-\frac{1}{2}+\epsilon} < |y_j + \frac{\mu_1}{\sqrt{n+1}}| \leq 4\tau$ for $\mathbf{h}$ values of j , and
- (ii) $|y_j + \frac{\mu_1}{\sqrt{n+1}}| \leq n^{-\frac{1}{2}+\epsilon}$ for $n - 1 - \mathbf{h}$ values of j .

Since $\mathbf{h} \geq 1$, we easily conclude from (i) that any $\mathbf{y} \in T(\mathcal{J}_{\mathbf{h}})$ satisfies either $|\mu_1| > n^\epsilon/2$ or $|y_j| > n^{-\frac{1}{2}+\epsilon}/2$ for at least $\mathbf{h}$ values of j . Thus $\mu_2 > \frac{1}{4}n^{-1+2\epsilon}$ holds for all $\mathbf{y} \in T(\mathcal{J}_{\mathbf{h}})$, which implies that

$$I''_3(\mathbf{h}) \leq \sqrt{n} \int_{\mathbb{R}^{n-1} \cap \{\mathbf{y} : \mu_2 > \frac{1}{4}n^{-1+2\epsilon}\}} \exp\left(-\frac{\beta}{8} n \mu_2\right) \prod_{j=1}^{n-1} dy_j \leq \sqrt{n} \left(\frac{8\pi}{\beta n}\right)^{\frac{n-1}{2}} \exp(-c'_3 n^{2\epsilon})$$

for large n and some $c'_3 > 0$, and we find

$$I_2(0) \leq \sum_{\mathbf{h}=1}^{n-1} I_3(\mathbf{h}) \leq 2^{\beta \frac{n(n-1)}{2}} e^{\sqrt{n}M(tg)} \sqrt{n} \left(\frac{8\pi}{\beta n}\right)^{\frac{n-1}{2}} \exp(-2c_3 n^{2\epsilon}) \leq \exp(-c_3 n^{2\epsilon}) I_1, \quad (3.18)$$

for some $c_3 > 0$ and all sufficiently large n . By (3.7), (3.9), (3.17), (3.18), we have

$$I\left(\frac{t}{\sqrt{n}}g\right) - I_1 \leq \exp(-c_4 n^{2\epsilon}) I_1 \quad (3.19)$$

for some $c_4 > 0$ and all sufficiently large n . Theorem 1.3 for $g \not\equiv 0$ now follows from (3.6). From (3.2), (3.3) and (3.5), we see that if $g \equiv 0$ then $\mathcal{O}(n^{-\frac{q}{2}+(1+q)\epsilon} + n^{-1+8\epsilon})$ in (3.6) can be replaced by $\mathcal{O}(n^{-1+8\epsilon})$. This proves Theorem 1.3 for $g \equiv 0$.

Furthermore, for $g \equiv 0$, by definition of I_1 we have

$$\begin{aligned} \frac{I_1}{I(0)} &= \mathbb{P}\left(|\theta_j - \theta_n| \leq n^{-\frac{1}{2}+\epsilon} \text{ for all } j \in \{1, \dots, n-1\}\right) \\ &\leq \mathbb{P}\left(|e^{i\theta_j} - e^{i\theta_n}| \leq n^{-\frac{1}{2}+\epsilon} \text{ for all } j \in \{1, \dots, n-1\}\right). \end{aligned}$$

Hence Theorem 1.1 directly follows from $I(0) - I_1 \leq \exp(-c_4 n^{2\epsilon}) I_1$ (which is (3.19) with $g \equiv 0$).

Data availability statement. There is no data associated to this work.

Conflict of interest statement. There is no conflict of interest.

Acknowledgements. The author is grateful to Brendan McKay for useful remarks and for pointing out [6]. The author is also grateful to two anonymous referees for their careful reading and excellent remarks. Support is acknowledged from the Swedish Research Council, Grant No. 2021-04626.

References

- [1] P. Billingsley, Probability and measure. Anniversary edition. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 2012.
- [2] J. Boursier, Optimal local laws and CLT for 1D long-range Riesz gases, arXiv:2112.05881.
- [3] C. Charlier, A point process on the unit circle with mirror-type interactions, arXiv:2212.06777.
- [4] D.J. Daley and D. Vere-Jones, *An introduction to the theory of point processes*, General theory and structure. Second edition Probab. Appl, Springer, New York, 2008.
- [5] P.J. Forrester, *Log-gases and random matrices*, London Mathematical Society Monographs Series, 34. Princeton University Press, Princeton, NJ, 2010.
- [6] M. Isaev and B.D. McKay, Complex martingales and asymptotic enumeration, *Random Structures Algorithms* **52** (2018), no. 4, 617–661.
- [7] K. Johansson, On Szegő's asymptotic formula for Toeplitz determinants and generalizations, *Bull. Sci. Math.* (2) **112** (1988), no. 3, 257–304.
- [8] B.D. McKay, The asymptotic numbers of regular tournaments, Eulerian digraphs and Eulerian oriented graphs, *Combinatorica* 10 (1990), no. 4, 367–377.
- [9] B.D. McKay and N.C. Wormald, Asymptotic enumeration by degree sequence of graphs of high degree, *European J. Combin.* **11** (1990), no. 6, 565–580.